Thermal projection system
for surface figure correction
of core optics in
Gravitational Waves interferometers

Tesi di Dottorato di Ricerca in Fisica Sperimentale
In Partial Fulfillment of the Requirements for the Ph.D. Degree in Experimental Physics

Candidate
Annalisa Allocca

Supervisor
Dr. Richard Day

Tutor
Dr. Carmela Marinelli

Ciclo di Dottorato XXVII
## Contents

**Introduction**  

1 Detecting Gravitational Waves  

1.1 Gravitational Waves: a new picture of the universe  
1.1.1 GW as solution of Einstein’s equations  
1.1.2 The effect of a GW  

1.2 GW signal from astrophysical sources  
1.2.1 GW generation and amplitude of the signal  
1.2.2 Indirect measurement of Gravitational Waves  

1.3 GW Detectors  
1.3.1 Michelson interferometry  
1.3.2 Increasing arm length with Fabry-Perot cavities  
1.3.3 The Pound–Drever–Hall technique for the lock-in detection  
1.3.4 Increasing laser power with Recycling cavities  
1.3.5 Enhancing the signal and shaping the detector response  
1.3.6 Stability conditions for the recycling cavities  

1.4 The era of Advanced Gravitational Waves detectors  
1.4.1 Detectors scheme  
1.4.2 Detector sensitivity  

2 Corrective thermal compensation devices  

2.1 Outline of the problem  
2.1.1 Mode mismatch  
2.1.2 Light scattering  
2.1.3 Frequency splitting  

2.2 The Thermal Compensation  
2.2.1 The Principle  

2.3 Thermal Compensation devices for GW Interferometers  
2.3.1 Cavity Matching improvement with TDM  
2.3.2 Mirror Radius of Curvature Correction
2.3.3 High Spatial Frequency defects reduction

3 Light scattering into High Order Modes

3.1 The coupling mechanism

3.1.1 Wavefront spoiling from an aberrated surface
3.1.2 Scattering into Higher Order Modes

3.2 HOMs in different cavity configurations

3.2.1 Marginally stable cavities
3.2.2 Stable cavities
3.2.3 Cavity locked on High Order Modes

3.3 Selectively suppress HOM

3.4 Reconstruct mirror maps from HOM’s frequency shifts

3.5 Conclusions

4 Thermal Compensation System: the principle

4.1 Thermal effects in optical materials
4.2 Heat equation

4.2.1 Solution of the heat equation
4.3 Thermoelastic deformations in the steady-state

4.3.1 Solution of the Hooke’s equations

4.4 The thermo-elastic problem in the Fourier domain

4.5 System transfer function

4.5.1 Numerical evaluation of mirror transfer function
4.6 Heat pattern calculation
4.7 Linearity with the absorbed power
4.8 Coating reflectivity and its alteration

4.9 Conclusions

5 The CHRAC design

5.1 General features

5.1.1 Source radiation distribution
5.1.2 Mirror absorption
5.1.3 Numerical aperture of the optical system
5.1.4 Optical telescope main features
5.1.5 Focusing the image on the mirror
5.1.6 Image deformation and intensity re-scaling

5.2 CHRAC as actuators matrix

5.2.1 Choice of pixel size in the image plane
5.2.2 Choice of actuator size in the object plane

5.3 CHRAC setup for table-top experiments

5.3.1 CHRAC setup using Digital Micromirror Device
Poiché del terribile il bello
non é che il principio, che ancora noi sopportiamo,
e lo ammiriamo così, ché quieto
disdegna di annientarci.

R. M. Rilke, Prima Elegia Duinese.
Introduction

The Theory of General Relativity developed by Einstein nearly a century ago foresees that an accelerated massive body loses energy by emitting \textit{Gravitational Waves}, which show up in the form of space-time ripples \cite{1, 2}. The existence of Gravitational Waves has been indirectly proved with the discovery of the binary pulsar PSR1913+16 \cite{3}, but their direct detection has not yet occurred: indeed, a Gravitational Wave induces a displacement on free-falling masses of the order of $10^{-18}$ m. Such small displacements can be sensed as a phase shift between two interfering laser beams. For this reason, the detection of GW is mainly entrusted to large scale Michelson interferometers, whose sensitivity is enhanced by using a high power laser, Km-long Fabry Perot cavities in the arms and high performance suspension systems to filter out seismic noise and bring the mirrors as close as possible to the free-fall condition.

\textit{First generation} detectors operating in the past years, like GEO600 in Germany \cite{4}, Virgo in Italy \cite{5}, the two LIGO detectors in the United States of America \cite{6} and TAMA300 in Japan \cite{7}, did not claim any detection. In order to improve the performance by a factor of ten and therefore widen the event horizon \cite{9}, \textit{second generation} detectors were designed: among them, Advanced Virgo \cite{10} and Advanced LIGO \cite{8} are currently in the installation and commissioning phase, and will be both fully operational by 2016.

The main requirement for a Gravitational Waves interferometer to reach the design sensitivity consists of a nearly perfect destructive interference between the two beams reflected by the interferometer arms at the output of the detector. This condition, called \textit{dark fringe} condition, is fulfilled only if
the beams’ wavefronts perfectly match and cancel out each other. One of the mechanisms responsible for introducing asymmetric wavefront aberrations is represented by core optics residual roughness: indeed, mirror imperfections can scatter light from the fundamental mode into spurious modes. This process can ruin the symmetry between the reflected beams’ wavefronts and, consequently, spoil the dark fringe condition.

The aim of this thesis is to present a method to reduce wavefront aberrations due to core optics imperfections through an active control system of thermal correction: the Central Heating Residual Aberration Correction (CHRAC). The CHRAC consists of a matrix of pixels emitting heat radiation with a black-body distribution: an optical telescope images the matrix onto the high reflectivity surface of the mirror which, in turn, absorbs the radiation and deforms according to the projected heat pattern.

After an introduction concerning Gravitational Waves and their detection, the thesis will deal with the problem of identifying mirror defects mainly affecting the interferometer sensitivity and their removal through the system of thermal compensation.

The outline of the thesis is reported hereafter:

**Chapter 1** is devoted to the derivation of Gravitational Waves and to a brief review of the main astrophysical sources. Therefore, the main features of an interferometric detector are described and an overview of Advanced Virgo is given.

In **Chapter 2** the main problems arising from mirror residual roughness are discussed and the existing systems of thermal compensation are considered, each one specifically designed to deal with a particular kind of mirror defect. At the end of the Chapter, the CHRAC thermal projection system is introduced, and its main features are outlined.

Chapter 3 onwards are devoted to the CHRAC system working principle and design, and the presentation of the first experimental results. In particular, **Chapter 3** is mainly centered on the identification of the mirror figure errors which introduce wavefront aberrations and excite high order modes in resonant Fabry-Perot cavities. In **Chapter 4** a complete analytical
description of the physical mechanism at the base of the CHRAC is given. In particular, the problem is solved in the spatial frequency domain, which allows to compute a system transfer function and to easily determine the heat pattern to project onto the mirror.

How to design from scratch a thermal compensation system able to perform the needed correction is the topic of Chapter 5. In particular, the setup is designed in two different sizes, in order to apply it to a large-scale experiment as well as on a table top one.

Chapter 6 reports the first experimental results obtained with these two CHRAC setups compared to the simulations. Finally, the feasibility of CHRAC is demonstrated by its application to a table-top experiment in order to improve the cavity beam quality of a LG$_{33}$ mode, and the experimental results are reported in Chapter 7.
Chapter 1

Detecting Gravitational Waves

1.1 Gravitational Waves: a new picture of the universe

Sky objects emit radiation over the whole EM spectrum. It allows to take pictures of the universe at multiple wavelengths, ranging from Gamma-rays to Radio waves, and each picture gives us a different kind of information about the universe composition (fig. 1.1). How much more information could we get if we were able to take pictures using not-EM waves, like Gravitational Waves?

The detection of Gravitational Waves is first of all an additional confirmation to the Theory of General Relativity, but could also tell us about the dynamics of large-scale events in the Universe like death of stars and birth of black holes, giving us the possibility to understand physical mechanisms in strong gravity regimes which can’t be jet fully foreseen by the theory.
1.1.1 GW as solution of Einstein’s equations

In Special Relativity Einstein describes the spacetime as a 4-dimensional manifold with a flat metric defined by the Minkowsky tensor $\eta_{\mu\nu}$.

In General Relativity the space-time is bent by the energy-mass distribution of the universe and, at the same time, its curvature acts on matter and manifests itself as gravity. In this new perspective, the interval of two separated
events in space-time is described with the no-more flat metric tensor $g_{\mu\nu}$ as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$  \hfill (1.1)

and is governed by *Einstein’s field equations* [13]:

$$G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}$$  \hfill (1.2)

Here $G_{\mu\nu}$ is the *Einstein tensor*, which is a function of the metric and its first and second derivatives, $T_{\mu\nu}$ is the *impulse-energy tensor* and $G$ is Newton’s constant. The Greek indexes run from 0 to 3, and the space-time can be expressed as $x^\mu = (ct, x, y, z)$.

Einstein’s equations elegantly describe the space-time distortion $g_{\mu\nu}$ as a function of the energy-mass distribution of the universe. Due to their non linearity, Einstein’s equations are difficult to be solved. However, in the far field (i.e. far from the sources) we can consider the weak-field approximation and write the metric tensor as a small perturbation to the flat metric:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$$  \hfill (1.3)

where $h_{\mu\nu} \ll 1$. Moreover, we can consider the harmonic gauge for the coordinates, and re-write Einstein’s equations as:

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_i^2} \right) h_{\mu\nu} = -16\pi T_{\mu\nu}$$  \hfill (1.4)

which in the far field approximation ($T_{\mu\nu} = 0$) become:

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_i^2} \right) h_{\mu\nu} = 0$$  \hfill (1.5)

This equation clearly shows that, far from the sources, a small perturbation of the flat metric can be described by the wave equation and moves at the speed of light $c$: that’s what is called a Gravitational Wave, a freely-propagating degree of freedom of the gravitational field which doesn’t require any local source to exist.
1.1.2 The effect of a GW

A general solution of the vacuum field equation (1.5) can be written as superposition of plane waves:

\[ h_{\mu\nu} = \varepsilon_{\mu\nu} \exp[i(\omega_{GW} t - k \cdot r)] \] (1.6)

where the tensor \( \varepsilon_{\mu\nu} \) describes the polarization of the wave, \( \omega_{GW} \) is the angular frequency of the Gravitational Wave and \( k \) is the wave vector.

Imposing the so called Transverse Traceless gauge and choosing the wave vector \( k_i \) along the z-axis, the polarization tensor can be written as:

\[ \varepsilon_{\mu\nu} = h_+ \varepsilon_+^{\mu\nu} + h_\times \varepsilon_\times^{\mu\nu} \] (1.7)

where:

\[
\varepsilon_+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
\varepsilon_\times = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (1.8)

\( \varepsilon_+ \) and \( \varepsilon_\times \) are called, respectively, plus and cross polarization. To get the feeling for the physical effect due to Gravitational Waves, it is useful to consider the motion that they induce on nearby free falling particles. Let us consider two particles, one placed at the origin of the coordinates, and the other at \( x^i = (L, 0, 0) \). If a Gravitational Wave propagating along \( z \) passes through the particles, their relative distance will be given by:

\[ L' = \int |ds^2|^{1/2} = \int_0^L |g_{11}|^{1/2} dx \approx L + \frac{1}{2} h_{11} L \] (1.9)

therefore, the distance between the two particles will vary by the quantity \( \delta L = L - L' \), which is:
\[ \delta L = \frac{1}{2} h_{11} L \]  

(1.10)

The displacement is proportional to the distance between the particles \( L \) and to the amplitude of the wave \( h \). More in general, let us call \( L = (0, l_x, l_y, 0) \) the quadrivector defining the distance between two particles. The vector \( \delta L \) describing the displacement of the particles from their equilibrium position due to a \( plus \) polarized GW is:

\[
\delta L = \begin{pmatrix}
0 \\
\delta l_x \\
\delta l_y \\
0
\end{pmatrix} = \frac{1}{2} h_+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
l_x \\
l_y \\
0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 \\
h_+ l_x \\
h_+ l_y \\
0
\end{pmatrix} 
\]

(1.11)

As shown in figure (1.1.2) top, free fall particles are distorted in a \( plus \) shape, hence the name \( plus \) for this polarization. Similarly, for the \( cross \) polarization we have:

\[
\delta L = \begin{pmatrix}
0 \\
\delta l_x \\
\delta l_y \\
0
\end{pmatrix} = \frac{1}{2} h_\times \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
l_x \\
l_y \\
0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 \\
h_\times l_y \\
h_\times l_x \\
0
\end{pmatrix} 
\]

(1.12)

and the test masses distortion is shown in the bottom of figure (1.1.2). From these equations we can also deduce that GW are transverse waves, since they vary the relative distance along the directions perpendicular to the wave propagation direction.

\section*{1.2 GW signal from astrophysical sources}

\subsection*{1.2.1 GW generation and amplitude of the signal}

To better understand the mechanism of generation of gravitational waves, we must go back to the equations coupled to matter (1.4). As for electro-
Figure 1.2: Effect of a GW plus polarized (top) and cross polarized (bottom) passing through a set of free falling particles.

magnetism, the solution can be obtained appealing to the Green’s function \( G(x_\alpha - y_\alpha) \), so that the general solution to equation (1.4) can be written as:

\[
h_{TT}^{\mu \nu}(x_\alpha) = -16\pi G \int G(x_\alpha - y_\alpha) T_{\mu \nu}(y_\alpha) d^4y \tag{1.13}
\]

Making the approximation that the source is isolated, far away from the observer and slow moving, we can express the the Gravitational Wave field as:

\[
h_{TT}^{ij}(r, t) = \frac{2G}{rc^4} \left[ \frac{d^2}{d^2 t} I_{ij} \right]_{t - r/c} \tag{1.14}
\]

where \( r \) is the distance from the source to the observer and \( I_{ij} \) is the quadrupole momentum tensor associated to the energy density of the source and calculated at the time \( t_r = t - r/c \):

\[
I_{ij}(x, t) = \frac{1}{c^2} \int_V d^3x T_{00} x_i x_j(x, t) \tag{1.15}
\]

To clarify why the quadrupole term is the lowest order contributing to the GW emission, we can again refer to electromagnetism for a comparison. In electromagnetism, a dipole changing in time corresponds to a motion of the
center of charge density, which can oscillate without violating any physical conservation law. On the other hand, in gravitation a changing dipole corresponds to a motion of the center of mass, which cannot happen unless also another center of mass, gravitationally interacting with the first, changes. If it’s not the case, the momentum conservation is violated. Equation (1.14) shows that the farther the source the smaller the amplitude of the gravitational wave detected by the observer. We can also calculate how much power is emitted by a source considering the luminosity of the system:

\[ \mathcal{L} = \frac{G}{5c^5} \sum \left| \frac{d^3 I_{jk}}{dt^3} \right|^2 \]  

(1.16)

To estimate the order of magnitude of the power emitted by a gravitational source, consider a body of mass M and radius R, and suppose that T is the time scale over which the quadrupole momentum varies on. Defining \( \epsilon \) the factor measuring mass asymmetry in its spatial distribution, the quadrupole moment can be approximated by \( Q \approx \epsilon MR^2 \). With these definitions, equation (1.16) becomes:

\[ \mathcal{L} = \epsilon \frac{c^5}{G} \left( \frac{r_{Sch}}{r} \right)^2 \left( \frac{v}{c} \right)^6 \]  

(1.17)

with \( r_{Sch} \) the Schwarzschild radius of the body.

With this formula in our hands, we can estimate the order of magnitude of the GW amplitude while reviewing the known sources and their expected signal.

**Monocromatic signal from spinning objects**

A possible source of GW is represented by spinning pulsars non-axisymmetric or with spin axis not aligned to the principal axis. Their spin frequency can be considered constant over a long time period, so that the emitted GW can be thought as monocromatic.

The amplitude of the wave depends on the momentum of inertia of the star along the spin axis \( I \), on the distance from the Earth and on the ellipticity
In this equation each parameter appears divided by a typical value that they can assume.

In our galaxy, a huge number \((10^8 - 10^9)\) of neutron stars can be found, but their speed is too slow or their ellipticity too small to give rise to a detectable signal. To give an example, for the Vela pulsar, expected to emit at \(f \approx 20\text{ Hz}\), an upper limit for the amplitude of emitted GW is \(h = 2 \times 10^{-24}\) \[15\].

### Bursts from supernovae explosion

The explosion of a very massive object like a supernova, if non axisymmetric, can generate a strong and hopefully detectable GW. The shape of the wave is not fully predictable, since it depends on the mass distribution of the exploding star, as well as the signal frequency, which is expected to be below \(1\text{ kHz}\), but is anyway strongly related to the mechanism leading to the star explosion. Nonetheless, having an idea of the star mass, it is possible to estimate the order of magnitude of the GW amplitude:

\[
h \approx 9.6 \cdot 10^{-20} \left(\frac{10\text{ kpc}}{r}\right)
\]  
(1.19)

which is in the detection region for present GW interferometers.

### Chirp from coalescing compact binaries

The coalescence of binary systems formed by neutron stars (NS-NS), black holes (BH-BH) or mixed systems (NS-BH) is defined as *Compact Binary Coalescence*. Since they are massive objects, they interact gravitationally with a very strong field, and their coalescence can produce detectable GW. The signal emitted during a coalescence can be divided into three phases, whose relative duration and importance depend upon the mass composition.
of the system. The first one is the “inspiral” phase: the orbit shrinks and the two stars rotate faster and closer to each other. This signal they produce is called chirp, since its shape resembles the twittering of birds. Then there is the “merger”, which takes place when the distance between the two stars becomes comparable to the sum of their radii. In this phase the waveform is very difficult to predict, since the week-field approximation is no more valid. Finally is the “ringdown” phase, in which the object resulting from the merging of the two original stars relaxes towards a stable state, emitting a damped sinusoid.

Defining the chirp mass \( \mathcal{M} = \mu^{3/5}M^{2/5} \), where \( M = m_1 + m_2 \) is the total mass of the system and \( \mu = m_1m_2/M \) is the reduced mass, we can get an idea of the order of magnitude of the signal amplitude:

\[
h \propto 10^{-19} \left( \frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left[ \frac{Mpc}{r} \right] \]

Moreover, we can estimate the frequency of the “innermost stable circular orbit” (ISCO) as:

\[
f_{ISCO} \approx \frac{4.4kHz}{(M/M_\odot)} \]

Figure 1.3: Chirp signal produced by NS-NS system.
Stochastic background

Stochastic background is a relic from the early evolution of the universe which arises from a large number of random, independent events. According to the inflation model, gravitational waves were produced between approximately $10^{-36}$ to $10^{-32}$ seconds after the Big Bang. For this reason, the detection of GW from the Stochastic background could tell us a lot about the very beginning of the universe.

Although it is not possible to identify the Stochastic background with a specific waveform, we can nonetheless specify its spectrum [16]:

$$\Omega_{GW}(f) = \frac{f}{\rho_c} \frac{d \rho_{GW}}{df}$$  \hspace{1cm} (1.22)

where $d \rho_{GW}$ is the energy density of gravitational radiation contained in the frequency range $f$ to $f + df$ and $\rho_c$ is the critical energy density required to close the universe:

$$\rho_c = \frac{3c^2H_0^2}{8\pi G}$$  \hspace{1cm} (1.23)

with $H_0$ Hubble expansion rate.

1.2.2 Indirect measurement of Gravitational Waves

So far, Gravitational Waves have never been directly detected, even though the discovery in 1974 of the pulsar binary system PSR B1913+16 represents an indirect evidence of their existence. The system is composed of a pulsar which together with another neutron star orbits around their common center of mass. The General Relativity foresees that the system loses energy and angular momentum by emitting Gravitational Waves, which translates into a decrease of the orbital period of the pulsar. This decrease has been measured by Hulse and Taylor and compared to the value predicted by the theory. The measurements have been carried on over more than 30 years, and the result is reported in fig. [L4]. The perfect agreement between the prediction and the experimental data represents an indirect proof of the existence of Gravitational Waves. Thanks to this discovery, Hulse and Taylor won the
Figure 1.4: Orbital decay of PSR B1913+16. The theoretically expected change in the epoch of periastron according to General Relativity is represented by the parabola, while the data points indicate the observed change up to 2005. Plot from [17].

Nobel prize in 1993.

1.3 GW Detectors

The difficulty in the direct detection of Gravitational Waves relies in their very small amplitude: a typical signal is expected to produce a displacement of the order of $10^{-18}$ m, and a very sophisticated instrument is needed to reveal such a small effect.

The first effort to detect gravitational waves was performed by Joe Weber, using resonant bars at room temperature: strains in space due to a Gravitational Wave can excite the bar resonance frequency, and such an excitation
can be amplified up to detectable levels. Modern versions of Weber’s bar are cryogenically cooled and use superconducting quantum interference devices to detect vibrations \[18\]. However, due to their higher sensitivity and to wider bandwidth, the interferometric detectors of Gravitational Waves have overcome the resonant bars detectors.

The reason why interferometric detectors are particularly suited for Gravitational Waves detection is explained in the next section.

1.3.1 Michelson interferometry

Let us consider figure \[1.5\]. A laser beam hits a partially reflecting mirror (beam splitter) and gets split in two. Each of the two beams travels along an independent path (called arm) until they get reflected back by a mirror. Then they recombine at the beam splitter and interfere with each other. This is the basic working principle of a Michelson interferometer. Since the phase accumulated by the laser beam traveling along a path is proportional to the length of the path \(\Delta \phi = \frac{2\omega}{c}L\), any small difference with respect to the initial length in terms of phase will be:

\[
\delta \phi = \frac{2\omega}{c} \delta L
\]

(1.24)

Thanks to its configuration, a Michelson interferometer is particularly well suited for the GW detection purpose because of the geometry of the strains that the wave produces. Indeed, as described above, the effect of a gravitational wave propagating along \(z\)-direction is to change the distance between free-falling masses in the \(x-y\) plane and, as shown in equations (1.11) and (1.12), the displacement is proportional to the amplitude of the GW:

\[
\frac{\delta L}{L} = \frac{h}{2}
\]

(1.25)

If we use the mirror at the end of each interferometer’s arm as free-falling test mass, we can write the length variation of the interferometer arm due to
the gravitational wave in terms of phase variation:

$$\delta \phi = \frac{\omega \chi}{c} L \mod \pi$$ \hfill (1.26)

This equation shows that a Michelson interferometer is a good candidate as detector of Gravitational Waves, as long as we can make the signal big enough to be detected and decoupled from any other external noise which could cause a shake of the mirrors.

![Figure 1.5: Schematic of a Michelson interferometer.](image)

![Figure 1.6: Sign conventions.](image)

To see where the GW signal is embedded into, we can write the field at the antisymmetric port of the interferometer (denoted as (ASY) in the figure) using the plane wave description. Before doing it, let’s state the sign conventions: Referring to figure (1.6), we have

$$E_{Rf} = -rE_f$$ \hfill (1.27)
$$E_{Rb} = rE_b$$

where $r$ is the mirror reflectivity. With this conventions the field at the antisymmetric port is:

$$E_{ASY} = E_0 t_{bs} r_{bs} \left( r_x e^{2i kl_x} - r_y e^{2i kl_y} \right)$$ \hfill (1.28)

Here $t_{bs}$ and $r_{bs}$ are, respectively, the transmissivity and the reflectivity of
the beam splitter, while \( r_x \) and \( r_y \) are the reflectivity of the mirror at the end of the \( x \) and \( y \) arm, respectively, and \( l_x \) and \( l_y \) are two arm lengths. Defining the quantities:

\[
l_- = \frac{l_y - l_x}{2} \quad \text{and} \quad l_+ = \frac{l_y + l_x}{2}
\]

we can rewrite equation (1.28) as:

\[
E_{ASY} = E_0 l_{bs} r_{bs} e^{2ikl_-} \left( e^{2ikl_-} - e^{-2ikl_-} \right)
= E_0 l_{bs} r_{bs} e^{2ikl_-} 2i \sin (2kl_-)
\]

(1.29)

The intensity of any signal coming from the antisymmetric port is sensitive only to the differential motion of the end mirrors. Similar arguments show that common modes appears only at the symmetric port.

The length difference between the two arms can be written as the sum of a macroscopic difference \( \Delta l_- \) and a microscopic difference \( \delta l_- \). The macroscopic difference can be fixed such as the beams coming from the two arms interfere destructively and no power is detected at the antisymmetric port (which for this reason is also called the dark port), unless there is a microscopic difference due, for example, to the effect of a GW. In this case, taking the square of equation (1.29) we can calculate the power detected at the dark port:

\[
P_{ASY} \propto P_0 \sin^2 (2k\delta l_-) \approx P_0 (2k\delta l_-)^2
\]

(1.30)

which is proportional to the square of \( \delta l_- \). As we already discussed in the first chapter, the signal we want to detect is very small, and the displacement produced is of the order of \( 10^{-18} \) m, about 3 orders of magnitude less than the classical electron radius. This numbers suggest that we need to improve the signal as much as possible. Firstly, our signal must be proportional to \( h \) instead of \( h^2 \). We'll see in next paragraph how to obtain a linear signal. Moreover, equations (1.26) and (1.30) can help in understanding which is the way to enhance the effect product by a gravitational wave: longer interferometer arms and higher input power \([19]\).
1.3.2 Increasing arm length with Fabry-Perot cavities

As shown in the last section, longer arms give larger signals. Ground based detectors cannot have arbitrary long arms, but their effective length can be increased by bouncing the light back and forth inside them. The way to afford it, is to insert Fabry-Perot cavities in each arm of the Michelson interferometer.

Static intra-cavity field

A Fabry-Perot cavity is composed by two mirrors $M_1$ and $M_2$, and $r_1$, $r_2$, $t_1$ and $t_2$ are their respective reflectivity and transmissivity. Referring to figure (1.8) and keeping in mind the sign convention of figure (1.6), we can compute the propagating fields:

\[ E_{cav} = t_1 E_{in} + r_1 r_2 e^{-2ikL} E_{cav} \]
\[ E_{ref} = r_1 E_{in} - t_1 r_2 e^{-2ikL} E_{cav} \]
\[ E_{trans} = t_2 e^{-ikL} E_{cav} \]
From the first equation, we get the expression for the cavity field, and therefore for the intracavity power:

\[ P_{\text{cav}} = |E_{\text{cav}}|^2 = \frac{|t_1|^2}{(1 - r_1 r_2 e^{-2ikL})^2} P_{\text{in}} \]  

(1.32)

The resonance condition occurs when \( e^{-2ikL} = 1 \). In this case, the cavity gain is given by:

\[ G_{\text{cav}} = \frac{|t_1|^2}{(1 - r_1 r_2)^2} \]  

(1.33)

The effect of a Gravitational Wave would induce a small detuning \( \delta L \) around the resonant position. Consequently, the power stored in the cavity will become:

\[ P_{\text{cav}} = \frac{|t_1|^2}{(1 - r_1 r_2)^2 + 4r_1 r_2 \sin^2 (k\delta L)} P_{\text{in}} = G_{\text{cav}} \frac{1}{1 + \left[ \frac{2\sqrt{r_1 r_2}}{1 - r_1 r_2} \sin \left( \frac{2\pi \delta L}{\lambda} \right) \right]^2} P_{\text{in}} \]  

(1.34)

Defining the cavity \textit{Finesse} as:

\[ F = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2} \]  

(1.35)

it is possible to re-write eq. (1.34) as:

\[ P_{\text{cav}} = G_{\text{cav}} \frac{1}{1 + \left( \frac{2F}{\pi} \right)^2 \sin^2 \left( \frac{2\pi \delta L}{\lambda} \right)} P_{\text{in}} \]  

(1.36)
Figure 1.9: Schematic of the Fabry-Perot cavity and fields at each point of the cavity.

On the other hand, the field reflected from the cavity can be written as:

\[
E_{\text{ref}} = \frac{r_1 + r_2(r_1^2 + r_2^2)e^{-2ikL}}{1 - r_1 r_2 e^{-2ikL}} E_{\text{in}}
\]  

(1.37)

and it can be shown [20] that, close to the resonance, the phase of the reflected field is given by:

\[
\phi = \frac{4\pi}{\lambda} \frac{2F}{\pi} \delta L
\]

(1.38)

which shows that the dephasing due to a length change is directly proportional to the cavity Finesse.

**Dynamical response**

So far we have considered the static response of a Fabry-Perot cavity, in other words we only looked at the power change as a function of the cavity detuning from its resonance condition. Another aspect which is important to consider is the cavity frequency response: indeed, the gravitational wave signal can be modeled as a sinusoidal signal shaking the mirrors at a certain frequency, and we are interested in knowing the cavity response at that particular frequency. Consider the schematic of figure (1.9). We can model the relative motion between the two mirrors as a sinusoidal waveform that applies only to the End mirror:

\[
x(t) = x_0 \cos \omega_s t
\]

(1.39)
Therefore, the field reflected by the mirror $M_e$ gets an additional dephasing:

$$E_4 = -r_e E_3 e^{-2ikx(t)} \quad (1.40)$$

If the displacement is very small with respect to the wavelength ($x_0 \ll \lambda$), we can expand the exponential and get:

$$E_4 = -r_e E_3 - 2i k r_e x_0 e^{-i \omega_s t} E_3$$

$$= -r_e E_3 - 2 \pi \frac{x_0}{\lambda} (e^{-i \omega_s t} + e^{i \omega_s t}) E_3 \quad (1.41)$$

Equation (1.41) shows that if the mirror is moved at a frequency $\omega_s$, two signal sidebands at the frequency of the signal are created inside the cavity. These additional sidebands propagate inside the cavity and undergo interference. It is possible to compute the field reflected from the cavity at $\omega_s$ we get [20]:

$$E_6(\omega_s) = -2 \pi \frac{r_e}{\lambda} \frac{1}{1 + F} \left( \frac{f_s}{c} \right)^2 E_1 x_0 \quad (1.42)$$

This equation shows that the cavity response as a function of the frequency has a simple pole at a frequency given by:

$$f_P = \frac{c^2}{4L} \quad (1.43)$$

where $\frac{c^2}{2L}$ is the cavity free spectral range and $F$ is the cavity finesse. Therefore, any sideband at a frequency lower than the cavity pole, gets an additional gain, while when its frequency is higher than the cavity pole, it is suppressed by the cavity filtering action.

### 1.3.3 The Pound–Drever–Hall technique for the lock–in detection

At the end of section 1.3.1 we saw that the power at the antisymmetric port is proportional to the square of the GW amplitude. It is possible to recover a signal linear in $h$ by using the *Pound-Drever-Hall technique* (PDH) [21][22]. This technique was invented to stabilize the frequency of a laser by locking it
to a Fabry-Perot reference cavity. In our case, we want to measure the small changes in the cavity length due to the effect of a Gravitational Wave.

To show how the technique works, let’s refer to figure ... . The laser beam incoming in a Fabry-Perot cavity, whose electric field can be written as $E_0 e^{i\omega t}$ is phase modulated through a Pockels cell, which is a crystal with an optical length that can be driven by changing an applied voltage, in such a way to have:

\[
E_{\text{inc}} = E_0 e^{i(\omega t + \beta \sin \Omega t)}
\]

\[
\approx E_0 \left[ J_0(\beta) + 2iJ_1(\beta) \sin \Omega t \right] e^{i\omega t}
\]

\[
= E_0 \left[ J_0(\beta)e^{i\omega t} + J_1(\beta)e^{i(\omega + \Omega)t} - J_1(\beta)e^{i(\omega - \Omega)t} \right]
\]

Here $\Omega/2\pi$ is the phase modulation frequency and $\beta$ is the modulation depth, and we expanded the exponential in terms of the Bessel functions $J_0$ and $J_1$.

This equation shows that there are actually beams with three different frequencies incident on the cavity: a carrier, with frequency $\omega/2\pi$, and two sidebands, with frequencies $(\omega \pm \Omega)/2\pi$. We are interested in the field reflected
from the cavity. The reflectivity for the carrier field is given by equation (1.37). However, for the sidebands the reflectivity is slightly different, due to the fact that they accumulate a different dephasing. In general, we can define \( R(\omega) \) as the reflectivity at the frequency \( \omega \). In this terms, the reflected field can be written as:

\[
E_{\text{ref}} = E_0 \left[ R(\omega)J_0(\beta)e^{i\omega t} + R(\omega + \Omega)J_1(\beta)e^{i(\omega+\Omega)t} - R(\omega - \Omega)J_1(\beta)e^{i(\omega-\Omega)t} \right]
\]  
(1.45)

and therefore the reflected power \( P_{\text{ref}} = |E_{\text{ref}}|^2 \) will be:

\[
P_{\text{ref}} = P_c|R(\omega)|^2 + P_s|\operatorname{Re}[R(\omega)R^*(\omega + \Omega) - R^*(\omega)R(\omega - \Omega)]| \cos \Omega t
+ \operatorname{Im}[R(\omega)R^*(\omega + \Omega) - R^*(\omega)R(\omega - \Omega)] \sin \Omega t
+ (2\Omega \text{ terms})
\]  
(1.46)

Apart from the DC term, we have oscillating terms which are called the in-phase term, which is proportional to \( \sin \Omega t \) and is in phase with the laser phase modulation, and 90-out-of-phase term, proportional to \( \cos \Omega t \). De-modulating the in-phase signal at the modulation frequency, we get the PDH signal:

\[
\varepsilon = 2\sqrt{P_cP_s}\operatorname{Im}[R(\omega)R^*(\omega + \Omega) - R^*(\omega)R(\omega - \Omega)]
\]  
(1.47)

which is a signal in DC.

Let us focus the initial problem. We want to relate the laser phase shift to a signal which is linear in \( \delta L \) and, therefore, in \( h \). If the carrier is near to the resonance and the sidebands are not, we can assume that the sidebands are totally reflected. In this condition eq. (1.47) becomes:

\[
\varepsilon \approx 2\sqrt{P_cP_s}\operatorname{Im}[R(\omega)]
\]  
(1.48)

Close to the resonance, for a small detuning \( \delta L \) the phase of the beam is given by:

\[
\phi \approx 2\pi N + 4\pi \frac{\delta L}{\lambda}
\]  
(1.49)
Rearranging the terms, we end up with an error signal given by:

$$\varepsilon = -16\sqrt{P_c P_s \frac{F}{\lambda}} \delta L$$  \hspace{1cm} (1.50)$$

which shows that we have an error signal linearly proportional to $\delta L$.

### 1.3.4 Increasing laser power with Recycling cavities

![Figure 1.11: Schematic of a Michelson interferometer with Power Recycling mirror.](image)

We have seen that we can increase the Michelson interferometer arm lengths by adding Fabry-Perot cavity inside them, and linearize the dependence of the signal upon the gravitational wave amplitude by using the PDH technique to extract the signal. Another trick we can use to increase the interferometer sensitivity is to boost the input power.

As already mentioned before, the interferometer working point is kept at the dark fringe condition. According to the energy conservation, if there’s no light at the antisymmetric port, we should find it reflected back towards the laser. Recycling this wasted power and sending it back into the interferometer can help us to improve the sensitivity of the instrument without
having to develop a more powerful laser. To do that, we can add a partially transmitting mirror between the beam splitter and the laser (fig. 1.11). The whole interferometer can be considered as a compound mirror, and we can use it as the end mirror of a Fabry-Perot cavity placing a second mirror between it and the laser. This second mirror acts as the input mirror, and controlling the distance between it and the interferometer we can build up a standing wave between the two, effectively increasing the power incident on the interferometer by a factor proportional to the finesse of the resulting cavity, said for this reason recycling cavity, while the new mirror introduced is called recycling mirror.

1.3.5 Enhancing the signal and shaping the detector response

![Figure 1.12: Schematic of a Michelson interferometer with Power Recycling mirror and Signal Recycling mirror.](image)

One of the main features which identifies an Advanced Detector is the presence of a mirror put between the interferometer and the dark port, which is called signal recycling mirror (fig. 1.12). An interferometer with both
power recycling and signal recycling mirrors is said dual-recycled. Introducing this last mirror, we have listed all the interferometer optics which are referred to as core-optics.

The presence of the signal recycling mirror does not affect the carrier field: indeed, the carrier beams reflected by the arms interfere destructively at the beam splitter and get completely reflected at the symmetric port, while the sidebands coming from arms differential displacements are completely transmitted to the antisymmetric port. Therefore, we can put a mirror at the antisymmetric port to create an additional recycling cavity which will be seen only by the audio-sidebands [20]. The presence of this mirror allows to shape the detector frequency response in terms of the signal recycling cavity tuning defined as:

$$\phi = kl_s + \frac{\pi}{4}$$  \hspace{1cm} (1.51)

where, referring to figure [1.12] we have $l_s = l_{SRC} + (l_N + l_W)/2$. To understand the effect of this phase, we must refer to the interferometer frequency response (fig. [1.13]). For a tuning $\phi = 0$ the detector band-width is increased, even if the optical gain is lower (broad-band signal recycling configuration). For $\phi = \pi/2$ the low frequency sensitivity is improved, but the bandwidth becomes narrower (tuned signal recycling configuration). Therefore, thanks to the signal recycling it is possible to tune in frequency the interferometer response, allowing to optimize the detector for different astrophysical sources.

1.3.6 Stability conditions for the recycling cavities

The stability conditions for a Fabry-Perot cavity are briefly summarized in (A.3). A cavity is said to be stable if the additional Gouy phase shift acquired by the cavity high order modes (equation [A.7]) is larger than the cavity line-width$^1$ in this way the high order modes will no more be resonant inside the cavity. The cavity line-width is defined as the Full Width at Half Maximum of the cavity peak:

$$\delta L_{FWHM} = \frac{\lambda}{4F}$$

with $F$ the cavity Finesse.
cavity. This effect is called mode cleaning of stable resonant cavities, and allow only one mode to resonate inside them.

If, instead, the Gouy phase shift is smaller than the cavity line-width, the high order modes can be amplified by the cavity and therefore become resonant. This is the case for marginally stable cavities, as Advanced Virgo recycling cavities, which are not able to completely filter out high order modes. To limit this effect, it becomes crucial to reduce optical defects, either by improving the quality of mirror surface or by introducing compensation systems able to correct aberrations and therefore avoid the high order mode generation. This topic will be widely discussed throughout this thesis.

1.4 The era of Advanced Gravitational Waves detectors

First generation kilometer-scale interferometers like Virgo (in Cascina, Italy) [5], LIGO (in Hanford, WA and Livingstone, LA) [6] and GEO600 (in Han-
Figure 1.14: Simplified optical layout of Advanced Virgo only showing the main elements. Schematic from [41].

Innov, Germany) [4] were operative for science runs lasting few months each between 2005 and 2011, also performing joint data taking. During this period no detection was claimed, but a lot of experience was accumulated and new upper limits were stated.

The second decade of the twenty-first century is the era of Advanced Detectors: Advanced LIGO and Advanced Virgo will be both fully operative by 2016, GEO600 will operate an upgrade which will bring it to a competitive sensitivity for High Frequency signals (GEO-HF). Finally, a 3 Km cryogenic underground detector called KAGRA is being built in Japan [11]. This network of detectors will allow to improve the accuracy in the reconstruction of the source angular position.
1.4.1 Detectors scheme

The optical configuration of the Advanced detectors has the general scheme of a dual-recycled Michelson interferometer with Fabry-Perot arm cavities with a Finesse of about 450. Here we will briefly describe the structure of Advanced Virgo, which is very similar to that of the LIGO detectors. Advanced Virgo optical scheme is shown in figure (1.14). The laser beam frequency is pre-stabilized on a reference cavity and therefore sent to the interferometer. An Electro-Optic Modulator (EOM) (which is a Pockels cell) provides five modulation frequency for the control of the interferometer. Therefore, the beam passes through a Faraday isolator to isolate the laser from the light reflected back from the interferometer, and finally is filtered by a triangular Fabry-Perot cavity called Input Mode Cleaner (IMC). Finally the beam is injected into the dual-recycled Fabry-Perot Michelson interferometer. In order to reproduce the condition of free-falling masses, all the core optics of the interferometer are suspended.

The goal of Advanced detectors is to increase the number of observable sources by a factor 1000, which implies that the sensitivity has to be improved by a factor 10. Below we will briefly see which are the main noise sources limiting the interferometer sensitivity and the solutions adopted to limit them.

1.4.2 Detector sensitivity

The signal $s(t)$ output of the detector is given by the sum of a gravitational wave signal $h(t)$ and a noise $n(t)$: the detector sensitivity $\tilde{h}(t)$ is actually defined in terms of the spectral density of this noise. Assuming that the noise is Gaussian and stationary, we can write the one-sided noise spectral density $S_n(f)$ as:

$$\frac{1}{2} S_n(f) = \int e^{i2\pi ft} C_n(t) dt$$

(1.52)
where \( C_n(t) \) is the noise autocorrelation between time 0 and \( t \), and the detector sensitivity is defined as:

\[
\tilde{h}(t) \equiv \sqrt{S_n(f)}
\] (1.53)

It is measured in \( 1/\sqrt{Hz} \).

Great part of the sensitivity band is limited by the fundamental noise: it arises from the quantum fluctuations of the laser power which translate into noise at the read-out photodiode on the one hand and in mirror displacement for radiation pressure noise on the other hand. In addition to that, environmental and technical noise sources must be taken into account.

**Quantum noise**

The detector sensitivity is limited by quantum noise. The first contribution to the quantum noise is given by the *shot noise*, which arises from the fact that the average number of photons impinging on a photodiode is fixed, but its statistics is Poissonian. In terms of the equivalent noise spectral density it translates into:

\[
\tilde{h}(t)_{sn} = \frac{\lambda}{2\pi L} \sqrt{\frac{\hbar \omega}{\eta P_0}} [1/\sqrt{Hz}] 
\] (1.54)

where \( \lambda \) is the laser wavelength, \( \hbar \omega \) is the energy of the incoming photon, \( \eta \) is the photodiode quantum efficiency and \( P_0 \) is the input power. This equation shows that the sensitivity is enhanced by increasing arm length and input power.

Another contribution to the quantum noise is given by the *radiation pressure noise*: the impact of the photons on the mirrors induces a mirror displacement. Unlike the shot noise, which is flat in frequency, the radiation pressure noise is proportional to \( 1/f^2 \). Again in terms of the equivalent noise spectral density it is:

\[
\tilde{h}(t)_{rp} = \frac{1}{mLf^2} \sqrt{\frac{\hbar P_0}{2\pi^3 c \lambda}} [1/\sqrt{Hz}] 
\] (1.55)

In this case, the radiation pressure noise increases with the square root of the input power, therefore it is expected to dominate the sensitivity at low
frequency. The effect of radiation pressure noise can be reduced by increasing the test masses weight. To go beyond the quantum limit, it is foreseen to employ squeezed light injection [23].

Seismic and gravity gradient noise

The dominant noise at frequencies below 10 Hz is represented by seismic noise originating from vibrations of the ground which are transferred to the test masses via their suspensions. The typical linear spectral density of seismic displacement can be modeled as:

\[ \tilde{x} \simeq \frac{\alpha}{f^2} \]  

where the coefficient \( \alpha \) depends on the site, and is about \( 10^{-7} \text{ m} \cdot \text{Hz}^{3/2} \) in Cascina. In order to attenuate seismic noise by at least 10 orders of magnitude, all test masses are suspended to a chain of 7 pendulums about 8-m long called Superattenuators (SA) (fig. 1.15). The suspension point of the chain is at the top stage of an inverted pendulum with a three-leg structure. The seismic noise is filtered above its resonance frequency of about 30 mHz. The inverted pendulum is used as passive pre-isolation stage to reduce the seismic motion of the top suspension point. For the very low frequency motion, the position of the suspension point is actively controlled through a system of position sensors and actuators.

Thermal noise

In the 10-200 Hz range one of the main noise sources is thermal noise. This includes several different types of thermal noise, from the Brownian noise in the mirror substrate and coating to the pendulum and violin modes of the mirror suspensions. To reduce mirror thermal noise, several solutions may be adopted [24]:

- the cooling of the mirrors using cryogenics: this solution will be adopted by the Japanese underground interferometer KAGRA, and is foreseen as possible choice for third generation detectors;
Figure 1.15: Advanced Virgo superattenuator. From top to bottom, we find the three legs of the inverted pendulum, the filter 0 surrounded by the top ring, the passive filters 1 to 4, and the mirror suspension. The last stage is represented by the marionetta and the actuation stage, devoted to the mirror positioning at frequencies above 10 mHz.
• the choice of materials with good mechanical properties and low absorption for the infrared

• the increase of the beam size on mirrors and the use of larger or non Gaussian beams.

The idea of using flat and wide non-Gaussian beams has been initially suggested for the so called mesa or flat-top beams, up to the proposal of using higher-order LG_{p}^{l} beams [25], [26, 27]: it is because, for the same mirror diameter and in equivalent conditions of diffraction losses and beam power, high order LG_{p}^{l} have a multi-ringed power distribution which is wider than the one of the fundamental Gaussian mode. It allows to strongly reduce the impact of mirror thermal noise on interferometer sensitivity. The use of LG_{p}^{l} modes has been foreseen for the third generation interferometer Einstein Telescope [12], but before their implementation the generation and the application of LG_{p}^{l} modes has to be carefully studied and explored. Many table-top experiments have been carried out to study the features of these modes [28, 29], and have highlighted the issues arising when operating the interferometer with these modes.

On the other hand, thermal noise also translates into suspension noise. To reduce it, fused silica suspensions are used in place of the steel wires. These suspensions are attached directly to the mirror, and are referred to as monolithic suspension. Thanks to this high quality material, excitation of the modes of the suspensions are confined to a narrow frequency band.

Stray light

The roughness of the surfaces encountered by the beam along its path as well as spurious reflections from non ideal coatings can generate scattered light at small and large angle which can spoil the dark fringe condition. To limit this effect, baffles and diaphragms are places in suitable positions inside the interferometer to screen rough surfaces and discontinuities. On the other hand, core optics aberrations are meant to be controlled through a suitable system of Thermal Compensation, as will be more widely explained in the
The accomplishment of all the requirements for each subsystem should bring Advanced Virgo to reach the design sensitivity shown in figure 1.16. Reaching this sensitivity will allow to have a detection rate about 4 order of magnitude higher as for the case of the Virgo detector, as reported in table 1.1.

<table>
<thead>
<tr>
<th>Source</th>
<th>AdV detection rate [yr⁻¹]</th>
<th>Virgo detection rate [yr⁻¹]</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS-NS</td>
<td>17</td>
<td>4.5·10⁻³</td>
</tr>
<tr>
<td>NS-BH</td>
<td>4.7</td>
<td>1.2·10⁻³</td>
</tr>
<tr>
<td>BH-BH</td>
<td>7.6</td>
<td>1.9·10⁻³</td>
</tr>
</tbody>
</table>

Table 1.1: Expected detection rates of compact binary coalescence for Advanced Virgo compared to that of Virgo [30].
Chapter 2

Corrective thermal compensation devices

The first Chapter concludes with the design sensitivity curve of Advanced Virgo. One of the main conditions for the interferometer to reach the target sensitivity is a nearly perfect destructive interference at the antisymmetric port. This simple condition supposes that the wavefront of the beams recombining at the Beam Splitter are not affected by aberrations, but almost perfectly matched and cancel each other out. The residual light impinging on the detector at the dark port should stay below a certain threshold, whose value is determined in such a way as to fulfill the sensitivity requirements. However, in the real world the beam circulating in the interferometer passes through many optics and is possibly spoiled by their defects. For this reason, it becomes crucial to control optical aberrations and correct them in order to have a beam wavefront as close as possible to the ideal one. This is the role of the thermal compensation systems (TCS) we are going to describe in the present chapter.

2.1 Outline of the problem

Mirror aberrations can be classified as cold and thermal defects. Small deviation from the nominal radius of curvature, defects in the coating
or in the polishing, inhomogeneities in the substrate are referred to as mirror cold defects.

At present, very sophisticated techniques have been developed to improve mirror surface smoothness, such as ion beam polishing and corrective coating [31]. These techniques both consist of reducing the mirror residual roughness below a certain spatial frequency, the main difference between the two basically being the way the polishing is performed. The corrective coating technique allows to reach a residual flatness of a fraction of a nanometer over a 150 mm diameter, but its further improvement is currently limited by the metrology accuracy.

On the other hand, thermal defects are all the effects due to high power beams passing through or being reflected off the optics. The optical absorption in the coating and in the substrate results in an increase in mirror temperature. This, in turn, induces both a change of the refractive index and a deformation due to the substrate thermal expansion.

Finally, in addition to defects of the mirror surface, additional strains and deformation can be added to the mirror during the suspension phase: for instance, Advanced Virgo mirrors are suspended using glass fibers attached to the mirror with ears, and this system induces small deformations of the mirror surface or wavefront distortions in transmission.

Cold and thermal defects and suspension strains sum up and determine what is commonly referred to as mirror aberrations, which cause mode mismatch and scattering, both responsible for power losses and unwanted light at the dark fringe, as we will see in the next sections.

2.1.1 Mode mismatch

When a beam has to be injected into an optical resonator, it is necessary that both its intensity profile and its phase match with the resonator fundamental
mode. The goodness of the matching is described by an overlap integral:

\[ \eta = \frac{\left| \int \Psi_0 \Psi_i dS \right|^2}{\left| \int \Psi_0 dS \right|^2 \ast \left| \int \Psi_i dS \right|^2} \]  

(2.1)

where \( \Psi_0 \) is the ideal cavity field, \( \Psi_i \) is the input field to match with the cavity resonant mode, and the integration is carried over the entire beam.

If the injection optics are affected by aberrations, the cavity input beam will be aberrated itself, resulting in a worse cavity coupling and in a lower power \( P_0 \) circulating in the cavity.

Mode mismatch can be due also to aberrations in the cavity optics: if, for example, the mirror Radius of Curvature is different from the nominal value, the input beam parameters (like the wavefront RoC) will no longer match with those of the cavity, resulting in less power coupled into the resonator.

### 2.1.2 Light scattering

Another mechanism responsible for power losses is represented by scattering due to mirror surface roughness.

The process of an incoming light beam incident on the rough surface of a mirror can be modeled as light diffracted by a grating. For a roughness of spatial frequency \( \rho \) and a normal incidence beam of wavelength \( \lambda \), the diffraction law is given by:

\[ \sin \theta_n = n\lambda\rho \]  

(2.2)

where \( n \) is the diffraction order and \( \theta_n \) is the angle at which the light is scattered. For the sake of simplicity, we will consider only the first order scattering, i.e. \( n = 1 \). For low-angle scattering, the proportion of light scattered by a mirror roughness of amplitude \( A \ll \lambda \) is given by [33]:

\[ \alpha \approx \left[ J_1 \left( \frac{4\pi A}{\lambda \cos \theta} \right) \right]^2 \]  

(2.3)

\[ \approx \frac{1}{2} \left( \frac{4\pi}{\lambda} \right)^2 \frac{A^2}{2} \cos \theta \]  

(2.4)
The Root Mean Square (RMS) of the sinusoidal surface is given by \( \sigma = \frac{A}{2} \). For small scattering angles \( \cos \theta \approx 1 \), and we can write \( \alpha \) as:

\[
\alpha = \left( \frac{4\pi}{\lambda} \right)^2 \sigma^2 \tag{2.5}
\]

which shows that there is a scaling factor between the proportion of scattered light and the surface RMS:

\[
\left( \frac{4\pi}{\lambda} \right)^2 = 140 \text{ppm/nm}^2 \tag{2.6}
\]

Therefore the RMS roughness of a mirror can be considered as a kind of scattering power [34].

These very simple equations show that different spatial frequency defects scatter light at different angles. In particular, with a roughness dominated by low spatial frequency defects, light is mainly scattered at small angles and can couple to High Order Modes which could be close to resonance. High Spatial frequency defects, instead, cause light scattering at larger angles, engendering losses in the intra-cavity power and scattered light. Medium Spatial frequency defects, finally, can scatter light into High Order Modes whose spatial distribution is comparable with the mirror size. In this case the loss mechanism is two-fold: indeed, HOM’s extract power from the fundamental mode and, at the same time, they generate clipping losses if their spatial extension is as wide as the mirror size.

The process of scattering into High Order Modes is very important to understand and to control, and the next Chapter will be entirely devoted to its development and understanding.

To give a quantitative idea of this process, let us calculate the scattering angle for Advanced Virgo 3 Km long arm cavities with 35 cm diameter mirrors (fig. 2.1). Suppose that mirror roughness scatters light from the center of the Input mirror: the maximum angle for which the scattered light will still reach the end mirror is given by:

\[
\theta = \arctan \left( \frac{r}{L} \right) \approx 60 \mu m \tag{2.7}
\]
where \( r \) is the radius of the mirror and \( L \) is the cavity length. From equation (2.2) we can calculate the threshold value of the spatial frequency above which scattering from mirror roughness will only cause stray light:

\[
\rho = \frac{\sin \theta}{\lambda} \approx 56 \text{ m}^{-1}
\]

This simple calculation gives an idea of what kind of power losses we must expect depending on the spatial frequency of mirror defects.

### 2.1.3 Frequency splitting

The resonance condition for a beam resonating in a cavity is mainly determined by the cavity length and by the Radius of Curvature of its mirrors. Indeed, as we already saw in Section 1.3.2, the total phase shift from the Input mirror to the End mirror of the cavity for a Gaussian mode of order \( n, m \) is given by [25]:

\[
\Phi_G(L) = kL - (n + m + 1) \times \arccos \sqrt{g_i g_e}
\]  

(2.8)

where the first term takes into account the propagation phase of the beam, and \( g \) is the cavity \textit{g-factor} for the Input and End mirror, which is related
to the mirror Radius of Curvature by:

\[ g = 1 - \frac{L}{R} \quad (2.9) \]

In real cavities, it can happen that, due to mirror figure defects, modes of the same order see a different overall radius of curvature: it induces what is called a *frequency splitting*: namely, degenerate modes resonate at different frequency. This mechanism is potentially risky: indeed, HOM’s whose design resonance frequency is far from that of the fundamental mode, might meet the resonance condition of the cavity and start resonating, introducing asymmetries between the arms and preventing a good beam cancellation at the dark port.

Mode mismatch, light scattering and frequency splitting are mechanisms which show in very simple equations how mirror roughness can reduce the interferometer sensitivity both by decreasing the power circulating in the cavities and by spoiling the dark fringe condition. In the next section we will see how it is possible to perform an *in-situ* mirror compensation which will depend on where the optics is positioned along the beam path and on the spatial frequency of the defects that we wish to correct.

### 2.2 The Thermal Compensation

One of the best solutions to reduce in-situ mirror aberrations relies on Adaptive Optics (AO).

The AO technique has been extensively used and validated in astronomical applications, but it has also been proposed as a good method to correct aberrations of Gaussian beams \[35\]. The usual methods for AO are based on Deformable Mirrors taking advantage of the Micro Electro Mechanical System (MEMS) technology, but they are either not vacuum compatible or too noisy to be installed along the GW detectors beam path \[40\].

An alternative solution is represented by the *Mirror Thermal Compensation*: the beam wavefront is modified by changing the mirror features through heat-
ing. This technique is favorable first because it allows to perform corrections introducing a very low noise and also because it makes possible to carry out a non-contact correction, which is very useful to apply for mirrors that make up the resonant cavities.

In the following we will briefly explain the basic principle of the Thermal Compensation.

### 2.2.1 The Principle

The general principle at the base of the Thermal Compensation System is to use an auxiliary heat source to induce controlled thermal effects in the mirror and therefore correct the beam phase aberrations. The principle of Thermal Compensation has been very clearly summarized by R. Lawrence in his thesis [52]. Here we will report the main steps useful to our description. Given an optic of thickness \( d \) and index of refraction \( n \) at room temperature \( T_0 \), the beam path through the optics is given by:

\[
S_0 = nd
\]  

(2.10)

When the optical element is heated up due to a high power beam passing through it or to any other heat source, a change is induced both in its thickness and in its refractive index. These modify the beam optical path, which therefore becomes:

\[
OPL = S_0 + n\Delta d + d\Delta n_T + d\Delta n_E
\]  

(2.11)

In this formula we recognize three different contributions:

- the **thermoelastic deformation**, which is the physical deformation of the mirror substrate related to the temperature change through the linear expansion coefficient \( \alpha \):

\[
\Delta d \propto \alpha \Delta T
\]  

(2.12)

- the **thermooptic effect**, more commonly known as thermal lensing, which is the change of the refractive index due to the temperature increase
(or decrease):
\[
\Delta n_T = \frac{dn}{dT} \Delta T
\]  
(2.13)

- the **elastooptic effect**, the refractive index change owing to strains in the substrate resulting from thermal expansion of the optics itself:
\[
\Delta n_E \approx -\alpha (1 + \nu) n \Delta T
\]  
(2.14)

with \( \nu \) the Poisson coefficient.

Starting from Eq. \( (2.11) \), we can compute the phase of a beam transmitted through a heated optic:
\[
\phi_t(x, y, t) = 2\frac{\pi}{\lambda} \int_0^{d(r)} n(x, y, z, t)dz \\
\simeq 2\frac{\pi}{\lambda} \left( n_0d_0 + \int_0^{d_0} \Delta n_T(x, y, z, t)dz + \int_0^{d_0} \Delta n_E(x, y, z, t)dz + \right. \\
+ \left. \frac{2\pi}{\lambda} (n_0s_z(x, y, d_0, t) - (n_0 - 1)s_z(x, y, 0, t)) \right) \\
= \phi_0(x, y, t) + \phi_T(x, y, t) + \phi_E(x, y, t) + \psi(x, y, t)
\]  
(2.15)

where we called \( s_z(x, y, z, t) \) the mirror deformation and we chose \( z = 0 \) as High Reflectivity surface of the mirror. In this formula, \( \phi_0(x, y, t) \) is the phase of the unperturbed beam, while in \( \phi_T(x, y, t) \) and \( \phi_E(x, y, t) \) are embedded the contribution due to the thermal lensing and to the elastooptic effect, respectively. At last, \( \psi(x, y, t) \) is related to the thermoelastic expansion of the mirror, and is the only term that counts when we consider the phase of a reflected beam:
\[
\psi_{ref}(x, y, t) = 2\frac{2\pi}{\lambda} s_z(x, y, 0, t)
\]  
(2.16)

Knowing the phase change experienced by a beam passing through or being reflected by a heated optic, we can induce a controlled heating process and suitably heat up the optics to bring the wavefront as close as possible to the ideal one.


2.3 Thermal Compensation devices for GW Interferometers

The goal of reaching the design sensitivity for Second Generation Detectors comes along with more and more strict requirements on mirror polishing and with the necessity of controlling the thermal effects induced by high power beams in the cavities. In this scenario, the role of the in-situ compensation of mirror defects becomes crucial.

The experience of the First Generation detectors allowed for many of the problems related to mirror aberrations to emerge. For this reason, some thermal compensation devices were already designed and applied with successful results [36, 37]. Moreover, the arise of new issues and more stringent requirements led to the necessity of having more complex compensation systems able to perform even finer corrections. In the following we will present some of the Thermal Compensation devices developed so far, and we will introduce the new Thermal Compensation system which is the main subject of this thesis work.

2.3.1 Cavity Matching improvement with TDM

An Adaptive Optics system developed in Virgo mainly to deal with the mode matching problem is the Thermally Deformable Mirror (TDM) [38, 39]. It consists of a standard mirror with an array of 61 resistors in contact with its rear side, which is its High Reflectivity surface. Each resistor can heat up and induce a local change of temperature in the substrate, which in turn produces an optical path length modification, according to the equations of section 2.2. The laser beam experiences this OPL modification passing through the substrate and being reflected back by the High Reflectivity surface of the mirror, as shown in figure (2.2).

Exploiting this principle, it is possible to suitably tune the power dissipated by each resistor into the mirror substrate in order to get the desired beam phase modification.

This setup is particularly suitable to improve the mode matching. Indeed, it
Figure 2.2: Schematic of the Thermally Deformable Mirror from [39]. An array of resistors is in contact with the HR surface of the mirror. The wavefront passes through the substrate and experiences thermal lensing and elastooptic effects.

has been demonstrated [40] that it is possible to control the matching of a laser beam with a resonant cavity using the high order mode content of the beam reflected by the cavity as an error signal.

2.3.2 Mirror Radius of Curvature Correction

The lowest order defect which can be corrected is the deviation of the mirror Radius of Curvature (RoC) from its ideal value. This correction can be performed by creating a temperature gradient inside the mirror by projecting heat radiation onto it. Two different methods have been adopted in the last generation detectors to perform this kind of correction: the Ring Heater [36] and the CHRoCC [37].

Ring Heater

The Ring Heater (RH) was first developed in GEO600 and afterwards implemented in the first and second generation GW detectors design. The first RH was made of a ring-like Duran glass rod wrapped with a 100 µm thick stainless steel ribbon which is heated by passing a DC current through it. A picture of the first prototype is shown in figure (2.3a). The device can induce
Figure 2.3: (a): First prototype of ring heater installed behind one of the cavity mirrors in GEO600. (b): Schematic showing the Ring Heater positioning in Advanced Virgo. Figure from [36].

A change in the ROC, making the surface to become more concave (what is commonly called a negative RoC correction), tuning its value closer to the nominal, therefore improving the cavity matching and decreasing the dark port power.

The subsequent forms of RH present variations in the material, while exploiting the same working principle. For instance, the Advanced detector RH’s will be placed around the mirror instead of behind it, as shown in figure (2.3b). Moreover, to optimize the power reaching the mirror, a golden shield will be placed all around the ring. The RH operating temperature will be a few hundred of degrees Celsius and will be tuned depending on the needs.

**CHRoCC**

During the commissioning of Virgo+ and the third Virgo Science Run (VSR3), an asymmetry of about 130m was found between the RoC’s of the North and West End mirrors. This asymmetry gave rise to a performance that was worse than the previous Science Run. In this case the RH solution was not compatible with the need to increase the radius of curvature. For this reason, another solution was adopted, called Central Heating Radius of Curvature Correction (CHRoCC) [37]. The CHRoCC consists of a vacuum-compatible 1 inch Alumina Ceramic Heater placed in the first focus of an ellipsoidal
The radiation emitted by the heater and collected by the reflector is focused on the High Reflectivity surface of the End cavity mirror. Therefore, the mirror temperature increases and a change in the RoC is induced. Unlike the RH, the CHRoCC is able to provide a positive RoC correction. Operating at a temperature of about 500\degree C it provides an increase of the mirror temperature of less than 3 degrees (figure (2.5)). The net result of this setup resulted in reducing the asymmetry in the RoC from about 130 m to about 16 m, allowing to restore good interferometer sensitivity.

2.3.3 High Spatial Frequency defects reduction

Higher spatial frequency defects need more complex compensation devices, in order to perform compensations with more complex patterns. We will see in the following two possible solutions, both based on the radiative heating principle.

In [52] the first demonstration of this kind of compensation was studied. It consists of using as a heat source a laser beam at a wavelength which is completely absorbed by the optics, namely a CO2 laser beam. In the following
we will describe the CO2-based compensation setup for Advanced Virgo. On the other hand, another solution has been presented in [42], and it consists of using, as actuators, single resistors emitting incoherent black-body radiation arranged in a matrix. We will describe both of them in the following.

**CO2 laser**

The use of the CO2 laser system ($\lambda = 10\mu m$) in Advanced Virgo is used mainly to reduce aberrations in the recycling cavities. Shining the corrective heat pattern on a Compensation Plate (CP) put right before each of the input test masses in the recycling cavities, the setup will modify the Optical Path Length of the beam, and will therefore correct the wavefront aberrations. The heat pattern to be projected on the CP can be split into a symmetric and an asymmetric part. It allows to perform two corrections independently: a Double Axicon system will provide a correction for the symmetric parts, while a scanning system will take care of the asymmetric defects compensation. A schematic of the system is shown in figure (2.6) [41].

**Double axicon system** The CO2 laser beam is split into the two arms of the Double Axicon System (DAS) using a polarizer: this configuration
Figure 2.6: Optical layout of the Advanced Virgo TCS. A zoom of the Double Axicon System is reported in the yellow square, while a zoom of the Scanning System is reported in the red square Figure from [41].
Figure 2.7: Optical layout of the Advanced Virgo TCS. A zoom of the Double Axicon System is reported in the yellow square, while a zoom of the Scanning System is reported in the red square. The *Aiming laser* reported in the lower corner is needed for a first coarse alignment of the DAS.

allows to maximize the power efficiency of the setup. Moreover, having two arms allows to divide the control over an inner and an outer radius intensity distribution. Therefore, for each arm a focusing lens and one axicon lens will provide control over the geometrical parameters of each ring. Finally, the rings from the two arms of the DAS are recombined by another polarizer and sent to the CP. The schematic of this system is reported in the yellow square of figure (2.6).

**Scanning system** A pick-off of the CO2 laser beam is used for the scanning system, devoted to the correction of higher spatial frequency defects. The power modulation will be provided by an acousto-optic modulator (AOM). The beam waist on the CP is about 0.5 cm, which is the same size as the aberrations to be corrected. The scan projects on the CP a heating pattern over a square $16 \text{ cm} \times 16 \text{ cm}$ such as the one shown in the red square of figure (2.6), and is performed by following one of the scan patterns in figure (2.7). The scanning of the laser beam is achieved using two galvonometer mirrors.

In the final design of the CO2 compensation system a third element will be implemented: the *central heating* (not shown in the schematic because still in the design stage at the time of writing). It consists of a round spot.
of the same size as the cavity beam which will provide the heating of the central region of the mirror in case of cavity unlock. This is foreseen in order to avoid temperature transients, and will be implemented taking a pick-off beam from the CO2 laser. The CO2 beam will be projected through a viewport located between the Beam Splitter and the Input towers. According to the simulations, the power required to compensate for 1 ppm coating absorption at 125 W input power is 18 W for the Double Axicon system and a few more W for the scanning system. The compensation setup will be able to reduce the residual optical path length RMS by a factor 20 for spatial frequencies of up to 40 \( m^{-1} \).

**CHRAC**

An alternative method to the CO2 laser system has been presented in \cite{42} and its first experimental validation is the main subject of this thesis work. The Central Heating Residual Aberration Correction (CHRAC) is a method to perform a thermal compensation acting directly on the mirror High Reflectivity surface and using resistors as heating sources. Instead of changing the Optical Path Length of the beam, as for the case of CO2 laser system, the CHRAC acts by modifying the mirror shape and, consequently, the phase of the reflected beam. For this reason, it is applied on the HR surface of the cavity End mirror, for the beam not to be affected by the mirror thermal lensing. The CHRAC consists of a matrix of ceramic elements, each of which is heated up and emits blackbody radiation. By tuning the power dissipated by each resistor, it is possible to generate a heat pattern which is imaged on the mirror surface using a telescope. The mirror absorbs the pattern and expands, accordingly to the induced temperature field. Therefore, the initial wavefront distortions are flattened due to the reflection from a deformed mirror surface, as described in equation 2.16.

Unlike the compensation systems previously presented, the CHRAC is meant to be applied to the End mirror of a Fabry Perot cavity. A schematic of the setup is shown in figure (2.9). Figure from \cite{42}.
Figure 2.8: Schematic of CHRAC setup: an array of heaters is imaged on the HR surface of the mirror to correct.

Figure 2.9: CAD schematic of the CHRAC setup. Figure from [42].
In the following chapters we will present all the steps of the CHRAC design and the experimental results obtained with the first CHRAC prototype. Finally, we will show the results obtained applying a table-top version of the CHRAC to a Fabry-Perot cavity, showing the validity of the principle.
Chapter 3

Light scattering into High Order Modes

In Chapter 2 we briefly described the scattering mechanism that can be induced by mirror roughness. Furthermore, we saw that scattering at small angles can cause power coupling into unwanted High Order Modes, which are automatically suppressed by the cavity filtering action if their resonance frequency is far enough from the cavity resonance condition. If, instead, their resonance frequency is very close to that of the fundamental mode, these spurious modes will inevitably start resonating causing round-trip-losses (RTL) inside the cavity and asymmetries between the two arms. The net result is that beams reflected from the arm cavities will not cancel out at the interferometer antisymmetric port and will cause spurious light spoiling the dark fringe.

In addition, if this process happens with the sideband fields, it will introduce extra issues in the locking procedure. For these reasons, it is crucial to be able to reduce the scattering, and it is for this reason that the requirements on mirror polishing are very stringent. For Advanced Virgo mirrors the residual roughness is fixed at 0.5 nm RMS, which is about one order of magnitude smaller than the requirements of Virgo+. This threshold becomes even lower for the mirrors of Third Generation detectors designed to be operated with the LG33 High Order Modes: indeed, in this case there are 10 modes with the
same resonance condition in the cavity, and a residual surface roughness of $10^{-2}$ nm is needed in order to reduce scattering into degenerate modes [43]. This threshold is beyond what can be achieved with the current state-of-art techniques, as already mentioned in the last Chapter. In this scenario, it becomes more and more important to develop a mechanism of in-situ correction of mirror defects.

In the present Chapter we will describe the low-angle scattering process and we will apply it to the case of marginally stable cavities, Stable cavities and cavities locked on High Order Modes. Furthermore, we will see the possible solutions devised for this problem.

### 3.1 The coupling mechanism

#### 3.1.1 Wavefront spoiling from an aberrated surface

Consider a distorted mirror surface and a beam incident on it, and let us call $\varphi_0(x, y)$ the incoming field. Calling $Z(x, y)$ the surface roughness and $r$ the mirror reflectivity, the reflected beam will be:

$$
\varphi_R(x, y) = re^{-2ikZ(x,y)}\varphi_0(x, y)
$$

(3.1)

A possible way to describe this process is to use the modal decomposition method described in [46]. In this representation, each optical element can be seen as a matrix operator acting on a complex space (the modal space). In this picture, we expand the electromagnetic field of a light beam as a superposition of orthonormal Gaussian modes:

$$
\varphi(x, y) = \sum a_{mn}\psi_{mn}(x, y)
$$

(3.2)

where $\psi_{mn}(x, y)$ are Gaussian modes (described for example by the Hermite-Gaussian functions), and $a_{mn}$ are vectors in the modal space. The $\psi_{mn}(x, y)$ are eigenstates of the unperturbed system, and all the distortions can be treated as perturbations that transfer energy from one eigenmode to another. Consider the case of a Fabry-Perot cavity and, to simplify, imagine that only
one of the two mirrors has a distorted surface. In the modal space, a rough mirror surface can be represented as:

\[ M_{nm,kl} = r\langle nm|e^{-2ikZ(x,y)}|kl \rangle \]  

(3.3)

As an example, consider the case of a Fabry-Perot cavity operated with a Gaussian mode. We are interested in the coupling between the cavity fundamental mode and the HOM’s, so the mirror operator will become:

\[ M_{nm,00} = r\langle nm|e^{-2ikZ(x,y)}|00 \rangle \]  

(3.4)

which in the approximation \(|Z(x,y)| \ll \lambda\) is:

\[ M_{nm,00} \simeq -2ikr\langle nm|Z(x,y)|00 \rangle \]

\[ = -2ikr \int \int \psi_{nm}^*(x,y)Z(x,y)\psi_{00}(x,y)dxdy \]

(3.5)

\[ \Gamma_{nm,00} \] is the coupling coefficient between the Gaussian mode and the high order mode \(|nm\).

If any of these scattering coefficients becomes large, the corresponding excited HOM can become resonant into the cavity. In some cases, it can prove to be

Figure 3.1: Diagram illustrating a mirror distorted surface. The values of \(Z(x, y)\) describing the roughness define the deviation of the surface with respect to the ideal one.
useful to decompose mirror maps as a sum of Zernike polynomials (Appendix B), which are a set of orthogonal polynomials on the unit radius very suitable to describe mirror aberrations. In this case we would get:

\[ Z(x, y) = \sum_{i,j} c_{ij} Z^j_i(x, y) \]  \(3.6\)

and the mirror operator of equation (3.5) can be rewritten as:

\[ M_{nm,00} = -2ikr \sum_{ij} c_{ij} \int \psi^*_{nm}(x, y) Z^j_i(x, y) \psi_{00}(x, y) dxdy \]  \(3.7\)

### 3.1.2 Scattering into Higher Order Modes

To better understand the connection between the mirror figure of error and the HOM’s which gets excited, we can recall the considerations made about scattering in Section 2.1.2 and write the coupling coefficients \(\Gamma\) in a more explicit form. For this purpose, as in Section 2.1.2, we can model mirror aberration as a grating with spatial wavelength \(\Lambda_M\). It is useful to define a spatial frequency normalized to the size of the beam waist \(w_0\) as:

\[ \Omega = \frac{\sqrt{2}\pi w_0}{\Lambda_M} \]  \(3.8\)

Therefore, we have to choose a base of modes and rewrite the coupling operator of eq. (3.5).

Consider, for example, the base of Hermite Gauss polynomials (Appendix A). Since we are interested in the spatial distribution of modes along the plane parallel to the mirror surface, we discard for the moment the terms depending on the \(z\) coordinate. Moreover, we can factorize the dependence on \(x\) and \(y\) coordinate, and consider the mode in one dimension suitably normalized as:

\[ \psi_n(x) = e^{-x^2/2} \frac{H_n(x)}{(n!2^n\sqrt{\pi})^{1/2}} \]  \(3.9\)
In these conditions, we can write the aberrated wavefront of the Gaussian mode $\psi_0(x)$ as:

$$\psi(x) = e^{-x^2/2} \sqrt{\frac{4}{\pi}} e^{-i[\theta x + A \cos(\Omega x + \Phi)]} \quad (3.10)$$

where $\theta$ is a term describing the tilt, $A$ is the mirror deformation amplitude and $\Phi$ is a spatial phase which takes into account for a symmetric ($\Phi = 0$) or antisymmetric ($\Phi = \pi/2$) deformation with respect to the beam axis.

If we recalculate the scattering coefficient into the mode $n$ with this kind of mirror deformation we will find [44]:

$$\Gamma_{n,0} = \int \psi(x)\psi_n(x)dx = -\frac{i}{\sqrt{n!}2^n} \left[ \delta_{1n}\theta + A \cos \left( \Phi + n\frac{\pi}{2} \right) \frac{\Omega^n}{n!} e^{-\Omega^2/4} \right] \quad (3.11)$$

where we used the approximation of small tilt $\theta \ll 1$ and small roughness amplitude $A \ll 1$.

This equation shows very clearly that, to first approximation, a tilt of the wavefront couples only in the mode of order 1. Moreover, thanks to the presence of the phase $\Phi$, we see that distortions that are symmetric with respect to the beam axis couple into even High Order Modes, and antisymmetric distortions into the odd ones. If we discard the part related to the tilt, we can calculate the power scattered into the mode $n \geq 2$:

$$P_n = \frac{1 - (-1)^n \cos 2\Phi A^2}{2} \left( \frac{\Omega^n}{n!} \right)^2 e^{-\Omega^2/2} \quad (3.12)$$

which shows that, for a given normalized spatial period roughness $\Lambda_M$, the maximum power is scattered into the mode:

$$n_{max} = \frac{\Omega^2}{2} = \left( \frac{\pi w_0}{\Lambda_M} \right)^2 \quad (3.13)$$
Moreover, if we remember that the spatial period of a HOM of order $n$ is given by $\Lambda_n \approx \frac{4w_0}{\sqrt{n}}$, we can rewrite eq. (3.13) as:

$$\frac{\Lambda_M}{\Lambda_{n_{\text{max}}}} = \frac{\pi}{4}$$

(3.14)

This equation connects the spatial period of mirror roughness to the spatial period of the HOM which is mainly excited by the scattering mechanism.

### 3.2 HOMs’ in different cavity configurations

With the scattering process clear in mind, we can analyze different cavity configurations to better understand what kind of issues we could have to deal with in each case.

#### 3.2.1 Marginally stable cavities

As described in (AdV optical configuration), Advanced Virgo recycling cavities are designed to be marginally stable. It means that the higher order modes gain a very small Gouy phase shift with respect to the carrier and their resonance condition is not too far from that of the fundamental mode, as schematically shown in figure (3.2). If the mirror figure error scatters light into some of these modes and makes them resonate, the filtering effect of the Fabry-Perot cavity is not able to cut them out, inducing a power subtraction from the fundamental mode and increasing the RTL. Nevertheless, this is not a problem for the carrier field: indeed, carrier HOM’s generated in the recycling cavities do not resonate in the arm cavities and are completely reflected back, acquiring a phase of $\pi$. Thanks to this additional phase, they are brought far from the carrier resonance condition and filtered out.

Conversely, this process can represent a problem for the sidebands which are already anti-resonant in the arms: therefore, if they get excited by mirror surface defects, they become resonant in the recycling cavities. Since the locking is performed using the sidebands as an error signal, beam aberrations can easily translate into locking issues.
Figure 3.2: Free Spectral Range of a Fabry-Perot cavity. The HOMs resonance frequencies (blue lines) overlap with the cavity linewidth: if an HOM is excited, it keeps resonating subtracting power from the fundamental mode. Plot from [20].

In this scenario, it is clear that the role of thermal compensation to reduce mirror aberrations becomes very relevant.

### 3.2.2 Stable cavities

High order modes can become resonant also in cavities designed as stable. It can occur if their resonance frequency happens to be very close to the resonance of the fundamental mode. Also in this case, light scattered into these specific modes can cause high RTL. This is the case, for instance, for Advanced Virgo arm cavities, where a small deviation from the nominal RoC can bring modes 8 and 9 into resonance. In figure 3.3 the power in HOM’s 8 and 9 is plotted versus the RoC deviation from the nominal value. It implies a very stringent tolerance on the RoC value. This configuration is examined in [42]. The requirements of mirror polishing for Advanced Virgo have been fixed in such a way as to keep the RTL below 75 ppm. It translates into a power spectral density (PSD) curve which corresponds to mirror maps with a residual roughness of 0.5 nm RMS. Nevertheless, if the mirror RoC deviates from the ideal value, the same PSD triggers the scattering process from the
fundamental mode to the HOM’s close to the resonance. It happens because modes 8 and 9 have a characteristic spatial frequency which is related to the mirror map spatial frequency that can easily excite them.

### 3.2.3 Cavity locked on High Order Modes

So far we only considered the case of Fabry-Perot cavities locked on a Gaussian mode. Actually, a Fabry-Perot cavity can be operated either with a Gaussian mode or a high order mode.

As already hinted in the first Chapter, the use of high order modes, like the 9th order $LG_{33}$ mode, is foreseen in the design of some of the third generation detectors, like the Einstein Telescope, in order to reduce mirror thermal noise which limits the sensitivity of second generation detectors for frequencies around 100 Hz. In this case, there are 10 modes of order 9 which can resonate at the same frequency in the cavity, and part of the power can be scattered into the degenerate modes. Moreover, a slight frequency splitting of their resonance frequency due to the mirror figure error can generate a power subtraction from the main mode circulating in the cavity [47], [43].
To describe the scattering mechanism, we start from the definition of the Laguerre-Gauss modes. Laguerre-Gauss modes are solutions of the paraxial wave equation in cylindrical polar coordinates. The LG mode associated with the Laguerre polynomial $L_p^{|l|}$ is given by:

$$u_{pl}(r,\phi,z) = \frac{1}{w(z)} \sqrt{\frac{2p!}{\pi(|l| + p)}} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|} \cdot L_p^{|l|} \left(\frac{2r^2}{w^2(z)}\right) \exp\left(i(2p + |l| + 1)\right) \Psi(z) \cdot \exp\left(-\frac{kr^2}{2R_c(z)} - \frac{r^2}{w^2(z)} + il\phi\right)$$

where $r$, $\phi$ and $z$ are the coordinates around the optical axis, $k$ is the wavenumber, $\Psi(z)$ the Gouy phase, $w(z)$ and $R_c(z)$ are, respectively, the beam spot size and its radius of curvature at the position $z$. The modes are characterized by the indexes radial $p$ and azimuthal $l$, and the mode order is given by $(2p + |l|)$. The phase factor multiplying the Gouy phase suggests that modes of the same order accumulate the same phase shift after a round trip, so they share the same resonance conditions in the cavity, which is why they are referred to as degenerate.

The mechanism relating cavity mirror imperfections to coupling between LG modes is dealt with in [48]. Here we only report the calculation results helpful for our purposes.

Consider a single Fabry-Perot cavity, this time operated with LG$_{33}$ modes, where one of the mirrors has a perfectly smooth surface, while the other has a figure error described by the function $Z(x,y)$ which, to simplify calculations, we will write in radial coordinates: $Z(r,\phi)$. Mirror distortions will couple power from the mode $(p,l)$ into the mode $(p',l')$. The mode coupling coefficient is given by:

$$\Gamma_{pl,p'l'} = \int_S u_{pd}(r,\phi,z) \exp(2ikZ(r,\phi))u_{p'd'}^* dS$$

where $S$ is the plane perpendicular to the optical axis. Decomposing the mirror map in terms of Zernike polynomials as in eq. [B.3], the coupling
coefficients of eq. (3.16) can be rewritten as:

\[ \Gamma_{pl,p' l'}^{nm} = \int_S u_{pl} \exp \left( 2ikZ_m^m \right) u_{p'l'}^\ast dS \]  
(3.17)

which for small mirror deformations becomes:

\[ \Gamma_{pl,p' l'}^{nm} = \int_0^{2\pi} \int_0^R u_{pl} u_{p'l'}^\ast (2ikZ_m^m) r dr d\phi \]  
(3.18)

Replacing the expression of Laguerre-Gauss (3.16) and Zernike polynomials (B.2) in the (3.18) we see that the integral is non-zero only when:

\[ m = |l - l'| \]  
(3.19)

This selection rule directly connects the mirror deformation to the coupling of one LG mode to another. All LG modes fulfilling this rule can be excited by the mirror deformations, but modes of order different from 9 cannot resonate inside the Fabry-Perot cavity and are automatically suppressed.

To better understand this coupling mechanism, let us consider an example. Imagine that one of our cavity mirror is affected by astigmatism, so it can be described by the Zernike polynomial \( Z_2^2 \). The azimuthal index \( m = 2 \) defines the only possible 9th order modes the LG can scatter into: LG\(_{33}^3\), \( 5 \) and LG\(_{43}^1\), \( 1 \). These two modes, in turn, couple to other modes within the same order following the ilk rule: they scatter back into the LG\(_{33}^3\) mode and, respectively, also into the LG\(_{17}^1\) and LG\(_{43}^{-1}\). Of course, in the second stage scattering the coupling coefficient is much smaller.

### 3.3 Selectively suppress HOM

The more direct and easy-to-think way to suppress the scattering mechanism described so far consists of reducing the amplitude of mirror defects. Indeed, referring to the equations of Section (3.1), the smoother the mirror surface the lower the value of the function \( Z(x, y) \) and, consequently, the smaller the
coupling coefficients $\Gamma$.
In general, applying a correction supposes the a-priori knowledge of the mirror figure error or, alternatively, a system allowing an in-situ computation of the correction. Therefore, the beam reflected by a “corrected” mirror acquires an additional phase compensating for the original aberrations:

$$E_{cor}(x, y) = \exp\{2ikZ(x, y) + Z_{cor}(x, y)\}E_0(x, y)$$

(3.20)

In some cases, as shown in Section (3.2.2), we already know by design which modes are susceptible to being excited. The solution to this problem is outlined again in [42]. The authors of the paper show that it is possible to find a correction map reducing the scattering into selected modes, in this specific case modes 8 and 9. In other words, the correction to perform must yield a field $E_b(x, y)$ which does not contain modes 8 and 9.
To find this kind of correction map, they minimize the merit function:

$$e = \int \int |E_b(x, y) - ie^{-2ik[Z_{cor}(x,y) - Z(x,y)]}E_0(x, y)|^2 dxdy$$

(3.21)

If mirror maps are written, as usual, as a sum of Zernike polynomials, this minimization problem can be solved with standard numerical algorithms.
This method holds either when the cavity fundamental mode is a Gaussian beam or a higher order mode.

So far we always assumed the knowledge of the mirror map. In the case of LG$_{33}$ modes it can represent a strong limitation since, as said before, the required residual roughness for mirrors is below metrology’s present capability. A map-less technique would therefore be needed to calculate mirror correction maps, as reported in [49] and [50]. In these papers it is shown that the merit function 3.21 is not the only possible error signal to use in order to compute the correction map, but the intensity distribution of the field itself can be used to estimate the mode composition. The strength of this method relies on the possibility of performing an adaptive sensing and control using, as error signal, nothing more than the information extracted.
from the intensity image of the cavity beam. Briefly, it consists of finding a set of mode creating maps orthogonal to each other which, added to the cavity mirrors, can reproduce any kind of beam shape. In this way, one can decompose the cavity beam as the sum of the fundamental mode and a contribution coming from high order modes:

$$\Psi(x, y) = \alpha_f \Psi_f(x, y) + \sum_m \alpha_m \Psi_m(x, y)$$ (3.22)

where the subscript $f$ indicates the fundamental mode resonating in the cavity, while the index $m$ runs over all the higher order modes or degenerate modes. Then it is possible to find the mirror maps generating exactly the desired mode intensity distribution and, finally, find the correction map to bring all the power back into the fundamental mode.

A possible technique to perform an in-situ correction of the mirror surface with great accuracy is the CHRAC system presented in [42] and already mentioned in chapter 2. In this thesis the working principle of CHRAC will be explained, then the design and realization of the first prototypes will be presented. Finally, we will show the effect that this corrective device introduces on a cavity beam shape, reducing the scattering coefficients into degenerate modes and improving the cavity beam quality and the interferometer contrast defect.

### 3.4 Reconstruct mirror maps from HOM’s frequency shifts

As already explained in Section (2.1.3), one of the problems which can be determined by the mirror deviation from its ideal shape is the change of the resonance conditions for the modes, which therefore resonate at frequencies different from the ideal ones. It happens because, due to their different spatial distribution, each mode impinges on a different region of the mirror, and the Radius of Curvature that they experience depends on the mirror figure of error averaged over the mode spatial extent.
In the perspective of correcting the mirror surface without the a-priori knowledge of the mirror map, it could be desirable to connect the resonance frequency shift to the mirror map features. For low order aberrations, like astigmatism, this connection proves to be quite easy: indeed, astigmatism can be regarded as a different radius of curvature along the \( x \) and \( y \) mirror axis directions, and starting from equation (2.8) it is easily possible to connect the frequency shift of the HOM of order 1 to the mirror aberration. Explicitly, the resonance frequency of the modes of order 1 are given by:

\[
\nu_{10} = \frac{c}{2\pi L} \arccos \sqrt{1 - \frac{L}{R_x}} \\
\nu_{01} = \frac{c}{2\pi L} \arccos \sqrt{1 - \frac{L}{R_y}}
\]  

(3.23)

Therefore, it is straightforward to extrapolate the measurement of the RoC along the two directions starting from the resonance frequency of the mode of order 1 and reconstruct the mirror astigmatic map. This measurement has been carried on and the results are presented in [51]. Nonetheless, for higher spatial frequency aberrations, the procedure becomes more complicated, the problem relying on connecting the information extrapolated from the frequency shift of each mode to the effective mirror map. To get this result we tried a new approach that we are going to present hereafter. This analytic derivation has not been compared to simulations nor to experimental measurements yet. However, it represents a new idea and a starting point for further investigations.

**Cavity resonance condition in the modal representation**

Consider a Fabry-Perot cavity of figure (3.4). In the modal representation already used before, define the operators \( R \) and \( T \) as:

\[
R = \langle nm | e^{-2ikZ(x,y)} | kl \rangle \\
T = \langle nm | e^{-2ikT(x,y)} | kl \rangle
\]  

(3.24)  

(3.25)
Figure 3.4: Schematic of a Fabry-Perot cavity. $\phi_i$ and $\phi_o$ are, respectively, the input and output fields, while $\phi$ is the field circulating in the cavity. The non-perfect mirror surfaces are described by mirror maps $Z_1(x,y)$ and $Z_2(x,y)$.

where $Z(x,y)$ is the deviation of mirror surface from the ideal shape and $T(x,y)$ is its transmission map, which takes into account for mirror substrate inhomogeneities.

Let us define $P$ as the operator of propagation in free space along the cavity of length $L$:

$$P = \langle nm|e^{-ikL}|kl\rangle$$ (3.26)

With these definitions we can write the cavity round trip operator as:

$$C = r_1r_2R_1PR_2P$$ (3.27)

Here the subscript 1 and 2 refer to input and end mirror. The cavity resonance frequency can be determined writing the equations for the field circulating in the cavity. Let $\phi_i$ and $\phi_o$ be the input and output fields, respectively, we can write the output field as a function of the input:

$$\phi = t_1T_1\phi_i + C\phi$$
$$\phi_o = t_2T_2P\phi$$ (3.28)

which yields:

$$\phi_o = t_1t_2T_1(1-C)^{-1}T_2\phi_i$$
$$= \frac{t_1t_2}{\det(1-C)}T_2\text{adj}(1-C)T_2\phi_i$$ (3.29)
If we consider this equation in the frequency domain, we see that the resonance frequencies of the cavity are given by its poles, which are found by solving the equation:

\[ \text{det}[1 - C] = 0 \]  

(3.30)

In general, they will be complex quantities, where the real part is associated to the central value of the resonance, while its imaginary part is related to the resonance bandwidth.

**Step 1: linearize in the deformation amplitude**

Since we imagine that cavity imperfection will slightly modify the cavity features, we can write the round trip operator as:

\[ C = C_0 + \delta C \]  

(3.31)

where we called \( C_0 \) the ideal cavity operator and \( \delta C \ll 1 \). In this approximation we can rewrite eq. (3.30) as:

\[
\text{det}(1 - C_0 - \delta C) = \text{det}(1 - C_0)\text{det}(1 - (1 - C_0)^{-1}\delta C)
\]

\[
= \text{det}(1 - C_0)\exp(\text{Tr}(\log(1 - (1 - C_0)^{-1}\delta C)))
\]

\[
\simeq \text{det}(1 - C_0)\exp(-\text{Tr}(1 - C_0)^{-1}\delta C)
\]

\[
\simeq \text{det}(1 - C_0)(1 - \text{Tr}(1 - C_0)^{-1}\delta C) = 0
\]

(3.32)

where we have used matrix algebra to reduce the expression. The last line of equation 3.32 shows that the new resonance frequencies are given by:

\[ \text{Tr}[(1 - C_0)^{-1}\delta C] = 1 \]  

(3.33)

which is:

\[
\sum_M \langle M | [(1 - C_0)^{-1}\delta C] | M \rangle = 1
\]

(3.34)
Introducing the identity operator and indicating with $|M\rangle$ the eigenmode of the unperturbed cavity $|n,m\rangle$, we can also write the equation 3.34 as:

$$\sum_M \langle M|[(1 - C_0)^{-1}]|N\rangle \langle N|\delta C|N\rangle = 1$$  \hspace{1cm} (3.35)

but since the ideal cavity operator is diagonal in the base of ideal cavity eigenstate, this equation becomes:

$$\sum_M \frac{\langle M|\delta C|M\rangle}{1 - r_1 r_2 e^{i\phi_M(k)}} = 1$$  \hspace{1cm} (3.36)

where

$$\phi_M(k) = 2(n + m + 1)\eta(k) - 2kL$$  \hspace{1cm} (3.37)

**Step 2: linearize in the frequency shift**

So far we made a first approximation, stating that the round trip operator deviates by a small amount from the ideal condition and we got the formula 3.36 containing the new resonance frequencies. We can therefore perform a second approximation, writing also the resonance frequencies as a sum of the ideal cavity resonance and a small deviation, namely:

$$k = \tilde{k}_P + \delta k_P$$  \hspace{1cm} (3.38)

Defining $\delta C_{MM} = \langle M|\delta C|M\rangle$ and using the approximation 3.38, we can rewrite the equation 3.36 as:

$$\sum_M \frac{\delta C_{MM}(\tilde{k}_P + \delta k_P)}{1 - r_1 r_2 e^{i\phi_M(\tilde{k}_P+\delta k_P)}} = 1$$  \hspace{1cm} (3.39)

and since

$$1 - r_1 r_2 e^{i\phi_M(\tilde{k}_P)} = 0$$  \hspace{1cm} (3.40)
because it satisfies the ideal cavity resonance conditions, for \( \delta k_P \ll 1 \) we can expand the phase \( \phi_M(\bar{k}_P + \delta k_P) \) as:

\[
\sum_M \frac{\delta C_{MM}(\bar{k}_P + \delta k_P)}{1 - r_1 r_2 e^{i\phi_M(\bar{k}_P)} + i r_1 r_2 e^{i\phi_M(\bar{k}_P)} \frac{d\phi_M(\bar{k}_P)}{dk} \delta k_P} = 1 \tag{3.41}
\]

and using the \( 3.40 \) we can write:

\[
\frac{\delta C_{PP}(\bar{k}_P)}{i r_1 r_2 e^{i\phi_M(\bar{k}_P)} \frac{d\phi_M(\bar{k}_P)}{dk}(\bar{k}_P)} = \delta k_P \left[ 1 - \sum_M \frac{\delta C_{MM}(\bar{k}_P)}{1 - r_1 r_2 e^{i\phi_M(\bar{k}_P)}} \right] \tag{3.42}
\]

Since we are only considering terms in the first order of the perturbation, we discard the second term on the right hand side and we get finally:

\[
\delta k_P = \frac{\delta C_{PP}(\bar{k}_P)}{i r_1 r_2 e^{i\phi_M(\bar{k}_P)} \frac{d\phi_M(\bar{k}_P)}{dk}(\bar{k}_P)} \tag{3.43}
\]

**Step 3: connect frequency shifts and mirror maps**

At this point, what is left is to connect the frequency shift of equation \( 3.43 \) with the mirror map. In particular, we must recall that the term describing the cavity deviation from its ideal shape can be written to the first order as:

\[
\delta C = r_1 r_2 [(\delta R_1)PP + P(\delta R_2)P] \tag{3.44}
\]

Therefore, the matrix element \( \delta C_{MM} \) can be written as:

\[
\delta C_{MM} = r_1 r_2 (\langle M|(\delta R_1)PP|M\rangle + \langle M|P(\delta R_2)P|M\rangle) = r_1 r_2 e^{2ikL+2i(m+n+1)\eta} \langle M|\delta R_1 + \delta R_2|M\rangle \tag{3.45}
\]

Remembering eq. \( 3.5 \) we can write:

\[
\langle M|\delta R|M\rangle = 2ik\langle M|Z|M\rangle \tag{3.46}
\]

where \( Z \) represents the deviation of the mirror surface from its ideal shape, i.e. the mirror map.
Replacing the 3.43 in the 3.45 we find:

$$\delta k_M = \frac{2\bar{k}_M}{d\phi_M dk(\bar{k}_M)} \langle M|Z_1 + Z_2|M \rangle \quad (3.47)$$

Evaluating the derivative at the denominator we see that:

$$\frac{d\phi_M}{dk} \langle \bar{k}_M | \bar{k}_M \rangle = 2(n + m + 1) \frac{dn}{dk}(\bar{k}_M) - 2L \simeq -2L \quad (3.48)$$

therefore the 3.47 becomes:

$$\delta k_M = -\bar{k}_M \frac{1}{L} \langle M|Z_1 + Z_2|M \rangle \quad (3.49)$$

and substituting in the initial expression of the new frequencies it becomes:

$$k_M = \bar{k}_M + \delta k_M = \bar{k}_M \left(1 - \frac{1}{L} \langle M|Z_1 + Z_2|M \rangle \right) \quad (3.50)$$

which in terms of frequency is:

$$\nu_M = \bar{\nu}_M \left(1 - \frac{1}{L} \langle M|Z_1 + Z_2|M \rangle \right) \quad (3.51)$$

This expression connects the new eigenfrequencies of the cavity to cavity mirror maps. More in particular, eq. (3.52) shows that it is not possible to disentangle the contribution due to each of the mirror figure errors, but we can define a new compound map given by the sum of the two $Z = Z_1 + Z_2$. Therefore, the frequency shift is given by:

$$\delta \nu = -\bar{\nu}_M \frac{1}{L} \langle M|Z|M \rangle \quad (3.52)$$

This equation allows to foresee the frequency shift of the mode $|M\rangle$ starting from the expectation value of the map over the selected mode. Nonetheless, the measurement of the frequency shift does not result in the reconstruction of the mirror map, since we lack of information about its spatial distribution.

To overcome this problem, we recall equation (3.7), where the mirror map is
decomposed in terms of Zernike polynomials. In particular, we have:

\[
\langle M|Z|M \rangle = -2ik \sum_{ij} c_{ij} \int \int \psi_{nm}^*(x,y) Z_j^i(x,y) \psi_{nm}(x,y) dx dy
\]  

(3.53)

Substituting this equation into 3.52 we get an expression connecting the frequency shift to the map decomposed in Zernike polynomials, where the unknowns are represented by the \( c_{ij} \) coefficients. Therefore, if we are able to measure \( p \) Higher order modes into the cavity, we can choose to decompose the map in \( p \) Zernike polynomials and write a linear system of \( p \) equations in the coefficients \( c_{ij} \) of the kind:

\[
\begin{align*}
\langle 00|Z|00 \rangle &= \sum_{ij} c_{ij} \int |\psi_{00}(x,y)|^2 Z_j^i(x,y) dx dy \\
\langle 01|Z|01 \rangle &= \sum_{ij} c_{ij} \int |\psi_{01}(x,y)|^2 Z_j^i(x,y) dx dy \\
&\vdots \\
\langle nm|Z|nm \rangle &= \sum_{ij} c_{ij} \int |\psi_{nm}(x,y)|^2 Z_j^i(x,y) dx dy
\end{align*}
\]  

(3.54)

In this way we can get an estimation of the \( p \) coefficients \( c_{ij} \) to reconstruct the mirror map. The more High Order Modes are detected in the cavity scan, the more Zernike polynomials are used to describe mirror imperfections and, therefore, the more accurate is the map reconstruction.

This approach, albeit needing more investigations, is potentially a very powerful tool: if validated, it will allow to reconstruct a mirror map by simply varying the cavity length and recording the resonance frequency of each high order mode.

### 3.5 Conclusions

In this section we have presented the scattering mechanism into HOMs’ due to mirror residual roughness and we looked at many cavity configurations to explain the problems that it can introduce in interferometric GW detectors.
Therefore, two possible solutions have been outlined: the first one consists of reducing the scattering in a selected mode by identifying the mirror figure of error responsible for the scattering process. The second one, which is still in early stages and needs more investigation, consists instead of shifting the resonance frequency of a high order mode by projecting on it a suitable map. In both cases, what is needed is a system of thermal compensation able to induce a controlled modification of the mirror surface. This is what the present thesis work is devoted to, as will be presented in the next Chapters.
Chapter 4

Thermal Compensation System: the principle

The basic principle of thermal compensation relies on the controlled change of optical material properties through heating which will cause a modification of the shape of the wavefront. In the present Chapter we will outline the analytical approach to the problem, mainly focusing on the aspects more relevant for the case of Central Heating Residual Aberration Correction, where a heat pattern is projected onto the high reflectivity surface of the mirror inducing its modification of its surface figure and changing the wavefront of the reflected beam.

4.1 Thermal effects in optical materials

In Section (2.2) the main effects due to mirror heating have been presented. In particular, Eq. (2.15) describes the phase change of a beam transmitted through a heated optics, while the phase modification of a beam reflected by the mirror surface is described by Eq. (2.16), which we recall here for clarity:

\[ \psi_{\text{ref}} (x, y, t) = -\frac{2\pi}{\lambda} s_z (x, y, 0, t) \]  

Here, \( s_z (x, y, 0, t) \) describes the displacement of the HR surface of the mirror with respect to its ideal shape determined by the temperature change inside
the mirror, as suggested by Equation (2.12). Therefore, in order to control mirror shape, we need to connect the temperature field ruled by the heat equation to the displacement field described by Hooke’s law. We will describe both of these physical processes following the guidelines of J.-Y. Vinet and P.Hello in [54] and [55].

Notice that throughout the present chapter we will only consider the heating of the mirror due to the thermal correction and we will discard the effect of the cavity beam heating, which has been widely treated elsewhere (see for example [56], [57]).

4.2 Heat equation

To understand the absorption mechanism of heat radiation in the mirror substrate, we can start by writing the electric field in a dielectric material. Defining $n = n_1 + in_2$ the complex refractive index and $x$ the spatial coordinate varying over mirror thickness, we have:

$$E(r,x) = E_0(r)e^{i\omega(t-\frac{nx}{c})}$$

$$= E_0(r)e^{-\frac{\omega n_2 x}{c}}e^{i\omega(t-\frac{n_1 x}{c})}$$

where $c$ is the speed of light and $E_0(r)$ is the amplitude of the electric field. The phase of the electric field $E(r,x)$ has an oscillating term and a damping term, where $\beta = \frac{\omega n_2}{c}$ is the absorption coefficient of the material for the wave of frequency $\nu = \frac{\omega}{2\pi}$. The inverse of the absorption coefficient $\delta = 1/\beta$ is known as the penetration depth. The absorption coefficient for glasses like Fused Silica or BK7 is very high in the wavelength range $[3\mu m, 10\mu m]$, as reported in literature [59]. Therefore, the penetration depth of heat radiation is very short and, consequently, all the power is absorbed by the coating and by a thin superficial layer of the substrate.

On the basis of these considerations, we can write the steady state heat equations for our mirror.

Consider a cylindrical mirror of radius $a$ and thickness $d$ placed in vacuum.
Figure 4.1: Heat pattern projected on the HR face of a cylindrical mirror. The mirror heats up in a layer very close to the mirror surface.

Its front surface is covered with a Highly Reflecting coating, and a heat pattern is projected on its front surface, denoted by \( z = 0 \) in figure (4.1).

Given \( I(r) \) the heat pattern intensity and \( K \) the substrate thermal conductivity, the steady-state heat equation

\[
K \Delta T(r, z) = -I(r) e^{-\beta(z)} \tag{4.3}
\]

reduces, in our case, to:

\[
\Delta T(r, z) = 0 \tag{4.4}
\]

The boundary conditions of these differential equations take into account for the heat flow on the surfaces and on the edge of the mirror. Since the mirror is in vacuum, it can only lose heat by radiation. For small temperature
changes, the heat flux radiated by the mirror can be linearized, giving:

\[ F = \sigma_e (T^4 - T_e^4) \]  
\[ \simeq 4\sigma_\varepsilon T_e^3 \delta T \]

where \( \sigma_e \) is the Boltzmann’s constant corrected with the emissivity of the mirror substrate. Therefore, the boundary conditions are given by:

\[-K \left[ \frac{\partial T}{\partial z} \right]_{z = -d/2} = \varepsilon I(r) - 4\sigma_e T_e^3 \delta T(r, z = 0) \quad (4.6)\]
\[-K \left[ \frac{\partial T}{\partial z} \right]_{z = d/2} = 4\sigma_e T_e^3 \delta T(r, z = d) \quad (4.7)\]
\[-K \left[ \frac{\partial T}{\partial r} \right]_{r = a} = 4\sigma_e T_e^3 \delta T(r = a, z) \quad (4.8)\]

where \( \varepsilon \) is the efficiency of conversion of radiation into heat power in the coating.

### 4.2.1 Solution of the heat equation

The general solution of equation (4.4) can be expressed as a Dini series:

\[ T(r, z) = \sum_m (A_m e^{k_m z} + B_m e^{-k_m z}) J_0(k_m r) \]  

where \( J_0(k_m r) \) is the Bessel function of the first kind of order 0. The boundary condition (4.8) shows that the coefficients \( \zeta_m = a \times k_m \) are the zeros of the function

\[ x J_1(x) - \tau J_0(x) = 0 \]

where \( \tau = \frac{4\sigma_e T_e^3 a}{K} \) is the reduced radiation constant.

The coefficients \( A_m \) and \( B_m \) can be found imposing the boundary conditions (4.6) and (4.7) on the two mirror surfaces. Substituting these expression into
where eq. \((4.9)\) yields the final expression for the temperature field:

\[
T(r, z) = \sum_m \frac{\varepsilon p_m a}{K} e^{-k_m da} \times \\
\times \frac{\left(\xi_m - \tau\right)e^{-k_m(2d-z)} + \left(\xi_m + \tau\right)e^{-k_mz}}{(\xi_m + \tau)^2 - (\xi_m - \tau)^2 e^{-4kd}} J_0(k_m r)
\]  

(4.10)

where the \(p_m\) are the coefficients of the Dini expansion of the intensity \(I(r)\):

\[
I(r) = \sum_m p_m J_0(k_m r)
\]  

(4.11)

Intuitively, the formal dependence of eq. \((4.10)\) from the \(p_m\) coefficients suggests that the exact expression of the temperature field depends on the intensity distribution of the heat pattern.

### 4.3 Thermoelastic deformations in the steady-state

The temperature field inside the mirror induces a mirror deformation which is, in turn, responsible for the cavity beam phase change described by equation \((4.1)\). We will briefly outline this mechanism in the following.

Let \(\vec{s}(r,z)\) be the displacement vector, describing the change of position of a particular point before and after the heating process. For isotropic media, the deformation process can be described using Hooke’s law, which allows to relate the strain tensor \(E_{ij}\) to the stress tensor \(\Gamma_{ij}\) in presence of a temperature field \(T\): \[53\]:

\[
\Gamma_{ij} = \delta_{ij}(\lambda E - \nu T) + 2\mu E_{ij}
\]  

(4.12)

where \(\nu\) is the stress temperature modulus, \(E\) is the trace of the strain tensor, \(\delta_{ij}\) is the well known Kronecker tensor, and \(\lambda\) and \(\mu\) are, respectively, the first and second Lamé coefficients which, expressed in terms of the Poisson
ratio $\sigma$ and of the Young modulus $Y$, are given by:

$$\lambda = \frac{Y\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{Y}{2(1+\sigma)} \quad (4.13)$$

Since we are assuming cylindrical symmetry, we can use cylindrical coordinates, and we find that the displacement vector has only two components: $s_r(r, z)$ and $s_z(r, z)$. It reduces the components of the strain and of the stress tensors to four:

$$\begin{align*}
E_{rr}(r, z) &= \frac{\partial s_z(r, z)}{\partial r} \\
E_{\phi\phi}(r, z) &= \frac{s_r(r, z)}{r} \\
E_{zz}(r, z) &= \frac{\partial s_r(r, z)}{\partial z} \\
E_{rz}(r, z) &= \frac{1}{2} \left[ \frac{\partial s_z(r, z)}{\partial r} + \frac{\partial s_r(r, z)}{\partial z} \right]
\end{align*}$$

$$\begin{align*}
\Gamma_{rr} &= -\nu T + \lambda E + 2\mu E_{rr} \\
\Gamma_{\phi\phi} &= -\nu T + \lambda E + 2\mu E_{\phi\phi} \quad (4.14) \\
\Gamma_{zz} &= -\nu T + \lambda E + 2\mu E_{zz} \\
\Gamma_{rz} &= 2\mu E_{rz}
\end{align*}$$

In equilibrium, the net force in every direction must vanish. This condition is embedded in the equilibrium equations:

$$\begin{align*}
\frac{\partial \Gamma_{rr}}{\partial r} + \frac{1}{r}(\Gamma_{rr} - \Gamma_{\phi\phi}) + \frac{\partial \Gamma_{rz}}{\partial z} &= 0 \\
\left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \Gamma_{rz} + \frac{\partial \Gamma_{zz}}{\partial z} &= 0
\end{align*} \quad (4.15)$$

Finally, the solution must satisfy the boundary conditions:

$$\begin{align*}
\Gamma_{zz}(r, 0) &= \Gamma_{zz}(r, d) = 0 \\
\Gamma_{rz}(a, z) &= 0 \\
\Gamma_{rz}(r, 0) &= \Gamma_{rz}(r, d) = 0 \\
\Gamma_{rr}(a, z) &= 0
\end{align*} \quad (4.16)$$
4.3.1 Solution of the Hooke’s equations

In case of a known heat source localized on the reflecting surface, the two components of the displacement are given by:

\[
\begin{align*}
    s_r(r, z) & = \frac{\nu}{2(\lambda + \mu)} r \int_0^r T(r', z) r' dr' \\
    s_z(r, z) & = \frac{\nu}{2(\lambda + \mu)} \left[ \int_0^z T(r, z') dz' + \Phi(r) \right]
\end{align*}
\]

where \(\Phi(r)\) is a function whose explicit form is determined in order to satisfy the boundary conditions and the equilibrium equations. The complete form of the solution is reported in Appendix (C). Here we will give an example of mirror deformation obtained with a given heat pattern obtained using these formulas.

Figure (4.2) shows the deformation of a BK7 substrate obtained by projecting a Gaussian shaped heat pattern with a radius \(w = 5 \times 10^{-4}\) m and a total power of 1 mW. The analytic result has been compared with the result obtained with a Finite Element Numerical Simulation carried out with COMSOL Multiphysics\textsuperscript{®}. The two methods show very good agreement.

Figure 4.2: Mirror surface deformation profile obtained from the analytical solution of the thermoelastic problem and with a FE Numerical Simulation. The deformation is induced by a Gaussian shaped heat pattern with a waist \(w = 5 \times 10^{-4}\) m and a total power of 1 mW.
4.4 The thermo-elastic problem in the Fourier domain

So far we have formulated the thermo-elastic problem in the space domain. Now we will approach the problem from another point of view, solving the same equations in the Fourier domain: in this way we will be able to find a frequency response of the system and to compute its transfer function, which is a compact description of the input-output relation for a linear system.

Transform all the equations defined on the mirror surface in the spatial frequency domain requires some approximation: we need to consider a mirror of infinite dimension along the radius direction. Moreover, since we are not interested in the effect of the temperature gradient through the substrate, we can further simplify the analytical formulation by considering a semi-infinite mirror\(^1\). In Cartesian coordinates it translates to:

\[
\begin{align*}
-\infty < x < +\infty \\
-\infty < y < +\infty \\
0 < z < +\infty
\end{align*}
\]

(4.19)

where the plane identified by \((x,y,0)\) is the mirror surface.

Temperature field

Equation (4.4) is the stationary equation for the field temperature:

\[ \Delta T(x, y, z) = 0 \]  

(4.20)

A function which satisfies this equation is said to be harmonic, and can be written in the form:

\[ T(x, y, z) = \frac{1}{4\pi^2} \int_{R^2} dpdq\theta(p, q)e^{ipx+iqy-kz}\text{where } k \equiv \sqrt{p^2 + q^2} \]

(4.21)

\(^1\)The case of a mirror with finite thickness is reported in [58], and leads to the same results for the calculation of mirror deformation along \(z\).
In this coordinate frame, the only boundary condition which survives out of the three (4.6-4.8) is the one expressing the balance of heat flux on the surface $z = 0$:

$$-\left[\frac{\partial T}{\partial z}\right]_{z=0} = \frac{1}{K} \Phi(x, y) - \frac{4\sigma_c T^3}{K} T(x, y, 0)$$  \hspace{1cm} (4.22)

where we call $\Phi(x, y)$ the incoming heat flux due to the thermal source, and the second term takes into account for the heat flux radiated by the mirror surface.

After the Fourier transform, eq. (4.22) becomes:

$$(Kk + 4\sigma_c T^3)\theta(p, q) = \frac{1}{4\pi^2}\tilde{\Phi}(p, q)$$  \hspace{1cm} (4.23)

Defining $\kappa_0 = \frac{4\sigma_c T^3}{K}$ we obtain for the temperature field in the Fourier domain:

$$\theta(p, q) = \frac{\tilde{\Phi}(p, q)}{4\pi^2 K(k + \kappa_0)}$$  \hspace{1cm} (4.24)

which yields:

$$\tilde{T}(p, q, z) = \frac{\tilde{\Phi}(p, q)}{K(k + \kappa_0)} e^{-kz}$$  \hspace{1cm} (4.25)

This equation connects the Fourier transform of the temperature field to the Fourier transform of the heat pattern.

**Displacement field**

For what concern the displacement field, we start from the Hooke’s equation (4.12) which must satisfy the boundary conditions expressing the absence of resulting forces on the surface and the 3 Navier-Cauchy equilibrium equations, which in our approximation of semi-infinite medium reduce to:

$$\partial_i \Gamma_{ij}(x, y, z) = 0$$  \hspace{1cm} (4.26)

$$\Gamma_{3i}(x, y, 0) = 0$$  \hspace{1cm} (4.27)
In this coordinate frame the strain tensor is given by:

$$E_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i), \forall i, j = 1, 2, 3$$  \hspace{1cm} (4.28)

In these conditions it is possible to verify that the Hooke equation (4.12) is satisfied by a displacement vector written in the form:

$$\begin{align*}
    s_x(x, y, z) &= -i \frac{\nu}{2(\lambda + \mu)} \frac{1}{4\pi^2} \int_{R^2} dp dq \frac{\theta(p, q)}{k^2} e^{ipx + iqy - kz} \\
    s_y(x, y, z) &= -i \frac{\nu}{2(\lambda + \mu)} \frac{1}{4\pi^2} \int_{R^2} dp dq \frac{\theta(p, q)}{k^2} e^{ipx + iqy - kz} \\
    s_z(x, y, z) &= \frac{\nu}{2(\lambda + \mu)} \frac{1}{4\pi^2} \int_{R^2} dp dq \frac{\theta(p, q)}{k^2} e^{ipx + iqy - kz}
\end{align*}$$

where we recall that $\frac{\nu}{2(\lambda + \mu)} = \alpha(1 + \sigma)$, where $\alpha$ is the linear expansion coefficient for the material and $\sigma$ is its Poisson ratio.

For our description, we are interested in the $z$ component of the displacement, which is the component along the beam axis direction. Therefore, we can write everything in Fourier Transform as:

$$\tilde{s}_z(p, q, 0) = \alpha (1 + \sigma) \tilde{T}(p, q, 0)$$  \hspace{1cm} (4.29)

Finally, substituting the expression for $\tilde{T}(p, q, z)$ of equation (4.25) we have an equation giving the system response to a heat pattern impinging on it in the spatial frequency domain:

$$\tilde{s}_z(p, q, 0) = \frac{\alpha (1 + \sigma)}{k[K(k + \kappa_0)]} \tilde{\Phi}(p, q)$$  \hspace{1cm} (4.30)

### 4.5 System transfer function

Equation (4.30) shows that the system response is directly proportional to the input, therefore is linear.

We can therefore calculate the system transfer function, which is the mirror deformation due to the projection of a heat pattern with a defined distribu-
Figure 4.3: Response of the system in the spatial frequency domain for a BK7 substrate, with $\kappa_0 \approx 4.95 \, \text{m}^{-1}$. Note that to pass from Fourier domain to spatial frequency domain we have $(p,q) = \frac{4\pi^2}{(f_x, f_y)}$, therefore equation [4.31] has to be divided by $4\pi^2$.

In this equation we can easily recognize a filter with a double pole: $k = 0$ and $k = \kappa_0$, as is shown in the plot 4.3.

The mirror behaves like a low pass filter: the amplitude of its response becomes lower and lower the higher the spatial frequency of the perturbation: in other words, under equal power condition, the surface deformation is bigger the lower the spatial frequency of the pattern.

We must take into account that this transfer function has been obtained...
under the approximation of a mirror infinite in the radial direction, which implies its translational invariance. In general, for a finite mirror the solution could be slightly different.

4.5.1 Numerical evaluation of mirror transfer function

To confirm the validity of the system transfer function analytically obtained in the last section, we want to compare it to the transfer function that we obtain numerically from a Finite Element Simulation.

The procedure is the following: first the system response to a Gaussian heat pattern projected in the center of the mirror has been numerically evaluated. Therefore, the input pattern $I(x,y)$ and the system output $s_z(x,y)$ have been transformed into the spatial frequency domain with a Fast Fourier Transform (FFT) algorithm implemented in Matlab®. Finally, the transfer function is given by the ratio:

$$T(K) = \frac{\mathcal{F}\{s_z(x, y)\}(K)}{\mathcal{F}\{I(x, y)\}(K)}$$

(4.32)

where we indicated with $\mathcal{F}$ the Fourier transform of the function and with $K = \sqrt{p^2 + q^2}$ the spatial frequency in the radial direction.

To further validate this procedure, we calculated the transfer function also with a semi-analytical approach, i.e. getting the system response from the analytic calculations described in section (4.3). In figure (4.5.1) we show the comparison of the transfer functions obtained with the three above-mentioned approaches. As expected, the numerical and the semi-analytical methods agree perfectly, the slight difference at low frequency due to the lack of points in that region. On the other hand, the analytic solution in the spatial frequency domain is in good agreement only for spatial frequencies higher than \( \approx 20 \text{ m}^{-1} \), while it follows a different trend at lower frequencies. This might be due to the fact that, as mentioned above, the approximation of a mirror infinite in the radial direction doesn’t account for the boundary conditions on the edges which would give rise to a more complex solution. Nonetheless, this is already a good approximation for our purposes: indeed,
as already pointed out, the curves match quite well for spatial frequencies higher than 20 m\(^{-1}\), which is 0.05 m in the spatial domain. It means that the analytic transfer function is correct when we take into account patterns with characteristic spatial frequencies higher than 20 m\(^{-1}\), which is comparable to the size of the cavity beam for Advanced detectors.

Finally, the analytic calculation allows us to say that, in the high frequency region, the system response to an actuator drops as 1/K\(^2\), with K spatial frequency of the correction: in other words, the mirror acts as a second order filter on the actuation.

### 4.6 Heat pattern calculation

The transfer function of a system is a very powerful tool. Indeed, the knowledge of mirror response at each spatial frequency allows to easily shape the heat pattern for each kind of deformation we wish to induce on the mirror.
Let us consider $m(x, y)$ the target deformation. Thanks to the Transfer function $T(K)$, we can get the intensity distribution $I(x, y)$ which induces the target deformation by rearranging equation (4.32):

$$I(x, y) = \mathcal{F}^{-1}\left\{ \frac{\mathcal{F}\{m(x, y)\}(K)}{T(K)} \right\} \tag{4.33}$$

where $\mathcal{F}^{-1}$ is the Fourier anti-transform.

To clarify this procedure, we will consider an example. Suppose we want to induce on the mirror a deformation such as the one shown in figure (4.5a). Following equation (4.33), we must calculate its Fourier transform and divide it by the mirror transfer function. Nonetheless, since the map is defined only over a small region of the mirror, its edge doesn’t vanish smoothly. Therefore, its Fourier transform would contain very high frequencies and would require a high power to be corrected, as suggested by the transfer function (4.3). To overcome this problem, an algorithm of fit and smoothing has been used to artificially reconstruct the map over a radius 20% bigger than the map radius, and bring the map edge to zero softly, as shown in figure (4.5b). On the other hand, also the map itself contains high spatial frequency structures. For the same reason as before, we apply a low pass filter to cut them out. The cutoff frequency of such a filter is a trade off between the maximum available power of the compensation system and the needed spatial frequency accuracy.

After this map processing, we obtain the map shown in figure (4.5c).

Finally, we find that the heat pattern necessary to achieve the target map (4.5c) is given by the one shown in figure (4.6).

As a check, this heat pattern has been used as absorption map in the Finite Element Simulation, in order to compare the induced deformation to the target one. The difference between the two is shown in figure (4.7a). This residual map has a PtV of about 1.15 nm. An iterative process can help to improve the quality of the correction: we can find the heat pattern which corrects the residual map and add it to the initial heat pattern. Figure (4.8b) shows the correction achieved after only 1 iteration compared to the target map (4.8a), and the corresponding residual shown in figure (4.7b) has
Figure 4.5: Deformation to be induced on the mirror surface through a heat pattern. The map is 0.075 m in radius and is placed in the center of a mirror 0.25 m in radius. The colorbar scale varies between -10 nm and 10 nm, while x and y axes range from -0.1 m to 0.1 m. (a) - initial map; (b) - map with smoothed edges in a ring 20% bigger than the map radius; (c) - map obtained after the application of a low pass filter of order 10 with a cutoff frequency of 0.03 m$^{-1}$. 
Figure 4.6: Heat pattern to be applied to the mirror in order to obtain the deformation shown in fig. 4.5c. The total power needed is 6.31 W. Colorbar in W/m².

a PtV of about 0.6 nm.

4.7 Linearity with the absorbed power

It is straightforward to show that mirror expansion is linear with the absorbed power. A reliable way to check it is to show that the system transfer function, calculated in the way described above, does not change while varying the input power. This is shown in figure (4.9), where we compare different transfer functions computed with different values of input power. In order to avoid edge effects due to the finite mirror size, we considered a mirror diameter of 2 m, which is much bigger than the heat pattern size (a Gaussian with a waist $w = 5 \times 10^{-4}$ m). The perfect superimposition of all transfer functions implies that the expansion is linear with the absorbed power: indeed the mirror response is always the same also when varying the input power.
Figure 4.7: Difference between the target map shown in figure (4.5c) and the deformation induced by the heat pattern: direct process (a) and after 1 iteration (b). The triangular structures visible in the map are due to the mesh used in the FE simulation. Colormap ranges between -0.5 nm and 0.5 nm.

Figure 4.8: Comparison between the target map from which the high spatial frequencies have been filtered out (figure 4.5c is reported here for clarity) and the deformation (b) obtained after only 1 iteration.
Figure 4.9: Comparison among mirror transfer functions obtained with different Input power for a mirror 1 m radius. Their matching shows that mirror deformation is linearly proportional to the Input power.

4.8 Coating reflectivity and its alteration

At the beginning of this Chapter it was shown that heat absorption happens in a layer very close to the mirror surface, in other words most of the power is deposited in the coating. Therefore, we might be interested in knowing how much the coating reflectivity changes with absorbed power. Indeed, coating reflectivity determines cavity Finesse value, and we don’t want it to be lowered in the process of thermal compensation.

The coating for mirrors of Advanced detectors is composed by a succession of thin layers of materials deposited on the surface of an optical component. It is designed in such a way that reflections of light from each surface undergo maximum constructive or destructive interference, depending on whether it is meant to enhance reflectivity (Highly Reflecting coating), or to suppress it (Anti-reflecting coating). In the case we are considering, we are mainly interested in the change of reflectivity in the HR coating, which consists of a periodic layer system composed of two materials, one with high refractive index (typically Ta$_2$O$_5$) and a layer with low refractive index (usually SiO$_2$).
Figure 4.10: Change of coating reflectivity (colormap) as a function of its temperature increase evaluated for different wavelengths centered around $\lambda_0 = 1064$ nm. Colorbar in percentage.

A temperature change in the coating induces a modification of the layers refractive index, as explained in section 2.2, changing in turn also its reflectivity. A simulation was carried out to check what is the change of reflectivity induced by an increase of temperature inside the coating. The simulation, which takes advantage of tools reported in [60], give the results shown in figure (4.10) and (4.11): for a coating temperature change of 1 degree we get a change of reflectivity of about 4.5 ppm. In order to evaluate whether this change can affect the sensitivity of Advanced Virgo, it is enough to say that such a modification in the reflectivity of the ETM, for example, induces a change of its transmissivity from 1 ppm to 3.5 ppm, which was already found to be negligible [61].
Figure 4.11: Reflectivity change as a function of the temperature increase in the coating for $\lambda_0 = 1064$ nm. Note that the reflectivity change is linear up to a temperature increase of few tens of degrees. In our case, a temperature change of few degrees is foreseen.

4.9 Conclusions

In this section we have shown analytically the behavior of a mirror substrate when it gets thermally heated and deforms. Solving the problem in the spatial frequency domain allows to compute the system transfer function, which highlights that the system response to an actuator drops as $1/K^2$, with $K$ spatial frequency of the correction. Moreover, the transfer function allows to determine the heat pattern needed to induce a mirror deformation without requiring too many iterations: this point is crucial, since it allows to save precious time during the Commissioning of the interferometer. Finally, the system transfer function allows to evaluate the amount of power necessary to achieve the desired mirror map: for this reason, it becomes fundamental to know it during the design phase of a thermal compensation system.
Chapter 5

The CHRAC design

The first Chapters of this thesis were devoted to the description of the main problems arising from the residual roughness of intra-cavity mirror surface: mode mismatch, scattering into HOM’s and frequency splitting can all endanger the interferometer dark fringe condition and, therefore, lower the detector sensitivity. To reduce the risk of introducing these effects, it is possible to resort to thermal compensation: in particular, in this thesis we propose and characterize a system which exploits the change of beam wavefront due only to the deformation of the mirror surface: the CHRAC system, whose physical working principle has been analytically described in Chapter 4. In the present Chapter we are going to introduce the feasibility study of the first CHRAC prototypes, mainly focusing on its design.

In particular, two different CHRAC prototypes are studied, their difference relying in the size of the experiment they should be applied to: the CHRAC as matrix of actuators, suitable for Advanced detectors size mirror, and the CHRAC for table-top experiments.

5.1 General features

To design the CHRAC from scratch each of its parts must be accurately analyzed. Mirror defect analysis defines the shape of the correction matrix,
while the spatial and the spectral distribution of the power emitted by the heat source determines the design of the projection telescope. After analyzing all of these aspects, we must be able to design an optimized setup suitable for our purposes.

These features will be studied in the course of the present Chapter.

5.1.1 Source radiation distribution

As already said in Chapter 2, one of the main innovations of the CHRAC method consists of the use of a heat source which emits incoherent radiation. In first approximation, we can consider this kind of source as a blackbody: indeed, it’s a body heated at temperature $T$ in thermal equilibrium which emits radiation over the whole spectrum according to the well known Planck law [62]:

\[
I_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}
\]  

(5.1)

where $I_\lambda$ is the spectral radiance, i.e. the power radiated per unit area of heated surface in the direction normal to the surface, in the unit solid angle at the wavelength $\lambda$, $h$ and $k_B$ are the Planck and Boltzmann constants, respectively, and $c$ is the speed of light.

The radiation emitted by the surface of a black body in thermodynamic equilibrium obeys Lambert’s cosine law. It states that the energy flux $\Phi$ from a given element of area $dA$ of the emitting surface, detected from a direction that makes an angle $\theta$ with the normal to $dA$, into a solid angle $d\Omega$ centered on the direction indicated by $\theta$, is proportional to the cosine of $\theta$:

\[
\Phi_\lambda(\theta) = I_\lambda dAd\Omega cos(\theta)
\]  

(5.2)

Finally, for a heat source of surface $A$ and emissivity $\varepsilon$ heated at a temperature $T$, the emitted energy density integrated over the whole electromagnetic spectrum is given by the Stefan-Boltzmann law:

\[
E = A\varepsilon\sigma_B(T - T_0)^4
\]  

(5.3)
where \( \sigma_B \) is Stefan-Boltzmann constant and \( T_0 \) is the room temperature. This formula suggests that in order to enhance the source power we can either augment the heated surface or increase its temperature. This kind of source has a double advantage: since heat sources like resistors are less sensitive to current variations, they are not as difficult to be power stabilized as laser sources. Secondly, they are way less expensive than a laser. This simple formulas will be very useful in the phase of the CHRAC design.

### 5.1.2 Mirror absorption

In order to calculate the amount of power absorbed by the mirror substrate, in addition to the black body spectrum emission and the emissivity spectrum of the heating element, we need to take into account the absorption spectrum of the mirror substrate. As an example, let us report the case of Advanced detectors which use Fused Silica substrates, whose absorption spectrum is reported in figure [5.2]: radiation is absorbed by the substrate starting from a wavelength higher than 3 \( \mu \text{m} \). As reported in [42], the final absorption spectrum is obtained by multiplying among them all of the spectra mentioned above.

The amplitude of mirror deformation depends on the linear expansion coef-
ficient of the substrate. For our first experimental tests, we used BK7 mirror substrate, whose linear expansion coefficient is about 10 times greater than that of Fused Silica: in this way the effect induced by the thermal radiation absorption is bigger and, therefore, easier to be detected.

5.1.3 Numerical aperture of the optical system

The Numerical Aperture ($\mathcal{NA}$) of an optical system is defined as:

$$\mathcal{NA} = \frac{d}{2f} \quad (5.4)$$

where $d$ is the system clear aperture and $f$ the optical system focal length. Geometric calculations show that the intensity of the image projected on the mirror is proportional to the $\mathcal{NA}$ of the optical system.

Consider the schematic of figure (5.3). $\delta A$ is the patch emitting radiation, while $\delta I$ is the image formed on the mirror surface through the optical system, that we schematize as a single lens for sake of clarity. Since the solid angle subtended by $\delta A$ is the same as the solid angle subtended by $\delta I$, we have
Figure 5.3: The power emitted by an element of area $dA$ is collected by the lens of numerical aperture $\mathcal{NA} = d/2f$ and focused into the element of area $dI$. Geometric calculations (reported in the text) show that the intensity of the projected image is proportional to the $\mathcal{NA}$ of the optical telescope. Here $\alpha$ is the angle between the vector normal to the patch $\delta A$ and the optical axis, while $\beta$ is the angle between the vector normal to the patch $\delta I$ and the optical axis. Finally, $\theta$ is the aperture angle of the optical system.

$$\delta A \cos \alpha \frac{p^2}{q^2} = \frac{\delta I \cos \beta}{q^2} \Rightarrow \frac{\delta A}{\delta I} = \frac{\cos \beta}{\cos \alpha} \left( \frac{p}{q} \right)^2 = \frac{\cos \beta}{\cos \alpha} \frac{1}{M^2}$$ \hspace{1cm} (5.5)

On the other hand, we can compute the solid angle seen by the patch $\delta A$ as:

$$\Omega = \left( \frac{d}{2} \right)^2 \frac{\pi}{p^2}$$ \hspace{1cm} (5.6)

The radiated power through $\Omega$ is given by:

$$\delta P = \varepsilon \sigma_B (T^4 - T_0^4) \delta A \cos \alpha \Omega \cos \theta$$ \hspace{1cm} (5.7)

$$= \varepsilon \sigma_B (T^4 - T_0^4) \delta A \cos \alpha \left( \frac{d}{2} \right)^2 \frac{\pi}{p^2} \cos \theta$$

From these relationships, we can derive an equation connecting the trans-
mitted intensity to the system numerical aperture:

\[ I = \frac{\delta P}{\delta I} \]

\[ = \frac{\delta A}{\delta I} \cos \alpha \varepsilon B (T^4 - T_0^4) \left( \frac{d}{2} \right)^2 \frac{1}{p^2} \cos \theta \]

\[ = \cos \beta \varepsilon B (T^4 - T_0^4) \left( \frac{d}{2} \right)^2 \frac{1}{p^2} \cos \theta \frac{1}{M^2} \]

where we used the (5.5) in last line. Therefore, remembering from geometric optics laws that \( p \) is related to the lens focal length \( f \) and to the magnification \( M \) by:

\[ p = f \left( 1 + \frac{1}{M} \right) \]

we can write the (5.9) as:

\[ I = \cos \beta \cos \theta \varepsilon B (T^4 - T_0^4) \left( \frac{d}{2f} \right)^2 \frac{1}{(1 + M)^2} \]

\[ = \cos \beta \cos \theta \varepsilon B (T^4 - T_0^4) \left( \frac{\mathcal{A}}{2} \right)^2 \frac{1}{(1 + M)^2} \]

This equation suggests that the bigger the \( \mathcal{A} \) the higher the transmitted intensity. In particular, the intensity is directly proportional to the square of the system \( \mathcal{A} \).

This formula will result very useful in the design stage, as will be shown later.

### 5.1.4 Optical telescope main features

The telescope has a twofold role: collecting more power as possible from the source minimizing losses and, at the same time, focusing a sharp image of the heat pattern onto the HR mirror surface without introducing aberrations.

As already shown in the previous paragraph, loosing less power as possible from a source emitting with a lambertian distribution requires a high \( \mathcal{A} \), i.e. big lens aperture and a short focal length.

At the same time, power loss minimization also relies in the choice of lenses
material: indeed, the telescope must be highly transmissive in the wavelength range absorbed by the mirror substrate. This requirement restricts the selection of materials to glasses like ZnSe, Ge, CaF$_2$, MgF$_2$.

Finally, for the thermal compensation to perform a good correction it is crucial for the telescope not to introduce aberrations. The ones which could occur with more probability are chromatic and spherical aberrations. Chromatic aberrations could occur since the heat source emits broadband radiation: therefore, the telescope could focus different wavelengths at different distances from the mirror surface, resulting in a blurred heat pattern. The reduction of chromatic aberration can be accomplished, for instance, by using lenses materials with high Abbe number or designing suitable optical compound systems, like achromatic doublets or triplets.

Secondly, the requirement of high $\mathcal{NA}$ system can easily imply the occurrence of spherical aberrations, which would deform correction maps and introduce a bias in the correction itself. This effect can be avoided by using aspheric lenses or doublets.

Unfortunately, sometimes it is difficult to fulfill all the requirements at the same time: indeed, in some cases the thermal correction system has to be installed on an experiment which is already in place, and space constraints could make it difficult to design the setup from scratch. What is important is to analyze the requisites and trade to find a good compromise.

To better clarify this description, we report an example.

We want to test the performance of a projection telescope by imaging on a screen a square resistor heated up to a temperature of about 800 K and looking at the screen heating with a thermal camera (FLIR i5). In image (5.10a) the telescope was made up of a Fresnel lens of a plastic material, with least absorption for radiation in the infrared region 8-14 $\mu$m and $\mathcal{NA} = 1.2$. In image (5.10b) a ZnSe lens was used, with a $\mathcal{NA} = 0.3$. In both cases a configuration 2f-2f was adopted for the telescope, in order to have a magnification $M=1$. The power transmitted was measured with a broadband power meter, resulting 80 mW in the first case and 50 mW in the latter. From the pictures it is clear that the image quality is significantly higher in image (b), even if less power is collected.
Figure 5.4: Image of a square resistor obtained with a Fresnel lens (a) and a ZnSe lens (b) as projection telescope. Refer to the text for more details.

5.1.5 Focusing the image on the mirror

Since the mirror we wish to correct is part of a Fabry-Perot cavity, the optical axis of the projection system cannot be perpendicular to the mirror. For this reason, we need to perform the heat pattern projection making an angle with respect to the cavity axis as shown in image (5.5). In this conditions, an extended object orthogonal to the projection optical axis and imaged on the mirror won’t give rise to a sharp figure: indeed, its sides will be focused either closer or further with respect to the mirror surface plane. Therefore, we need to find a direct relationship between tilt angle of the projection system, indicated with $\beta$ in figure (5.5), and the object tilt with respect to the projection optical axis the for the image to be fully focused. It can be determined with geometric optics calculations.

Referring to figure (5.7), the object is focused on the image plane if:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'}$$  \hspace{1cm} (5.12)

For the tilted system, we can write the relationship between $p$ and $q$ and between $p'$ and $q'$:

$$\begin{cases} p' = p - L \sin \alpha \\ q' = q + LM' \cos \alpha \tan \beta \end{cases}$$
Figure 5.5: Schematic of the cavity and the projection system. If the object is not tilted with respect to the optical axis (a), its image won’t be fully focused on the mirror surface, while it has to be tilted by an angle $\beta'$ related to $\beta$ as shown in the text for the image to be fully focused (b).

Figure 5.6: Image formation through one lens using Zemax. (a): starting object; (b): image obtained without tilting the object as in the configuration shown in figure 5.5 (a); (c): image obtained when the object is tilted according to equation (5.15) as in the configuration shown in figure 5.5 (b).
Substituting eq. (5.13) into eq. (5.12) we end up with the following equation system:

\[
\begin{cases}
\frac{1}{p-L \sin \alpha} + \frac{1}{q+LM' \cos \alpha \tan \beta} = \frac{p+q}{pq} \\
M' = \frac{-q- LM' \cos \alpha \tan \beta}{p-L \sin \alpha}
\end{cases}
\]

It yields as solution:

\[
\begin{align*}
\tan \beta &= \frac{-q}{p} \tan \alpha \\
M' &= \frac{pq}{-p^2 + Lp \sin \alpha + Lq \sin \alpha}
\end{align*}
\] (5.15)

In absence of tilt, eq. (5.15) gives the usual magnification formula \(M = -\frac{q}{p}\). Tilting the object with respect to the optical axis according to eq. (5.15), we get an image fully in focus on the mirror surface, even if each point of the object is affected by a different value of the magnification, which means that the image will be deformed. It must be taken into account and consequently corrected.
5.1.6 Image deformation and intensity re-scaling

An object tilted with respect to the optical axis gives rise to two consequences. First of all the image will be distorted because the magnification is different point by point, and we need to reshape it. For example, we could design heating element whose size progressively changes in order to obtain equal sized pixels on the image plane.

Secondly, we need to tune the power emitted by each heater: indeed, the telescope collects more power from the heaters closer to the first lens, and it will result in a higher intensity on the image plane for closer pixels. Moreover, each pixel experiences a different magnification, which affects the image pixel intensity. All these effects must be taken into account.

5.2 CHRAC as actuators matrix

5.2.1 Choice of pixel size in the image plane

In Chapter 3 the problem of scattering into HOM due to mirror figure of error was described. In order to remove the scattering into selected modes it is necessary to apply to the mirror a correction map reducing defects at a certain spatial frequency. Here we will see that this spatial frequency is directly related to the pixel size of our correction map and, consequently, to the actuator size.

To make everything more clear, we will study the case reported in Section 3.1 and studied in [42] to dimension the pixel size for Advanced Virgo arm cavity end mirrors. In this specific case, the scattering into the modes of order 8 and 9 has to be reduced. Therefore, according to what was described in Section 3.2.2, the correction map will have a specific spatial frequency. Indeed, looking at the mirror PSD before and after the correction (figure 5.8), we see that a dip appears at the spatial frequency of the correction map, i.e. at about 16 m$^{-1}$. To further confirm that it is related to the mirror figure scattering into modes of order 8 and 9 we recall equation (3.13). For the case of end cavity mirror, $w_0 = 58$ mm, therefore we have that the spatial frequencies which scatter more into these modes are:
Figure 5.8: PSD of Advanced Virgo arm cavity End mirror. After the correction is applied a dip appears at a frequency corresponding to the characteristic frequency of the correction map. Plot from [42].

\[
\rho_8 = \frac{\sqrt{2}}{\pi w_0} = 15.5 \text{m}^{-1} \quad \rho_9 = \frac{\sqrt{3}}{\pi w_0} = 16.5 \text{m}^{-1}
\]  

(5.16)

where \( \rho_8 \) and \( \rho_9 \) are the mirror roughness spatial frequencies scattering into the mode of order 8 and 9, respectively.

In order to define the right pixel size, we invoke the Nyquist theorem [63]. This theorem originates from Signal Processing to digitize analog signals. Any analog signal can be decomposed into many frequencies, and the highest frequency component determines the bandwidth of the signal. The Nyquist theorem states that the sampling rate must be at least twice the highest frequency component (2\( f_{\text{max}} \)). In our case, we consider 24 m\(^{-1} \) as the highest spatial frequency to correct, as shown in plot 5.8, and it automatically determines the maximum pixel size, which comes out to be about 21 mm.

Finally, we need to determine the number of pixels to be used. This parameter depends on the size of the cavity beam, whose radius on the End mirror is 58 mm. Therefore, the heat pattern will be 3 times bigger than the waist and, with a pixel size of 21 mm, we will need a correction matrix of
5.2.2 Choice of actuator size in the object plane

Once the size of the pixel in the image plane is defined, the size of the actuators in the object plane has to be set. In particular, the size has to be chosen in such a way to maximize the power collected by the telescope and sent onto the mirror. For this reason, this choice progresses together with the telescope design: indeed, a bigger actuator will require a telescope with a stronger magnification for the image to keep a fixed size. To better understand the problem we can recall equation (5.11), which summarizes the dependence of the transmitted intensity from all the parameters. To simplify, imagine that the emitting surface and the image plane are both perpendicular to the optical axis. In this case the intensity is given by:

$$I(T, NA, M) = \pi \varepsilon \sigma_B (T^4 - T_0^4) NA^2 \frac{1}{(1 + M)^2} \cos \left[ \arctan \left( \frac{NA}{(1 + 1/M)} \right) \right]$$

(5.17)

where we wrote $\theta$ as a function of the system numerical aperture and magnification.

The trend of the intensity as a function of the actuator size is reported in figure 5.2.2, where it is plotted versus $1/M$. Fixing the pixel size in the image plane and the $NA$ of the projection system, we vary the actuator size in the object plane and adjust its distance from the lens in order to have a focused image on the mirror.

The curve shows that increasing the actuator size represents a gain in terms of transmitted intensity up to a certain value, after which it becomes asymptotic and approaches the value

$$I \simeq \pi \varepsilon \sigma_B (T^4 - T_0^4) NA^2$$

as $M$ goes to zero: the transmitted intensity follow the usual Plank law and is proportional to the square of the numerical aperture, as it was already
Figure 5.9: Transmitted intensity as a function of the inverse of the system magnification. In this simulation we are using a single lens with an aperture of 200 mm and a focal length of 270 mm. As actuator we are considering a square element with an emissivity of $\varepsilon = 0.95$, heated at a temperature of $800^\circ C$, while the pixel size in the image plane is set to 21 mm.

Albeit this plot is crucial in the design stage, the final project of the whole system has also to take into account space constraints. Indeed, sometimes the optimal actuator choice leads to a telescope design that does not match the real space availability. Moreover, another limit is represented by the minimum distance of the telescope from the mirror to be corrected: indeed, the smaller the telescope magnification, the closer it is to the mirror, and we must take care of not clipping the cavity beam.

Therefore, the general rule summarized in equation (5.17) must be taken as a guideline and a trade off process is needed to choose the best configuration conform to the external constraints.
5.3 CHRAC setup for table-top experiments

As already mentioned before, sometimes the CHRAC setup has to be installed on an already existing experiment. On large scale interferometric detectors, the mirrors where we want to apply the correction have a diameter of 35 cm, and resistors are a good choice to be used as actuators.

When the optics where the correction map has to be projected onto is smaller, applying the correction would become more difficult: indeed, the portion of the mirror where the correction has to be applied is small, the cavity beam radius being fraction of a millimeter big. If resistors are used as actuators, the projection telescope should operate a strong magnification in order to perform the required correction. In this condition, two kind of problems would arise: first of all, we would introduce aberrations in the projected map. Moreover, the higher the spatial frequency of the defects to be corrected, the higher the number of actuators to use, the worse the aberrations problems. Secondly, we would need a big space availability to perform the desired magnification, and most of the times this is not the case, especially for table-top experiments. To give an example, imagine that we want to use a $10 \times 10$ matrix of 5 mm side actuators: for a cavity beam 1 mm diameter we would need a magnification $M = 0.05$. Using an optical telescope with a reasonable $\mathcal{N}A$ not to introduce strong aberrations, such a setup would take up a space of several meters, and the last telescope lens could end up too close to the mirror to correct, possibly intercepting the cavity beam.

Here we present an alternative solution for the CHRAC setup for use on table-top experiments. Also in this case we start from the principle that we want to project a correction map on the HR surface of the mirror, but instead of using single actuators as pixels, we take advantage of the Digital Light Processing technology by Texas Instruments® which utilizes a Digital Micromirror Device (DMD): the correction map is created by the micromirrors of the DMD and the DMD screen is illuminated by a heat source. The light reflected by the micromirrors is then collected by the projection telescope which images the DMD screen onto the mirror HR surface.

In the following we will describe what we will call from now on the table-top
5.3.1 CHRAC setup using Digital Micromirror Device

DLP is a MEMS technology from Texas Instruments® which uses a Digital Micromirror Device. It consists of an array of digitally switchable aluminum micromirrors laid out on a semiconductor chip. DLP is employed in many applications. Basically it constitutes the working principle for certain projectors, indeed it is used in Digital Cinemas, Televisions and so on. In the DMD chip the mirrors are organized in a two-dimensional matrix, and each micromirror is switchable between two discrete angular positions: −12° (corresponding to the OFF state) and +12° (which is called the ON state). The angular positions are measured relative to a 0° "flat state", which is parallel to the array plane. The hinge-axis around which each micromirror tilts is diagonal with respect to the overall array. Every single micromirror is positioned over a corresponding CMOS memory cell and its angular position is determined by the binary state (logic 0 or 1) of the corresponding CMOS memory cell contents. To produce greyscales, the micromirror is toggled on and off very quickly, with a frequency ranging from 50kHz to 500kHz with a binary pulse-width modulation, so that the ratio of ON time to OFF time determines the shade of gray produced.

Many models are available on the market: they differ each other basically for the number of micromirrors and their size. They are designed to be used with broadband visible light (400nm - 700nm), which can represent a problem in our application: therefore we may have to make some modifications in order to use it for our aims. Mainly, we have to take into account two things: the micromirror size, which may generate diffraction effects when illuminated with thermal radiation, and the protective glass on the top of the chip, which could absorb in the same wavelength range as the mirror we wish to correct.

The model that we found more suitable for our purposes is the DLP7000, which is an array of 786432 mirrors organized in a two-dimensional matrix of 1024 columns by 768 rows. In this model the micromirror pitch (defined as...
the distance between two consecutive centers) is 13.68 µm. This size was chosen in order to minimize diffraction effects for the incident radiation, whose wavelength distribution is peaked around 2 µm, and has a black body distribution, as already mentioned above. Moreover, the glass window covering the chip was removed and replaced with a CaF$_2$ glass in order to avoid radiation absorption in the wavelength range of interest. The first prototype of the table-top CHRAC experiment was made using the DLP board contained in the television model WD-52327 by Mitsubishi Digital Electronic®. The board was removed from the TV and mounted on the optical table using articulated posts to allow any kind of positioning. The DMD was oriented in such a way that the light reflected from its ON state was parallel to the optical table and directed towards the optical axis of the projection system.

### 5.3.2 Illumination system

As already described before, the DMD has to be illuminated with a IR source: this light will be then reflected by the DMD, collected by the projection system and absorbed by the mirror substrate. Since the projection telescope will image the DMD plane onto the mirror, it is important that the DMD is illuminated evenly, and that the illumination system doesn’t image the
source on the DMD plan. The risk is that the correction map shown on the DMD is not uniformly illuminated, and the image projected onto the mirror will contain the illumination source details, that we want to avoid.

Illumination uniformity is the typical issue arising, for example, in optical microscopy: the specimen to be analyzed has to be uniformly illuminated, such as to avoid the source details being focused on the sample plane. Many microscopes use the Koehler illumination to illuminate the sample at the field point. This kind of design could be suitable also to illuminate the DMD. It must be taken into account, though, that it requires many optical elements, and choosing the right optics and substrate we may end up with a very cumbersome (and perhaps also very expensive) optical system, which may not match with the space needs. For this reason, we decided to start studying more compact systems, allowing to collect more power as possible, as explained in paragraph (5.1.3) and at the same time fulfilling the aforesaid evenness requirements.

Finally, the design of the illumination system walks along with the projection system planning, as we will see in the next section.

5.3.3 Projection system

The main features required by the projection system have been already discussed in paragraph 5.1.4. Some additional considerations need to be made concerning the telescope aperture in the case of the table-top CHRAC. First of all, it is important that all the power reflected by the DMD is collected by the projection system. This implies that the divergence of the heat beam from the illumination system must match the telescope aperture. Secondly, the light from the OFF state should be completely cut out from the optical system, and only light from the ON state must be collected in order to guarantee a better image contrast: therefore, the telescope must be designed in such a way as to collect all the light from the ON state and reject all the light from the OFF state.

The considerations reported so far will become clearer with the two examples below.
Ellipsoidal reflector

The first illumination system tested considers the use of an aluminum ellipsoidal reflector as optical element to collect the power from the heat source. The advantage of using this system lies in the fact that, in the ideal case, all the radiation emitted by the source positioned in the first focus is collected and ducted in the second focus. Thanks to this system no other optics are interposed between the source and the DMD, and it results in less overall losses. The DMD can be positioned in the second focus or very close to it, in order to avoid to be exactly in the image plane of the source. However, this latter problem can be overcome by using an extended and uniform source. In our case, a reflector 22.4 cm in diameter was used, with a the distance between the two foci of 50.8 cm. For what concerns the projection system, we have to fulfill the following requirements:

1. matching the system aperture to the heat beam divergence, which puts a constraint on the distance between the DMD and the first lens of the telescope \( D = r \tan \alpha \), with \( \alpha = 15^\circ \).

2. the light from the OFF state must be cut out, and only light from the ON state must be collected.

3. it must fulfill the requirements described in 5.1.3.

The final telescope design is composed of two lenses: an achromatic doublet 30 mm diameter × 60 mm focal length, and a ZnSe lens 50 mm diameter × 75 mm focal length as second lens, with an overall magnification of \( M = -1.142 \). The schematic of the setup is reported in figure 5.11. Note that, to better highlight the heat beam shape along its path, the DMD is considered as a transmitting element: indeed, since the DMD is just a mirror, it doesn’t affect the beam shape, unlike lenses and reflectors.

Parabolic reflector

A second configuration tested used a 1 inch parabolic reflector in place of the ellipsoidal one. In this case, putting the heat source as close as possible
Figure 5.11: Schematic of the table-top CHRAC setup using a ellipsoidal reflector as illumination system and a two lenses telescope. See text for more details.

to the reflector focus, produces an almost collimated beam. The projection system of this second setup consists of a ZnSe lens 50 mm diameter × 75 mm focal length and a CaF$_2$ lens 38.1 mm diameter × 75 mm focal length. The advantage of this setup relies on the fact that it transmits visible light, making the alignment procedure easier, as will be shown later. A schematic of the setup is shown in figure (5.12), where we used the same convention as in figure (5.11).

5.4 Conclusions

In this Chapter we gave the guidelines to design a possible system of thermal compensation able to perform the mirror figure error correction: the CHRAC. Each aspect concerning this thermal projection system has been analyzed: power optimization, projection telescope aberration reduction, accomplishment of the space constraint. The study reported in this Chapter is the starting point to the realization of two CHRAC prototypes which will be presented in Chapter [6]. Finally, the effectiveness of the table-top CHRAC device on a cavity beam is illustrated in Chapter [7].
Figure 5.12: Schematic of the table-top CHRAC setup using a parabolic reflector as illumination system and a two lenses telescope. See text for more details.
Chapter 6

First experimental tests

In the last Chapter the design of CHRAC setup was extensively explained. The present Chapter will be devoted to the description and preliminary characterization of the first two prototypes: the CHRAC matrix which has been realized using resistors as actuators, and the DMD-based table-top CHRAC.

6.1 CHRAC matrix prototype assembling and testing

In order to choose the actuator better responding to the needs listed in the last Chapter, many different kinds of actuators were characterized. The final choice converged on a resistor which was used to assemble the actuator matrix. However, even if only preliminary measurements of the mirror deformation induced by CHRAC have been carried out with this prototype, they represent a proof of the principle.

6.1.1 Actuator choice

The CHRAC matrix prototype is intended as a device to correct defects of mirrors whose size is comparable to that of Advanced detector mirrors. Maximize the intensity transmitted through the projection system and project a uniform image of each pixel are the requirements we want to fulfill. The test
of the actuators has been carried out by performing two kinds of measurements: an optical telescope made up of a 2 inch ZnSe lens of 75 mm focal length in a 2f-2f configuration was used to image the heater on a screen. The transmitted power was measured using a broadband power meter and the heater image was detected by a thermal camera (FLIR i5) in order to check its uniformity.

Many commercial resistors were tested in order to find the most suitable for our purposes. The features of the most significant ones are summarized below.

**Induceramic**

It is an Alumina metallic heating element made up of high-temperature cofired ceramic. The resistor is a 10×10 mm square, and the heated area corresponds to about 5 × 9 mm². The main characteristic of this resistor is that its surface is very uniform, so it can be a good candidate for a pixel of the actuator matrix. It can reach a temperature of 1000°C. Figure (6.1a).

**Mini-lite 12-20**

This heater has a Silicon Nitride heating element 2 mm in diameter. It is usually used to light a gas top burner, so it has a fast ignition time and is very long lasting even if it is kept at high temperature for long periods. Its rated temperature is about 1200°C. Figure (6.1b).

**Helioworks EK-8520**

The heating element has a coiled Kanthal Filament with a Gold plated parabolic reflector 1 cm in diameter behind it. Its operating temperature is about 950°C. The parabolic reflector helps in collimating the emitted power. Figure (6.1c).

**OSRAM Halogen 64425**

This is an halogen light bulb with a diameter of about 5 mm and capable of heating up to a nominal temperature of 2500°C. Because of this high
temperature, the peak wavelength of the emitted radiation is in the visible range, so most of the emitted power is not absorbed by the mirror substrate. Moreover, the glass bulb absorbs the light in the wavelength range we are interested in. Therefore our target only sees the temperature of the glass bulb and not the filament itself. Figure (6.1).

**Scitec IR-12K**

This heater is a coiled Kanthal filament with a 1 inch parabolic mirror behind and can reach temperatures of about 1000°C. Moreover, the radiating
<table>
<thead>
<tr>
<th>Heater</th>
<th>Operating power (W)</th>
<th>Measured T (°C)</th>
<th>Detected intensity (W/m²)</th>
<th>Actuator area (m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Induceramic</td>
<td>8.5</td>
<td>700</td>
<td>500</td>
<td>5×10⁻⁵</td>
</tr>
<tr>
<td>Mini-lite 12-20</td>
<td>20</td>
<td>900</td>
<td>6370</td>
<td>3.14×10⁻⁶</td>
</tr>
<tr>
<td>Mini-lite 12-20 with parabolic reflector</td>
<td>20</td>
<td>800</td>
<td>550</td>
<td>5.1×10⁻⁴</td>
</tr>
<tr>
<td>Helioworks EK-8520</td>
<td>5.5</td>
<td>750</td>
<td>710</td>
<td>1.27×10⁻⁴</td>
</tr>
<tr>
<td>OSRAM Halogen 64425</td>
<td>20</td>
<td>more than 2000</td>
<td>2390</td>
<td>2×10⁻⁵</td>
</tr>
<tr>
<td>OSRAM Halogen 64425 with parabolic reflector</td>
<td>20</td>
<td>more than 2000</td>
<td>650</td>
<td>5.1×10⁻⁴</td>
</tr>
<tr>
<td>Scitec IR-12K</td>
<td>11</td>
<td>750</td>
<td>830</td>
<td>5.1×10⁻⁴</td>
</tr>
</tbody>
</table>

Table 6.1: Main features of the tested resistors. The power is measured with a broadband power meter by focusing the resistor through a ZnSe 2 inch lens in 2f-2f configuration.

element has a high emissivity in the infrared region, allowing to have more power in the region of interest.

Most of the resistors presented here emit radiation over the whole solid angle. If a parabolic mirror is put behind the resistor, the power emitted at large angle is recollected (see table (6.1)). This is to the detriment of the uniformity of the pixel image, therefore the actuator loses its eligibility as actuator. However, this kind of actuator can still be used as a pixel when the optical system has a strong demagnification which would blur out the source detail. This is shown in fig. (6.2): starting from the left, the first picture shows the heater in the focus of the parabolic mirror, the second is the image of the heater obtained in a 2f-2f configuration and the third is obtained when the optical telescope has a magnification M≈ 0.1. In the second image all the source details are clearly visible, while they start blurring out when the image is made smaller, mostly due to lens aberrations.
Figure 6.3: Comparison between the intensity transmitted through a telescope made up of a single lens 50 mm aperture × 75 mm focal length for the Mini-lite 12-20 igniter heated up at 900°C and the Scitec IR-12K heated at 750°C.

Figure 6.4: Schematic of the setup needed to project a 2 mm diameter spot using a Mini-lite igniter (top) or a Scitec IT-12K (bottom) as actuator. In the drawing only the emitted radiation in the solid angle subtended by the lens is represented.
6.1.2 Example of actuator choice

The following example will better clarify how to perform the choice of an actuator.
Imagine we want to project a round pixel 2 mm in diameter on a mirror using a 2 inch diameter lens with a focal length of 75 mm, and we want to compare the performance of the Mini-lite 12-20 Igniter to the Scitec IR-12K. Returning to the formula (5.17) we can get the transmitted intensity for both the heaters, which are shown in the plot (6.3).

A magnification of M=1 is required to make an image of the correct size with the Mini-lite igniter, while a magnification M=0.078 is required for the Scitec IR-12K. The plot (6.3) shows that the transmitted intensity is slightly higher for the second actuator. On the other hand, in the second case the total space required for the setup is much higher, and the lens position gets closer to mirror to be corrected the smaller the magnification, which sometimes it is not feasible. A schematic is shown in figure (6.4).

A good compromise between emitted power and image uniformity makes our choice to converge on the Induceramic square resistor as actuator in the CHRAC matrix (fig. 6.5). Moreover, the ceramic material has good filtering properties and limits the effect of current fluctuations which can introduce noise in the system to be corrected.

Figure 6.5: Induceramic resistor cold (left) and heated up at about 700°C (right). Right image taken with a thermal camera.
6.1.3 Probe setup to measure the applied deformation

In order to detect the effect of the CHRAC on the mirror HR surface, a probe setup was developed. A Nd:YAG laser beam ($\lambda = 1064\text{nm}$) is fiber coupled in order to reduce wavefront aberrations, and it is expanded almost up to the size of the whole mirror. A telescope images the mirror surface onto a wavefront sensor based on the lateral shearing interferometer principle (PHASICS SID4) [67]. The mirror deformations due to heat pattern projection induce a change in the probe beam wavefront, and the phase modification is detected with the Phasics.

A schematic of the setup is shown in figure (6.6).

More exactly a relative measurement is performed: it consists in a first zeroing phase, where the probe beam wavefront with no projected pattern is taken as a reference, then the deformation induced by the CHRAC with respect to the new reference is detected.

In order to better detect the effect, it is preferable to limit air flows and environmental light. This condition is afforded by covering the setup with a dark enclosure.
Figure 6.6: Probe setup to reveal the mirror deformation induced by the heat pattern. More details in the text.

Figure 6.7: Picture of the setup used for the full characterization of one single pixel.
6.1.4 One pixel full characterization

In chapter 4 we analytically described the way the substrate expands. In this section we will compare the simulations to the experimental results. The setup reported schematically in figure (6.6) is shown in picture (6.7): the Mini-lite 12-20 igniter is used to project a round spot onto a 1 inch BK7 mirror. The heater is imaged onto the HR mirror surface using a 2 inch ZnSe PLCX lens with a magnification $M = 1$ as shown in the schematic on the top of figure (6.8).

Figure 6.8: Top: schematic of the setup. (a): starting heat pattern; (b) aberrated heat pattern reaching the mirror surface. It is used as absorption map in the Finite Element simulation. Optical simulation obtained with Zemax.
Figure 6.9: Figure (a): mirror deformation due to 5.92 mW total power absorbed by a 1 inch BK7 mirror substrate with an absorption map shown in figure (6.8b). The heat absorption also induces an overall change in the radius of curvature of \(7.66 \times 10^3\). Simulation carried out with COMSOL Multiphysics. Deformation amplitude is in meter. Figure (b): mirror deformation due to 5.92 mW total power absorbed by a 1 inch BK7 flat mirror revealed using the probe setup described in section (6.1.3).

To reproduce in simulation the same conditions as in the experiment, the heater’s temperature is measured in order to estimate the power spectral distribution of the projected heat pattern, which is important to take into account the effect of chromatic aberration through the projection system. A simulation carried out with Zemax\textsuperscript{R} allowed to determine the exact pattern impinging on the mirror (figure 6.8), which can be used in the Finite Element simulation as absorption map, intended as the pattern whom with heat is absorbed by the mirror. Moreover, the power really absorbed by the mirror is measured in order to use the correct value in the simulation.

The simulation result (fig. 6.9a) is compared to the actual mirror deformation obtained experimentally, shown in figure (6.9b). To better compare the results, we cut a line along the mirror diameter and look at the deformation along the line: simulation and experimental results (fig. 6.10) show a very good agreement, confirming the validity of the model. Finally, figure (6.11a) shows a temperature change of the mirror of about 1.2 degree.
Celsius. The mirror temperature is measured using a thermal camera FLIR sc300 and is shown in fig. (6.11b). Also in this case the change in temperature of the mirror substrate is about 1 degree Celsius. The small difference between simulation and experiment is attributed to the cooling down effect of air flow.

![Surface deformation graph](image)

**Figure 6.10:** Comparison between simulated and experimental results. The figure shows a section of figure 6.9a superimposed to a section of figure 6.9b. They show a very good agreement.
Figure 6.11: Temperature change induced in the mirror substrate due to the absorption of 5.92 mW. (a): Simulation carried out with COMSOL Multi-physics. (b): Picture taken with a FLIR thermal camera. In both cases the change in temperature of the mirror substrate is about 1 degree Celsius.

6.1.5 CHRAC matrix prototype

As mentioned at the end of section 6.1.1, the first matrix prototype was assembled using Induceramic resistors as actuators. A steel frame was designed to host 61 resistors. The complete matrix is shown in picture (6.12).

The resistors are connected to a custom-made driver with 64 channels, which is based on the Pulse Width Modulation (PWM) power control method [68]. Each channel of the driver can supply a maximum current of 1A and a voltage of 24V which is provided by an external power supply, for a maximum dissipated power of 24W. Power dissipated in each channel is a fraction of maximum power and is given by:

\[
P_d = \frac{PWM_{value}}{4096} P_{max}
\]

where \( P_d \) is the power dissipated in the single resistor and \( PWM_{value} \) is a number in the range \([0, 4096]\).

The output frequency of the waveforms can be changed in range of about 50Hz to 3000Hz. It is given by:

\[
f_{out} = \frac{50000000}{4096 \cdot (PV + 1)} \tag{6.1}
\]
Figure 6.12: CHRAC matrix assembled with Induceramic resistors.
where $PV$ is a Prescaled Value in the range $[3, 255]$. Equation (6.1) is important because it allows to identify the range of frequencies where this setup can introduce noise.

### 6.1.6 Experimental results

To detect the deformation induced by the CHRAC matrix a probe setup as the one described in section 6.1.3 was used. The CHRAC was tested on a 140 mm diameter BK7 mirror, with a projection system composed of a 2 inch ZnSe lens in a 2f-2f configuration. A picture of the setup is shown in figure (6.13).

![Figure 6.13: Probe setup used to sense the mirror deformation induced by CHRAC. The fiber coupled laser beam (A) impinges on the mirror surface (B) and gets reflected. A two lenses telescope images the mirror surface onto the PHASICS (C). The CHRAC matrix (D) is imaged onto the HR mirror surface through a lens.](image)
Referring to the example reported in [42], the deformation induced by 5 pixels was examined. As for the single pixel analysis, the measurements were compared to the simulation results. Each one of the five pixels was powered at about 10W, for a total intensity of 500 W/m² impinging on the mirror. The results were compared to the Finite Element simulation (fig. 6.14b), where the image obtained with Zemax in fig. (6.14a) was used as absorption map.

Figure 6.14: (a): Image projected onto the mirror surface obtained with a Zemax simulation. Chromatic aberrations through the projection system not taken into account. (b): Result of the Finite Element simulation carried out with COMSOL Multiphysics using image (a) as absorption map.

A picture of the heaters switched on and of their image on the mirror surface was taken with the thermal camera FLIR i5 is shown in figure (6.15).

Mirror deformation was revealed with the probe setup described above. Different configurations have been tried to show that changing the heat pattern it is really possible to induce the desired deformation. An example is given in figure (6.17): here many resistors are switched on in a smiley-like pattern. The resulting deformation follows the same shape. These preliminary measurements represent the proof that it is possible to use a source of incoherent infrared radiation like a resistor to generate a controlled heat pattern. Square and even resistors can be used as pixels of the
Figure 6.15: Left: resistors heated up on the CHRAC frame. Right: image of the heaters on a paper put in front of the mirror. The target is not centered on the 5 pixels to better show their image. Images taken with a thermal camera.

Figure 6.16: Probe beam wavefront deformation revealed with the setup shown in fig. (6.13). Colorbar in m.

actuation matrix and the image of the heat pattern on the mirror induces a deformation which is well predicted by simulations. This step represents the foundation to start a new system of thermal compensation.

6.1.7 Next steps towards the thermal compensation

The preliminary experimental results reported in the last section show the feasibility of the method. The next step consists of controlling the actuators
Figure 6.17: Many pixels of the CHRAC matrix are heated up (top) and induce a mirror deformation (bottom). The induced deformation shows a distortion due to the magnification as in equation (5.15). Colorbar in nm.
in order to obtain the target mirror deformation and, therefore, the wavefront correction. A well experimented control algorithm to reach this target is reported in [40]. The method consists of a first calibration phase, in which all the $N$ actuators are switched on one after another and their responses, called influence functions, are collected in a $N \times M$ interaction matrix $M$. Therefore, given $\psi_t$ the target wavefront deformation, the algorithm aims at minimizing the difference:

$$\psi_e = \psi_t - Ma$$

(6.2)

where $a$ the $N \times 1$ vector contains all the actuators coefficients. This minimization can be carried out through a Least Square Algorithm. However, it yields a non bounded vector that can have either positive or negative values. Therefore, this method is replaced by a Singular Value Decomposition method (SVD), which allows to diagonalize non-square matrices [69]. It consists of writing the $M$ matrix as:

$$M = U \cdot \Sigma \cdot W^T$$

(6.3)

where $\Sigma$ is a matrix of eigenvalues $\lambda_i$, $U$ is a matrix of eigenvectors being a basis in the $M \times M$ space, and $W^T$ is a matrix of eigenmodes, basis of the $N \times N$ space. This matrix is invertible, therefore it is possible to calculate the vector of actuation values $a$ to correct the incident wavefront $\psi_i$ as:

$$a = M^+ \cdot \psi_i$$

(6.4)

On the other hand, the knowledge of the system transfer function allows to obtain the correct heat pattern much faster. Indeed, it is enough to measure the system response to a single actuator to compute the mirror transfer function and obtain the heat pattern according to the procedure explained in section [4.6]. However, in this case the transfer function cannot take into account the defects of single actuators, as for the case of influence functions, and it may cause that finding the correct heat pattern could take longer. Moreover, since each resistor experiences a different magnification through the projection system (as shown in section [5.1.5]) this scaling factor must be
taken into account to determine the correct power emitted by each actuator.

\subsection{6.2 Table-top CHRAC characterization}

In section 5.3 the table-top CHRAC prototype was briefly described. Here we will present the first setup test and characterization which was carried out by projecting a round spot on the mirror, as was done for the CHRAC matrix. To do it, a circle was displayed on the DMD screen and imaged onto the mirror. In order to enhance the efficiency of the setup, the map displayed on the DMD are shown in greyscale, which allows us to tune the amount of light transmitted through the projection setup: indeed, a white pixel on the DMD gives the maximum reflected light, and the opposite happens for the black pixel.

The mirror deformation was sensed using a setup as the one described in section 6.1.3. This measurement represents a good prototype test, useful to compare the induced deformation with the simulation predictions. Moreover, it highlights many issues related to this kind of setup which were corrected in the final version.

\subsubsection{6.2.1 Setup installation and characterization}

The table-top CHRAC setup is schematically shown in figure 6.18, while a picture is reported in figure 6.20. The design has already been described in section 5.3.3 (Parabolic reflector), the only difference being the second lens of the projection telescope.

One of the main issues encountered during the realization of the setup was the alignment of the projection system: for this reason, a coarse alignment was performed using the Mini-lite igniter as heat source, as shown in figure 6.18 where the projection system is marked in yellow. Red lines in the same picture show the probe laser path. After the telescope alignment, the heater was replaced with the DMD (figure 6.20).
Figure 6.18: Schematic of the setup used to perform the first DMD characterization. A Mini-lite igniter is placed close to the focus of a parabolic mirror and is used as illumination source for the DMD. The image to project on the end cavity mirror is displayed on the DMD screen. The heat radiation reflected by the DMD is collected by the projection telescope, which is composed of an achromatic doublet 30 mm aperture × 60 mm focal length and a ZnSe lens 50 mm aperture × 75 mm focal length, with an overall magnification of $M = 1.14$. The correction is applied on a 1 inch BK7 mirror with a RoC= 0.5 m
Figure 6.19: Probe setup (red lines) and projection system (yellow line). The alignment of the projection system on the mirror is performed by using the hot tip of the Mini-lite igniter as heater: it allows to collect more power and, consequently, have a more visible effect.

Figure 6.20: Left: parabolic reflector illuminating the DMD. Right: table-top CHRAC aligned to perform the correction on the mirror.
To characterize the table-top CHRAC the following procedure was followed: a round spot 4.2 mm in diameter is showed on the DMD. After experiencing the magnification of the projection system it becomes a spot 4.8 mm in diameter on the mirror surface. The total power absorbed by the mirror substrate was measured to be about 8 mW. An estimation of the expected deformation is given by a Finite Element simulation. Mirror deformation is shown in figure (6.21), while in figure (6.22) the mirror deformation profile is compared to the simulation result.

![Figure 6.21: Mirror deformation induced by the table-top CHRAC setup.](image)
The pupil on which the probe beam phase analysis is carried out has a diameter of about 5 mm: for this reason, experimental data correspond only to a part of the whole mirror deformation.

As in the case of the CHRAC matrix, this measurement shows the feasibility of the method. Moreover, this first characterization allowed to highlight one of the issues of this setup, which is the heat beam positioning on the mirror and its focusing: indeed, the achromatic doublet of the projection system, used to reduce spherical and chromatic aberrations, absorbs the radiation in the visible spectrum. To align the heat pattern to the cavity beam, a paper target was put in front of the mirror surface and its heating was checked with a thermal camera: it indicated the heat pattern footprint on the mirror surface, and
helped in comparing its position to the cavity beam position. After this characterization, the projection system was redesigned in order to be transmissive in the visible range and reduce difficulties in the alignment phase.

In the next chapter we will see an application of the table-top CHRAC to an interferometer 30 cm Fabry-Perot arms operated with the Laguerre Gauss modes. It will be shown that this setup will be effectively able to improve the beam quality.

### 6.3 Noise contributions

It is important to evaluate the main noise contributions introduced by the CHRAC setup and estimate whether they affect the interferometer sensitivity. In particular, in this section we will investigate the mechanisms that convert CHRAC intensity noise into displacement noise. Fluctuations in locally deposited heat can give rise to different kinds of effects, which can be mainly summarized in:
Thermo-elastic noise

Fluctuations in the local thermal expansion. This effect mostly changes the cavity length.

Thermo-refractive noise

Local change of the refractive index. This effect changes the OPL of the beam passing through the optic.

Radiation pressure

A fluctuation of the power impinging on the mirror induces a fluctuation in the radiation pressure acting on the mass, therefore inducing a displacement of its position.

Aiming at applying the CHRAC on the End mirror of the arm cavity, we can temporarily discard the Thermo-refractive noise contribution, which mainly affects of the beam passing through the heated optic. We therefore focus onto the remaining two.

In [66] it was shown that the thermo-refractive contribution to displacement noise averaged over the main laser beam intensity profile is given by:

\[ \Delta z_{TE} = \frac{P}{2\pi f C \rho} (1 + \eta) \alpha RIN \]  

with \( P \) the total power of the heat pattern, \( C \) the heat capacity of the substrate, \( \rho \) the substrate density, \( \eta \) the Poisson coefficient and \( \alpha \) the linear expansion coefficient. What we denoted with \( RIN \) is the equivalent to the Relative Intensity Noise for the CHRAC, which we will evaluate later.

On the other hand, the radiation pressure contribution is given by:

\[ \Delta z_{RP} = \frac{P}{c m (2\pi f)^2} RIN \]  

Comparing equation (6.5) to (6.6) it is easy to see that the prevailing contri-
bution is given by the Thermo-elastic noise which decays as $1/f$ against the $1/f^2$ of the Radiation Pressure noise.

At this point, we need to evaluate the Residual Intensity Noise, which is affected by two contributions. The first one is the fluctuation in the photon number for each field mode, which for a black-body source is given by (70):

$$\frac{\Delta P}{P} = \sqrt{\frac{8k_B T}{P}}$$  \hfill (6.7)

The second contribution is due to the current fluctuation in the actuator, which translates to a fluctuation of the power reaching the mirror surface. In order to evaluate that, we need to compute the transfer function between current and power fluctuation:

$$\frac{dP}{dI} = \frac{dP}{dT} \frac{dT}{dI}$$  \hfill (6.8)

The two derivatives can be evaluated separately.
From the Stefan-Boltzmann law (eq. 5.3) we have:

$$\frac{dP}{dT} = 4\zeta T^3$$

(6.9)

where $\zeta = A\varepsilon\sigma_B$.

The contribution of $\frac{dT}{dI}$ can be evaluated starting from the fact that a variation of the temperature is due to a fluctuation of the power $P$ dissipated in the resistor. Therefore, remembering that $P = RI^2$, where $R$ is the resistance and $I$ the current flowing through it, we have that $\Delta T \propto \Delta P$ and therefore:

$$(T - T_0) = \gamma I^2$$

(6.10)

where we called $\gamma$ the proportionality constant and:

$$\frac{dT}{dI} = 2\gamma I$$

(6.11)

Substituting (6.10) and (6.11) into (6.8) we finally get:

$$\frac{dP}{dI} = 8\gamma \zeta I (\zeta I^2 + T_0)^3$$

(6.12)

Moreover, in evaluating this transfer function, we must take into account that the resistor acts on the current fluctuations as a lowpass filter, whose cutoff frequency is determined by its thermalization constant. We put the cutoff frequency at $f_c = 8$ Hz for a ceramic resistor: this value has been chosen by measuring the thermalization constant of the resistor.

At this point, in order to evaluate the displacement noise due to the current noise, we should measure the power supply current noise. However, the specific power source used for the CHRAC prototypes are based on the PWM principle which can easily introduce a frequency dependent noise, as shown in section 6.1.5. Therefore, we are more interested in evaluating the maximum allowed current noise to stay a factor of ten below the design sensitivity of Advanced Virgo. To do that, we start with the sensitivity curve of Advanced Virgo, and going backwards we deduce the maximum allowed current noise, which is shown in figure (6.24). A block schematic of the noise coupling
Figure 6.24: Maximum allowed spectral current noise to stay a factor of 10 below the interferometer sensitivity curve. It is calculated for 1 A of current and 1 W of power absorbed by the mirror.

mechanism for CHRAC is reported in figure (6.23). This requirement of spectral current noise is easy to satisfy [71], and a new different driver can be designed in order to fulfill the noise requirements.

Finally, to have an idea of the amplitude of the different noise contributions, we plot them one by one together with the design residual displacement curve of Advanced Virgo (fig. 6.25).

6.4 Conclusions

In this section we have proved the CHRAC working principle: following the guidelines given in Chapter 5, we carried out the choice of the actuator which has been then used to assemble the CHRAC matrix. The mirror response has been simulated re-creating as much as possible the experimental conditions: indeed, we reproduced the aberrations introduced by the optical projection system and the amount of power actually absorbed by the mirror substrate. This accurate analysis allowed us to get a very good agreement between simulations and experimental results. We also showed that, suitably
Figure 6.25: Displacement noise induced by Radiation Pressure noise (green curve) calculated for a power $P = 1$ W and a mass of weight $m = 42$ Kg compared to sensitivity curve of Advanced Virgo (red curve). The dashed line represents the safety margin for all technical noises. AdV design curve from [10].

designing the power source, the noise introduced by CHRAC is compatible with the requirements for technical noises of Advanced Virgo. Finally, to further validate the method, we realized a prototype whose functioning can be tested on a table-top experiment in order to prove the feasibility of the method. This will be the subject of next Chapter.
Chapter 7

Application of CHRAC system to LG33 interferometers

In the previous chapters we presented the working principles of CHRAC, the basic aspect for its design and the characterization of two CHRAC prototypes. Here we will describe the results of the application of the table-top CHRAC to a Fabry-Perot cavity being part of a Michelson interferometer operated with LG₃₃ modes and previously realized at the Astroparticle and Cosmology (APC) Laboratory in Paris [29]. After describing briefly the LG₃₃ interferometer, we will present the experimental results confirming the validity of the CHRAC principle and its benefits [72].

7.1 The LG₃₃ table-top interferometer

7.1.1 Experiment description

The table-top interferometer operated with LG₃₃ modes is schematically represented in figure (7.1). A complete description of the experiment is reported in [24, 73]. Here we will highlight its main aspects. The interferometer can be divided into three parts. The first is the LG₃₃ mode generator (right part in figure 7.1): it is mainly
Figure 7.1: Optical scheme of the LG$_{33}$ interferometer setup. Schematic from [73].
composed of a diffractive phase plate which produces a pseudo-LG$_{33}$ (so called because of its limited purity), and a monolithic mode-cleaner, a 30 cm long plano-concave cavity of Finesse 100 locked with the Pound Drever Hall technique [21]. The mode-cleaner filters the beam and gives the LG$_{33}$ mode in output shown in figure (7.2) left.

To have an idea of how far this beam is from the ideal distribution, we can calculate its intensity overlap integral with the ideal intensity distribution. Intensity overlap integral is defined as:

$$\gamma = \int \int |\psi_{\text{meas}}|^2 \cdot |\psi_{\text{th}}|^2 dS$$ (7.1)

where $|\psi_{\text{meas}}|^2$ is the measured normalized transverse intensity distribution and $|\psi_{\text{th}}|^2$ is the normalized theoretical distribution. Since in the intensity overlap integral the phase of the two beams doesn’t appear, it represents an upper limit for the beam purity. In this case, it results in being grater than 99% [73].

Secondly, a suitable injection system is designed in order to inject the beam into the interferometer (left in figure (7.1)). It consists of an electro-optic modulator (EOM), needed to generate two 6.25 MHz sidebands which are used to lock the interferometer arms with the PDH technique, a Faraday isolator and a 4-lens mode matching telescope to match the beam shape to
<table>
<thead>
<tr>
<th>Reflectivity</th>
<th>ITM</th>
<th>ETM</th>
</tr>
</thead>
<tbody>
<tr>
<td>RoC</td>
<td>+∞</td>
<td>0.5m</td>
</tr>
<tr>
<td>Spot size</td>
<td>288 µm</td>
<td>455 µm</td>
</tr>
</tbody>
</table>

Table 7.1: Reflectivity and Radius of Curvature (RoC) and LG<sub>33</sub> beam spot size for Input Mirror (ITM) and End Mirror (ETM)

the arm optical cavities.

Finally, there is the Michelson interferometer with 30 cm long Fabry-Perot arm cavities. Each arm is a plano-concave cavity of Finesse 200 composed of 1 inch diameter mirrors. Data relative to cavity mirrors are reported in table 7.1. Three degrees of freedom have to be controlled to keep the interferometer in the dark fringe condition: two arm cavity lengths and the arm-length difference. The error signal for the control of the Michelson arm-length difference is generated by a dithering technique: the Michelson length is modulated at 1-kHz using a Piezoelectric actuator (PZT) and the corresponding signal is detected at the interferometer output. Low frequency corrections are then sent to the same PZT actuator.

In order to enhance the effect of the CHRAC thermal compensation, both the mirrors of the cavity subject to the correction (which hereafter we will refer to as cavity 2 of the interferometer) were replaced with BK7 mirrors: indeed, the linear expansion coefficient for BK7 is about 10 times greater than for Fused Silica. It allows to have a bigger deformation and therefore a more visible effect with the same absorbed heating power.

The performance of the interferometer is described in detail in [73]: in that paper a complete analysis of the beam composition is carried out. Such analysis highlights that, for cavities with Fused Silica mirror substrate, the main mirror aberration responsible for the scattering mechanism into modes of order 9 other than the LG<sub>33</sub> is astigmatism. Mismatching and misalignment give rise to coupling into modes of different order, which are not resonant in the cavity, but result anyway in the reflected beam and can spoil the quality of the dark fringe.
In the following we will carry out a similar analysis for cavity 2 with BK7 mirror substrates, whose transmitted and reflected beams are shown in figure (7.3), in order to get information about mirror defects which we aim to correct by CHRAC.

7.1.2 Cavity modes analysis

In Section 3.2.3 we described the scattering mechanism into HOM’s for a cavity locked on the LG_{33} mode. In particular, the selection rule of equation (3.19) indicates how cavity defects introduce a phase shift in the fundamental mode and cause a power transfer to the degenerate modes.

Here we want to identify the main defects affecting cavity 2 and correct them with the table-top CHRAC setup. In particular, we want to carry out a reverse process and recognize mirror defects starting from the cavity beam composition.

The only information we can access in our experimental conditions is the beam intensity profile $I(x,y)$. Therefore, in order to find the beam composition as a function of spurious modes, we followed the procedure described in [49]: given a basis $\Psi_i(x,y)$ with $i = (1,\ldots,n)$, we look for the power image

\begin{figure}[h]
\centering
\subfloat[]
\hspace{1cm}
\subfloat[]
\caption{Normalized intensity profile of the transmitted beam (a) and reflected beam (b).}
\end{figure}
\[ P(x, y; c) = \sum_{i=1}^{n} |c_i \Psi_i(x, y)|^2 \]  

(7.2)
given by minimization of the least square problem:

\[ E[\alpha] = \iint |I(x, y) - P(x, y; \alpha)|^2 \, dx \, dy \]  

(7.3)

The first problem to solve is, then, the choice of basis. Since the transmitted beam contains information about the modes resonant into the cavity, it can be decomposed in terms of the cavity eigenstates. To first approximation, we can consider the 10 modes of order 9 as a basis for the cavity. This statement holds as long as cavity defects are small and do not allow modes of higher order to become resonant: indeed, if it is the case, modes of order 9 cannot be considered a basis anymore.

The minimization of equation (7.3) is carried out using the gradient descent numerical algorithm [74]. At first, the modes of order 7, 8, 9, 10 and 11 are used as a basis to perform the power image decomposition of the transmitted beam. Since the coefficients amplitude for modes of order different from 9 was negligible, we concluded that using the modes of order 9 as a basis was a good approximation for our case.

The decomposition of cavity 2’s transmitted beam is shown in fig. [7.4]: the LG\(_{3,3}\) mode couples mainly into modes LG\(_{2,5}\) and LG\(_{4,1}\). According to the selection rule of equation (3.19), to a first approximation, the mirror defect which affects the most the scattering process is astigmatism, described by the Zernike polynomial \(Z_{2}^{2}\). Moreover, the secondary scattering process implies that these two modes will scatter back into the LG\(_{3,3}\) mode and also into the LG\(_{1,7}\) and LG\(_{4,-1}\), respectively.

What appears strange at first sight in the beam decomposition is the imbalance between the coefficient amplitudes corresponding to the modes LG\(_{2,5}\) and LG\(_{4,1}\), for which we would expect the same amplitude. This break of symmetry can be explained adding a small tilt to the mirror, described by the Zernike polynomial \(Z_{1}^{1}\). According to the selection rule, this cavity defect determines a scattering from the LG\(_{3,3}\) to modes of order different from the
Figure 7.4: Amplitude of the coefficients of the beam intensity decomposition in terms of the 10 polynomials of 9th order.

9th: in particular, modes LG\textsubscript{2,4} and LG\textsubscript{3,2} of order 8, and modes LG\textsubscript{3,4} and LG\textsubscript{4,2} of order 10. Nonetheless, as long as these modes are not resonant in the cavity, their coefficients will not appear in the transmitted beam decomposition. On the other hand, the tilt coupled to the astigmatism has the effect to produce an asymmetry in the scattering into the modes of the same order, in our case LG\textsubscript{2,5} and LG\textsubscript{4,1} and, consequently, LG\textsubscript{1,7} and LG\textsubscript{4,−1}.

To confirm this description, an FFT simulation was carried out using FOG, and the simulation results are shown in figure (7.5). The values used for the mirror map are reported in table (7.2). It is important to notice that in the description of mirror map we limit ourselves to the lowest order spatial defects, like tilt, astigmatism and defocus, which are not enough to fully reconstruct the mirror surface and, therefore, the beam shape. Nonetheless, this simplified analysis allows us to reproduce well enough mirror astigmatism.

This preliminary analysis provides an initial indication of the thermal correction to apply in order to improve the beam purity.
Figure 7.5: (a): Experimental measurement of the transmitted beam intensity profile. (b) Transmitted beam obtained from simulations, where the parameters reported in table (7.2) have been used to reproduce mirror defects. The window side is 6mm. Same colorbar for the two figures.

Figure 7.6: Beam intensity decomposition in terms of the 10 polynomials of 9th order. Comparison between real beam and simulated beam.
<table>
<thead>
<tr>
<th>Defect</th>
<th>Zernike polynomial</th>
<th>PtV Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astigmatism</td>
<td>$Z_2^2$</td>
<td>12.5 nm</td>
</tr>
<tr>
<td>Tilt</td>
<td>$Z_1^1$</td>
<td>0.38 $\mu$m</td>
</tr>
<tr>
<td>Defocus</td>
<td>$Z_0^2$</td>
<td>2 nm</td>
</tr>
</tbody>
</table>

Table 7.2: Zernike polynomials added to the mirror map over a diameter of 6 mm to reproduce the cavity transmitted beam.

![mode intensity profile](image)

Figure 7.7: Simulated LG$_{3,3}$ intensity profile at the end mirror.

### 7.1.3 First estimation of the heat pattern

Once the main mirror defects have been identified, it is necessary to devise the heat pattern to perform the correction.

The heat pattern size is related to the cavity beam size: indeed, the mirror defects affecting the beam shape are the ones “seen” by the cavity beam. For this reason, we refer to the beam intensity profile on the end mirror shown in figure (7.7). At about 2 mm from the center the power of the beam is just a few ppm than the maximum power, and we can therefore consider a region of 4 mm in diameter as the part of the mirror actually “seen” by the beam. It suggests that the heat pattern diameter must be at least 4 mm. We chose
In the previous section it was shown that to reproduce the cavity transmitted beam it is necessary to add to the mirror map astigmatism with 13 nm PtV. To correct this defect, we need to choose a heat pattern inducing a mirror deformation which exactly cancels it out. A Finite Element simulation demonstrated that two white circles can induce an astigmatic mirror deformation, as it is shown in figure (7.8). Therefore, for the beginning we will use this as a heat pattern.

7.2 Table-top CHRAC setup to improve the beam quality

7.2.1 Setup installation

The table-top CHRAC setup installed on the interferometer bench has a parabolic mirror with the Mini-lite igniter in the focus. As already mentioned in Section 6.2, one of the main problems arising in the table-top CHRAC installation is the heat beam alignment. For this reason, we chose as projection setup a two lenses telescope with substrates transmitting vis-
Figure 7.9: Picture of the table-top CHRAC setup installed on the LG$_{33}$ interferometer bench.
7.2.2 Estimation of the achievable correction

In this section we want to make an estimation of the deformation that we are able to perform with the setup illustrated in section (7.2.1). This evaluation has been made as follows.

A white circle of known size on the DMD screen was imaged through the projection system and the power absorbed by the mirror was measured with a broadband power meter. In this way, it was possible to calculate the heat intensity absorbed by the mirror substrate, which was found to be 550 W/m². This information was used to perform a simulation and evaluate the amplitude of the induced astigmatic deformation, which was found to have a PtV value of 7.5 nm over a region of 6 mm. This value is about half that needed. Nonetheless, even if astigmatism will not be corrected completely,
it still allows to induce a visible effect.

### 7.3 CHRAC alignment and focus

For the correction to be effective on the beam shape, we need the correction pattern to overlap with the Nd:YAG beam resonating in the cavity and to be in focus on the HR mirror surface.

A first coarse alignment can be performed using the light in the visible and NIR ranges which can be detected with a piece of paper or a viewer IR card, respectively.

Then, a fine alignment is carried out using the transmitted beam intensity as error signal to sense the mirror deformation. The beam was detected with a beam profiling camera Beamage - 3.0 from Gentec. To increase the sensitivity to the effect performed by the thermal correction, we take the transmitted beam intensity distribution without compensation as a reference.

The focusing is performed by fine tuning the position of the projection system mounted on translation stages in order to maximize the detected effect on the transmitted beam.

We then need to identify the astigmatism axes on the DMD screen. This was done by comparing the measurements with the simulation results, as shown in figure 7.11. Here a spot of about 3.8 mm diameter is projected on the south west side of the cavity beam center, for a total absorbed power of about 8 mW. The bottom left of the Figure 7.11 shows the simulation of the expected intensity shape of the residual deformation in the cavity beam. Note that the non-orthogonality of the axes is compensated by the magnification factor reported in equation (5.15) induced by the tilted DMD. Moreover, also the heat pattern is affected by the same magnification factor. Nonetheless, at this level of sensitivity we couldn’t appreciate the difference when the pattern reshaped according to equation (5.15) was projected.
Figure 7.11: (a): image shown on the DMD screen. Dashed lines show astigmatism axes; (b) induced mirror deformation with about 8 mW absorbed power. Colorbar ranges from 0 to $18 \times 10^{-9}$ m, while the axes cover 6 mm. Simulated (c) and experimental (d) residual beam intensity change due to the map (a) projection on the mirror. In both cases, the intensity of the residual beam is about $1/3$ of the intensity of the reference beam (which is the transmitted beam without correction).
7.4 Thermal correction of cavity mirror defects

Once the alignment and focusing procedure is completed, it is possible to project the correction pattern on the mirror and observe the change in the beam shape. Moreover, to further verify the method, we showed that it was possible to both reduce and increase mirror astigmatism, applying the correction map and its negative. The results have also been validated by simulations. The applied heat patterns are shown in fig. (7.12). They induce, respectively, a deformation of the mirror (as the one shown in fig. (7.10)) which partially reduces (7.12a) and increases (7.12b) mirror astigmatism.

7.4.1 Effect on a single Fabry-Perot cavity

Transmitted beam

In figure (7.13) the transmitted beams before (a) and after the correction ((b) and (c)) are shown: in particular, images (7.13b) and (7.13c) show the transmitted beam when the two maps (7.12a) and (7.12b) are applied in turn. Astigmatism correction has a visible effect on beam intensity distribution: indeed, power from the inner ring of the mode is better distributed along the whole ring. On the other hand, lateral peaks on the inner ring rise up when astigmatism is enhanced.

The beam decomposition analysis carried out in section 7.1.2 is applied to all the transmitted beam intensity distributions (left column of figure (7.13)) and reported in figure (7.14). As described in section 7.1.2, also in this case the induced astigmatic deformation mainly affects the mode $LG_{4,1}$ and its scattering into the mode $LG_{4,-1}$, while the effect on the mode $LG_{2,5}$ is less visible, confirming that the scattering into this mode involves more complex coupling mechanisms. Nonetheless, in the case of astigmatism reduction more power is coupled into the fundamental mode while, on the contrary, power into $LG_{3,3}$ is reduced in the case of astigmatism enhancement.

A model of the cavity defects helped to better estimate the amplitude of astigmatism induced with the CHRAC setup and compare it with the ex-
Figure 7.12: Correction maps displayed on the DMD and imaged on the mirror surface. (a): astigmatism correction; (b): astigmatism enhancement. The orange circle shows the position of the cavity beam with respect to the heat pattern. It should be noticed that the heat pattern is not perfectly centered on the cavity beam due to mechanical limitations. Figures (c) and (d) show the heat pattern superimposed to a qualitative indication of the mirror deformation represented with the colorscale.
Table 7.3: Zernike polynomials added to the mirror map over a diameter of 6 mm to reproduce in simulation the cavity transmitted beam in case of astigmatism reduction and astigmatism enhancement.

<table>
<thead>
<tr>
<th>Defect</th>
<th>Zernike polynomial</th>
<th>PtV Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astigmatism</td>
<td>$Z_2^2$</td>
<td>8 nm</td>
</tr>
<tr>
<td>Tilt</td>
<td>$Z_1^1$</td>
<td>0.25 $\mu$m</td>
</tr>
<tr>
<td>Defocus</td>
<td>$Z_0^0$</td>
<td>5 nm</td>
</tr>
</tbody>
</table>

Expected effect.
The simulated beams are shown in the right column of figure (7.13), while the beam decompositions are shown in figure (7.15). As already pointed out before, this simplified model only allows to control the effect of astigmatism on the beam mode content, but a more complex analysis would be required to exactly reproduce the cavity beam intensity. The values used to reproduce the mirror map are reported in table (7.4.1).
Figure 7.13: Normalized intensity distribution for cavity 2 transmitted beam. Real beams (left column) are compared to the simulated beams (right column). Figures (a) and (d) show the beam when no correction is applied, while in figure(b)-(e) and figure(c)-(f) astigmatism is, respectively, reduced and enhanced. Same colorscale.
Figure 7.14: Decomposition of the real transmitted beam in modes of order 9. The thermal correction reduces the scattering into modes $LG_{2,5}$ and $LG_{4,1}$ which are excited by astigmatism, while the same coupling is enhanced when the the existent astigmatism is increased.
Figure 7.15: Comparison between coefficients of the beam decomposition for the case of astigmatism reduction (top) and astigmatism enhancement (bottom).
Figure 7.16: Coefficients amplitude of the real transmitted beam decomposition as a function of the dissipated power. Coefficients relative to the mode LG\(_{4,1}\) and LG\(_{4,-1}\) decrease monotonically, as better highlighted in figure (7.17).

To better highlight the effect of the correction, the variation of the beam modes composition has been studied as a function of the power dissipated by the heat source and, consequently, absorbed by the mirror substrate. In particular, the change can be seen in the amplitude of the coefficient relative to the LG\(_{4,1}\) and LG\(_{4,-1}\) modes, as shown in figure (7.16) and in figure (7.17). In the latter, the linear fits, whose parameters are reported in table (7.4.1) show that the coefficients amplitude could be reduced to zero with a power about two times greater than what the setup could provide in this configuration.

On the other hand, due to the coupling between astigmatism and tilt and to mirror defects which would require a more complex description, the coefficients relative to the modes LG\(_{2,5}\) and LG\(_{4,-1}\) don’t follow the same trend.
\[ f(x) = p_1 x + p_2 \]

<table>
<thead>
<tr>
<th>$p_1 [1/W]$</th>
<th>$p_2$</th>
<th>$-p_2/p_1 [1/W]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{4,1}$</td>
<td>-0.0059</td>
<td>0.3269</td>
</tr>
<tr>
<td>$c_{4,-1}$</td>
<td>-0.0017</td>
<td>0.0864</td>
</tr>
</tbody>
</table>

Table 7.4: Parameter for the two linear fit of figure (7.17). The value of $R^2$ is greater than 0.99 for both the fits.

**Reflected beam**

The effect of thermal compensation is even more evident in the reflected beam intensity distribution, as shown in figure (7.18). In this case, though, the beam decomposition wouldn’t be very meaningful: indeed, the field reflected by the input cavity mirror interferes with the input field, and the detected intensity pattern contains information about the aberrations of both the beams. Therefore, a complete analysis of the beam intensity pattern as the one performed for the transmitted beam would require information about amplitude and phase of both input beam and reflected beam, which we don’t have.

### 7.4.2 Effect on the interferometer dark-fringe

The effect of the thermal compensation was finally tested on the performances of the Michelson interferometer. An asymmetric Michelson configuration was examined: at the dark fringe we got the interference between the beam reflected from cavity 2 and the beam reflected from the input mirror of cavity 1, whose transmission had been blocked for the beam not to resonate in the cavity.

We define the interferometer *contrast* or *visibility* as:

\[
C = \frac{P_{BF} - P_{DF}}{P_{BF} + P_{DF}} \quad (7.4)
\]

163
Table 7.5: Interferometer contrast change with mirror thermal compensation

<table>
<thead>
<tr>
<th></th>
<th>Contrast</th>
</tr>
</thead>
<tbody>
<tr>
<td>No correction</td>
<td>50%</td>
</tr>
<tr>
<td>Astigmatism correction</td>
<td>62.5%</td>
</tr>
<tr>
<td>Astigmatism enhancement</td>
<td>35%</td>
</tr>
</tbody>
</table>

where $P_B F$ and $P_D F$ are the power of the bright fringe detected with interferometer unlocked, and the power of the dark fringe in the locked configuration, respectively.

The effect of thermal correction on the interferometer contrast is summarized in table (7.5): the visibility increases by about 25% with respect to its initial value in the case of astigmatism reduction, while it is decreases by 30% when astigmatism is enhanced.

Interferometer visibility change is clearly visible also from the intensity beam profile at the dark fringe (fig. 7.19).

7.5 Conclusions

In this Chapter we have shown a successful application of the CHRAC setup to a table-top interferometer.

A preliminary analysis of the cavity transmitted beam allowed to identify the astigmatism as the main defect affecting the cavity. With the aid of simulations, we could find the astigmatism axes orientation and finally we were able to change the defect PtV amplitude.

Cavity astigmatism was not completely corrected: the accomplishment of this target was related to the need of more heating power and, therefore, to a change of the heat source used to illuminate the DMD. Having more available power can also allow to correct higher spatial frequency defects, which is critical in interferometers operated with higher order modes.

If the CHRAC projection device could be applied on both the cavities, it would be possible to significantly improve the interferometer visibility, which represents a crucial point in Gravitational Waves interferometry.
These measurements represent the first experimental proof of the CHRAC system working principle, and show that the CHRAC can be a good candidate to perform the thermal compensation of mirror defects in interferometric devices.
Figure 7.17: Amplitude of the coefficients relative to the mode LG\(_{4,1}\) (a) and LG\(_{4,-1}\) (b) as a function of the power dissipated by the heater. The error on the coefficients is given by repeated measurements. The linear fit shows that having twice the power allows to completely correct astigmatism.
Figure 7.18: Reflected beam from cavity 2. Figure (a) shows the beam when no correction is applied, while in figure (b) and figure (c) astigmatism is, respectively, reduced and enhanced.

Figure 7.19: Dark fringe intensity beam. Figure (a) shows the beam when no correction is applied, while in figure (b) and figure (c) astigmatism is, respectively, reduced and enhanced.
Conclusions

In Advanced detectors of Gravitational Waves the good quality of the optics is crucial for the interferometer to reach the design sensitivity. Indeed, cavity optics defects enhance high order modes generation in the cavities, and the different mode content in the two arms impede the nearly perfect destructive interference of the two reflected beams at the interferometer output, with a consequent sensitivity reduction. For this reason, the control of optical aberrations and their reduction represents a hot-topic in the field of Advanced detectors, and led to the development of many different techniques.

Mirror surface roughness and its effects on the interferometer represent the starting point of this thesis. As a solution to this issue, a new system of thermal compensation has been proposed, designed, realized and tested.

To identify the figure errors which scatter light from the cavity fundamental mode into unwanted modes the prior knowledge of the mirror map is required. However, if the mirror map is not known a-priori, it is useful to have a way to compute it. In Chapter 3 a new approach has been proposed to reconstruct mirror maps starting only from the cavity modes frequency shift. This method has still to be proved, but represents the starting point to a map-less approach. Moreover, the connection between the mirror map to the modes frequency shift can be also used in a reverse way: we can use it to find the map which induces a shift of the HOM resonance frequency far from the cavity line-width, bringing it out of resonance.

In order to fully control the mirror response to heat absorption, its transfer function has been analytically computed, as reported in Chapter 4. This
analytical approach has been validated with numerical simulations, showing a good agreement in the spatial frequency range of interest for our purposes. The knowledge of the system transfer function represents a very powerful tool, since it allows to compute the correct heat pattern able to induce the wanted deformation in only a few iterations, allowing to save precious time during the commissioning of the interferometer.

After we identified the mirror figure error to be reduced and computed the heat pattern to induce the desired correction, we finally designed a possible thermal compensation system able to perform the correction: the CHRAC. Chapter 5 is devoted to the analysis of each aspect concerning this thermal projection system: power optimization, projection telescope aberration reduction, accomplishment of the space constraint. Starting from these considerations, the CHRAC setup has been designed in two sizes, in order to be able to apply it to a large-scale experiment as well as on a table top experiment.

A very good agreement between the mirror induced deformations and the simulated predictions represents a first proof of the CHRAC working principle, as shown in Chapter 6. The possibility of inducing a controlled modification of the mirror surface is presented. Moreover, we found that suitably designing the power source, the noise introduced by CHRAC is compatible with the requirements for technical noises of Advanced Virgo.

To test the CHRAC ability in improving the cavity beam quality, the setup has been applied to a table-top experiment operated with Laguerre-Gauss modes. Even if an intrinsic limitation due to lack of power did not allow to perform the complete correction of the mirror defect, the CHRAC setup potential was shown by reducing and enhancing mirror astigmatism, inducing a visible effect on the intensity distribution of the transmitted beam as well as on the reflected one and, therefore, on the dark-fringe.

Proving the CHRAC principle, the work contained in this thesis lays the
foundations for the complete development of a new compensation system which can be used in Advanced interferometers to perform in-situ thermal compensation of mirror aberrations.
Appendix A

Gaussian optics and high order modes generation

In the description of Fabry-Perot cavity we carried out in Chapter 1, we used the plane wave approximation to describe the laser field. However, we should take into account that mirrors have a finite size and a radius of curvature which affect the circulating beam shape.

A.1 Gaussian beams in Fabry-Perot cavities

The propagation of light inside an optical system is described by Maxwell equations. For a laser beam inside an interferometer, the propagation is well described in the paraxial approximation. The complete derivation can be found in most of Optics textbooks ([25, 76]). Here we just report the fundamental Gaussian solution of the paraxial wave equation propagating along \( z \) under the assumption of cylindrical symmetry:

\[
\Psi(r, z) = \frac{1}{\sqrt{1 + \left( \frac{z}{z_R} \right)^2}} e^{-\frac{x^2+y^2}{w^2(z)}} e^{-ik\frac{x^2+y^2}{2R(z)}} e^{i\arctan x \frac{z}{zR} e^{-ikz}} \quad (A.1)
\]

where

\[
w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \quad (A.2)
\]
is the beam size as a function of the beam waist $w_0$, where the transverse beam size is minimum, and of the Rayleigh range:

$$z_R = \frac{kw_0}{2} \quad (A.3)$$

which gives the length scale of the beam expansion. The wavefront radius of curvature is defined as:

$$R(z) = z \left(1 + \frac{z_R^2}{z^2}\right) \quad (A.4)$$

Finally, when propagating along $z$, the beam experiences an additional dephasing, which is called Gouy phase:

$$\phi_G = - \arctan \frac{z}{z_R} \quad (A.5)$$

These parameters completely characterize the longitudinal and transverse shape of a beam inside a resonant cavity.

### A.2 High Order Modes generation

Higher order solutions can be written in different basis, the most common is the Hermite-Gauss basis, for which holds a rectangular symmetry, while in case of cylindrical symmetry we recur to Laguerre-Gauss polynomials.

The Hermite-Gauss modes, or Electromagnetic Transverse modes (TEM) are characterized by two indexes $m$ and $n$:

$$TEM_{m,n}(x, y, z) = N_{m,n}(z)e^{ikz}H_m\left(\frac{\sqrt{2}x}{w(z)}\right)H_n\left(\frac{\sqrt{2}y}{w(z)}\right)$$

$$\times e^{-i(m+n+1)\arctan(z/z_R)} e^{i\left(\frac{x^2+y^2}{2w^2(z)}\right)} e^{-\left(\frac{x^2+y^2}{w^2(z)}\right)} \quad (A.6)$$

where

$$N_{m,n} = \sqrt{\frac{2}{\pi w^2(z)2^{n+m}m!n!}}$$
is a normalization factor and $H_n(t)$ are the Hermite polynomials defined as:

$$H_n(t) = e^{t^2} \left( -\frac{d}{dt} \right)^n e^{-t^2}$$

Notice that each of the $TEM_{m,n}$ has a different Gouy phase depending on the mode order:

$$\phi_G = -(m + n + 1) \arctan \frac{z}{z_R}$$  \hspace{1cm} (A.7)

It implies that modes of different order resonate at slightly different frequency in the cavity.

### A.3 Cavity stability condition

As shown in equation (A.4), the beam wavefront is not flat, but has a radius of curvature that changes with the distance. For a Fabry-Perot cavity to admit resonating modes, it is necessary that the mirror radius of curvature matches that of the beam. More in detail, the curvature of the wavefront at the two mirror positions are:

$$R_1 = L_1 \left( 1 + \frac{z_R^2}{L_1^2} \right)$$

$$R_2 = L_2 \left( 1 + \frac{z_R^2}{L_2^2} \right)$$  \hspace{1cm} (A.8)

where both radii of curvature are positive and $L_1$ and $L_2$ are the distances of the mirrors from the waist. Defining the cavity $g$ factors as:

$$g_1 = 1 - \frac{L}{R_1}$$

$$g_2 = 1 - \frac{L}{R_2}$$  \hspace{1cm} (A.9)
where \( L = L_2 - L_1 \) is the cavity length, it is possible to write the cavity resonance condition in the form [76]:

\[
0 \leq g_1 g_2 \leq 1 \tag{A.10}
\]

If this condition is satisfied, the cavity is said to be stable.

In terms of the cavity g-factor, we can define the transverse mode spacing among high order modes, which is given by:

\[
TMS = FSR \times \frac{1}{\pi} \cos^{-1}(\pm \sqrt{g_1 g_2}) \tag{A.11}
\]

where \( FSR = \frac{c}{2L} \) is the cavity free-spectral range.
Appendix B

Zernike polynomials

It result useful in some applications to describe mirror distortions in terms of a base of polynomials, called Zernike polynomials, which represent a complete orthogonal basis over the unit circle. Zernike polynomials are identified by the radial and the azimuthal indexes \( n \) and \( m \), with \( 0 \leq m \leq n \). For any azimuthal index \( m \) we have one even and one odd polynomial:

\[
Z^m_n(\rho, \phi) = N^m_n R^{|m|}_n(\rho) \cos(m\phi) \quad (B.1)
\]

\[
Z^{-m}_n(\rho, \phi) = N^{-m}_n R^{|m|}_n(\rho) \sin(m\phi)
\]

where \( \rho \) is the normalized radial coordinate, \( \phi \) is the azimuthal angle, \( N^\pm_n \) is the amplitude and

\[
R^{|m|}_n = \sum_{k=0}^{(n-|m|)/2} \frac{(-1)^k (n-k)! r^{n-2k}}{k! \left( \frac{n+|m|}{2} - k \right)! \left( \frac{n-|m|}{2} - k \right)!}
\]  

(B.2)

is the radial function.

It is straightforward to perform a coordinate change from polar to cartesian, and write any mirror map as a superposition of Zernike polynomials:

\[
Z(x, y) = \sum_{i,j} c_{ij} Z^j_i(x, y)
\]

(B.3)
Appendix C

Hooke’s equation full solution

In Chapter 4 we stated Hooke’s equations and all the boundary conditions and the Navier-Cauchy equilibrium equations, whose solution should be of the form:

\[ s_r(r, z) = \frac{\nu}{2(\lambda + \mu)} \frac{1}{r} \int_0^r T(r', z) r' dr' \]  
\[ s_z(r, z) = \frac{\nu}{2(\lambda + \mu)} \left[ \int_0^z T(r, z') dz' + \Phi(r) \right] \]

where \( T(r, z) \) is the temperature field reported in equation (4.10), that we can write for sake of clarity as:

\[ T(r, z) = \sum_s T_s(z) J_0(\zeta_s r/a) \]

and \( \Phi(r) \) is a function whose explicit form is determined in order to satisfy the boundary conditions and the equilibrium equations.

\[ \Phi(r) = -\int_0^r \frac{dr'}{r'} \int_0^{r'} \frac{\partial T}{\partial z}(r'', -h/2) r'' dr'' + C \]
with $C$ arbitrary constant. This solution satisfies all the boundaries but the last, for which this solution brings to:

$$\Gamma_{rr}(a, z) = -\frac{\mu}{\lambda + \mu} \sum_{s>0} T_s(z) \frac{J_1(\zeta_s)}{\zeta_s}$$

(C.5)

instead of $\Gamma_{rr}(a, z) = 0$, which means that the stress in the radial direction is not compensated. For this term to vanish, it is necessary to introduce an extra two-components displacement vector:

$$\delta s_r(r, z) = \frac{\lambda + 2\mu}{2\mu(3\lambda + 2\mu)} (\omega_0 r + \omega_1 rz)$$

(C.6)

$$\delta s_z(r, z) = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} (\omega_0 z + \omega_1 z^2/2) - \frac{\lambda + 2\mu}{4\mu(3\lambda + 2\mu)} \omega_1 r^2$$

(C.7)

where the constants $\omega_0$ and $\omega_1$ are found minimizing the quadratic error:

$$Q(\omega_0, \omega_1) = \int_0^d [\Gamma_{rr}(a, z) + \omega_0 + \omega_1 z]^2 \, dz$$

(C.8)

Their explicit forms are given by:

$$\omega_0 = \frac{\alpha Y \tau \varepsilon P}{\pi K d} \sum_s p_s \frac{J_0(\zeta_s) \sinh \gamma_s}{\zeta_s^3} c_{1,s}$$

(C.9)

$$\omega_1 = -\frac{12\alpha Y \tau \varepsilon a P}{\pi K d^3} \sum_{s>0} p_s \frac{J_0(\zeta_s) \gamma_s \cosh \gamma_s - \sinh \gamma_s}{\zeta_s^4} c_{2,s}$$

(C.10)

where $\alpha$ is the linear expansion coefficient related to $\lambda$, $\mu$ and $\nu$ by

$$\alpha = \frac{\nu}{2(\lambda + \mu)(1 + \sigma)}$$

$P$ is the total power of the heat pattern, and we have introduced the following definitions:

$$\begin{align*}
\gamma_s &= \zeta_s z/a \\
c_{1,s} &= \zeta_s \sinh \gamma_s + \tau \cosh \gamma_s \\
c_{2,s} &= \zeta_s \cosh \gamma_s + \tau \sinh \gamma_s
\end{align*}$$

(C.11)
Finally, the complete solution is given by:

\[
s_r(r, z) = \alpha (1 + \sigma) \sum_{s > 0} \frac{T_s(z)}{\zeta_s} J_1(\zeta_s r/a) \tag{C.12}
\]

\[
\delta s_r(r, z) = \frac{1 - \sigma}{Y} (\omega_0 + \omega_1 z) r \tag{C.13}
\]

for the component along \( r \), and

\[
s_z(r, z) = \frac{\alpha (1 + \sigma) \varepsilon P}{2\pi K} \sum_s \frac{p_s}{\zeta_s} \left[ \frac{1}{c_{2,s}} + \left( \frac{\sinh \gamma_s}{c_{1,s}} - \frac{\cosh \gamma_s}{c_{2,s}} \right) J_0(\gamma_s) \right] \tag{C.14}
\]

\[
\delta s_z(r, z) = -\frac{2\sigma}{Y} \left( \omega_0 + \frac{1}{2} \omega_1 z \right) z - \frac{(1 - \sigma)}{2Y} \omega_1 r^2 \tag{C.15}
\]

for the \( z \) component of the displacement vector.
Acknowledgments

My PhD was carried out around the world. It gave me the possibility to meet many people, all different, and everybody left me something which I’ll carry along with me for the rest of the life.

I will start acknowledging people working on the Virgo site, whom with I spent longer and who had to bear me for longer. Gabriele, Eric, Maddalena, Paolo, Bas, Gabriel, “il Magazzú”, Henrich: thanks for helping me in learning the job, simplifying my way to the understanding of the small secrets of the interferometer. Mostly, thanks for trying to teach me how to smash at ping pong (unsuccessfully!). Thanks to Marie for sharing her knowledge about thermal compensation. Thanks to Adam Kutinya, whose great experience in electronics made possible the crazy project of deforming a mirror with a TV, and who patiently repaired components every time I broke something. Thanks to Flavio, Federico and the electronic team, always ready to support my work both with the welding and with the coffee. Thanks to Severine, whose honesty and sweetness have made of her a reference point for me. Thanks for taking time to teach me the basics of French grammar, which allowed me to survive in Paris for three months! A special thanks goes to Antonino for being the “doubts spreader” who increased the quality of my work, and mostly for being a good friend present any time I need. Thanks to Richard for guiding me along the way of thermal compensation. Mostly, thanks for showing me how crazy ideas can turn into wonderful and unexpected Physics results, and for helping me coming out of desperation in some of the toughest moment of my PhD; thanks to Armelle and the three
“little Days’” for adopting me as the fourth child of the family.
Thanks to Franco Frasconi for silently taking care of me in spite of his loud
voice, to Giancarlo Cella for his precious help with the mathematical formu-
lation of the new physical problems I had to deal with, to Andrea Basti for
taking his time to realize mechanical devices for my out-of-size experiments.

One of the most intense period during this PhD was the one that I spent
at Caltech. There I had my first real approach with experimental life. I’d
like to thank Rana Adhikari for showing me that, aside from the formulas,
it is possible to “feel” the Physics, and Koji Arai for his patient and clear
explanations about the secrets of the 40 m. Thanks to Hiro Yamamoto, for
his kindness in answering any kind of question. Thanks to Manasa, Matt,
Vivien, Jenne, Nic, because “everybody feels lost sometimes”, and its so nice
to have good friends around you can count on!
I would like to thank people from the APC lab: Matteo Barsuglia, Matteo
Tacca, Alberto Gatto and Crystelle Buy. I shared with you the craziest part
of my thesis work: thanks for hosting me and my cumbersome TV in your
lab. A special thanks goes to Jean-Yves Vinet, for his precious hints about
thermal compensation theoretical foundations.

Thanks to Prof. Pier Simone Marrocchesi, for following my roaming PhD
across continents, and to Dr. Carmen Marinelli, for the seriousness in fol-
lowing my thesis.
Thanks to Enrico Calloni, always present for giving suggestions and for ex-
citing me about Physics topics other than Gravitational waves.

Last but not the least, thanks to the good friends who, by far or from too
close, walked along with me encouraging me any time I needed to: thanks to
Claudio, Teresa and Nicola for being my support in the black french nights;
thanks to Eliana, for being the silent shadow always walking alongside me,
to Simona, for being ready to run in my help any time I needed, to Federica,
for warming my days with Sicilian spirit, to Ciccio, Carlo, Antonia, Luca
and Teresa for being always present friends, patient with me disappearing
for months.

Thanks to my wonderful family, who encouraged me step by step, even when disapproved my choices. Thanks to Alessandra for taking the part of the elder sister when I couldn’t be the more mature. Thanks to Francesco: every single day you were my main source of strength. This thesis is dedicated to you.
Bibliography


