ENTANGLEMENT FROM DISSIPATIVE DYNAMICS INTO OVERLAPPING ENVIRONMENTS

Tesi di Laurea in Quantum Information

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Entanglement is synonymous of quantum correlations that cannot be explained by any local classical theory. Despite it was initially relegated to foundational issues, it is nowadays well established that entanglement represents a fundamental resource for quantum information tasks [1].

Being entanglement a purely quantum phenomenon, it was considered fragile under contamination of environment noise in open quantum systems and thus, the standard pursued approach relies in 'working against environment', i.e. avoid as much as possible such effects. However, recently it has been put forward the alternative idea of working 'with environment'. In particular, a dissipative approach to quantum information processing may lead to forms of cooperation whereby the environment enhances some coherent tasks performed on the system [2].

This alternative avenue was paved by studies showing that a common dissipative environment is able to induce entanglement, even without any interaction among subsystems [3, 4]. In this regard, in [5] it was shown that, considering an arbitrary number of qubits dissipating into the same environment, the inter-qubit entanglement can persist for long time up to stationary conditions. Moreover in [6] it has been proved that two qubits (not directly interacting), although dissipating in two different environments, can become entangled through the dissipative dynamic, if there is a third qubit that dissipates into both environments.

Here along this line, we consider a more general scenario, namely two ensembles containing arbitrary number of qubits, dissipate into overlapping environments, that is, a number of qubits are in common to both environments. The evolution of the system is then described by a Markovian master equation whose analytical solution for an arbitrary number of qubits, is found considering an initial states with a single excitation i.e. only a qubit in the excited state. This initial condition allowed us to treat the problem in a Hilbert spaces of reduced dimension. In this case, rather than inter-qubit entanglement it is worth studying the bipartite entanglement between the two ensembles by excluding the common qubits. This was carried out by using
a computable measure of entanglement called Negativity [18] and we showed that the largest amount of entanglement can be created when excitations are initially located among side (non common) qubits. Moreover, the stationary entanglement exhibits a monotonic (resp. non-monotonic) scaling versus the number of common (resp. side) qubits [7].

Finally, we partially extended the problem to the case of a non-Markovian evolution, where we proved that the dissipative dynamics is able to create quantum superposition, and it is reasonable to think that also entanglement could be generated.

The original work behind this thesis is summarized in the following papers:

• Riccardo Mengoni, Laleh Memarzadeh, Stefano Mancini, Entanglement from dissipative dynamics into overlapping environments, Phys. Rev. A 90, 062341 (2014)

• Riccardo Mengoni, Laleh Memarzadeh, Stefano Mancini, Non-Markovian dynamics of entanglement in overlapping environments, in preparation

The thesis is organized as follows. In Chapter 1 we introduce the notion of entanglement and the Negativity as its measure. In Chapter 2 we derive the master equation describing the dissipative dynamics of open systems. Then in Chapter 3 we present the model of overlapping environments and we study its Markovian evolution, distinguishing between the cases where the initial excitation is located among common and side qubits. Going on, in Chapter 4 we evaluate the amount of achievable entanglement and discuss the main results. In Chapter 5 we try to extend the problem into the non-Markovian scenario and finally conclusions are drawn in the last chapter.
Chapter 1

Quantum Entanglement

In this chapter, after a brief historical introduction, the phenomenon of Quantum Entanglement will be discussed.

At first it will be defined for pure states, where Schmidt decomposition provides a useful mathematical characterization.

Then, we will define entanglement in the case of mixed states. It will be treated the problem of separability, focusing on the PPT criterion, and finally a computable measure of entanglement called Negativity will be presented.

1.1 Historical note

Quantum entanglement is a physical phenomenon that implies the existence of global states of composite system which cannot be written as a product of the states of individual subsystems (or a convex linear combination of them).[8]

Entanglement is synonymous with strong correlations of purely quantum nature that are non-local, i.e. such that the action on a subsystem instantaneously influence another far away subsystem.

This counter intuitive feature of quantum mechanics was first discussed by Albert Einstein in 1935, in a joint paper with Boris Podolsky and Nathan Rosen [9]. In this study, they formulated the EPR paradox, a thought experiment that attempted to show that “quantum-mechanical description of physical reality given by wave functions is not complete” and hence, it must contain hidden variables.

It was Erwin Schrödinger, in a successive paper, who coined the word Entanglement to name this correlations described in the EPR experiment[10][11].

Even though Schrödinger was dissatisfied with the concept of entanglement, since it seemed to violate the speed limit on the transmission of infor-
mation (i.e. speed of light), implicit in the theory of relativity, he was the first to underline its importance: “I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.”[10][11]

However, despite the interest, only in 1964, John Stewart Bell proved that one of the key assumptions in EPR’s argument, the principle of locality, was mathematically inconsistent with the predictions of quantum theory. [12]

Finally, in the ’70 and ’80, correlations associated with the phenomenon of quantum entanglement have actually been observed in laboratories. [13] [14]

Nowadays entanglement is considered as a uniquely quantum mechanical resource and, although there is yet no complete theory of entanglement, it plays a key role in many of the most interesting applications of quantum computation and quantum information. [15]

1.2 Pure states

In this section it will be defined entanglement in the case of pure states, but first, a recall of the postulates of quantum mechanics is in order. In the following, it will be used the bra-ket notation introduced by Dirac. [16]

1.2.1 Postulates of Quantum Mechanics

Postulate 1.2.1.1 To an isolated quantum system is associated a space of states $\mathbb{H}$ which is a Hilbert space. The state of the system is then represented by a unit norm vector $|\Psi\rangle$ on such Hilbert space.

The simplest quantum system we can consider has associated the Hilbert space $\mathbb{C}^2$ and it is called quantum bit (or shortly qubit).[15]

In this space it is common to define the canonical basis as

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

hence a generic qubit state is described by a vector

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1.$ However being a state $|\Psi\rangle$ a projective ray in $\mathbb{C}^2$ we can parametrize $\alpha, \beta$ in such a way that

$$|\Psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle.$$
where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. This means that the set of states $\{ |\Psi(\theta, \varphi)\rangle \}_{\theta, \varphi}$ corresponds to the surface of a 2-sphere $S^2 \subset \mathbb{R}^3$ of radius one (known as the Bloch sphere).

Postulate 1.2.1.2 The space of states of a composite physical system is the tensor product of the states spaces (Hilbert spaces) of its subsystems.

Postulate 1.2.1.3 The state change of an isolated physical system with associated Hilbert space $\mathbb{H}$ is described by a unitary operator $U : \mathbb{H} \to \mathbb{H}$

$$|\Psi\rangle \rightarrow |\Psi'\rangle = U |\Psi\rangle$$

with $U^\dagger U = UU^\dagger = I$ being $U^\dagger$ the adjoint of $U$ and $I$ the identity operator on $\mathbb{H}$.

Time evolution of the state of a closed quantum system is described by Schrödinger equation

$$i \frac{d |\Psi\rangle}{dt} = H |\Psi\rangle$$

where $H$ is an hermitian operator called Hamiltonian, that represent the energy of the system. It is straightforward to see that the solution to Schrödinger equation is

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

and, since the operator $H$ is hermitian, the transformation $U = e^{-iHt}$ is unitary, as asserted before.

Postulate 1.2.1.4 To an observable physical quantity $A$ is associated an Hermitian operator in the space of states (Hilbert space $\mathbb{H}$), i.e. $A : \mathbb{H} \to \mathbb{H}$ such that $A = A^\dagger$. The possible measurement outcomes are the eigenvalues $\{ a_j \}_j$ ($a_j \in \mathbb{R}$) of $A$. The probability that the outcome is $a_j$, given that the system was in the state $|\Psi\rangle$, is

$$p_{\Psi}(a_j) = \langle \Psi | P_j | \Psi \rangle,$$

with $P_j = |a_j\rangle \langle a_j|$ the projector onto the subspace of eigenvector $|a_j\rangle$ of $A$ corresponding to the eigenvalue $a_j$.

\textsuperscript{1} Planck constant $\hbar$ is set to one
As a consequence of a measurement outcome \( a_j \), the state of the system changes as follows

\[
|\Psi\rangle \rightarrow |\Psi'\rangle = \frac{P_j |\Psi\rangle}{\sqrt{p_\Psi(a_j)}}. \quad (1.2.3)
\]

### 1.2.2 Entangled pure states

Let us consider two Hilbert spaces \( \mathbb{H}_A \) and \( \mathbb{H}_B \) related to the quantum systems A and B, with basis states respectively \( \{|i_A\rangle\} \), and \( \{|j_B\rangle\} \).

According to the first and second postulate of quantum mechanics, the space of states of the bipartite composite system \( \mathbb{H}_{AB} \) is given by the tensor product of its two subsystems i.e. \( \mathbb{H}_A \otimes \mathbb{H}_B \), having as basis states the set \( \{|i_A\rangle \otimes |j_B\rangle\}_{i,j} \) or in a more compact way \( \{|i_Aj_B\rangle\}_{i,j} \).

From the second postulate follows that it is possible to write any vector state of the composite system, \( |\Psi_{AB}\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B \) as

\[
|\Psi_{AB}\rangle = \sum_{i,j} c_{i,j} |i_A\rangle \otimes |j_B\rangle = \sum_{i,j} c_{i,j} |i_Aj_B\rangle \quad (1.2.4)
\]

Then we have to distinguish two possible kinds of states.

**Definitions 1.2.2**

- **Pure states** \( |\Psi_{AB}\rangle \in \mathbb{H}_{AB} \) that can be written as \( |\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \) with \( |\Psi_A\rangle \in \mathbb{H}_A \) and \( |\Psi_B\rangle \in \mathbb{H}_B \) are called factorable (or product) states.
- **Those states** \( |\Psi_{AB}\rangle \in \mathbb{H}_{AB} \) that are not factorable, i.e. \( |\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle \) for any \( |\Psi_A\rangle \in \mathbb{H}_A \) and \( |\Psi_B\rangle \in \mathbb{H}_B \), are called entangled.

Notice that for factorable states it is assigned a precise state to each subsystem A and B (besides the state of the global system AB).

This is no longer true for entangled states for which the state is precisely assigned only to the global system AB.

Hence, if the two systems are entangled, there is no way to characterize one without referring to the other.
1.2.3 Schmidt decomposition

The Schmidt decomposition is a useful mathematical description that gives an insight on entanglement.[8]

Let us consider a generic state \( |\Psi_{AB}\rangle \in \mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B \) as in (1.2.2) and put \( \dim \mathbb{H}_A = d_A \) and \( \dim \mathbb{H}_B = d_B \), then we can write

\[
|\Psi_{AB}\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |i_A\rangle |j_B\rangle.
\] (1.2.5)

The coefficients \( c_{ij} \in \mathbb{C} \) are the so called probability amplitudes, they are such that \( \sum_{i,j} |c_{ij}|^2 = 1 \) and can be rearranged in a matrix form as follows

\[
C = \begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1d_B} \\
c_{21} & c_{22} & \cdots & c_{2d_B} \\
\vdots & \vdots & \ddots & \vdots \\
c_{d_A1} & c_{d_A2} & \cdots & c_{d_Ad_B}
\end{pmatrix}.
\]

For a generic matrix with complex entries we can always write down the Singular Values Decomposition [19]. If we apply this decomposition to \( C \), we obtain

\[
C = UDV
\] (1.2.6)

where

- \( U \) is a \( d_A \times d_A \) unitary matrix,
- \( V \) is a \( d_B \times d_B \) unitary matrix,
- \( D \) is a \( d_A \times d_B \) diagonal matrix with non-negative real numbers on the principal diagonal. The diagonal entries, \( \sqrt{\lambda_a} \), of \( D \) are known as the singular values of \( C \).

Equation (1.2.4) for the components \( c_{ij} \) reads

\[
c_{ij} = \min\{d_A,d_B\} \sum_{a=1}^{\min\{d_A,d_B\}} U_{ia} \sqrt{\lambda_a} V_{aj}
\] (1.2.7)

and we can use this result to rewrite the global state of the composite system \( |\Psi_{AB}\rangle \) as follows

\[
|\Psi_{AB}\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \sum_{a=1}^{\min\{d_A,d_B\}} U_{ia} \sqrt{\lambda_a} V_{aj} |i_A\rangle |j_B\rangle.
\]
Hence we have

\[ |\Psi_{AB}\rangle = \min\{d_A,d_B\} \sum_{a=1}^{d_A} \sqrt{\lambda_a} \left( \sum_{i=1}^{d_A} U_{ia} |i_A\rangle \right) \left( \sum_{j=1}^{d_B} V_{aj} |j_B\rangle \right) \]

where the vectors \( \sum_{i=1}^{d_A} U_{ia} |i_A\rangle = |u_a\rangle_A \) and \( \sum_{j=1}^{d_B} V_{aj} |j_B\rangle = |v_a\rangle_B \) form two new basis, respectively for \( \mathbb{H}_A \) and \( \mathbb{H}_B \).

Finally we can write the Schmidt decomposition of a pure bi-partite quantum state:

\[ |\Psi_{AB}\rangle = \sum_{a=1}^{\min\{d_A,d_B\}} \sqrt{\lambda_a} |u_a\rangle_A |v_a\rangle_B \tag{1.2.8} \]

where the coefficients \( \sqrt{\lambda_a} \), called the Schmidt coefficients, are such that \( \sum_a \lambda_a = 1 \) and they are also eigenvalues of \( C^\dagger C \).

**Definition (Schmidt rank)** The Schmidt rank is equal to the number of non-zero Schmidt coefficients (i.e. equals the rank of the matrix \( C \)).

The Schmidt decomposition as it was defined for the state \( |\Psi_{AB}\rangle \) of our bipartite system, give us a mathematical characterization of the entanglement:

- \( |\Psi_{AB}\rangle \) is a factorable state if and only if has Schmidt rank equal to one
- \( |\Psi_{AB}\rangle \) is an entangled state if and only if it has Schmidt rank strictly greater than 1

It is worth noting that the maximum Schmidt rank equals \( d = \min\{d_A,d_B\} \).

**Definition (Maximally entangled state)** States \( |\Psi_{AB}\rangle \) having maximum Schmidt rank and Schmidt coefficients all equal to \( 1/d \) are called maximally entangled states:

\[ |\Psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{a=1}^{d} |u_a\rangle_A |v_a\rangle_B . \tag{1.2.9} \]

**Example.** For bipartite systems, the Hilbert space \( \mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \) is spanned by the four Bell’s maximally entangled basis states

\[ |\psi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle \pm |1\rangle |0\rangle) \quad |\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle \pm |1\rangle |1\rangle) . \tag{1.2.10} \]
These states have remarkable properties. Suppose that the bipartite system is in the state $|\psi^+\rangle_{AB}$ and that we perform a measurement of the observable $Z_A = |0\rangle_A \langle 0| - |1\rangle_A \langle 1|$ on the subsystem $A$.

The outcomes of the measurement will be $+1$ and $-1$, with probabilities $Pr(+1) = Pr(-1) = \frac{1}{2}$ and a posteriori states $|0\rangle_A$ and $|1\rangle_A$ respectively.

This measurement affects instantaneously the state of the subsystem $B$, that will collapse into state $|0\rangle_B$ (if we got $-1$ as outcome) or $|1\rangle_B$ (if, instead, we got $+1$) depending on the outcome of the measurement of the observable $Z_A$ on the subsystem $A$ and on the particular Bell state of the whole system.

Thus, the result of the measurements for both subsystems are perfectly correlated: we know nothing at all about the subsystems, although we have maximal knowledge of the whole system because the state is pure.

### 1.3 Mixed states

Suppose we want to describe a quantum system that is in a statistical mixture, i.e. can be in one of a series of states $|\psi_i\rangle$ (not necessarily orthogonal), with corresponding probabilities $p_i$ i.e. $p_i \geq 0$ such that $\sum_i p_i = 1$.

**Definition (Density Operator)** Given a statistical mixture (ensemble) of quantum states $\{p_i, |\psi_i\rangle\}$ on an Hilbert space $\mathbb{H}$, the corresponding density operator is defined as

$$
\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|.
$$

If we denote with $\mathcal{L}(\mathbb{H})$ the space of linear operators acting on the Hilbert space $\mathbb{H}$, then the class of density operators are characterized by the following theorem

**Theorem 1.3.1** An operator $\rho \in \mathcal{L}(\mathbb{H})$ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if satisfies the conditions

1) $Tr(\rho) = 1$;

2) $\forall |\gamma\rangle \in \mathbb{H}$, $\langle \gamma | \rho |\gamma\rangle \geq 0$.

The proof of this theorem is straightforward.
In order to describe subsystems of a composite quantum system, the reduced density operator is introduced.

**Definition (Reduced Density Operator)** Given a bipartite system $AB$ with associated Hilbert space $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$ in the state $\rho_{AB}$, the state of the subsystem $A$ is

$$\rho_A = Tr_B(\rho_{AB}).$$

Where $Tr_B$ stands for partial trace over $B$, i.e.

$$Tr_B(\rho_{AB}) = \sum_{j=1}^{\dim \mathbb{H}_B} \langle j_B | \rho_{AB} | j_B \rangle$$

($|j_B\rangle$ is an orthonormal basis of $\mathbb{H}_B$).

It is worth noticing that for a maximally entangled pure state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{a=1}^{d} |u_a\rangle_A |v_a\rangle_B,$$

where $d = \min\{d_A, d_B\}$, the reduced density operator of subsystem $A$ is

$$\rho_A = Tr_B(|\Psi_{AB}\rangle \langle \Psi_{AB}|) = I_A/d.$$  

(similarly for subsystem $B$ we find $\rho_B = I_B/d$).

Hence, the reduced density operator of a maximally entangled state is called maximally mixed state i.e. a mixed state describing an ensemble of orthogonal states spanning $\mathbb{H}_A$, occupied with a flat probability distribution.
1.3.1 Postulates of Quantum Mechanics revisited

It is possible to reformulate the Postulates of Quantum Mechanics using the density operator formalism as follows.

**Postulate 1.3.2.1** To an isolated quantum system is associated a Hilbert space $\mathbb{H}$. The state of the system is then represented by a density operator $\rho$ on such Hilbert space.

**Postulate 1.3.2.2** The space of states of a composite physical system is the tensor product of the states spaces (Hilbert spaces) of its subsystems.

**Postulate 1.3.2.3** The state change of an isolated physical system with associated Hilbert space $\mathbb{H}$ is described by a unitary operator $U : \mathbb{H} \rightarrow \mathbb{H}$

$$\rho \rightarrow \rho' = U\rho U^\dagger$$

with $U^\dagger U = UU^\dagger = I$, being $U^\dagger$ the adjoint of $U$ and $I$ the identity operator on $\mathbb{H}$.

Differentiating this equation with respect to time and recalling that the unitary operator $U = e^{-iHt}$, we immediately get an equation of motion for the density operator,

$$\frac{\partial \rho}{\partial t} = -i[H, \rho]$$

(1.3.2)

which is often referred to as the Liouville equation

**Postulate 1.3.2.4** A measurement process on $\mathbb{H}$ is described by a set of operators $\{M_m\} \subset \mathcal{L}(\mathbb{H})$ such that $M_m^\dagger M_m \geq 0$ and $\sum_m M_m^\dagger M_m = I$.

The possible measurement outcomes are labelled by the index $m$ and occur with probability

$$Pr(m) = Tr(M_m^\dagger \rho M_m).$$

As consequence of a measurement outcome $m$ the state of the system changes as follows

$$\rho \rightarrow \rho' = \frac{M_m^\dagger \rho M_m}{\sqrt{Pr(m)}}.$$  

(1.3.3)
1.3.2 Entangled mixed states

Let us suppose that a mixed state of a bipartite composite system is described by the density operator $\rho_{AB}$ that, for the second postulate of quantum mechanics, acts on the Hilbert space $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$.

**Definitions 1.3.3**

- States $\rho_{AB}$ that can be written as the tensor product of states $\rho_A$ on $\mathbb{H}_A$ and $\rho_B$ on $\mathbb{H}_B$, i.e. $\rho_{AB} = \rho_A \otimes \rho_B$ are called factorable (or product) states.

- States $\rho_{AB}$ that can be written as convex combination of states $\rho^k_A$ on $\mathbb{H}_A$ and $\rho^k_B$ on $\mathbb{H}_B$, i.e. $\rho_{AB} = \sum_k p_k \rho^k_A \otimes \rho^k_B$ are called separable states (coefficients $p_k$ are such that $p_k \geq 0$ and $\sum_k p_k = 1$). Obviously, separable states include as particular case factorable states.

- Those states that are not separable, i.e. cannot be expressed as a convex combination of any state $\rho^k_A$ on $\mathbb{H}_A$ and $\rho^k_B$ on $\mathbb{H}_B$

$$\rho_{AB} \neq \sum_k p_k \rho^k_A \otimes \rho^k_B$$  \hspace{1cm} (1.3.4)

are called entangled states.

In practice it is hard to decide if a given states is separable or entangled basing on the definition itself. Therefore one of the fundamental problems concerning entanglement is the so called separability problem, that is, given a state bipartite, determine whether it is entangled or separable.

### 1.4 Separability problem: PPT criterion

In general the problem of separability of mixed states appears to be extremely complex. Some operational criteria are known, but they work only in special cases. Here we will focus on a criterion for the separability of a bipartite system that is called the Positive Partial Transpose (PPT) criterion. [17]

**Definition (Partial transpose)** Consider a density operator $\rho_{AB}$ acting on $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$. The partial transpose $\rho^T_A$ of $\rho$ with respect to the subsystem $A$, is the hermitian, trace-normalized operator defined to have matrix elements

$$\langle i_A, j_B | \rho^T_A | k_A, l_B \rangle = \langle k_A, j_B | \rho | i_A, l_B \rangle$$  \hspace{1cm} (1.4.1)

for a fixed orthonormal basis $|i_A, j_B \rangle = |i_A \rangle \otimes |j_B \rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$. 


Theorem (PPT Criterion) Given a density operator $\rho_{AB}$ on $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$, a necessary condition for its separability is that

$$\rho_{AB}^{T_A} \geq 0.$$ 

Proof. By hypothesis, if $\rho_{AB}$ is separable, it can be written as

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k.$$ 

In this case, the effect of the partial transposition is trivial:

$$\rho_{AB}^{T_A} = \sum_k p_k (\rho_A^k)^T \otimes \rho_B^k.$$ 

where $(\rho_A^k)^T$ denotes the transpose of $\rho_A^k$.

Since the transposition map preserves eigenvalues and $\rho_A^k$ is positive by definition, also $(\rho_A^k)^T$ will be positive. Hence the spectrum of $\rho_{AB}^{T_A}$ is the same as the spectrum of $\rho_{AB}$ and, in particular, $\rho_{AB}^{T_A}$ must still be positive semidefinite.

This proves the necessity of the PPT criterion. ■

It is possible to prove that the criterion of positive partial transpose is necessary and sufficient for a system of two qubits i.e. Hilbert space $\mathbb{H}_{AB}$ isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^2$ [8] [17].

In other words, PPT criterion states that if $\rho_{AB}^{T_A}$ has at least one negative eigenvalue, then the state $\rho_{AB}$ will surely be entangled.

The result is independent of which part is transposed since $\rho_{AB}^{T_A} = (\rho_{AB}^{T_B})^T$.

1.5 A measure of Entanglement

As we have seen in section (1.3), with the example of Bell maximally entangled states, not only it is worth asking whether a certain state is entangled or not, but also what is its degree of entanglement.

While for pure states the Schmidt rank can be used to this purpose, a general measure of entanglement should be introduced in the context of mixed states.

In this section, Negativity will be discussed, it is a measure of entanglement that can be computed effectively for any mixed state of a bipartite system having arbitrary dimension. [18]
CHAPTER 1. QUANTUM ENTANGLEMENT

1.5.1 Negativity

Negativity $N$ is based on the trace norm of the partial transpose $\rho^{T_A}$ of the bipartite mixed state $\rho_{AB}$.

It essentially measures the degree to which $\rho^{T_A}$ fails to be positive, and therefore it can be regarded as a quantitative version of the PPT criterion for separability introduced before.

Consider a generic state $\rho$ of a bipartite system with finite dimensional Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$ and recall that $\rho^{T_A}$ denotes the partial transpose of $\rho$ with respect to the subsystem A.

The trace norm of any hermitian operator $A$ is defined as

$$\|A\|_1 = Tr \sqrt{A^\dagger A},$$

which is equal to the sum of the absolute values of the eigenvalues of $A$, when $A$ is hermitian. For density matrices all eigenvalues are positive and thus $\|\rho\|_1 = Tr(\rho) = 1$.

The partial transpose $\rho^{T_A}$ also satisfies $Tr(\rho^{T_A}) = 1$, but since it may have negative eigenvalues $\mu_i < 0$, its trace norm reads in general

$$\|\rho^{T_A}\|_1 = 1 + 2 \left| \sum \mu_i \right|.$$  \hspace{1cm} (1.5.2)

Definition (Negativity) The Negativity $N(\rho)$ of $\rho$ over $\mathbb{H}_A \otimes \mathbb{H}_B$ is the absolute value of the sum of the negative eigenvalues of $\rho^{T_A}$, that is $N(\rho) = | \sum \mu_i |$, from which

$$N(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{2}.$$  \hspace{1cm} (1.5.3)

In other words, Negativity measures by how much the partial transpose $\rho^{T_A}$ fails to be positive definite.
1.5.2 Properties of Negativity

Negativity provides a useful tool to quantify entanglement and in order to be considered a reliable measure, some conditions must hold true:

I) $N(\rho) \geq 0$, where the equality holds if and only if the state is separable.\(^2\)

II) $N(\rho)$ is maximum for Bell’s states and maximally entangled states in general.

III) $N(\rho)$ is a convex function, i.e.

$$N(\sum_i p_i \rho_i) \leq \sum_i p_i N(\rho_i) \quad (1.5.4)$$

whenever the $\rho_i$ are hermitian and $p_i \geq 0$ with $\sum_1 p_i = 1$.

IV) $N(\rho)$ is an entanglement monotone, which means that it does not increase under local transformations e.g. $U_A \otimes U_B$

\(^2\)This can be easily checked by noticing that the partial transpose of a separable state has all non-negative eigenvalues, that is $\|\rho_s^{T_A}\|_1 = 1$, which means, as it is clear from the definition (1.5.3), that $N(\rho) = 0$. 

Chapter 2

Open Systems Dynamics

An open quantum system is a quantum-mechanical system which interacts with an external quantum system called the environment.

In contrast to the case of a closed systems, the quantum dynamics of an open system cannot, in general, be represented in terms of a unitary time evolution.

In many cases it turns out to be useful to formulate, instead, the dynamics of an open system by means of an appropriate equation of motion for its density operator, a quantum master equation.

In this chapter, a derivation of the master equation will be given in the case of Markovian dissipative dynamics of the open quantum system.\[20\] [23]

2.1 Derivation of the Master Equation

Consider a quantum system $S$ and an environment, also called bath (or reservoir), denoted with $B$, each possessing a Hamiltonian written in terms of operators acting only on the respective subspace of the total Hilbert space.

These Hamiltonians govern the internal dynamics of the uncoupled system and bath. Consider also a third Hamiltonian that contains products of system and bath operators, thus coupling the system and bath.

The most general form of the total Hamiltonian is

$$H_{\text{tot}} = H_{\text{sys}} + H_{B} + H_{\text{int}} \quad (2.1.1)$$

where the constituent Hamiltonians describe respectively the system, the bath and their interaction.
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The total density operator for the system and the reservoir\(^1\), \(\rho_{tot}^S\) satisfies the equation of motion (1.3.2)

\[
\frac{d\rho_S^S}{dt} = -i[H_{sys} + H_B + H_{int}, \rho_S^S].
\]  
(2.1.2)

We want an equation for a density operator \(\rho_{sys}^S\) which describes only the quantum system and this is, of course, given by the reduced density operator

\[
\rho_{sys}^S(t) = Tr_B\{\rho_{tot}^S(t)\}. 
\]  
(2.1.3)

a) Transformation to interaction picture

It is more convenient to transform the equation of motion into the Dirac or interaction picture\(^2\), defined by setting

\[
\rho_{tot}(t) = U\rho_{tot}^S(t)U^\dagger
\]  
(2.1.4)

where

\[
U = e^{i(H_{sys}+H_B)t}. 
\]  
(2.1.5)

It is straightforward to show that the total density operator in the Interaction picture \(\rho_{tot}(t)\) obeys the following equation of motion

\[
\frac{d\rho_{tot}(t)}{dt} = -i[H_{int}(t), \rho_{tot}(t)]
\]  
(2.1.6)

where \(H_{int}(t) = UH_{int}U^\dagger\). Notice that, applying the partial trace over the bath, we get the density operator describing the system, in Dirac picture

\[
\rho_{sys}(t) = Tr_B\{\rho_{tot}(t)\}.
\]  
(2.1.7)

\(^1\)Subscript \(S\) applied to a \(\rho\) means that the density operator is expressed in Shrödinger picture; when there is no subscript on the density operators, it is intended to be in Interaction picture.

\(^2\)Given the Hamiltonian \(H = H_0 + V\) (where \(H_0\) is the free part and \(V\) is the interaction) and a generic operator \(M\), expressed in Shrödinger picture, then the corresponding operator in Interaction picture is defined by the unitary transformation \(M(t) = UMU^\dagger\) where \(U = e^{iH_0t}\).
b) Initial conditions

Let us assume that the system and the reservoir are initially independent, i.e. are uncorrelated, so that the total density operator at time $t = 0$ can be written as a tensor product

$$\rho_{\text{tot}}(0) = \rho_{\text{sys}}(0) \otimes \rho_B(0)$$

where $\rho_B$ stands for bath density operator.

In addition, we assume that the reservoir is so large that its statistical properties are unaffected by the weak coupling to the system [20], so that we can actually write

$$\rho_B(0) = \rho_B(t) = \rho_B$$

and hence

$$\rho_{\text{tot}}(0) = \rho_{\text{sys}}(0) \otimes \rho_B.$$ (2.1.8)

c) Integration of the equations of motion

Integrating equation (2.1.6) from 0 to $t$ we obtain

$$\rho_{\text{tot}}(t) = \rho_{\text{tot}}(0) - i \int_0^t dt' \left[ H_{\text{int}}(t'), \rho_{\text{tot}}(t') \right]$$ (2.1.9)

that substituted again in (2.1.6) gives the following integro–differential equation

$$\dot{\rho}_{\text{tot}}(t) = -i[H_{\text{int}}(t), \rho_{\text{tot}}(0)] - \int_0^t dt' \left[ H_{\text{int}}(t)[H_{\text{int}}(t'), \rho_{\text{tot}}(t')] \right].$$ (2.1.10)

d) Trace over the bath variables

If we trace both sides of the previous equation over the bath variables, taking into account (2.1.7), we obtain

$$\dot{\rho}_{\text{sys}}(t) = -\int_0^t dt' \, Tr_B \{[H_{\text{int}}(t)[H_{\text{int}}(t'), \rho_{\text{tot}}(t')]]\}$$ (2.1.11)

where we have assumed,

$$Tr_B \{H_{\text{int}}(t)\rho_{\text{tot}}(0)\} = 0$$
and used $\rho_{\text{tot}}(0) = \rho_{S}^{\text{tot}}(0)$ as in (2.1.8).

This means that the interaction has no diagonal elements in the representation in which $H_B$ is diagonal. In practice this is not a great restriction, since we can always redefine $H_{\text{sys}}$ and $H_{\text{int}}$ in order to include such diagonal elements in $H_{\text{sys}}$.

e) Weak Coupling: Born Approximation

Let us now assume weak coupling between the system and the bath, i.e. we assume that the environment is sufficiently large compared to the system to allow that the coupling with the latter does not alter its eigenstates. If this is the case, we might be justified in replacing $\rho_{\text{tot}}(t')$ by a factorised approximation\(^3\)

$$\rho_{\text{tot}}(t') \approx \rho_{\text{sys}}(t') \otimes \rho_B,$$

(2.1.12)

where we have allowed the system density operator to change significantly, by the fact that the system is much smaller than the bath, and hence the fractional effect of the interaction is much bigger on the system.

Simplifying equation (2.1.11) using (2.1.12) we get

$$\frac{d\rho_{\text{sys}}(t)}{dt} = -\int_{0}^{t} dt' \text{Tr}_B\{[H_{\text{int}}(t)[H_{\text{int}}(t'), \rho_{\text{sys}}(t') \otimes \rho_B]\}.$$  

(2.1.13)

This is the most general form of the Master Equation.

However, such an integro-differential equation is very difficult to handle as the dynamics at time $t$ depends on the state of the system in all previous times.

\(^3\)Actually, a much weaker assumption is sufficient: since in practice the interaction can be written as a sum of term like $X_{\text{sys}}A_B$, made of simple operators acting respectively only on the system and bath Hilbert spaces, all we need is an approximation like

$$\text{Tr}_B\{[A_B(t)[A_B(t'), \rho_{\text{tot}}(t')]] = \rho_{\text{sys}}(t') \otimes \text{Tr}_B\{[A_B(t)[A_B(t'), \rho_B]]\}.$$  

Essentially, this means that the correlation functions for the bath operators (that will be defined later) are not significantly changed by interaction.
2.2 Markov Approximation

In equation (2.1.13), it is clear that the operator $\rho_{\text{sys}}(t)$, which determines the statistical properties of the system of interest, depends on its past history. In fact it depends on $\rho_{\text{sys}}(t')$, with $0 < t' < t$, as we can see from the right side of equation (2.1.13).

Let us apply the Markov approximation in order to turn (2.1.13) into a differential equation. This is done by making an assumption on the bath correlation functions.[20]

Definition (Time Correlation Function) The time correlation function for a generic reservoir operator $A_B(t)$ is defined as

$$G_B(t, t') = \text{Tr}_B \{ A_B(t) A_B(t') \rho_B \}.$$ 

It tells us how the value of a microscopic variable at a given position and time $t$, represented by the operator $A_B(t)$, influences the value of the same microscopic variable at a later time, $t'$ (and at the same position).

We can assume that, since the interaction is weak, the rate of change of the interaction picture density operator $\rho_{\text{sys}}(t)$ will be quite slow compared to that of the bath operators, such as $A_B(t)$ and $A_B(t')$.

This means that we can assume the correlation function $G_B(t, t')$ to be picked about $t = t'$ so that the factor $\rho(t')$ in equation (2.1.13) changes insignificantly over the time taken for the correlation functions to vanish.

In this case we can do the following

- $\rho_{\text{sys}}(t') \longrightarrow \rho_{\text{sys}}(t)$

  This approximation is known as the Markov approximation and it yields to a first order differential equation for $\rho_{\text{sys}}$. This means that the knowledge of the system at a given point in time $t = t_0$ is sufficient to determine the state of the system for all $t > t_0$ or, in other words, the evolution of the system doesn’t take into account memory effects.

- We may replace the upper bound of the integration in (2.1.13) by infinity ($t \rightarrow \infty$), since these memory effects are short-lived and therefore the integrand decays very quickly for $t$ much greater than characteristic time-scale over which correlations in the bath decay.
We finally obtain the formula for a Markovian Master equation:

\[
\frac{d\rho_{\text{sys}}(t)}{dt} = -\int_0^\infty dt' \text{Tr}_B \{ [H_{\text{int}}(t) [H_{\text{int}}(t'), \rho_{\text{sys}}(t) \otimes \rho_B] \}. \tag{2.2.1}
\]

It's worth noticing that this equation is quite general, since it makes no restriction on the precise kind of bath or indeed on the bath state.

### 2.3 Dissipative Master Equation

In order to study the dissipative dynamics of the system, we restrict to the case of a thermal bath.\cite{20}

Let us suppose that the reservoir is in a thermal state defined by the density operator \( \rho_B \), such that

\[
\rho_B = \frac{e^{-\beta H_B}}{Z},
\]

where \( Z = \text{Tr} \{ e^{-\beta H_B} \} \) and \( \beta = \frac{1}{K_B T} \),

where \( K_B \) is Boltzmann constant and \( T \) is temperature.

We can consider the bath Hamiltonian \( H_B \) to be arbitrary, and attempt to derive a master equation for a quite general interaction \( H_{\text{int}} \).

Let us assume that the Schrödinger picture \( H_{\text{int}} \) can be written as

\[
H_{\text{int}} = \sum_m (X_m^+ \Gamma_m + X_m^- \Gamma_m^\dagger)
\]

where \( \Gamma_m \) and \( \Gamma_m^\dagger \) are respectively destruction and creation operators of the bath quanta and \( X_m^\pm \) are the operators that, together with the Hamiltonian of the system, characterize the algebra of the system, and satisfy the relation

\[
[H_{\text{sys}}, X_m^\pm] = \pm \omega_m X_m^\pm.
\]

We may note that the operators \( X_m^\pm \) play the role of ladder operators for the eigenstates of the Hamiltonian.

Substituting (2.3.2) inside the master equation (2.2.1) we obtain a number of different terms.

For simplicity, we only show here how to treat the following term

\[
- \int_0^\infty dt' \sum_{m,n} X_m^+ e^{i\omega_m t} X_n^- e^{-i\omega_n t'} \rho(t) \text{Tr}_B \{ \Gamma_m(t) \Gamma_n^\dagger(t') \rho_B \}. \tag{2.3.4}
\]
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Let us suppose that the thermal bath is in equilibrium, that is its state, as it was defined by the density operator \( \rho_B \) in (2.3.1), is a stationary state, and hence commutes with the Hamiltonian \( H_B \).

As a result of this, the general bath correlation function

\[
G_{mn}^B(t, t') = Tr_B\{\Gamma_m(t)\Gamma_n(t')\rho_B\} \tag{2.3.5}
\]

is invariant under translation in time i.e. it is a function of \( \tau = t - t' \), \( G_{mn}^B(t, t') = G_{mn}^B(\tau) \). This means that terms with \( \omega_n \neq \omega_m \) are rapidly varying functions of \( t \), and they will make many oscillations during the typical time scale over which \( \rho(t) \) alters significantly.\(^4\)

By doing this, the term (2.3.4) now reads

\[
- \sum_m X_m^+X_m^-\rho(t) \int_0^\infty d\tau e^{i\omega_m \tau} Tr_B\{\Gamma_m(\tau)\Gamma_m(0)\rho_B\}. \tag{2.3.6}
\]

Proceeding similarly with other terms we finally get

\[
\int_0^\infty d\tau e^{i\omega_m \tau} Tr_B\{\Gamma_m(\tau)\Gamma_m(0)\rho_B\} \equiv \frac{1}{2}K_m + i\delta_m \tag{2.3.7}
\]

\[
\int_0^\infty d\tau e^{-i\omega_m \tau} Tr_B\{\Gamma_m(0)\Gamma_m(\tau)\rho_B\} \equiv \frac{1}{2}K_m - i\delta_m \tag{2.3.8}
\]

\[
\int_0^\infty d\tau e^{i\omega_m \tau} Tr_B\{\Gamma_m^\dagger(\tau)\Gamma_m(0)\rho_B\} \equiv \frac{1}{2}G_m + i\epsilon_m \tag{2.3.9}
\]

\[
\int_0^\infty d\tau e^{-i\omega_m \tau} Tr_B\{\Gamma_m^\dagger(0)\Gamma_m(\tau)\rho_B\} \equiv \frac{1}{2}G_m - i\epsilon_m \tag{2.3.10}
\]

where integrals involving correlations with terms like \( \Gamma \Gamma \) and \( \Gamma^\dagger \Gamma^\dagger \) are assumed to be zero for a thermal bath in the steady state.

The master equation describing the system in Interaction picture finally takes the form\(^5\):

\[
\frac{d\rho(t)}{dt} = -i \sum_m \left[ \delta_m X_m^+X_m^- + \epsilon_m X_m^-X_m^+, \rho \right] + \frac{1}{2} \sum_m K_m(2X_m^\dagger\rho X_m^+ - \{X_m^+,X_m^-,\rho\}) \tag{2.3.11}
\]

\[
+ \frac{1}{2} \sum_m G_m(2X_m^\dagger\rho X_m^- - \{X_m^-,X_m^+,\rho\}).
\]

where \( \{,\} \) stands for the anti-commutator.

\(^4\)This is the so called Rotating Wave Approximation (RWA).

\(^5\)the subscript \( \text{sys} \) on the density operator of the system will be omitted from now on.
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Observations

1. The effect of the $\delta_m$ and $\epsilon_m$ terms is to add a small perturbation Hamiltonian term, these are called Lam\'b and Stark shift terms and are usually neglected.

2. Consequently to the assumption that the reservoir is in a thermal stationary state, and since $\Gamma_m$ are indeed destruction operators of the bath quanta, we can notice from equations (2.3.7-10) that $G_m$ should vanish at $T = 0$, while $K_m$ should not.

Thus, the terms that involve $K_m$ in (2.3.11) are present at the absolute zero, and represent transitions from upper to lower energy levels. The term involving $G_m$ instead represent transition to higher energy levels.

3. The master equation describing the dissipative evolution of a quantum system at absolute zero, can be written as

$$
\frac{d\rho(t)}{dt} = \frac{1}{2} \sum_{m} K_m(2X_m^-\rho X_m^+ - \{X_m^+, X_m^-, \rho\}).
$$

(2.3.12)

4. Referring to the algebra of the operators of the system described by relation (2.3.3), let $|0\rangle_m$ be the eigenstate of the Hamiltonian with the lowest energy and thus, such that $X_m^-|0\rangle_m = 0$.

Then the state $\rho = |0\rangle_m \langle 0|_m$ is a fixed point for the evolution (2.3.12) and all the remaining eigenstates of the Hamiltonian evolve decaying towards the state $|0\rangle_m$. This is the reason why this master equation is called dissipative.
2.4 Lindblad Master Equation

Quite generally, we can think of a master equation as a generalization of the Liouvillian equation (1.3.2) and thus be written in the following form:

\[ \frac{d\rho(t)}{dt} = \mathcal{L}(t)\rho(t). \]  

where \( \mathcal{L}(t) \) is the time-dependent Liouvillian superoperator. Clearly, this latter reduces to \( \mathcal{L}(t)\rho = [H,\rho] \) in case of purely (time-independent) Hamiltonian dynamics. The formal solution of such an equation is given in terms of the time-ordering operator

\[ \rho(t) = T \text{Exp}\left\{ \int_0^t dt' \mathcal{L}(t') \right\} \rho(0). \]  

However, for our purpose, we do not have any explicit time dependence in the Liouvillian superoperator \( \mathcal{L} \) and thus the solution reads

\[ \rho(t) = e^{t\mathcal{L}}\rho(0). \]  

Consider now the (one parameter) map \( \Lambda_t \) defined through

\[ \rho(0) \mapsto \rho(t) = \Lambda_t\rho(0) = e^{t\mathcal{L}}\rho(0). \]  

This map is (for any \( t \geq 0 \)) completely positive and trace-preserving (CPTP map) [21, 22]:

- Any extension of \( \Lambda_t \) is positive, i.e. \( (\Lambda_t \otimes I) A \) is positive for any positive operator \( A \) in \( \mathbb{H} \otimes \mathbb{H'} \) and for any \( \mathbb{H'} \).

- The map \( \Lambda_t \) preserves the trace of the operator to which it is applied.

It can be shown that, in order to satisfy the CPTP map conditions for \( \Lambda_t \), the associated Liouvillian superoperator must have the following Lindblad form [23]

\[ \frac{\partial\rho(t)}{\partial t} = \sum_i \left[ 2L_i\rho L_i^\dagger - \{L_i^\dagger L_i, \rho\} \right] = \mathcal{L}\rho \]  

where \( L_i \) are the operators of the system, called Lindblad operators.

The master equation in Lindblad form is the most general type of Markovian master equation describing a non-unitary evolution of a density operator \( \rho \).

Note that it is straightforward to see that master equation (2.3.12) satisfies the Lindblad form.
2.5 Markovian dissipative dynamics example

Consider a single qubit dissipating into a thermal bath and assume that the initial state of the qubit is

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle). \tag{2.5.1}$$

Rewriting the state of the qubit in the density operator formalism, we get

$$\rho = |\psi\rangle \langle \psi| = \frac{1}{2} (|1\rangle \langle 1| + |1\rangle \langle 0| + |0\rangle \langle 1| + |0\rangle \langle 0|), \tag{2.5.2}$$

with associated density matrix (in the canonical basis representation)

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.5.3}$$

The off diagonal terms in the previous matrix are called coherences, which are present in the case of superposition of states.

The Markovian dissipative evolution of the qubit is governed by the following Lindblad master equation

$$\frac{\partial \rho(t)}{\partial t} = D[\rho(t)] \equiv 2\sigma \rho(t) \sigma^\dagger - \{\sigma^\dagger \sigma, \rho(t)\} \tag{2.5.4}$$

where \(\{,\}\) is the anti-commutator and \(\sigma = |0\rangle \langle 1|\) and \(\sigma^\dagger = |1\rangle \langle 0|\) are respectively the lowering and rising operators related to the qubit.

The density operator describing the qubit at an arbitrary time \(t\) can be written as a linear combination of the elements of the canonical basis:

$$\rho(t) = \alpha(t) |1\rangle \langle 1| + \gamma(t) |0\rangle \langle 1| + \gamma^*(t) |1\rangle \langle 0| + \beta(t) |0\rangle \langle 0|. \tag{2.5.5}$$

Substituting (2.5.5) back into (2.5.4) we get the following set of differential equations for the coefficients

$$\dot{\alpha}(t) = -2\alpha(t),$$

$$\dot{\beta}(t) = 2\alpha(t),$$

$$\dot{\gamma}(t) = -\gamma(t). \tag{2.5.6}$$
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having as solutions

\[ \alpha(t) = \frac{e^{-2t}}{2}, \]

\[ \beta(t) = 1 - \frac{e^{-2t}}{2}, \]

\[ \gamma(t) = \frac{e^{-t}}{2}. \] (2.5.7)

As we can see, the coefficients \( \alpha(t) \) and \( \gamma(t) \) start from the initial value of \( \frac{1}{2} \) and then decrease exponentially to zero with time: the stationary state i.e. the state as \( t \) goes to infinity, will be simply the pure state \( \rho(\infty) = |0\rangle \langle 0| \).

Hence we can conclude that the Markovian dissipative dynamic causes an exponential decrease in time of the coherences (see Fig.2.1), "killing" the quantumness of the qubit state.

![Figure 2.1: Coherence \( \gamma(t) \) as a function of time \( t \).](image-url)
Chapter 3

Overlapping Environments Model

If we consider a system with more than one qubit, then the works on this topic fall into two main categories: one where all qubits are plunged in the same environment and the other where each qubit is plunged in its own environment. However, also intermediate situations can be thought and indeed it is studied a model that is a paradigm of them.

In this chapter such kind of model is presented, namely two ensembles containing arbitrary number of qubits subject to dissipative dynamics into overlapping environments (see Fig.3.1) \[7\]. This means that a number of qubits will be common to both environments.

Thus, the dissipative dynamics of the system it is studied by looking for analytical solutions of the master equation for an arbitrary number of qubits, with the condition of single excitation initial states.

In section 2.6 it has been shown that a dissipative dynamic decoherently affects the quantum state of a single qubit. As a consequence, one may expect that entanglement, since it is the result of a superposition of states, it is not created by dissipative dynamics. However, the contrary it has been proved in \[5\] and \[6\]. Proceeding along this line, the model studied in this work comes as an intermediate between those developed in \[5\] and \[6\].

Figure 3.1: (Color online) Pictorial representation of the system under study. An ensemble of qubit $A \cup C$ dissipates into one environment (depicted by solid line); another ensemble of qubit $B \cup C$ dissipates into another environment (depicted by solid line as well). The ensemble of qubit $C$ results dissipating into both environments.
3.1 The model

Let us consider ensembles $A$ and $B$, each containing a number $N$ of qubits, and a further ensemble $C$ containing $n$ qubits.\footnote{The case of $A$ containing a different number of qubit with respect to $B$ can be straightforwardly considered, however it leads to much more involved expressions without adding anything relevant with respect to the situation presented here.}

Let the qubits belonging to $A \cup C$ dissipate into one environment and those belonging to $B \cup C$ dissipate into another environment (see Fig.3.1). Thus, the ensemble of qubit $C$ results dissipating into both environments.

Given the total number of qubits $N_T = 2N + n$, the associated Hilbert space will be $\mathcal{H} \cong \mathbb{C}^{2^{N_T}}$ spanned by $\otimes_{i=1}^{N_T} \{|0\rangle_i,|1\rangle_i\}$ with $|0\rangle_i$ and $|1\rangle_i$ representing the ground and excited state of the $i$th qubit.

If we assume, for the sake of simplicity, that the temperature is maintained at the absolute zero, the purely dissipative dynamics of the entire system will be described by a Lindbladian master equation of the same form of equation (2.3.12)

$$\frac{\partial \rho}{\partial t} = D[\rho] \equiv 2\sigma_{AC}\rho\sigma_{AC}^\dagger - \{\sigma_{AC}^\dagger \sigma_{AC},\rho\} + 2\sigma_{BC}\rho\sigma_{BC}^\dagger - \{\sigma_{BC}^\dagger \sigma_{BC},\rho\}$$

(3.1.1)

where $\{ , \}$ denotes the anti-commutator.

The operators $\sigma_{AC}$ and $\sigma_{BC}$ are the lowering operators of the ensembles of qubits $A \cup C$ and $B \cup C$ respectively, and are defines as

$$\sigma_{AC} = \sum_{i \in A \cup C} \sigma_i, \quad \sigma_{BC} = \sum_{i \in B \cup C} \sigma_i,$$  

(3.1.2)

where $\sigma_i = |0\rangle_i\langle 1|$ and $\sigma_i^\dagger = |1\rangle_i\langle 0|$ are respectively the lowering and rising operators related to the $i$th qubit, for which the following equalities apply

$$\sigma_i |1\rangle_i = |0\rangle_i \quad \text{and} \quad \sigma_i^\dagger |0\rangle_i = |1\rangle_i.$$  

(3.1.3)

Note that the master equations (3.1.1) give rise to a competition between two dissipative dynamics (induced by the two environments) because there are $n$ qubits that appear in both $\sigma_{AC}$ and $\sigma_{BC}$.

In other words, the dissipative dynamics of the set $A \cup C$ can not be independent from that of the set $B \cup C$. 

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3.2 Dissipative dynamics

To solve the master equation (3.1.1) we follow the strategy put forward in Ref.[5]. Starting from the formal solution

\[ \rho(t) = e^{tD} \rho(0) \]  

(3.2.1)

and resorting to the Taylor expansion of the exponential super operator,

\[ e^{tD} = I + tD + \frac{t^2}{2}D^2 + \frac{t^3}{3!}D^3 + .. \]  

(3.2.2)

we may notice that repeated applications of \( D \) to \( \rho(0) \) will leave the state within a subspace \( \mathbb{H}_{\rho(0)} \subset \mathbb{H} \) of the Hilbert space \( \mathbb{H} = \mathcal{H} \otimes \mathcal{H}^* \) (here \( \mathcal{H}^* \) stands for the dual of \( \mathcal{H} \)).

After having identified \( \mathbb{H}_{\rho(0)} \), i.e. a set of operators on \( \mathcal{H} \) spanning \( \mathbb{H}_{\rho(0)} \), one can write down \( \rho(t) \) as linear combination of such operators with unknown time dependent coefficients.

Then a set of linear differential equations for such coefficients can be derived by inserting the expansion back into Eq.(3.1.1).

It’s worth noticing from (3.1.1) that there exist non trivial operators (i.e. not multiple of identity) commuting with the Lindblad operators \( \sigma_{AC} \) and \( \sigma_{BC} \), hence the stationary solution will not be unique [29] and we should expect different steady states depending on \( \rho(0) \).

The advantage of this procedure is that, given a small number of initial excitations \( e \) \( (e \ll N_T) \) we have the following inequality:

\[ \dim \mathbb{H}_{\rho(0)} \leq \left[ \sum_{i=0}^{e} \binom{N_T}{i} \right]^2 \ll [2^{N_T}]^2 = \dim \mathbb{H}. \]  

(3.2.3)

Hence, we confine our attention to the dynamics arising from an initial state containing at most one excitation. Then given an ensemble \( \bullet \) of qubits \( (A, C \text{ or } B \text{ whatever it is}) \), the relevant states will be

\[ |g\rangle^\bullet = |0\rangle \ldots |0\rangle_1 \ldots |0\rangle \quad \text{and} \quad |e_i\rangle^\bullet = |0\rangle \ldots |1\rangle_1 \ldots |0\rangle. \]

We shall distinguish two cases, one in which such excitation is located in \( C \) and the other in which is located in \( A \) (or equivalently in \( B \)).
### 3.2.1 Single excitation initially in $C$

Here we assume that in the initial state there exist one excitation among those qubits dissipating energy into both environments (say it is located in the $k$th qubit of ensemble $C$).

We start introducing the following states for the total $N_T$ qubit system:

\[
\begin{align*}
|g\rangle &= |g\rangle^A|g\rangle^C|g\rangle^B, \\
|a\rangle &= \left(\sum_{i \in A} |e_i\rangle^A\right) |g\rangle^C|g\rangle^B, \\
|b\rangle &= |g\rangle^A|g\rangle^C \left(\sum_{i \in B} |e_i\rangle^B\right), \\
|c\rangle &= |g\rangle^A \left(\sum_{i \in C \setminus k} |e_i\rangle^C\right) |g\rangle^B, \\
|k\rangle &= |g\rangle^A|e_k\rangle^C|g\rangle^B. 
\end{align*}
\]  

(3.2.4)

Actually, $|a\rangle$ (resp. $|b\rangle$) is a uniform superposition of single excitations of qubits belonging to $A$ (resp. $B$) and $|c\rangle$ is a uniform superposition of single excitation of qubits belonging to $C$, excluding the $k$th site.

By applying $\mathcal{D}$ to the states (3.2.4) we find the following set of equations:

\[
\begin{align*}
\mathcal{D}[|g\rangle\langle g|] &= 0, \\
\mathcal{D}[|a\rangle\langle a|] &= 2N^2|g\rangle\langle g| - N(2|a\rangle\langle a| + \Omega_{ak} + \chi_{ac}), \\
\mathcal{D}[|b\rangle\langle b|] &= 2N^2|g\rangle\langle g| - N(2|b\rangle\langle b| + \Omega_{bk} + \chi_{bc}), \\
\mathcal{D}[|c\rangle\langle c|] &= 4(n-1)^2|g\rangle\langle g| - (n-1)(4|c\rangle\langle c| + 2\Omega_{ck} + \chi_{ac} + \chi_{bc}), \\
\mathcal{D}[|k\rangle\langle k|] &= 4|g\rangle\langle g| - \Omega_{ak} - \Omega_{bk} - 2\Omega_{ck} - 4|k\rangle\langle k|. 
\end{align*}
\]  

(3.2.5)
where we have defined the following independent operators

\[
\Omega_{ak} = |a\rangle\langle k| + |k\rangle\langle a|, \quad \chi_{ab} = |a\rangle\langle b| + |b\rangle\langle a|,
\]
\[
\Omega_{bk} = |b\rangle\langle k| + |k\rangle\langle b|, \quad \chi_{bc} = |b\rangle\langle c| + |c\rangle\langle b|,
\]
\[
\Omega_{ck} = |c\rangle\langle k| + |k\rangle\langle c|, \quad \chi_{ac} = |a\rangle\langle c| + |c\rangle\langle a|.
\]

To find a set of closed relations, we then apply \(D\) to these operators obtaining

\[
D[\Omega_{ak}] = 4N|g\rangle\langle g| - 2N|k\rangle\langle k| - 2|a\rangle\langle a|

- (N + 2)\Omega_{ak} - N\Omega_{ck} - 2\chi_{ac} - \chi_{ab},
\]
\[
D[\Omega_{bk}] = 4N|g\rangle\langle g| - 2N|k\rangle\langle k| - 2|b\rangle\langle b|

- (N + 2)\Omega_{bk} - N\Omega_{ck} - 2\chi_{bc} - \chi_{ab},
\]
\[
D[\Omega_{ck}] = 8(n - 1)|g\rangle\langle g| - 4|c\rangle\langle c| - 2n\Omega_{ck}

- (n - 1)(4|k\rangle\langle k| + \Omega_{ak} + \Omega_{bk}) - \chi_{ac} - \chi_{bc},
\]
\[
D[\chi_{ab}] = -2N\chi_{ab} - N(\Omega_{ak} + \chi_{ac} + \Omega_{bk} + \chi_{bc}),
\]
\[
D[\chi_{ac}] = -N(2|c\rangle\langle c| + \Omega_{ck} + \chi_{ac}) + (n - 1)

\times (4N|g\rangle\langle g| - 2|a\rangle\langle a| - 2\Omega_{ak} - \chi_{ab} - 2\chi_{ac}),
\]
\[
D[\chi_{bc}] = -N(2|c\rangle\langle c| + \Omega_{ck} + \chi_{bc}) + (n - 1)

\times (4N|g\rangle\langle g| - 2|b\rangle\langle b| - 2\Omega_{bk} - \chi_{ab} - 2\chi_{bc}),
\]

The set of closed relations in (3.2.5) and (3.2.7) guarantees that the density operator describing the system at arbitrary time \(t\) leaves in the space

\[
\mathbb{H}_{\rho(0)} = \text{span} \{ |g\rangle\langle g|, |k\rangle\langle k|, |a\rangle\langle a|, |b\rangle\langle b|, |c\rangle\langle c|, \Omega_{ak}, \Omega_{bk}, \Omega_{ck}, \chi_{ab}, \chi_{ac}, \chi_{bc} \}.
\]
Thus, expanding $\rho(t)$ as
\[
\rho(t) = c_0(t)|g\rangle\langle g| + c_1(t)|k\rangle\langle k| + c_2(t)|a\rangle\langle a|
+ c_3(t)|b\rangle\langle b| + c_4(t)|c\rangle\langle c| + c_5(t)\Omega_{ak} + c_6(t)\Omega_{bk}
+ c_7(t)\Omega_{ck} + c_8(t)\chi_{ab} + c_9(t)\chi_{ac} + c_{10}(t)\chi_{bc},
\]
and inserting it back to (3.1.1), it yields the following set of differential equations for the time dependent coefficients, describing the dynamics of the system:
\[
\begin{align*}
\dot{c}_0 &= 4c_1 + 2N^2(c_2 + c_3) + 4(n-1)^2c_4 + 4N(c_5 + c_6) + 8(n-1)c_7 \\
&+ 4N(n-1)(c_9 + c_{10}), \\
\dot{c}_1 &= -4c_1 - 2N(c_5 + c_6) - 4(n-1)c_7, \\
\dot{c}_2 &= -2Nc_2 - 2c_5 - 2(n-1)c_9, \\
\dot{c}_3 &= -2Nc_3 - 2c_6 - 2(n-1)c_{10}, \\
\dot{c}_4 &= -4(n-1)c_4 - 4c_7 - 2N(c_9 + c_{10}), \\
\dot{c}_5 &= -c_1 - Nc_2 - (N + 2)c_5 - (n - 1)c_7 - Nc_8 - 2(n - 1)c_9, \\
\dot{c}_6 &= -c_1 - Nc_3 - (N + 2)c_6 - (n - 1)c_7 - Nc_8 - 2(n - 1)c_{10}, \\
\dot{c}_7 &= -2c_1 - 2(n - 1)c_4 - N(c_5 + c_6) - 2nc_7 - N(c_9 + c_{10}), \\
\dot{c}_8 &= -c_5 - c_6 - 2Nc_8 - (n - 1)(c_9 + c_{10}), \\
\dot{c}_9 &= -Nc_2 - (n - 1)c_4 - 2c_5 - c_7 - Nc_8 - (N + 2n - 2)c_9, \\
\dot{c}_{10} &= -Nc_3 - (n - 1)c_4 - 2c_6 - c_7 - Nc_8 - (N + 2n - 2)c_{10}. \quad (3.2.10)
\end{align*}
\]
The initial conditions read $c_j(0) = \delta_{j,1}$ with $j = 0, \ldots, 10$. 
The solution of the set of differential equations (3.2.10) can be easily found. First notice that due to symmetry between $A$ and $B$, it is $c_2 = c_3$, $c_5 = c_6$, $c_9 = c_{10}$, furthermore it results

$$c_0 = \frac{2e^{-2(N+2n)t}}{(N+2n)} \left[ e^{2(N+2n)t} - 1\right],$$

$$c_1 = \frac{e^{-2(N+2n)t}}{(N+2n)^2} \left[ (N+2n - 2)e^{(N+2n)t} + 2 \right]^2,$$

$$c_2 = c_8 = \frac{e^{-2(N+2n)t}}{(N+2n)^2} \left[ e^{(N+2n)t} - 1 \right]^2,$$

$$c_4 = 4c_2,$$

$$c_5 = \frac{e^{-2(N+2n)t}}{(N+2n)^2} \left[ -(N+2n - 2)e^{2(N+2n)t} + (N+2n - 4)e^{(N+2n)t} + 2 \right],$$

$$c_7 = 2c_5, \quad c_8 = c_2, \quad c_9 = 2c_2. \quad (3.2.11)$$

### 3.2.2 Single excitation initially in $A$

We now assume that the initial excitation is in the ensemble $A$ at $k$th site. Proceeding like in the previous section, we introduce, in addition to (3.2.4), the following notation for $N_T$ qubit states:

$$|a'\rangle = \left( \sum_{i \in A \backslash k'} |e_i\rangle^A \right) |g\rangle^C |g\rangle^B,$$

$$|c'\rangle = |g\rangle^A \left( \sum_{i \in C} |e_i\rangle^C \right) |g\rangle^B,$$

$$|k'\rangle = |e_{k'}\rangle^A |g\rangle^C |g\rangle^B. \quad (3.2.12)$$
CHAPTER 3. OVERLAPPING ENVIRONMENTS MODEL

Actually, $|a'\rangle$ is a uniform superposition of single excitations of qubits belonging to $A$ excluding the initial excitation at $k'$th and those in $C$. Furthermore, $|c'\rangle$ is a uniform superposition of single excitation of qubits in $C$.

Next we use Eq.(3.2.12) similarly to Section 3.2.1. That is, we apply $\mathcal{D}$ of (3.1.1) on $\{|g\rangle\langle g|, |a'\rangle\langle a'|, |b\rangle\langle b|, |c'\rangle\langle c'|, |k'\rangle\langle k'|\}$ and on the independent operators arising as outcome to find the following set of relations:

\[
\begin{align*}
\mathcal{D}[|a'\rangle\langle a'|] &= 2(N-1)^2|g\rangle\langle g| - (N-1)(2|a'\rangle\langle a'| + \Omega'_{ak} + \chi'_{ac}), \\
\mathcal{D}[|b\rangle\langle b|] &= 2N^2|g\rangle\langle g| - N(2|b\rangle\langle b| + \chi'_{bc}), \\
\mathcal{D}[|c'\rangle\langle c'|] &= 4n^2|g\rangle\langle g| - n(4|c'\rangle\langle c'| + \Omega'_{ck} + \chi'_{ac} + \chi'_{bc}), \\
\mathcal{D}[|k'\rangle\langle k'|] &= 2|g\rangle\langle g| - \Omega'_{ak} - \Omega'_{ck} - 2|k'\rangle\langle k'|, \\
\mathcal{D}[\Omega'_{ak}] &= 4(N-1)|g\rangle\langle g| - 2(N-1)|k'\rangle\langle k'| - 2|a'\rangle\langle a'| - N\Omega'_{ak} - (N-1)\Omega'_{ck} - \chi'_{ac}, \\
\mathcal{D}[\Omega'_{bk}] &= -(N+1)\Omega'_{bk} - N\Omega'_{ck} - \chi'_{ab} - \chi'_{bc}, \\
\mathcal{D}[\Omega'_{ck}] &= 4n|g\rangle\langle g| - n\Omega'_{ak} - (2n+1)\Omega'_{ck} - n\Omega'_{bk} - 2n|k'\rangle\langle k'| - 2|c'\rangle\langle c'| - \chi'_{ac}, \\
\mathcal{D}[\chi'_{ab}] &= -(2N-1)\chi'_{ab} - (N-1)(\Omega'_{bk} + \chi'_{bc}) - N\chi'_{ac}, \\
\mathcal{D}[\chi'_{ac}] &= (N-1)(4n|g\rangle\langle g| - 2|c'\rangle\langle c'| - \Omega'_{ck}) - n(2|a'\rangle\langle a'| + \Omega'_{ak} + \chi'_{ab}) - (N+2n-1)\chi'_{ac}, \\
\mathcal{D}[\chi'_{bc}] &= n(4N|g\rangle\langle g| - 2|b\rangle\langle b| - \Omega'_{bk} - \chi'_{ab}) - 2n\chi'_{bc} - N(2|c'\rangle\langle c'| + \chi'_{bc}), \\
\end{align*}
\]

(3.2.13)
where, similarly to (3.2.6), we have defined

\[ \Omega'_{ak} = \lvert a' \rangle \langle k' \rvert + \lvert k' \rangle \langle a' \rvert, \quad \chi'_{ab} = \lvert a' \rangle \langle b \rvert + \lvert b \rangle \langle a' \rvert, \]

\[ \Omega'_{bk} = \lvert b \rangle \langle k' \rvert + \lvert k' \rangle \langle b \rvert, \quad \chi'_{bc} = \lvert b \rangle \langle c' \rvert + \lvert c' \rangle \langle b \rvert, \]

\[ \Omega'_{ck} = \lvert c' \rangle \langle k' \rvert + \lvert k' \rangle \langle c' \rvert, \quad \chi'_{ac} = \lvert a' \rangle \langle c' \rvert + \lvert c' \rangle \langle a' \rvert. \]

(3.2.14)

The set of closed relations in (3.2.13) guarantees that the density operator describing the system at arbitrary time leaves in the space

\[ \mathbb{H}_{\rho(0)} = \text{span} \{ \lvert g \rangle \langle g \rvert, \lvert k' \rangle \langle k' \rvert, \lvert a' \rangle \langle a' \rvert, \lvert b \rangle \langle b \rvert, \lvert c' \rangle \langle c' \rvert, \Omega'_{ak}, \Omega'_{bk}, \Omega'_{ck}, \chi'_{ab}, \chi'_{ac}, \chi'_{bc} \}. \]

(3.2.15)

Thus, expanding the density operator as

\[
\rho(t) = a_0 \lvert g \rangle \langle g \rvert + a_1 \lvert k' \rangle \langle k' \rvert + a_2 \lvert a' \rangle \langle a' \rvert + a_3 \lvert b \rangle \langle b \rvert + a_4 \lvert c' \rangle \langle c' \rvert + a_5 \Omega'_{ak} + a_6 \Omega'_{bk} + a_7 \Omega'_{ck} + a_8 \chi'_{ab} + a_9 \chi'_{ac} + a_{10} \chi'_{bc},
\]

(3.2.16)

leads (upon insertion into (3.1.1)) to a set of differential equations for the time dependent coefficients of the density operator (3.2.16).

Then, the dynamics is governed by the following differential equation:

\[
\dot{a}_0 = 2a_1 + 2(N - 1)^2a_2 + 2N^2a_3 + 4n^2a_4 + 4(N - 1)a_5 + 4na_7 + 4n(N - 1)a_9 + 4Na_{10},
\]

\[
\dot{a}_1 = -2a_1 - 2(N - 1)a_5 - 2na_7,
\]

\[
\dot{a}_2 = -2(N - 1)a_2 - 2a_5 - 2na_9,
\]

\[
\dot{a}_3 = -2Na_3 - 2na_{10},
\]

\[
\dot{a}_4 = -4na_4 - 2a_7 - 2(N - 1)a_9 - 2Na_{10},
\]

\[
\dot{a}_5 = -a_1 - (N - 1)a_2 - Na_5 - n(a_7 + a_9),
\]

\[
\dot{a}_6 = -(N + 1)a_6 - na_7 - (N - 1)a_8 - na_{10},
\]
\[ \dot{a}_7 = -a_1 - na_4 - (N - 1)a_5 - Na_6 + -(2n + 1)a_7 - (N - 1)a_9, \]

\[ \dot{a}_8 = -a_6 - (2N - 1)a_8 - n(a_9 + a_{10}), \]

\[ \dot{a}_9 = -(N - 1)a_2 - na_4 - a_5 - a_7 - Na_8 + -(N + 2n - 1)a_9, \]

\[ \dot{a}_{10} = -Na_3 - na_4 - a_6 - (N - 1)a_8 - (N + 2n)a_{10}. \]  

(3.2.17)

The solutions, with initial conditions \( a_j(0) = \delta_{j,1} \) with \( j = 0, \ldots, 10 \), read

\[ a_0 = \frac{e^{-2Nt}}{2N(N + 2n)} \left[ 2(N + n)e^{2Nt} - (N + 2n) - Ne^{-4nt} \right], \]

\[ a_1 = \frac{(N^2 + 2Nn - N - n) e^{-Nt}}{N^2(N + 2n)^2} \times \left[ (N^2 + 2Nn - N - n) e^{Nt} + (N + 2n) + Ne^{-2nt} \right] \]

\[ + \frac{e^{-2Nt}}{4N^2(N + 2n)^2} \left[ (N + 2n) + Ne^{-2nt} \right]^2, \]

\[ a_2 = \frac{e^{-2Nt}}{4N^2(N + 2n)^2} \left[ N + 2n + Ne^{-2nt} - 2e^{Nt}(N + n) \right]^2, \]

\[ a_3 = \frac{e^{-2Nt}}{4N^2(N + 2n)^2} \left[ -2ne^{Nt} + (N + 2n) - Ne^{-2nt} \right]^2, \]

\[ a_4 = \frac{e^{-2(N+2n)t}}{(N + 2n)^2} \left[ e^{(N+2n)t} - 1 \right]^2, \]

\[ a_5 = \frac{(N^2 + 2Nn - 2N - 2n) e^{-Nt}}{2N^2(N + 2n)^2} \times \left[ -2(N + n)e^{Nt} + (N + 2n) + Ne^{-2nt} \right] \]

\[ + \frac{e^{-2Nt}}{4N^2(N + 2n)^2} \left[ (N + 2n) + Ne^{-2nt} \right]^2, \]
\[ a_6 = \frac{e^{-Nt}}{4N^2(N+2n)^2} \left[ 4n \left( N^2 + 2Nn - N - n \right) e^{Nt} - 2(N + 2n)^2(N - 1) - (N + 2n)^2 e^{-Nt} + 2N^2 (N + 2n - 1) e^{-2nt} + N^2 e^{-(N+4n)t} \right] , \]
\[ a_7 = \frac{e^{-Nt}}{2N(N+2n)^2} \left[ 2 \left( n - N^2 - 2Nn + N \right) e^{Nt} - (N + 2n) + (2N^2 - 2n + 4Nn - 3N) e^{-2nt} + (N + 2n)e^{-(N+2n)t} + Ne^{-(N+4n)t} \right] , \]
\[ a_8 = \frac{e^{-2Nt}}{4N^2(N+2n)^2} \left[ -4n(N + n)e^{2Nt} + 2(N + 2n)^2 e^{Nt} - (N^2 + 2n)^2 - 2N^2 e^{(N-2n)t} - N^2 e^{-4nt} \right] , \]
\[ a_9 = \frac{e^{(N+2n)t} - 1}{2N(N+2n)^2} \times \left[ 2(N + n)e^{Nt} - (N + 2n) - Ne^{-2nt} \right] , \]
\[ a_{10} = \frac{e^{(N+2n)t} - 1}{2N(N+2n)^2} \times \left[ -2ne^{Nt} + (N + 2n) - Ne^{-2nt} \right] . \]

(3.2.18)
Chapter 4
Entanglement Dynamics

In this chapter it is described the entanglement dynamics between ensembles $A$ and $B$ (see Fig.3.1). In order to accomplish this goal, we have to first trace out the ensemble $C$ and then use the negativity introduced in section 1.5 to evaluate the amount of entanglement between the two ensembles.

4.1 Tracing out the common qubits

We distinguish between the two cases defined in the previous chapter.

4.1.1 Single excitation initially in $C$

Taking into account (3.2.4), (3.2.6) and tracing $C$ away from them we get:

$$\text{Tr}_C(|a\rangle\langle a|) = \sum_{i,j \in A} |e_i^A\langle e_j| \otimes |g^B\langle g| =: |\tilde{a}\rangle\langle \tilde{a}|,$$

$$\text{Tr}_C(|b\rangle\langle b|) = |g^A\langle g| \otimes \sum_{i,j \in B} |e_i^B\langle e_j| =: |\tilde{b}\rangle\langle \tilde{b}|,$$

$$\text{Tr}_C(|c\rangle\langle c|) = (n-1)|g^A\langle g| \otimes |g^B\langle g| =: (n-1)|\tilde{g}\rangle\langle \tilde{g}|,$$

$$\text{Tr}_C(|g\rangle\langle g|) = |g^A\langle g| \otimes |g^B\langle g| = |\tilde{g}\rangle\langle \tilde{g}|,$$

$$\text{Tr}_C(|k\rangle\langle k|) = |g^A\langle g| \otimes |g^B\langle g| = |\tilde{g}\rangle\langle \tilde{g}|,$$

$$\text{Tr}_C(\chi_{ab}) = |\tilde{a}\rangle\langle \tilde{b}| + |\tilde{b}\rangle\langle \tilde{a}| =: \tilde{\chi}_{ab}, \quad (4.1.1)$$

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while all the other terms in \( \{ \Omega_{ak}, \Omega_{bk}, \Omega_{ck}, \chi_{ac}, \chi_{bc} \} \) are zero when the \( \text{Tr}_C \) is applied to them.

At the end, thanks to (4.1.1), the trace over \( C \) of the density operator (3.2.9) gives the following bipartite state

\[
\rho_{AB} = \beta(t)|\tilde{g}\rangle \langle \tilde{g}| + c_2(t) \left( |\tilde{a}\rangle \langle \tilde{a}| + |\tilde{b}\rangle \langle \tilde{b}| + \tilde{\chi}_{ab} \right),
\]  

(4.1.2)

where we have taken into account that \( \text{Tr}(\rho) = c_0 + c_1 + 2Nc_2 + c_4(n-1) = 1 \)
and defined

\[
\beta(t) = 1 - 2Nc_2(t).
\]  

(4.1.3)

### 4.1.2 Single excitation initially in \( A \)

Taking into account of (3.2.12), (3.2.14) and tracing \( C \) away from them we get:

\[
\text{Tr}_C(|a'\rangle \langle a'|) = \sum_{i,j \in A \setminus k'} |e_i\rangle^A \langle e_j| \otimes |g\rangle^B \langle g| =: |\tilde{a}'\rangle \langle \tilde{a}'|,
\]

\[
\text{Tr}_C(|b\rangle \langle b|) = |g\rangle^A \langle g| \otimes \sum_{i,j \in B} |e_i\rangle^B \langle e_j| =: |\tilde{b}\rangle \langle \tilde{b}|,
\]

\[
\text{Tr}_C(|c'\rangle \langle c'|) = n|g\rangle^A \langle g| \otimes |g\rangle^B \langle g| =: n|\tilde{g}\rangle \langle \tilde{g}|,
\]

\[
\text{Tr}_C(|g\rangle \langle g|) = |g\rangle^A \langle g| \otimes |g\rangle^B \langle g| =: |\tilde{g}\rangle \langle \tilde{g}|,
\]

\[
\text{Tr}_C(|k'\rangle \langle k'|) = |e_{k'}\rangle^A \langle e_{k'}| \otimes |g\rangle^B \langle g| =: |\tilde{k}'\rangle \langle \tilde{k}'|,
\]

\[
\text{Tr}_C(\chi_{ab}') = |\tilde{a}'\rangle \langle \tilde{b}| + |\tilde{b}\rangle \langle \tilde{a}'| =: \tilde{\chi}_{ab},
\]

\[
\text{Tr}_C(\Omega_{ak}') = |\tilde{a}'\rangle \langle \tilde{k}'| + |\tilde{k}'\rangle \langle \tilde{a}'| =: \tilde{\Omega}_{ak},
\]

\[
\text{Tr}_C(\Omega_{bk}') = |\tilde{b}\rangle \langle \tilde{k}'| + |\tilde{k}'\rangle \langle \tilde{b}| =: \tilde{\Omega}_{bk},
\]  

(4.1.4)

while all the other terms in \( \{ \Omega_{ck}, \chi_{ac}, \chi_{bc} \} \) are zero when the \( \text{Tr}_C \) is applied to them.
At the end, thanks to (4.1.4), the trace over $C$ of the density operator (3.2.16) gives the following bipartite state

$$
\rho_{AB} = \text{Tr}_C(\rho) = \beta' |\tilde{g}\rangle \langle \tilde{g}| + a_1 |\tilde{a}\rangle \langle \tilde{a}| + a_2 |\tilde{b}\rangle \langle \tilde{b}| + a_3 |\tilde{a}'\rangle \langle a'| + a_4 |\tilde{b}'\rangle \langle b'| + a_5 \tilde{\Omega}_{ak} + a_6 \tilde{\Omega}_{bk} + a_8 \chi_{ab}',
$$

(4.1.5)

where we have used the relation $\text{Tr}(\rho) = a_0 + a_1 + (N-1)a_2 + Na_3 + na_4 = 1$ and defined

$$
\beta' = 1 - a_1 - (N-1)a_2 - Na_3.
$$

(4.1.6)

### 4.2 Negativity calculation

In order to calculate the negativity between ensembles $A$ and $B$, we have to find first the partial transposition $\rho_{AB}^{T_B}$. Again we distinguish two situations, according to Sections 3.2.1 and 3.2.2.

#### 4.2.1 Single excitation initially in $C$

We first derive from (4.1.1) the following result:

$$
\left( |\tilde{a}\rangle \langle \tilde{b}| \right)^{T_B} = \sum_{i \in A} \sum_{j \in B} |e_i^A \rangle \langle g^B | \otimes (|g^A \rangle \langle e_j^B|)^{T_B} = |\tilde{a}b\rangle \langle \tilde{g}|,
$$

(4.2.1)

where we have introduced the state

$$
|\tilde{ab}\rangle = \sum_{i \in A} \sum_{j \in B} |e_i^A \rangle \otimes |e_j^B|. \tag{4.2.2}
$$

Then, using (4.2.1) in (4.1.2) we get

$$
\rho_{AB}^{T_B} = \beta |\tilde{g}\rangle \langle \tilde{g}| + c_2 \left( |\tilde{a}\rangle \langle \tilde{a}| + |\tilde{b}\rangle \langle \tilde{b}| + |\tilde{g}\rangle \langle \tilde{ab}| + |\tilde{ab}\rangle \langle \tilde{g}| \right), \tag{4.2.3}
$$

with $\beta$ defined in (4.1.3).

The negativity is equal, by definition, to the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_B}$.

In order to find these eigenvalues, recalling the definition of $|\tilde{g}\rangle$, $|\tilde{ab}\rangle$, $|\tilde{a}\rangle$, $|\tilde{b}\rangle$ given in (4.1.1), we can represent $\rho_{AB}^{T_B}$ in the basis
\{ |\tilde{g}\rangle, |e_i\rangle^A \otimes |e_j\rangle^B, |e_i\rangle^A \otimes |g\rangle^B, |g\rangle^A \otimes |e_j\rangle^B \}_{i \in A, j \in B} \text{ which includes } 1 + N^2 + 2N \text{ vectors. Then, } \rho_{AB}^{T_B} \text{ takes the following block matrix form}

\[ \rho_{AB}^{T_B} = \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}, \tag{4.2.4} \]

where \( \tau \) and \( \omega \) are matrices of dimensions \((1+N^2) \times (1+N^2)\) and \((2N) \times (2N)\) respectively, made in the following way:

\[ \tau = \begin{pmatrix} \beta & e_2 & \ldots & e_2 \\ e_2 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ e_2 & 0 & \ldots & 0 \end{pmatrix}, \tag{4.2.5} \]

and

\[ \omega = \begin{pmatrix} \omega^{(1)} & 0 \\ 0 & \omega^{(1)} \end{pmatrix}, \tag{4.2.6} \]

with \( \omega^{(1)} \) a \( N \times N \) matrix having all entries equal to \( c_2(t) \), i.e.

\[ \omega^{(1)} = \begin{pmatrix} c_2 & c_2 & \ldots & c_2 & c_2 \\ c_2 & c_2 & \ldots & c_2 & c_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ c_2 & c_2 & \ldots & c_2 & c_2 \\ c_2 & c_2 & \ldots & c_2 & c_2 \end{pmatrix}. \tag{4.2.7} \]

For the property of block diagonal matrix determinant, the eigenvalues of \( \rho_{AB}^{T_B} \) satisfy the relation

\[ \det (\tau - \lambda I_{N^2+1}) \det (\omega^{(1)} - \lambda I_N) \det (\omega^{(1)} - \lambda I_N) = 0, \tag{4.2.8} \]

\((I_N \text{ denotes the } N \times N \text{ identity matrix). Thus, the non-zero eigenvalues of } \rho_{AB}^{T_B} \text{ are those of the two matrices } \tau \text{ and } \omega^{(1)}, \text{ namely}\)

\[ \lambda(\tau) = \frac{1}{2} \left( \beta(t) \pm \sqrt{(2Nc_2(t))^2 + \beta(t)^2} \right), \]

\[ \lambda(\omega^{(1)}) = Nc_2(t). \tag{4.2.9} \]

The only negative eigenvalue is the one of \( \tau \) with the minus in front of the square root, hence the negativity results

\[ N(t) = \sqrt{(Nc_2(t))^2 + \left( \frac{1}{2} - Nc_2(t) \right)^2} - \left( \frac{1}{2} - Nc_2(t) \right), \tag{4.2.10} \]

where the relation \( \beta(t) = 1 - 2Nc_2(t) \) has been used (see (4.1.3)).
4.2.2 Single excitation initially in $A$

We first derive from (4.1.4) the following results:

\[
\left( |\tilde{a}'\rangle \langle \tilde{b}| \right)^T_B = \sum_{i \in A \setminus k'} \sum_{j \in B} |e_i^A \rangle \otimes |g^B \langle e_j| \right)^T_B = |\tilde{a}'b\rangle \langle \tilde{g}|
\]

\[
\left( |\tilde{k}'\rangle \langle \tilde{b}| \right)^T_B = \sum_{j \in B} |e_k^A \rangle \otimes |g^B \langle e_j| \right)^T_B = |\tilde{k}'b\rangle \langle \tilde{g}|
\]

(4.2.11)

where we have introduced the states

\[
|\tilde{a}'b\rangle = \sum_{i \in A \setminus k'} \sum_{j \in B} |e_i^A \rangle \otimes |e_j^B\rangle,
\]

\[
|\tilde{k}'b\rangle = \sum_{j \in B} |e_k^A \rangle \otimes |e_j^B\rangle.
\]

(4.2.12)

Finally, using (4.2.11) in (4.1.5), we get

\[
\rho_{AB}^{T_B} = \beta' |\tilde{g}\rangle \langle \tilde{g}| + a_1 |\tilde{k}'\rangle \langle \tilde{k}'| + a_2 |\tilde{a}'\rangle \langle \tilde{a}'| + a_3 |\tilde{b}\rangle \langle \tilde{b}| + a_5 \tilde{\Omega}_{ak} + a_6 \left( |\tilde{g}\rangle \langle \tilde{k}'b| + |\tilde{k}'b\rangle \langle \tilde{g}| \right) + a_8 \left( |\tilde{a}'b\rangle \langle \tilde{g}| + |\tilde{g}\rangle \langle \tilde{a}'b| \right),
\]

(4.2.13)

with $\beta'$ defined in (4.1.6).

The negativity is equal, by definition, to the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_B}$.

In order to find these eigenvalues, recalling the definition of $|\tilde{g}\rangle, |\tilde{a}\rangle, |\tilde{b}\rangle$ given in (4.1.1), we can represent $\rho_{AB}^{T_B}$ in the basis

\[
\{ |\tilde{g}\rangle, |e_i^A \rangle \otimes |e_j^B\rangle, |e_i^A \rangle \otimes |g^B \langle e_j|, |g^A \otimes |e_j^B\rangle \} \quad i \in A, j \in B
\]

which includes $1 + N^2 + 2N$ vectors. Then, $\rho_{AB}^{T_B}$ takes the following block matrix form

\[
\rho_{AB}^{T_B} = \begin{pmatrix} \tau' & 0 \\ 0 & \omega' \end{pmatrix},
\]

(4.2.14)

where $\tau'$ and $\omega'$ are matrices of dimensions $(1+N^2) \times (1+N^2)$ and $(2N) \times (2N)$ respectively, made as follows:
\[ \tau' = \begin{pmatrix} \beta' & a_8 & \ldots & \ldots & a_8 & a_6 & \ldots & a_6 \\ a_8 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ a_8 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ a_6 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ a_6 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad (4.2.15) \]

(there are \( N(N - 1) \) elements equal to \( a_8 \) and \( N \) elements equal to \( a_6 \) in the first row and column) and

\[ \omega' = \begin{pmatrix} \omega'(1) & 0 \\ 0 & \omega'(2) \end{pmatrix}, \quad (4.2.16) \]

where \( \omega'(1) \) and \( \omega'(2) \) are \( N \times N \) matrices made as follows

\[ \omega'(1) = \begin{pmatrix} a_1 & a_5 & a_5 & \ldots & a_5 & a_5 \\ a_5 & a_2 & a_2 & \ldots & a_2 & a_2 \\ a_5 & a_2 & a_2 & \ldots & a_2 & a_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_5 & a_2 & a_2 & \ldots & a_2 & a_2 \\ a_5 & a_2 & a_2 & \ldots & a_2 & a_2 \end{pmatrix}, \quad (4.2.17) \]

and

\[ \omega'(2) = \begin{pmatrix} a_3 & a_3 & \ldots & a_3 & a_3 \\ a_3 & a_3 & \ldots & a_3 & a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ a_3 & a_3 & \ldots & a_3 & a_3 \\ a_3 & a_3 & \ldots & a_3 & a_3 \end{pmatrix}. \quad (4.2.18) \]

For the property of block diagonal matrix determinant, the eigenvalues of \( p_{AB}^{TN} \) satisfy the equation

\[ \det (\tau' - \lambda I_{N^2 + 1}) \det (\omega'(1) - \lambda I_N) \det (\omega'(2) - \lambda I_N) = 0. \quad (4.2.19) \]
Thus the non-zero eigenvalues of $\rho_{AB}^{TB}$ are those of the matrices $\tau'$, $\omega'(1)$ and $\omega'(2)$, namely

$$
\lambda(\tau') = \frac{1}{2}(\beta') \pm \frac{1}{2}\sqrt{(\beta')^2 + 4N(a_6^2 + (N-1)a_8^2)},
$$

$$
\lambda(\omega'(1)) = \frac{1}{2}(a_1 + (N-1)a_2)
\pm \sqrt{(a_1 - (N-1)a_2)^2 + 4(N-1)a_5^2},
$$

$$
\lambda(\omega'(2)) = Na_3.
$$

(4.2.20)

The only negative eigenvalue is the one of $\tau'$ with the minus in front of the square root, hence the negativity results

$$
\mathcal{N}(t) = \frac{1}{2}\sqrt{(a_1 + a_2(N - 1) + a_3N - 1)^2 + 4N(a_6^2 + (N-1)a_8^2)}
- \frac{1}{2}(1 - a_1 - a_2N + a_2 - a_3N).
$$

(4.2.21)
CHAPTER 4. ENTANGLEMENT DYNAMICS

4.3 Stationary Entanglement

Let us now comment the main results which stem from the negativity expressions (4.2.10) and (4.2.21), obtained in the cases of single excitation initially in C and in A respectively.

4.3.1 Stationary Negativity

The negativity (4.2.10) results monotonically increasing vs time up to a stationary value. In the limit $t \to \infty$ it becomes

$$
N(\infty) = \frac{1}{2} \sqrt{\frac{8N^2}{(2n+N)^4} - \frac{4N}{(2n+N)^2} + 1 + \frac{N}{(2n+N)^2} - \frac{1}{2}}. \quad (4.3.1)
$$

Coming to the negativity (4.2.21), it also results monotonically increasing vs time up to a stationary value. In the limit $t \to \infty$ it becomes

$$
N(\infty) = \frac{1}{2N^2(N+2n)^2} \left[ N^6 + 4N^5n(n+2) + 16N^4n^2(n+1) + 16N^3n^4 - 8N^2n^4 \right]^{1/2} \quad (4.3.2)
$$

This means that in both cases, without any direct interaction between the qubits, a common dissipative environment is able to induce entanglement.

For the system considered, the entanglement is created (starting from a single initial excitation) between the qubits of the ensembles A and B, and it remains stationary in time.

Furthermore notice that the two negativities coincide only in the case of $N = n = 1$. 
\section*{4.3.2 Negativity for fixed $N$}

For a fixed number of qubits $N$ inside ensemble $A$ (and $B$), the negativity (4.2.10) monotonically decreases with the number of qubits shared by both environments $n$ (see Fig.4.1).

This can be explained by the fact that the initial single excitation for increasing $n$ tends to persists in the common part rather than being shared by the side parts.

In contrast to (4.3.1), the negativity (4.3.2) monotonically increases vs $n$ and reaches a saturation value only for $n \gg N$ (see e.g. Fig.4.1).

This can be explained by the fact that the initial single excitation for increasing $n$ is more easily shared by the side parts – due to the increasing common part.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{negativity_graph}
\caption{(Color online) Stationary negativity when the initial excitation is in the ensemble $C$ (bottom-blue-dots) and in the ensemble $A$ (top-red-crosses) versus $n$ for $N = 4$.}
\end{figure}
4.3.3 Negativity for fixed $n$

The negativity in (4.3.1) is non monotonic vs $N$ (see e.g. Fig.4.2). Actually it has a maximum for $N = 2n$.

Also (4.3.2) is non monotonic vs $N$ (see e.g. Fig.4.2) and it has a maximum for $N = 2$, whatever the value of $n$ is.

The non monotonic behavior of both (4.2.10) and (4.2.21) vs $N$ should be ascribed to competing effects of side and common qubits.

Entanglement between the two side parts is due to inter-qubit entanglement created between the side qubits. On one side, as $N$ increases, the inter-qubits entanglement increases and so does the amount of entanglement between the two side parts; on the other side instead, the strength of entanglement decreases as $N$ increases, since it is always determined by a single initial excitation.

Figure 4.2: (Color online) Stationary negativity when the initial excitation is in the ensemble $C$ (bottom-blue-dots) and in the ensemble $A$ (top-red-crosses) versus $N$ for $n = 1$. 
Chapter 5

Non-Markovian Dynamics

Although most of the results on open systems dynamics are based on the weak coupling and Markovian approximations, recent studies have shown the limits of the Markovian description of quantum computation and quantum error correction. [25, 26, 27, 28]

Outside the region of validity of the Markovian approximation, the master equation describing the dynamics cannot be usually cast in the Lindblad form presented in Section 2.4 [23, 24].

In this chapter, after a derivation of the non-Markovian master equation [23, 20], the dissipative dynamics are studied.

For this purpose, we first examine the evolution of a single qubit in a thermal bath and then we try to extend it to the overlapping environment model, studied in chapters 3 and 4.

We will prove that in the overlapping environments model, starting from a pure state, superposition will be generated due to the non Markovian dissipative dynamics.

However, formation and calculation of entanglement will be the subject of future studies.
5.1 Generalized Master Equation with memory

In Section 2.1 we derived a master equation for a system interacting with a reservoir assuming weak coupling.

Thus, it was shown that the most general form of the Master Equation is given by equation (2.2.1) [20, 23, 24],

As we did in section 2.3, since we want to study the dissipative dynamics of the system, we consider a thermal bath\(^1\), defined by equation (2.3.1), and an interaction Hamiltonian of the form (2.3.2).

As in the markovian case, the substitution of equation (2.3.2) into (2.2.1) leads, with the same assumption of time translational invariance of the bath correlation function, to some terms of this kind

\[
- \sum_m \int_0^t d\tau X_m^+ X_m e^{i\omega_m \tau} \rho(t - \tau) \text{Tr}_B \{ \Gamma_m(\tau) \Gamma_m^\dagger(0) \rho_B \}.
\]

(5.1.1)

Notice however that in this case the density operator depends on \(\tau\) and so, it can not be taken outside the integral.

After a few algebraic steps, it is possible to rewrite the whole master equation as follows [24]

\[
\frac{\partial \rho(t)}{\partial t} = \int_0^t d\tau \mathcal{L}\rho(t - \tau) k(\tau)
\]

(5.1.2)

where \(\mathcal{L}\) is a Lindblad super-operator like the one defined in (2.4.5).

The term \(k(\tau)\), called memory kernel, is the responsible of the non-Markovianity of the equation and it is basically the bath correlation function

\[
k(\tau) = G_B^{B} (\tau, 0) = \text{Tr}_B \{ \Gamma_n(\tau) \Gamma_n^\dagger(0) \rho_B \} = \int d\omega J(\omega) e^{i(\omega - \omega_n)\tau}.
\]

(5.1.3)

In the last equality of equation (5.1.3) we have introduced the so-called spectral density of the reservoir \(J(\omega)\), which is therefore simply the Fourier transform of the correlation function \(k(\tau)\).

To a Lorentzian spectral density, typical of cavity quantum electrodynamics for a reservoir in the vacuum state, corresponds a memory kernel of the form [23, 24]

\[
k(\tau) = \gamma_0 \lambda e^{-\lambda \tau}.
\]

(5.1.4)

\(^1\)Focussing in particular, for the sake of simplicity, on the zero temperature reservoir.
The parameter $\lambda$ defines the spectral width of the coupling, which is connected to the reservoir correlation time $\tau_B$ by the relation $\tau_B = \lambda^{-1}$.

The parameter $\gamma_0$ is the strength of the coupling between the system and the reservoir, and $\tau_s = \gamma_0^{-1}$ gives the time scale over which the state of the system changes.

Such a memory kernel leads to the following form of the generalized master equation for the system density operator, [23]

$$\frac{d\rho(t)}{dt} = \frac{1}{2} \gamma_0 \lambda \int_0^t d\tau e^{-\lambda(t-\tau)} \sum_m (2X_m^- \rho(\tau)X_m^+ - \{X_m^+X_m^-, \rho(\tau)\}).$$

(5.1.5)

Finally, differentiating with respect to $t$ we obtain the non-Markovian dissipative master equation

$$\dot{\rho}(t) = -\lambda \dot{\rho}(t) + \frac{1}{2} \gamma_0 \lambda \sum_m (2X_m^- \rho(t)X_m^+ - \{X_m^+X_m^-, \rho(t)\}).$$

(5.1.6)

Notice that the non-markovian behaviour of this equation is valid for weak coupling, i.e

$$\frac{\gamma_0}{\lambda} \ll 1.$$  

(5.1.7)

### 5.2 Single Qubit Non-Markovian Dynamic

As we did in section 2.5, we consider a single qubit dissipating into a thermal bath at absolute zero, assuming that the initial state of the qubit is given by the following quantum superposition

$$\rho = \frac{1}{2} (|1\rangle \langle 1| + |1\rangle \langle 0| + |0\rangle \langle 1| + |0\rangle \langle 0|).$$

(5.2.1)

The evolution of the system is expressed by the non-Markovian master equation (5.1.6), that in this situation takes the form

$$\dot{\rho}(t) = -\lambda \dot{\rho}(t) + \frac{1}{2} \gamma_0 \lambda \left(2\sigma \rho(t)\sigma - \{\sigma^\dagger \sigma, \rho(t)\}\right)$$

(5.2.2)

where $\{,\}$ is the anti-commutator and $\sigma = |0\rangle \langle 1|$ and $\sigma^\dagger = |1\rangle \langle 0|$ are respectively the lowering and rising operators related to the qubit.

The density operator describing the qubit at an arbitrary time $t$ can be written as a linear combination of the elements of the canonical basis:

$$\rho(t) = \alpha(t) |1\rangle \langle 1| + \delta(t) |1\rangle \langle 0| + \delta^*(t) |0\rangle \langle 1| + \beta(t) |0\rangle \langle 0|.$$  

(5.2.3)
Substituting (5.2.3) back into the master equation (5.2.2) we obtain a set of equations for the coefficients:

\[ \ddot{\alpha}(t) = -\lambda \dot{\alpha}(t) - \gamma_0 \lambda \alpha(t), \]

\[ \ddot{\beta}(t) = -\lambda \dot{\beta}(t) + \gamma_0 \lambda \alpha(t), \]

\[ \ddot{\delta}(t) = -\lambda \dot{\delta}(t) - \frac{1}{2} \gamma_0 \lambda \delta(t). \] (5.2.4)

Since this is a set of second order differential equations, we also need to know the condition on the first derivative over time of the density operator, in order to have a unique solution.

However, it is impossible to say a priori which is the value of \( \frac{d\rho}{dt} \big|_{t=0} \) and thus we will obtain a solution that depends on \( \dot{\alpha}(0) \), \( \dot{\beta}(0) \) and \( \dot{\delta}(0) \).

The solutions of (5.1.4) are therefore

\[ \alpha(t) = e^{-\frac{1}{2} (\lambda + \sqrt{\lambda(-4\gamma_0 + \lambda)}) t} \times \]

\[ \frac{1}{4\lambda \sqrt{\lambda(-4\gamma_0 + \lambda)}} e^{-\lambda t} \times \]

\[ \left( 4\dot{\alpha}(0) + \lambda \right) \left( -1 + e^{\sqrt{\lambda(-4\gamma_0 + \lambda)} t} \right) + \frac{1}{4} \left( 1 + e^{\sqrt{\lambda(-4\gamma_0 + \lambda)} t} \right), \]

\[ \beta(t) = \frac{1}{4\lambda \sqrt{\lambda(-4\gamma_0 + \lambda)}} e^{-\lambda t} \times \]

\[ \left( \sqrt{\lambda(-4\gamma_0 + \lambda)} + 4e^{\lambda t} (\dot{\beta}(0) + \lambda) - 4\dot{\beta}(0) \right) + \]

\[ -e^{\frac{1}{2} (\lambda - \sqrt{\lambda(-4\gamma_0 + \lambda)}) t} \lambda + e^{\frac{1}{2} (\lambda + \sqrt{\lambda(-4\gamma_0 + \lambda)}) t} \lambda \]

\[ + 4\alpha \left( e^{\frac{1}{2} (\lambda - \sqrt{\lambda(-4\gamma_0 + \lambda)}) t} \lambda - e^{\frac{1}{2} (\lambda + \sqrt{\lambda(-4\gamma_0 + \lambda)}) t} \lambda + (1 + e^{\lambda t}) \sqrt{\lambda(-4\gamma_0 + \lambda)} \right), \]

\[ \delta(t) = e^{-\frac{1}{2} (\lambda + \sqrt{\lambda(-2\gamma_0 + \lambda)}) t} \times \]

\[ \frac{1}{4\lambda \sqrt{\lambda(-2\gamma_0 + \lambda)}} e^{-\lambda t} \times \]

\[ \left( 4\dot{\delta}(0) + \lambda \right) \left( -1 + e^{\sqrt{\lambda(-2\gamma_0 + \lambda)} t} \right) + \frac{1}{4} \left( 1 + e^{\sqrt{\lambda(-2\gamma_0 + \lambda)} t} \right). \] (5.2.5)
Despite the fact that these solutions depend on the constants $\dot{\alpha}(0)$, $\dot{\beta}(0)$ and $\dot{\delta}(0)$, it is still possible to calculate the density operator for time going to infinity.

In order to do so, we recall that the non-Markovianity condition implies $\frac{\gamma_0}{\lambda} \ll 1$ and thus, choosing a ratio $\frac{\gamma_0}{\lambda} = 0.05^2$, we obtain the following stationary state coefficients:

\[
\begin{align*}
\lim_{t \to \infty} \alpha(t) &= 0, \\
\lim_{t \to \infty} \delta(t) &= 0, \\
\lim_{t \to \infty} \beta(t) &= 1 + \dot{\alpha}(0) + \dot{\beta}(0).
\end{align*}
\] (5.2.6)

Note from the last limit that, since the trace must be preserved, i.e. $\alpha(t) + \beta(t) = 1$ for every $t$, we have to imply

\[
\dot{\alpha}(0) + \dot{\beta}(0) = 0,
\] (5.2.7)

hence, this is the only condition that must hold true for the first order derivative over time of the coefficients.

Relation (5.2.7) tells us that in the stationary state, the density operator is the pure state $\rho(\infty) = |0\rangle \langle 0|$ and thus, in the non-Markovian case, the dissipation "kills" the quantum superposition in the long run.

However, the behaviour of the coherence $\delta(t)$ of the density operator (5.2.3) for time $0 < t < \infty$ is not ensured to be exponentially decreasing as in the case of Markovian evolution, shown in section 2.6, but it depends on the choice of the initial condition $\dot{\delta}(0)$.

As an example, it was calculated the trend of the coherence $\delta(t)$ in time, for two different choices of $\dot{\delta}(0)$ (Figs 5.1 and 5.2).

When $\dot{\delta}(0) = 0.5$, Fig. 5.1, the dissipative dynamics initially increases the value of the coherence $\delta(t)$ up to a maximum value, then it makes it decreasing to zero, for $t \to \infty$.

This means that the quantum superposition of the qubit states persists in short time and it is indeed promoted by the non-Markovian dissipative dynamic; however, in the long run decoherence effects prevail.

In the case $\dot{\delta}(0) = 0$ instead, Fig. 5.2, we have a behaviour similar to the one seen for the Markovian dynamics, where the dissipation simply reduces the coherence monotonically until it falls down to zero, "killing" the quantumness.

\footnote{This is a good choice for the ratio $\frac{\gamma_0}{\lambda}$, as shown in [24], p.6}

\footnote{See Section 2.5 for the definition of coherences}
\[ \dot{\delta}(0) = 1/2 \]

![Graph of \( \delta(t) \) vs. \( t \) for \( \dot{\delta}(0) = 0.5 \)](image1)

Figure 5.1: Coherence \( \delta(t) \) as a function of time \( t \), for \( \dot{\delta}(0) = 0.5 \)

\[ \dot{\delta}(0) = 0 \]

![Graph of \( \delta(t) \) vs. \( t \) for \( \dot{\delta}(0) = 0 \)](image2)

Figure 5.2: Coherence \( \delta(t) \) as a function of time \( t \), for \( \dot{\delta}(0) = 0 \)
5.3 Overlapping Environments

Assume that the overlapping environments model described in Section 3.1 undergoes a non-Markovian dissipative dynamics.

The time evolution of the system will be thus described by the master equation (5.1.6), that in this case takes the form

\[
\frac{\partial^2 \rho}{\partial t^2} = -\lambda \frac{\partial \rho}{\partial t} + \frac{1}{2} \gamma_0 \lambda \left[ 2\sigma_{AC}\rho\sigma_{AC}^\dagger - \{\sigma_{AC}\sigma_{AC}^\dagger\rho\} + 2\sigma_{BC}\rho\sigma_{BC}^\dagger - \{\sigma_{BC}\sigma_{BC}^\dagger\rho\} \right]
\]

(5.3.1)

where the operators \(\sigma_{AC}\) and \(\sigma_{BC}\) are defined by relations (3.1.2) and (3.1.3).

To solve this master equation, we first note that the second term on the right side of (5.3.1) is exactly the Lindblad super-operator \(D\) defined in (3.1.1). Thus, we can follow the same strategy described in section 3.2, i.e. of a single initial excitation, and write \(\rho(t)\) as a linear combination of the operators spanning \(H_{\rho(0)}\).

Again, as done before, we should distinguish between two cases, depending on the position of the initial excitation.

5.3.1 Initial excitation in \(C\)

Using the same notation introduce in eq.(3.2.4) and applying \(D\) to the states in (3.2.4) as it was done in (3.2.5) and (3.2.7), we can expand \(\rho(t)\) as in (3.2.9) and finally, inserting it into the non-Markovian master equation (5.3.1), we get the following set of second order differential equations for the time dependent coefficients, describing the dynamics of the system:

\[
\ddot{c}_0 = -\lambda \dot{c}_0 + \frac{1}{2} \gamma_0 \lambda \left[ 4c_1 + 2N^2(c_2 + c_3) + 4(n - 1)^2 c_4 + 4N(c_5 + c_6) + 8(n - 1)c_7 \\
+ 4N(n - 1)(c_9 + c_{10}) \right],
\]

\[
\ddot{c}_1 = -\lambda \dot{c}_1 + \frac{1}{2} \gamma_0 \lambda \left[ -4c_1 - 2N(c_5 + c_6) - 4(n - 1)c_7 \right],
\]

\[
\ddot{c}_2 = -\lambda \dot{c}_2 + \frac{1}{2} \gamma_0 \lambda \left[ -2Nc_2 - 2c_3 - 2(n - 1)c_9 \right],
\]

\[
\ddot{c}_3 = -\lambda \dot{c}_3 + \frac{1}{2} \gamma_0 \lambda \left[ -2Nc_3 - 2c_6 - 2(n - 1)c_{10} \right].
\]

\footnote{It is trivial to see that the two derivatives over time \(\frac{\partial^2 \rho}{\partial t^2}\) and \(\frac{\partial \rho}{\partial t}\) leave the state in the same subspace.}
\[ \dot{c}_4 = -\lambda \ddot{c}_4 + \frac{1}{2} \gamma_0 \lambda \left[ -4(n-1)c_4 - 4c_7 - 2N(c_9 + c_{10}) \right], \]

\[ \dot{c}_5 = -\lambda \ddot{c}_5 + \frac{1}{2} \gamma_0 \lambda \left[ -c_1 - Nc_2 - (N+2)c_3 - (n-1)c_7 - Nc_8 - 2(n-1)c_9 \right], \]

\[ \dot{c}_6 = -\lambda \ddot{c}_6 + \frac{1}{2} \gamma_0 \lambda \left[ -c_1 - Nc_3 - (N+2)c_6 - (n-1)c_7 - Nc_8 - 2(n-1)c_{10} \right], \]

\[ \dot{c}_7 = -\lambda \ddot{c}_7 + \frac{1}{2} \gamma_0 \lambda \left[ -2c_1 - 2(n-1)c_4 - N(c_5 + c_6) - 2nc_7 - N(c_9 + c_{10}) \right], \]

\[ \dot{c}_8 = -\lambda \ddot{c}_8 + \frac{1}{2} \gamma_0 \lambda \left[ -c_5 - c_6 - 2Nc_8 - (n-1)(c_9 + c_{10}) \right], \]

\[ \dot{c}_9 = -\lambda \ddot{c}_9 + \frac{1}{2} \gamma_0 \lambda \left[ -Nc_2 - (n-1)c_4 - 2c_5 - c_7 - Nc_8 - (N+2n-2)c_9 \right], \]

\[ \dot{c}_{10} = -\lambda \ddot{c}_{10} + \frac{1}{2} \gamma_0 \lambda \left[ -Nc_3 - (n-1)c_4 - 2c_6 - c_7 - Nc_8 - (N+2n-2)c_{10} \right]. \]

(5.3.2)

The initial conditions read \( c_j(0) = \delta_{j,1} \) with \( j = 0, \ldots, 10 \).

5.3.2 **Initial excitation in** \( A \)

Using the same notation introduce in eq.(3.2.12) and applying \( D \) to the states in (3.2.12) as it was done in (3.2.13), we can expand \( \rho(t) \) as in (3.2.16) and finally, inserting it into the non-Markovian master equation (5.3.1), we get the following set of second order differential equations for the time dependent coefficients, describing the dynamics of the system:

\[ \ddot{a}_0 = -\lambda \dot{a}_0 + \frac{1}{2} \gamma_0 \lambda \left[ 2a_1 + 2(N-1)^2a_2 + 2N^2a_3 + 4n^2a_4 + 4(N-1)a_5 + 4na_7 + 4n(N-1)a_9 + 4Nna_{10} \right], \]

\[ \ddot{a}_1 = -\lambda \dot{a}_1 + \frac{1}{2} \gamma_0 \lambda \left[ -2a_1 - 2(N-1)a_5 - 2na_7 \right], \]
\[\ddot{a}_2 = -\lambda \dot{a}_2 + \frac{1}{2} \gamma_0 \lambda \left[-2(N-1)a_2 - 2a_5 - 2na_9\right],\]
\[\ddot{a}_3 = -\lambda \dot{a}_3 + \frac{1}{2} \gamma_0 \lambda \left[-2Na_3 - 2na_{10}\right],\]
\[\ddot{a}_4 = -\lambda \dot{a}_4 + \frac{1}{2} \gamma_0 \lambda \left[-4na_4 - 2a_7 - 2(N-1)a_9 - 2Na_{10}\right],\]
\[\ddot{a}_5 = -\lambda \dot{a}_5 + \frac{1}{2} \gamma_0 \lambda \left[-a_1 - (N-1)a_2 - Na_5 - n(a_7 + a_9)\right],\]
\[\ddot{a}_6 = -\lambda \dot{a}_6 + \frac{1}{2} \gamma_0 \lambda \left[-(N+1)a_6 - na_7 - (N-1)a_8 - na_{10}\right],\]
\[\ddot{a}_7 = -\lambda \dot{a}_7 + \frac{1}{2} \gamma_0 \lambda \left[-a_1 - na_4 - (N-1)a_5 - Na_6 + -(2n+1)a_7 - (N-1)a_9\right],\]
\[\ddot{a}_8 = -\lambda \dot{a}_8 + \frac{1}{2} \gamma_0 \lambda \left[-a_6 - (2N-1)a_8 - n(a_9 + a_{10})\right],\]
\[\ddot{a}_9 = -\lambda \dot{a}_9 + \frac{1}{2} \gamma_0 \lambda \left[-(N-1)a_2 - na_4 - a_5 - a_7 - Na_8 + -(N+2n-1)a_9\right],\]
\[\ddot{a}_{10} = -\lambda \dot{a}_{10} + \frac{1}{2} \gamma_0 \lambda \left[-Na_3 - na_4 - a_6 - (N-1)a_8 - (N+2n)a_{10}\right].\]

(5.3.3)

The initial conditions read \(a_j(0) = \delta_{j,1}\) with \(j = 0, \ldots, 10\).
5.4 Three qubits case

The sets of equations (5.2.2) and (5.2.3), found in the previous two sections, are sets of second order differential equations and hence, also a condition on the derivative in time of the density operator i.e. \( \frac{d\rho(t)}{dt} \bigg|_{t=0} \) is needed to find a unique solution. For simplicity, we restrict to the case where there are only three qubits dissipating, that is, \( N = n = 1 \).

5.4.1 Single excitation in \( C \)

If we only have three qubits, the density matrix (3.2.9) reduces to

\[
\rho(t) = c_0(t)|g\rangle\langle g| + c_1(t)|k\rangle\langle k| + c_2(t)|a\rangle\langle a|
+ c_3(t)|b\rangle\langle b| + c_5(t)\Omega_{ak} + c_6(t)\Omega_{bk} + c_8(t)\chi_{ab},
\]

(5.4.1)

The set (5.2.2) becomes

\[
\begin{align*}
\ddot{c}_0 &= -\lambda \dot{c}_0 + \frac{1}{2}\gamma_0 \lambda [4c_1 + 2(c_2 + c_3) + 4(c_5 + c_6)] \\
\ddot{c}_1 &= -\lambda \dot{c}_1 + \frac{1}{2}\gamma_0 \lambda [-4c_1 - 2(c_5 + c_6)] \\
\ddot{c}_2 &= -\lambda \dot{c}_2 + \frac{1}{2}\gamma_0 \lambda [-2c_2 - 2c_3] \\
\ddot{c}_3 &= -\lambda \dot{c}_3 + \frac{1}{2}\gamma_0 \lambda [-2c_3 - 2c_6] \\
\ddot{c}_5 &= -\lambda \dot{c}_5 + \frac{1}{2}\gamma_0 \lambda [-c_1 - c_2 - 3c_5 - c_8] \\
\ddot{c}_6 &= -\lambda \dot{c}_6 + \frac{1}{2}\gamma_0 \lambda [-c_1 - c_3 - 3c_6 - c_8] \\
\ddot{c}_8 &= -\lambda \dot{c}_8 + \frac{1}{2}\gamma_0 \lambda [-c_5 - c_6 - 2c_8]
\end{align*}
\]

(5.4.2)

with initial condition \( c_i(0) = 1 \) and \( c_i(0) = 0 \) for \( i = 0, 2, 3, 5, 6, 8 \).

Note that, although the solutions of the set (5.3.2) will depend on the choice of the constants \( \dot{c}_i(0) \), we can calculate these solutions with the help of a mathematical computation program like *Mathematica* or *MatLab* and, moreover, it is possible to find the stationary state of the system, i.e. \( \rho(t) \) in the limit \( t \to \infty \).

Assuming, as it was done in section 5.1, a ratio \( \gamma_0/\lambda = 0.05 \), we obtain

\[
\lim_{t \to \infty} c_0(t) = \dot{c}_0(0) + \frac{2}{3}(1 + \dot{c}_1(0) + \dot{c}_2(0) + \dot{c}_3(0) + \dot{c}_5(0) + \dot{c}_6(0) - \dot{c}_8(0))
\]
\[ \lim_{t \to \infty} c_1(t) = \frac{1}{9} (1 + \dot{c}_1(0) + \dot{c}_2(0) + \dot{c}_3(0) - 2\dot{c}_5(0) - 2\dot{c}_6(0) + 2\dot{c}_8(0)) \]

\[ \lim_{t \to \infty} c_2(t) = \lim_{t \to \infty} c_3(t) = \lim_{t \to \infty} c_6(t) = \lim_{t \to \infty} c_1(t) \]

\[ \lim_{t \to \infty} c_5(t) = \lim_{t \to \infty} c_6(t) = -\left( \lim_{t \to \infty} c_1(t) \right) \quad (5.4.3) \]

where we must impose the condition

\[ 1 + \dot{c}_0(0) + \dot{c}_1(0) + \dot{c}_2(0) + \dot{c}_3(0) = 1 \]

in order to have a proper density operator with unitary trace.

As we can see, although we have shown in section 5.1 that decoherence effects arise from the non Markovian dissipative dynamic of a single qubit in a thermal reservoir, equations (5.3.3) suggest that, for a proper choice of the initial conditions \( \frac{d\rho(t)}{dt} \bigg|_{t=0} \), coherences \(^5\) of the density operator could rapidly increase in time up to a stationary value (see Fig. 5.3).

This means that common dissipative environments, under the non Markovian evolution, are able to create superposition of the qubits states and hence it is reasonable to expect that it also create entanglement between qubits.

However, the presence of entanglement will not be proved in this work, but it could be the aim of future studies.

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\(^5\) The coherences are by definition the off diagonal elements of density matrix (5.3.1), namely \( c_5(t), c_6(t) \) and \( c_8(t) \).
5.4.2 Single excitation in $A$

If we have three qubits, the density matrix (3.2.16) reduces to

$$\rho(t) = a_0 |g\rangle\langle g| + a_1 |k\rangle\langle k'\rangle + a_3 |b\rangle\langle b| + a_4 |c\rangle\langle c'| + a_6 \Omega'_b k + a_7 \Omega'_c k + a_{10} \chi_{bc}, \quad (5.4.4)$$

The set (5.2.3) becomes

$$\ddot{a}_0 = -\lambda \dot{a}_0 + \frac{1}{2} \gamma_0 \lambda [2a_1 + 2a_3 + 4a_4 + 4a_7 + 4a_{10}],$$

$$\ddot{a}_1 = -\lambda \dot{a}_1 + \frac{1}{2} \gamma_0 \lambda [-2a_1 - 2a_7],$$

$$\ddot{a}_3 = -\lambda \dot{a}_3 + \frac{1}{2} \gamma_0 \lambda [-2a_3 - 2a_10],$$

$$\ddot{a}_4 = -\lambda \dot{a}_4 + \frac{1}{2} \gamma_0 \lambda [-4a_4 - 2a_7 - 2a_{10}],$$

$$\ddot{a}_6 = -\lambda \dot{a}_6 + \frac{1}{2} \gamma_0 \lambda [-2a_6 - a_7 - a_{10}],$$

$$\ddot{a}_7 = -\lambda \dot{a}_7 + \frac{1}{2} \gamma_0 \lambda [-a_1 - a_4 - a_6 - 3a_7],$$

$$\ddot{a}_{10} = -\lambda \dot{a}_{10} + \frac{1}{2} \gamma_0 \lambda [-a_3 - a_4 - a_6 - 3a_{10}]. \quad (5.4.5)$$

with initial conditions $a_j(0) = \delta_{j,1}$ with $j = 0, 1, 3, 4, 6, 7, 10$.

Also in this case, it is possible to compute the solution using a computation program, and see what is the stationary state of the system. Assuming again a ratio $\gamma_0/\lambda = 0.05$, we obtain

$$\lim_{t \to \infty} a_0(t) = \dot{a}_0(0) + \frac{2}{3} (1 + \dot{a}_1(0) + \dot{a}_3(0) + \dot{a}_4(0) + \dot{a}_7(0) + \dot{a}_{10}(0) - \dot{a}_6(0))$$

$$\lim_{t \to \infty} a_1(t) = \frac{1}{9} (1 + \dot{a}_1(0) + \dot{a}_3(0) + \dot{a}_4(0) - 2\dot{a}_{10}(0) - 2\dot{a}_7(0) + 2\dot{a}_6(0))$$
\[ \lim_{t \to \infty} a_3(t) = \lim_{t \to \infty} a_4(t) = \lim_{t \to \infty} a_{10}(t) = \lim_{t \to \infty} a_1(t) \]
\[ \lim_{t \to \infty} a_6(t) = \lim_{t \to \infty} a_7(t) = -\left( \lim_{t \to \infty} a_1(t) \right) \]

(5.4.6)

where we must impose the condition

\[ 1 + \dot{a}_0(0) + \dot{a}_1(0) + \dot{a}_3(0) + \dot{a}_4(0) = 1 \]

in order to have a proper density operator with unitary trace.

Also in this case, equations (5.3.6) suggest that, for a proper choice of the initial conditions \( \frac{d\rho(t)}{dt}\bigg|_{t=0} \), coherences\(^6\) of the density operator could rapidly increase in time up to a stationary value (see Fig. 5.4).

Therefore, also in this case, common dissipative environments are able to create superposition under the non Markovian evolution, and it is reasonable to expect that entanglement between qubits it is created.

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\(^6\)The coherences are by definition the off diagonal elements of density matrix (5.3.4), namely \( c_6(t), c_7(t) \) and \( c_{10}(t) \).
Conclusions

In this thesis we have studied entanglement arising from the Markovian dynamics of two ensembles of qubits dissipating into overlapping environments. That is, having a certain number of qubits in common that dissipates to both environments. We have then computed, using Negativity, the bipartite entanglement between the two ensembles by excluding such common qubits and assuming a single initial excitation.

It was shown that the dynamics of the entanglement crucially depends on the initial condition, especially on whether the single initial excitation is in the common qubits $C$ or in the side qubits $A$ or $B$.

We proved that the dissipatively created stationary entanglement exhibits a monotonic scaling versus the number $n$ of common qubits. In particular, the largest amount of entanglement is created when excitations are initially located among side qubits. This can be explained by the fact that an initial single excitation among common qubits tends to persist in the common part rather than being shared by the side parts.

Moreover stationary entanglement exhibits a non-monotonic behaviour versus the number of side qubits $N$. This should be ascribed to competing effects of side and common qubits.

Finally we extended the problem to the case of non-Markovian evolution, where we showed that the dissipative dynamics is able to create quantum superposition, and thus it is reasonable to think that also entanglement could be generated. However, the presence of entanglement was not proved as we did in the Markovian case, but this will be the aim of future studies.

The examined model might be of interest for several physical systems.

Simulation of dissipative dynamics of small ensembles of qubits has been already engineered [30]. Furthermore, in extending the setup of [31] one could face up with the situation of using more than one laser beam (and related vacuum fluctuations), hence ending up with overlapping environments. A similar situation can arise in cavity QED experiments with cavities hosting ensembles of atoms and connected by fibres which play the role of an environment [32]. Quite generally, when one has an array of atomic ensembles,
like array optical traps loaded with neutral atoms [33] or array of quantum dots [34], it may happen that the bath affecting one site can extend its effect over the neighbours sites.

In addition, the study of the non-Markovian dissipative dynamics, presented in this work, might be completed by investigating the formation of entanglement and it may lead to the publication of a new paper. Anyway, the analysed model can be further extended in the future, for example, looking for solutions with more general initial conditions or considering additional dissipative environments or, assuming a temperature of the thermal bath higher than the absolute zero.

Finally, we are confident that the present study sheds further light on dissipative quantum dynamics and on how entanglement is induced by the interplay of environments, thus enriching the knowledge on topics that are becoming increasingly exploited in quantum information processing.
Bibliography


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