On the Lagrangian formulation of gravity as a double copy of two Yang-Mills theories
Ai miei genitori,

per il loro costante supporto e il loro amore incondizionato.

A Claudia,

per la leggerezza e l’amore che ha portato nella mia vita.
Abstract

Yang-Mills gauge theories and Einstein’s General Relativity represent the main pillars on which our understanding of the fundamental interactions is presently based. Although originally formulated on account of completely independent intuitions and procedures, many connections have been later uncovered between gauge theories and gravity, stemming in particular from the inspection of scattering amplitudes in the two theories. In addition to the long-known KLT relations new correspondences were recently discovered, the so-called double-copy relations, eventually leading to recognize that, with some provisos, gravity can be understood as a suitably defined “square” of two Yang-Mills theories. While being essentially a theorem at tree level, this insight has been more recently extended to loop-level scattering amplitudes and to classical solutions, while still being unclear both at the Lagrangian level and in its ultimate geometrical meaning.

In this Thesis we investigate the possibility of a Lagrangian formulation of this correspondence between gauge theories and gravity, focusing on the non-supersymmetric case, in which the gravitational multiplet resulting from the product of two spin-one fields also contains a two-form field and a dilaton. To this end, we exploit the definition of “double-copy field” given by Duff et al. in order to build a quadratic Lagrangian, which has a neat interpretation as the “square” of two Yang-Mills quadratic Lagrangians, while also discussing its geometrical meaning. The free theory is then extended to the interacting level by means of the Noether procedure, building its off-shell cubic vertices. The interactions allowed by the gauge symmetry at the cubic level contain a sector which reproduces the results of the double copy for the three-point amplitudes, while also possessing a number of notable features. In particular, the resulting Lagrangian is invariant under a twofold Lorentz symmetry and is equivalent, up to suitable field redefinitions, to the so-called $\mathcal{N} = 0$ Supergravity at the cubic level, which provides the natural outcome of the double copy of two pure Yang-Mills theories. Last, we discuss the deformation to the gauge transformation of the fields in the gravitational multiplet which is determined from the Noether procedure. At this level we observe the need to modify the definition of the scalar and of the graviton fields in terms of the double-copy field by also allowing for quadratic terms amenable, as we also highlight, of a tantalizing geometrical interpretation.
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Notation and conventions

We adopt the “mostly plus” metric convention for the Minkowski metric:

\[ \eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1). \]

The symmetrization (antisymmetrization) of indices is performed with round (square) brackets, without normalization factors. For instance:

\[ X_{(ij)} := X_{ij} + X_{ji}, \]
\[ X_{[ij]} := X_{ij} - X_{ji}. \]

We use greek indices \( \mu, \nu, \ldots \) as Lorentz indices in flat Minkowski spacetime or as tensor indices in curved spacetime in \( D \) dimensions. We use latin indices \( i, j, \ldots \) as little group, \( SO(D-2) \) indices, thus with values \( 1, \ldots, D-2 \).

We adopt the following sign convention for the Riemann tensor (without assuming a torsion-free connection):

\[ R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \]

For the gravitational coupling constant in \( D \) dimensions, we use \( \kappa^2 = 8\pi G^{(D)}. \)

The Fourier transform is denoted with the symbol \( \mathcal{F} \), with normalization and sign defined as follows:

\[ f(x) = \int \frac{d^Dp}{(2\pi)^D} e^{-ip\cdot x} \mathcal{F}\{f\}(p), \]

with

\[ \mathcal{F}\{f\}(p) = \int d^Dxe^{ip\cdot x} f(x). \]

We denote with \( \mathcal{A} \) scattering amplitudes in Yang-Mills theory, while with \( \mathcal{M} \) we denote
amplitudes in gravitational theories (yet not necessarily pure graviton amplitudes).

When computing scattering amplitudes, we denote the scalar product between two Lorentz vectors $v_\mu$ and $w_\nu$ as $(vw) := v_\mu w^\mu$.

When dealing with rank-two symmetric tensors $h_{\mu\nu}$, we use the shorthands $\partial \cdot h_\mu := \partial^\alpha h_{\mu\alpha}$ for the divergence and $h := h^\alpha_\alpha$ for the trace.

We denote the convolution product of two functions with the symbol $\circ$:

$$[f \circ g](x) = \int d^D y f(y) g(x - y).$$

We denote with $\frac{1}{\Box}$ the inverse of the d’Alembert operator $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$, to be understood as the Green’s function $G$ of $\Box$:

$$\left[ \frac{1}{\Box} j \right](x) := \int d^D y G(y) j(x - y) = \int \frac{d^D p}{(2\pi)^D} \frac{\mathcal{F}\{j\}(p)}{-p^2} e^{-ip \cdot x}.$$ 

We shall sometimes use the notation

$$d^D k = \frac{d^D k}{(2\pi)^D}.$$
Chapter 1

Introduction

The use of non-observable quantities in our description of physical phenomena represents a cornerstone of modern Theoretical Physics. This is particularly evident for Quantum Field Theory (QFT), whose most fundamental objects are quantum fields and correlation functions, which are not necessarily in themselves physically observable quantities. The standard formulation of QFT is in terms of a Lagrangian density, defined as a function of the fields and their derivatives, which is particularly convenient for the study of symmetries. This formulation, however, typically brings about a lot of redundant information, the main instance of which being incarnated in the local symmetry termed “gauge invariance”, unavoidable in particular to provide a covariant field-theoretical description of massless particles. Gauge invariance is ubiquitous in phenomenological and theoretical models, including the Standard Model of particle physics and the Einstein-Hilbert description of the gravitational interaction, which provide our best established, yet incomplete, understanding of the laws of Nature.

From the Lagrangian formulation one can then compute scattering amplitudes, which can be viewed as an essential link between our theoretical description of the fundamental interactions and the experimental data, providing the dynamical information required to compute the cross sections. Scattering amplitudes are physically measurable quantities, therefore their values cannot depend on arbitrary choices such as the gauge-fixing or the field basis, thus being insensitive to the redundancies of the Lagrangian formulation. Already in the sixties, inspired by ideas of Heisenberg, the so-called $S$-matrix program [1] was initiated, with the aim of calculating $S$-matrix elements (i.e. scattering amplitudes). The latter were to be calculated without relying on intermediate states, requiring the introduction of non observable quantities, but only building on fundamental
principles comprising unitarity, causality, analyticity, the superposition principle and Lorentz-invariance. Originally motivated by the insufficiency of theories like QED to account for the phenomenology of strong interactions, the $S$-matrix program was then progressively abandoned after the advent of QCD in the seventies. It was eventually revived from a novel perspective in recent years, with the idea of disposing of the redundancies intrinsic of the field-theoretical description of the interactions. Several techniques were developed in order to directly calculate scattering amplitudes, leading to a tremendous progress in our computational power as well as to a better understanding that, especially in gauge theories, the result of such calculations can be amazingly simpler than what the direct evaluation of the Feynman diagrams would actually suggest. A remarkable example is provided by the Parke-Taylor formula \cite{2} for the $n$-gluon tree-level partial amplitude\footnote{We shall briefly introduce the concept of partial amplitudes in Section 2.2.4. It provides the purely kinematic part of a gluon amplitude, without the color part.} in four dimensions for the maximally helicity violating case, where all the gluons except for two have the same helicity\footnote{Formula (1.1) is written in the spinor-helicity formalism (see \cite{3} or \cite{4} for a review) where, in particular, for real light-like momenta $|\langle ij \rangle|^2 = 2p_i \cdot p_j$.}:

$$A_n[1^+ 2^+ ... i^- ... j^- ... n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}.$$ \hfill (1.1)

The appealing feature of (1.1) is that it provides a compact, elegant formula for a class of $n$-gluon amplitudes at tree level. By contrast, the number of Feynman diagrams required to compute the same amplitude with standard techniques displays a tremendous growth with $n$, as shown in table 1.1 for the first values of $n$ \cite{5}.

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td># diagrams</td>
<td>4</td>
<td>25</td>
<td>220</td>
<td>2485</td>
<td>34300</td>
<td>559405</td>
<td>10525900</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 1.1: Number of Feynman diagrams contributing to the $n$-gluon tree-level amplitude.

One reason for the huge difference between the Feynman diagrams calculation and the simplicity of the final expressions for amplitudes in gauge theories can be traced back to the fact that the individual Feynman diagrams are not gauge-invariant, together with the fact that their evaluation involves off-shell intermediate states in the propagators. On the contrary, the amplitudes are gauge-invariant quantities which only involve on-shell physical states: the passage from Feynman diagrams to scattering amplitudes eliminates
all the unphysical amount of information introduced together with the local symmetry of the theory. However, this observation in itself would not suffice to account for the compelling simplicity of scattering amplitudes. Indeed, it was recently realized that the latter, at least in some theories, organize themselves according to hidden symmetries which vastly constrain their expressions: these symmetry relations are additional with respect to the ones that guided the very construction of the theory and are anyway not at all evident from the Lagrangian formulation [3].

Such observations, together with the necessity of precise computations to be compared with the experimental data, collected in particular at the Large Hadron Collider, led to the study of the so-called on-shell techniques (see [4] for a modern review), especially employed in gauge theories, which helped to shed light on the hidden properties of scattering amplitudes. The main tool employed include the aforementioned spinor-helicity formalism, the decomposition in partial amplitudes for (possibly supersymmetric) gauge theories and the unitarity methods [6–11], which allow to build loop amplitudes starting from tree-level ones. A crucial boost in the development of the field was the discovery of the so-called BCFW recursion relations [12, 13] in 2004. These relate, in specific theories (including Yang-Mills theories and Einstein Gravity, together with their supersymmetric extensions), $n$-point tree-level amplitudes to sums of products of lower points amplitudes, with the result that, in such theories, all the tree-level amplitudes can be derived from the three-point ones. This is an impressive result, which implements the idea that in theories with a local symmetry a large part of the Lagrangian (all the vertices with more than three fields, from this perspective), while needed to guarantee gauge-invariance, may be regarded as being not strictly necessary to the goal of building physical quantities. The BCFW recursion relations actually are the main tool which is employed to prove, from the QFT perspective, two properties of certain scattering amplitudes which are of major importance for our work: the BCJ relations and the double-copy relations (both conjectured in [14] and proven by means of the BCFW relations in [15] and [16], respectively).

Among the hidden symmetries of scattering amplitudes that do not have a clear Lagrangian counterpart, of central interest in this Thesis will be the so-called color-kinematics (CK) duality [14, 17], a correspondence between the color factors and the kinematic factors of a particular set of cubic diagrams in Yang-Mills (YM) theories, whose
sum results in a given gluon amplitude. This symmetry, amazingly, points out a set of
relations between YM and gravitational theories: when the color factors of the YM dia-
grams are replaced by another copy of kinematic factors (possibly from a different YM
theory), the result is a gravitational amplitude. Correspondences of this type are collect-
tively known as double-copy (DC) relations [14, 17], and provide the central theme of
our work.

Both the CK duality and the DC relations have been proven to hold at tree level (while
their realization at loop level is still a conjecture) and have been generalized to various
different theories, including the supersymmetric extensions of YM theory and of Ein-
stein Gravity. However, in our work we shall focus on the simplest, yet fully non-trivial
case: the DC of pure YM theory which gives rise to a gravitational theory often referred
to as $\mathcal{N}=0$ Supergravity, due to its field content being the same as for the low-energy
limit of the closed bosonic String Theory, i.e. a graviton coupled to a two-form field and
to a scalar.

It must be mentioned that the idea of relations between gravitational and gauge the-
ory amplitudes was already present in the String-Theory literature since the discovery of
the Kawai-Lewellen-Tye (KLT) relations [18]. These are relations between scattering am-
plitudes in String Theory which, in their low-energy limit, reduce to relations between
tree-level gravitational amplitudes and products of tree-level gauge theory partial ampli-
tudes. However, the DC relations are, from several perspectives, a considerable novelty.
First, the KLT relations display rather cumbersome expressions for high number of par-
ticles, while the DC relations, crucially relying on the validity of the CK duality, are far
simpler in their structure. Moreover, they provide a more drastic implementation of the
idea that gravity is, in some sense to be clarified, the “square” of two spin-one gauge
theories, in particular since they do not rely on the structure of the corresponding UV
completion. Second, while the KLT relations only hold at tree level, the DC relations can
be straightforwardly generalized to loop level, consistently with a number of non-trivial
checks, although a complete proof is still lacking in the general case. Finally, the DC re-
lations have also been extended to correspondences relating classical solutions in Yang-
Mills theories and in General Relativity (possibly coupled to a two-form and a dilaton)
[19–28]. For these reasons, one may regard the DC relations as being more fundamental
than the KLT relations at the field-theoretical level (despite the two being equivalent
for tree-level scattering amplitudes), at least in the sense that the DC “paradigm” can be extended beyond tree-level scattering amplitudes.

Besides the theoretical interest in trying to understand the origin of such compelling new properties of quantum gravity, whose UV structure is one of the deepest and most puzzling open issues in Theoretical Physics, the DC relations own many aspects of phenomenological relevance. The first, most tangible one, is that they significantly simplify the calculations in perturbative quantum gravity, usually rather involved owing to the complicated structure of the Einstein-Hilbert Lagrangian expressed in terms of the graviton field. Calculations in YM theories are instead much simpler, also by virtue of the aforementioned modern techniques for the computation of scattering amplitudes. Therefore, the possibility of trading a quantity in gravity for the product of two quantities in YM theory may lead to relevant simplifications. Another significant advantage of the DC relations, still concerning the idea of “disentangling” the calculations in gravity, is that their extension to classical, radiative solutions [24–29] might be useful to the goal of computing quantities of interest for the study of gravitational radiation. This has become a central issue after the recent gravitational waves detections by LIGO and VIRGO ([30–36], and will be more and more important as the sensitivity of the two observatories increases. Finally, it is worth mentioning an open problem, whose relevance is both theoretical and phenomenological: the question about the UV-finiteness of $\mathcal{N} = 8$ Supergravity in $D = 4$. Indeed, it was observed that this theory displays a UV behavior softer than expected, with the current consensus being that it is finite up to seven loops [37–43]. The analysis up to four loops, using modern on-shell techniques, shows that the critical dimension for the divergence of $\mathcal{N} = 8$ Supergravity is compatible with the value $D_c = 6/L + 4$ [44–48], with $L$ being the number of loops, pointing to its finiteness in $D = 4 (L = \infty)$\(^3\). The DC relations have recently been employed [53–55], together with the unitarity method [6–11], in order to understand the UV properties of $\mathcal{N} = 8$ Supergravity to higher loops and it is hoped that they may help in leading to the final answer on this issue\(^4\).

\(^3\)Indeed, the same value of $D_c$ has been proven to hold for $\mathcal{N} = 4$ Super-Yang-Mills (SYM) [44, 49–52].

\(^4\)It must be mentioned that the DC relations at loop level relate the loop integrands of scattering amplitudes, not the results of the integrals: this explains why, for instance, in $D = 4$ pure YM theory is renormalizable, while the $\mathcal{N} = 0$ Supergravity, as well as pure Einstein Gravity, is not. Therefore, the finiteness of $\mathcal{N} = 4$ SYM can only suggest a soft UV behaviour for $\mathcal{N} = 8$ Supergravity, but it does not directly imply its finiteness.
1.1 Goals and motivations

In spite of all the progress made on the side of understanding and implementing the DC relations at various levels, very little is presently known about what could be regarded as a rather natural side of the problem: what is, if any, the Lagrangian counterpart of the DC structures derived at the level of amplitudes? Besides being a natural issue to raise, there are several motivations, in our opinion, to address such a question:

- As already mentioned, the existence of DC relations among amplitudes is a theorem at tree level but still a conjecture at loop level; we believe that a Lagrangian understanding of these properties might help to prove (or, possibly, disprove) their loop-level validity. In the case of a positive answer the DC relations would be useful in several additional respects, e.g. in order to study the UV behavior of the gravitational theories, and especially of the aforementioned $\mathcal{N} = 8$ Supergravity in $D = 4$, whose UV finiteness has not yet been disproved.

- Basic pieces of insights into YM and gravitational theories derive from an understanding of the underlying geometry[56]. We find it hard to believe that the situation may be radically different when it comes to assessing the actual meaning of the DC relations, if they are to be regarded as providing more than just a bunch of useful tricks helping to perform some computations. Understanding the geometry, however, requires relating the symmetries of the theories under scrutiny, and this naturally calls for trying to explore the DC structures at the Lagrangian level. The goal of the present work is to perform the first steps of this exploration, building on the few results available to date.

- A Lagrangian formulation would allow to address the issue of asymptotic symmetries, with the corresponding impact on soft amplitudes and memory effect (see e.g. [57]) from the perspective of the DC relations.

As of today, the two main contributions to the comprehension of the Lagrangian origin of the DC relations came from Bern et al. [16] and from Duff et al. [58]. In particular, a Lagrangian was proposed in [16] which reproduces the CK duality and the DC relations only up to the five-point amplitudes at tree level, though with the introduction of non-localities (which can be avoided only by introducing additional fields). A possible drawback of the approach of [16], which we wish to address, is the partial gauge-fixing
required by the proposed DC procedure. By contrast, in [58] a completely off-shell implementation of the DC relations on the fields was proposed, including a study of the corresponding off-shell symmetries, although only at the linearized level and without arriving at a proposal for a DC Lagrangian. Let us also notice that in [59] the problem of connecting the YM equations and the (extended) linearized gravity equations of motion was addressed, but again the introduction of non-localities proved necessary, this time when dealing with the coupling to sources.

In this Thesis we address the problem of the Lagrangian formulation of the DC relations from a novel perspective, building on the results of [58].

1.2 Results

As a starting point for our work we consider the definition of the product between two YM fields given in [58], which was shown to correctly reproduce the linearized (extended) gravitational symmetries from the linearized YM ones. The “double-copy field” resulting from this product, $H_{\mu\nu}$, carries the degrees of freedom of the $\mathcal{N} = 0$ Supergravity multiplet, both off shell and on shell. We added to this construction an off-shell definition of the scalar field in terms of $H_{\mu\nu}$, which was lacking in [58]. During the preparation of the present work, however, the paper [59] appeared, where an analogous definition of the scalar field was also given. With an appropriate identification of the field content of the $\mathcal{N} = 0$ Supergravity in terms of YM fields we proceed to build a quadratic Lagrangian for the field $H_{\mu\nu}$, to be interpreted as a “square”, in a concrete sense that we specify, of two YM Lagrangians. The resulting Lagrangian corresponds to the quadratic part of the $\mathcal{N} = 0$ Supergravity Lagrangian, being the sum of the free Lagrangians for a graviton, a two-form and a scalar. Moreover, when the field $H_{\mu\nu}$ is expressed in terms of the YM fields, our quadratic Lagrangian can be viewed as the “product”, in some sense to be specified, of two YM free Lagrangians, thus confirming our interpretation of a Lagrangian for the DC. Let us also mention that, when expressed in terms of a linearized field strength for $H_{\mu\nu}$, it resembles the quadratic Lagrangians built in [60] for the Maxwell-like massless higher spin fields. The transverse part of our quadratic Lagrangian matches the kinetic term obtained in [16] for the double-copy field $H_{\mu\nu}$, but we would like to stress that our construction is completely off shell and also contains pure gauge terms, which in [16] were discarded because of a preliminary gauge
fixing performed in the YM theory.

Then, we extend such quadratic Lagrangian to the interacting level with the addition of cubic vertices. This is achieved by means of the Noether procedure: a perturbative algorithm which allows to deform free gauge theories to interacting ones, while also consistently building the deformation of the gauge transformation. Our cubic vertex contains two sectors, identified by two arbitrary parameters. The selection of only one of the two sectors allows to correctly reproduce the results obtained in the DC of two three-gluon tree-level YM amplitudes and therefore we analyze this part in more detail.

We show that our Lagrangian extends the result of [16] at the cubic level, with the inclusion of longitudinal terms, and that it matches (modulo field redefinitions) the off-shell cubic vertices of the $\mathcal{N} = 0$ Supergravity Lagrangian. This is a non-trivial result if we observe that, in the latter, the coupling of the scalar field is not implied by the local symmetry of the theory and thus is not an output of the Noether procedure. Therefore, since in our case a specific value of the coupling between the two-form and the scalar is selected, with a completely fixed coefficient that matches the same coupling of $\mathcal{N} = 0$ Supergravity, this must be seen as a result of the fact that we work in terms of $H_{\mu\nu}$ as a fundamental variable. In other words, we provide a proof of the fact that the cubic couplings of $\mathcal{N} = 0$ Supergravity are indeed predicted from the assumption that the theory is a square of two YM Lagrangians, being especially non-trivial that we also obtain those couplings that are not implied by the linearized gauge symmetry. Finally, we also find that the part of the cubic vertex which reproduces the DC relations has, with respect to the remaining part of the vertex, an enhanced symmetry, which is also shared by the quadratic Lagrangian. This corresponds to the possibility of performing two independent Lorentz transformations on the left and right indices of $H_{\mu\nu}$, together with a discrete $Z_2$ symmetry, defined as the exchange of left and right indices. This result matches similar properties observed at the level of DC relations, in which the gravity amplitudes are built from two copies of the YM ones, each with an independent Lorentz symmetry. Moreover, we wish to stress that such symmetry is also shared by the terms in the Lagrangian which do not contribute to scattering amplitudes, which for the case of the three-point amplitudes comprise the pure gauge terms of the quadratic Lagrangian and the longitudinal parts of the cubic vertex. Such terms are precisely what we added to the results of [16].
In addition, we analyze the other main output of the Noether procedure at the cubic level, computing the first correction to the free gauge transformation of $H_{\mu\nu}$. Its symmetric and antisymmetric parts correspond, as expected, to the Lie derivative of a rank-two symmetric tensor and of a two-form, respectively. However, the correction to the gauge transformation of the scalar gives rise to something different from the Lie derivative of a scalar field, at least as long as we stick to the definition of the scalar field itself given at the linear level, thus raising a possible issue on its validity. As a solution to this puzzle we observe that our initial definition of the scalar was meant to guarantee its gauge invariance at the free level, and there is no reason why such definition should not receive corrections at the interacting level. Therefore, we propose to interpret the scalar field as an unknown, non-linear function of $H_{\mu\nu}$, of which we defined only the first order in its perturbative expansion. From the requirement that the full non-linear scalar transforms with its Lie derivative in a putative complete theory, interpreted perturbatively, we derive a system of equations which we solve at the second order in $H_{\mu\nu}$.

The result has a clear interpretation from the geometrical viewpoint, since the correction to the definition of the scalar is compatible with it being the convolution of the Green’s function of the Laplace-Beltrami operator with the Ricci scalar of a manifold with metric $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}H_{(\mu\nu)}$ and Levi-Civita connection. It would be interesting to investigate the physical meaning of this result. Similar considerations also apply to the definition of the graviton in terms of $H_{\mu\nu}$, that we also modify accordingly.

1.3 Further remarks

Before summarizing our work, we would like to highlight part of the additional progress made in the study of the CK duality and of the DC relations, that is not of primary interest for our work and thus will not be discussed in this Thesis.

The first remark is that, although the CK duality was originally discovered as a property of (possibly supersymmetric) YM theory, it was successively extended at least in two main directions. The first concerned the inclusion of gauge-coupled matter. Matter in the fundamental representation was introduced in [61], with the aim of removing, by means of a particular ghost prescription, unwanted states in theories resulting from the DC, in order for example to obtain pure Einstein Gravity as a result. In [62] the CK
duality was generalized to the case of QCD amplitudes, where the matter content is provided by the quarks. Furthermore, the CK duality was extended to other theories, such as (supersymmetric) Chern-Simons theories with matter [63, 64] and the non-linear sigma model (NLSM), where relations totally analogous to the ones of the CK duality undergo the name of flavor-kinematics duality [65–67]. Interestingly, in the latter case the duality between flavor and kinematics was derived from the symmetries of an action [68]: although at the moment there is no understanding of an analogous situation in YM theory, this is surely encouraging for the program of the Lagrangian implementation of the CK duality.

The second remark we wish to make is that also the DC relations, initially conceived as connecting YM and gravitational theories, have been observed several different contexts, both in QFT and in String Theory. For instance, starting from the cubic diagrams in YM theory, it was noticed that one could replace kinematic factors by additional color factors, leading to amplitudes in a theory for a scalar field transforming in the biadjoint representation of two gauge groups [69]. This construction was termed the zeroth copy [70]. Furthermore, after the discovery of the flavor-kinematics duality, it was realized [65, 71, 72] that the replacement of flavor factors by new copies of kinematic factors in the NLSM amplitudes leads to amplitudes in the special Galileon theory [73, 74]. In this case, as for the flavor-kinematics duality, these relations have been understood from the Lagrangian perspective [68]. Also, DC-like formulas for amplitudes in the Born-Infeld [75] and Dirac-Born-Infeld theories [76] were derived [67, 71, 72]. Such a universality of the DC paradigm, which can be summarized in a “multiplication table” [77] (tab. 1.2), is not fully understood. As of today, the only hint of a unifying principle behind all these theories can be traced back to the scattering equations of Cachazo, He and Yuan [70, 71, 78], who proposed a compact way of constructing the S-matrices for all the theories in tab. 1.2. Other generalizations of the DC paradigm comprise Einstein-Maxwell and Einstein-Yang-Mills theories [71, 72, 79–84], certain constructions in String Theory [85–89], conformal gravity [90] and gauge theories and gravity with the inclusion of higher derivative operators [91–93]. Another extension of the DC relations is their generalization to curved backgrounds, both for scattering amplitudes [94] and for classical solutions [20, 21].

Moreover, as we mentioned several times, the CK duality and the DC relations also
Table 1.2: Multiplication table of QFTs. BS=biadjoint scalar theory, NLSM=non-linear sigma model, YM=Yang-Mills theory, SG=special Galileon, BI=Born-Infeld theory, G=gravity.

hold for the supersymmetric extensions of YM theory and Einstein Gravity [14, 16]. In particular, $\mathcal{N} = 4$ SYM provides the simplest theory in which to test the loop-level CK duality conjecture. Such theory is also of great interest since the DC of two $\mathcal{N} = 4$ SYM amplitudes is conjectured, also at loop level, to give rise to amplitudes in $\mathcal{N} = 8$ Supergravity, which is worth studying because of the issue on the UV finiteness of the theory. In the general case, the DC of $\mathcal{N}_L$ SYM with $\mathcal{N}_R$ SYM gives rise to $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$ Supergravity$^5$.

We view all these developments as corroborations of the relevance of the CK duality and of the DC relations, thus further motivating our interest in a deeper understanding of such universal features relating theories that appear otherwise independent in all respects. However, since this work is devoted to the quest for an off-shell implementation of these relations, which has hitherto proven to be rather difficult, we shall work in the simplest possible setup, with the idea that what is understood in this case might be generalized to more complicated scenarios in the future. Therefore, as mentioned, we shall only consider pure YM theory, whose DC is the so-called $\mathcal{N} = 0$ Supergravity.

1.4 Plan of the work

The material is organized as follows:

- In chapter 2 we review the CK duality and the DC relations for scattering amplitudes. We also present some explicit calculations, aiming both at clarifying some general statement about these properties and at having a reference result, namely the DC for the tree-level, three-point amplitudes, which will be useful throughout

$^5$For the allowed values of $\mathcal{N}_L$ and $\mathcal{N}_R$ in the various dimensions where supersymmetry exists, the resulting Supergravity always have $\mathcal{N} \leq \mathcal{N}_{\text{max}}$, with $\mathcal{N}_{\text{max}}$ the maximal number of supercharges.
• In chapter 3 we discuss the main results obtained so far in the attempt of understanding the CK duality and the DC relations from the Lagrangian perspective. In particular, we focus on [58], in which the definition of the product of two YM fields is given, and on [16], also in order to make a comparison with our results at the end of the work. Last, we briefly review some results obtained in the study of the DC for classical solutions, focusing on [19].

• In chapter 4 we collect our original results. In the first part, we introduce a quadratic Lagrangian for the DC and we discuss its interpretation, as well as the equations of motion which result from it together with their gauge-fixing. Next, we move to consider interactions and we review the Noether procedure, presenting as explicit examples of its application the cases of YM theory and of the cubic vertex of the Einstein-Hilbert action. We then apply the procedure to the cubic vertex for the self-interaction of the DC field $H_{\mu \nu}$, comparing our solution with the results of [16]. Last, we discuss the correction to the gauge transformation of $H_{\mu \nu}$, providing its decomposition into transformations for the fields in the $\mathcal{N} = 0$ Supergravity multiplet. We find necessary to correct the definition of the scalar field and of the graviton with terms which are quadratic in $H_{\mu \nu}$. In particular, the correction to the scalar field points to an amusing geometrical interpretation, in which the field itself is interpreted as proportional to the Ricci scalar of the corresponding manifold.

• In chapter 5 we summarize our main results and we provide an outlook on the possible extensions and applications of our work.

• In appendix A we include, for pedagogical reasons, details on the evaluation of the graviton cubic vertices by means of the Noether procedure.

• In appendix B we include analogous details for the cubic vertices of the DC Lagrangian, also proving its equivalence, up to field redefinitions, to the three-point couplings of the $\mathcal{N} = 0$ Supergravity Lagrangian.
In this chapter we review some progress in our understanding of Yang-Mills (YM) theories and of their connections to theories of gravity, recently achieved by looking at the corresponding scattering amplitudes from a novel perspective. First, we will focus on the so-called color-kinematics (CK) duality \[14, 17\], a peculiar symmetry observed in pure gluon amplitudes in YM theories. Secondly, we will introduce the main theme of this Thesis: the double-copy (DC) relations \[14, 17\], expressing gravitational amplitudes as some squares of YM amplitudes, in a sense to be clarified, order by order in perturbation theory. We will consider a first, yet instructive, example, which will turn out to be useful throughout the whole exposition, before reviewing the DC relations in the general case. In order for this review to be self-contained, we shall also briefly recall some basics in the computation of scattering amplitudes, deriving the Feynman rules from YM and Einstein-Hilbert (EH) Lagrangian as well as the perturbative expansion of the latter.

### 2.1 Yang-Mills theory: Lagrangian and Feynman rules

A Yang-Mills theory is a gauge theory of several interacting massless spin-one fields $A^a_{\mu}$, with the index $a$ labelling the various fields. As we will review in chapter 4, the possible interactions are dictated by the gauge symmetry (up to higher derivative terms), resulting in the Lagrangian:

\[
\mathcal{L}_{YM} = \frac{1}{2} A^a_{\mu} \Box A^{a\mu} + \frac{1}{2} (\partial \cdot A^a)^2 - g f^{abc} (\partial_\mu A^c_\nu) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{cde} A^a_\mu A^b_\nu A^c_\rho A^{d\sigma},
\]  

(2.1)
where $g$ is the coupling constant of the theory, and where consistency demands the quantities $f^{abc}$ to be completely antisymmetric in their indices and to satisfy the Jacobi identity:

$$f^{abc} f^{cde} + f^{bcd} f^{eac} + f^{ace} f^{beb} = 0.$$  \hfill (2.2)

Indeed, they provide the structure constants of a Lie algebra, with the gauge fields transforming in the adjoint representation:

$$\delta A^a_\mu = \partial_\mu a^a + g f^{abc} A^b_\mu A^c_\mu.$$  \hfill (2.3)

The geometrical interpretation of $A^a_\mu$ is that of a connection on the principal bundle of the gauge group over spacetime and as such it allows to build a covariant derivative:

$$(D_\mu)_{ac} = \delta_{ac} \partial_\mu - g f^{abc} A^b_\mu,$$  \hfill (2.4)

as well as the curvature tensor of the principal bundle (or field strength):

$$F^a_{\mu\nu} = g^{-1} [D_\mu, D_\nu]^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu.$$  \hfill (2.5)

In terms of the curvature, the YM Lagrangian can be rewritten in a more geometrical fashion:

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a.$$  \hfill (2.6)

The local symmetry of the theory can be viewed as a redundancy in the description, that as such should not affect the value of physical quantities. Thus, one needs to implement specific tools allowing to dispose of this redundancy in the computation of scattering amplitudes in perturbation theory. To this end, we will exploit the Faddeev-Popov procedure (see e.g. [95]). In short, the procedure proposed by Faddeev and Popov basically amounts to the insertion of an appropriate delta-function in the path integral of the theory, avoiding the redundant integration over all the equivalent field configurations connected by gauge transformations. In practice, it corresponds to the addition of a term to the Lagrangian (known as the gauge-fixing Lagrangian, $\mathcal{L}_{GF}$) which explicitly breaks the gauge symmetry and makes the derivative operator of the quadratic Lagrangian invertible. Moreover, the procedure requires the introduction of ghost fields...
(anticommuting bosons, which only enter as internal lines in loop diagrams). To our purposes however, since we will only compute tree-level diagrams, we will neglect the ghost contribution and only keep $L_{GF}$, both in the case of the YM theory and for gravity.

A common choice for a covariant gauge-fixing in electrodynamics is the Lorenz gauge ($\partial \cdot A = 0$). The corresponding gauge choice in the covariant quantization of YM theories is the Lorenz $\xi$-gauge, where the gauge-fixing Lagrangian has the form:

$$
L_{GF} = -\frac{1}{2\xi} (\partial \cdot A^a)^2.
$$

(2.7)

For the sake of computational simplicity, we will further specify to the Feynman gauge ($\xi = 1$), where the Lagrangian becomes:

$$
L_{YM} + L_{GF}(\xi = 1) = \frac{1}{2} A^a_{\mu} \Box A^{a\mu} - gf^{abc} (\partial_\mu A^a_{\nu}) A^b_{\nu} A^{c\mu} - \frac{1}{4} g^2 f^{abc} f_{cde} A^a_{\mu} A^b_{\nu} A^e_{\mu} A^d_{\nu},
$$

(2.8)

while the propagator assumes a particularly simple form:

$$
\begin{align*}
&\left(\begin{array}{c}
\not{a}, \mu \\
\not{b}, \nu
\end{array}\right) \\
&= -i \frac{\delta_{ab}\eta_{\mu\nu}}{p^2},
\end{align*}
$$

where $p_{\mu}$ is the momentum of the propagating gluon.

Both here and in the computation of scattering amplitudes, we will consider all the particles as “entering” the diagram, so that momentum conservation reads:

$$
\sum_{i=1}^{n} p_{i\mu} = 0.
$$

(2.9)

With this convention the Feynman rules for YM theory are the following:

$$
V^\mu_1 \mu_2 \mu_3_{a_1 a_2 a_3} = gf^{a_1 a_2 a_3} \left\{ \eta^\mu_1 \mu_2 (p_1 - p_2)^{\mu_3} + \eta^\mu_2 \mu_3 (p_2 - p_3)^{\mu_1} + \eta^\mu_3 \mu_1 (p_3 - p_1)^{\mu_2} \right\},
$$

where $a_i, \mu_i$ are the indices for the $i$th particle.
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\[ V_{\mu_1 \mu_2 \mu_3 \mu_4}^{a_1 a_2 a_3 a_4} = -ig^2 \{ f^{a_1 a_2} f^{a_3 a_4} (\eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} - \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3}) \\
+ f^{a_1 a_4} f^{a_2 a_3} (\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} - \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4}) \\
+ f^{a_1 a_3} f^{a_2 a_4} (\eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4}) \} . \]

In the following section we will often employ these rules in order to compute scattering amplitudes.

### 2.2 Tree-level Yang-Mills amplitudes and color-kinematics duality

In this section we will compute some tree-level gluon scattering amplitudes aiming to illustrate both the color-kinematics (CK) duality and the double-copy (DC) relations in the simplest cases, where analytical calculations are still manageable. We recall that scattering amplitudes are obtained dressing the connected and amputated Green’s functions with the polarizations of the external particles. In the case of spin-one massless particles in \( D \) spacetime dimensions, the physical polarizations are states in the fundamental representation of \( SO(D - 2) \) (therefore, there are \( D - 2 \) of them). In covariant notation, they can be embedded into \( D \)-dimensional vectors \( \varepsilon_\mu(p) \) (with \( p \) the momentum of the particle) subject to the constraint \( p \cdot \varepsilon = 0 \) and, as we will detail in Section 2.4.1, to the equivalence relation \( \varepsilon_\mu \sim \varepsilon_\mu + cp_\mu \). The latter condition is equivalent to a gauge transformation on the spin-one field in momentum space and identifies an equivalence class of polarization vectors: Lorentz-invariance of the \( S \)-matrix requires the scattering amplitudes to be unaffected by the choice of the representative in the equivalence class.

#### 2.2.1 Three gluons amplitude

The scattering amplitude among three massless particles, that we shall denote with \( A_3 \), is obtained by dressing the Feynman rules for the cubic vertex with the external polarizations, as well as imposing on-shell conditions (\( p_i^2 = 0 \)) for each particle together with momentum conservation (\( \sum_i p_i = 0 \)). In the case of the three particles-kinematics these conditions are so constraining that, in \( D = 4 \), they only allow for complex momenta.
However, to our purposes it is interesting to work out the expression of the amplitude in arbitrary $D$, as it will turn out to be our main explicit tool to investigate the double copy.

Exploiting transversality of the polarization vectors it is straightforward to derive the following expression for $A_3$:

$$A_3 = \varepsilon_1^{\mu_1} \varepsilon_2^{\mu_2} \varepsilon_3^{\mu_3} \sqrt{\delta_{\alpha_1 \alpha_2 \alpha_3}} =$$

$$= g f^{a_1 a_2 a_3} \{(\varepsilon_1 \varepsilon_2)(\varepsilon_3 \cdot (p_1 - p_2)) + (\varepsilon_2 \varepsilon_3)(\varepsilon_1 \cdot (p_2 - p_3)) + (\varepsilon_3 \varepsilon_1)(\varepsilon_2 \cdot (p_3 - p_1))\}$$

$$= -2g f^{a_1 a_2 a_3} \{(\varepsilon_1 \varepsilon_2)(\varepsilon_3 p_2) + (\varepsilon_2 \varepsilon_3)(\varepsilon_1 p_3) + (\varepsilon_3 \varepsilon_1)(\varepsilon_2 p_1)\}$$

$$= -2g f^{a_1 a_2 a_3} \varepsilon^{\mu_1} \varepsilon^{\mu_2} \varepsilon^{\mu_3} \left\{ \eta^{\mu_1 \mu_2} p_3^\mu_3 + \eta^{\mu_2 \mu_3} p_1^\mu_1 + \eta^{\mu_3 \mu_1} p_2^\mu_2 \right\}. \quad (2.10)$$

Here and in what follows we use a notation in which scalar products between polarizations and momenta are written in parentheses and without the dot: for example instead of $\varepsilon_1 \cdot \varepsilon_2$ we will write $(\varepsilon_1 \varepsilon_2)$. We will come back to this expression in Section 2.4.2, when we will employ it to build gravitational amplitudes.

### 2.2.2 Four gluons amplitude

The tree-level scattering amplitude with four gluons provides the simplest example of color-kinematics duality. The amplitude is given by the sum of the Feynman diagrams in figure 2.1.

![Feynman diagrams](image)

**Figure 2.1**: Feynman diagrams contributing to 4-gluon scattering

We will refer to the diagrams 2.1a, 2.1b and 2.1c as $s-$, $t-$ and $u$-channel diagram respectively, from the names of the corresponding Mandelstam variables, defined as follows:

$$s := q_s^2 := (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_1 \cdot p_2 = 2p_3 \cdot p_4, \quad (2.11)$$
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In terms of these variables, the Jacobi identity of the structure constants reads:

\[ t := q_1^2 := (p_1 + p_4)^2 = (p_2 + p_3)^2 = 2p_1 \cdot p_4 = 2p_2 \cdot p_3, \quad (2.12) \]
\[ u := q_a^2 := (p_1 + p_3)^2 = (p_2 + p_4)^2 = 2p_1 \cdot p_3 = 2p_2 \cdot p_4, \quad (2.13) \]

where we specialised to the case of massless particles. In this case the general identity \( s + t + u = -\sum_i m_i^2 \), reduces to \( s + t + u = 0 \).

The contributions of the three Mandelstam channels to the tree-level scattering amplitude are:

\[ 2.1a = e_1 \varepsilon_2 \varepsilon_3 \varepsilon_4^2 f^{a_1a_2b} f^{a_3a_4c} \{ \eta^{\mu_1\mu_2} (p_1 - p_2)_{\nu} + \eta^{\mu_2\nu} (p_2 + q_s)_{\mu_1} - \eta^{\mu_1\nu} (q_s + p_1)_{\mu_2} \} \times \frac{(-i)}{s} \delta_{bc}^{\eta_\mu_\nu} \eta^{\mu_3\mu_4} (p_3 - p_4)_{\rho} + \eta^{\mu_4\rho} (p_2 - q_s)_{\mu_3} + \eta^{\mu_3\rho} (q_s - p_3)_{\mu_4} \]
\[ = -i \frac{g^2}{s} f^{a_1a_2e} f^{a_3a_4e} \{ (\varepsilon_1 \varepsilon_2) (p_1 - p_2)_{\alpha} + 2 \varepsilon_2^\alpha (\varepsilon_1 p_2) - 2 \varepsilon_1^\alpha (\varepsilon_2 p_1) \} \times \{ (\varepsilon_3 \varepsilon_4) (p_3 - p_4)_{\alpha} + 2 \varepsilon_4^\alpha (\varepsilon_3 p_4) - 2 \varepsilon_3^\alpha (\varepsilon_4 p_3) \}, \quad (2.14) \]

\[ 2.1b = -i \frac{g^2}{u} f^{a_1a_4e} f^{a_3a_3e} \{ (\varepsilon_1 \varepsilon_4) (p_1 - p_4)_{\alpha} + 2 \varepsilon_4^\alpha (\varepsilon_1 p_4) - 2 \varepsilon_1^\alpha (\varepsilon_4 p_1) \} \times \{ (\varepsilon_2 \varepsilon_3) (p_2 - p_3)_{\alpha} + 2 \varepsilon_3^\alpha (\varepsilon_2 p_3) - 2 \varepsilon_2^\alpha (\varepsilon_3 p_2) \}, \quad (2.15) \]

\[ 2.1c = -i \frac{g^2}{u} f^{a_1a_3e} f^{a_2a_2e} \{ (\varepsilon_1 \varepsilon_3) (p_1 - p_3)_{\alpha} + 2 \varepsilon_3^\alpha (\varepsilon_1 p_3) - 2 \varepsilon_1^\alpha (\varepsilon_3 p_1) \} \times \{ (\varepsilon_4 \varepsilon_2) (p_4 - p_2)_{\alpha} + 2 \varepsilon_2^\alpha (\varepsilon_4 p_2) - 2 \varepsilon_4^\alpha (\varepsilon_2 p_4) \}. \quad (2.16) \]

Let us notice that the three channels share a similar structure: the same power of the coupling constant divided by a Mandelstam variable, a color factor composed of two structure constants and a purely kinematic factor, containing Lorentz-invariant combinations of polarizations and momenta. It is possible to name the color factors according to the Mandelstam channel they appear in:

\[ \bullet \ c_s := f^{a_1a_2e} f^{a_3a_4e}, \quad \bullet \ c_t := f^{a_1a_4e} f^{a_2a_3e}, \quad \bullet \ c_u := f^{a_1a_3e} f^{a_4a_2e}. \quad (2.17) \]

In terms of these variables, the Jacobi identity of the structure constants reads:

\[ c_s + c_t + c_u = 0. \quad (2.18) \]

In the contribution stemming from the contact term (2.1d), on the other hand, all the three color factors are present but there are no propagators. Upon multiplying and
dividing each color factor by the corresponding Mandelstam variable, one can write the corresponding contribution to the amplitude as follows:

\[
2.1d = ig^2 \left\{ \frac{\hat{s}}{s} c_s \left[ (\epsilon_1 \epsilon_4)(\epsilon_2 \epsilon_3) - (\epsilon_1 \epsilon_3)(\epsilon_2 \epsilon_4) \right] + \frac{\hat{t}}{t} c_t \left[ (\epsilon_1 \epsilon_3)(\epsilon_2 \epsilon_4) + (\epsilon_1 \epsilon_2)(\epsilon_3 \epsilon_4) \right] + \frac{\hat{u}}{u} c_u \left[ (\epsilon_1 \epsilon_2)(\epsilon_3 \epsilon_4) - (\epsilon_1 \epsilon_4)(\epsilon_2 \epsilon_3) \right] \right\}.
\] (2.19)

thus leading to a structure similar to the contributions (2.14), (2.15) and (2.16), as it contains three terms of the type \( \frac{nc}{p} \), where \( c \) is one of the three color factors defined in eq. (2.17), \( p \) are denominators corresponding to Mandelstam variables (therefore they are scalar propagators) and \( n \) are purely kinematic factors at the numerator, containing momenta and polarizations in Lorentz invariant combinations. This allows us to perform a *splitting* of the contact term contributions, assigning to each of the three Mandelstam channels one of the addends in eq. (2.19), according to their color and propagator structure (e.g. the term proportional to \( \frac{c_s}{s} \) gets combined with the \( s \)-channel and similarly for the other two channels).

This splitting procedure, namely the multiplication and division by appropriate propagators in order to reproduce the same color and propagator structures of the Feynman diagrams with only trivalent vertices, can be implemented for amplitudes with more than four particles as well. Therefore, it is possible to rearrange the amplitude in terms of a new set of diagrams (which are *not* Feynman diagrams) with only trivalent vertices. In the case of the four-particle amplitude they are represented in fig. 2.2: where straight lines are used to represent the gluons, as opposed to the curly lines of the usual Feynman diagrams. Each of these new diagrams subsumes the sum of the Mandelstam channel diagram with a given color structure and the part of the contact term Feynman diagram with the same color structure.

After the splitting of the contact term, properly multiplied and divided by propagators, the tree-level four-gluon amplitude can be written in the simple form:

\[
\mathcal{A}_4 = g^2 \left\{ \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right\}.
\] (2.20)

where the kinematic factors at the numerator are given by:

\[
n_s = -i \left\{ (\epsilon_1 \epsilon_2)(\epsilon_3 \epsilon_4)(u - t) + 2(\epsilon_1 \epsilon_2)\left\{ (\epsilon_4 p_3)\epsilon_3 \cdot (p_2 - p_1) + (\epsilon_3 p_4)\epsilon_4 \cdot (p_1 - p_2) \right\} \right\}.
\]
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Via direct calculation, exploiting momentum conservation and transversality of the polarization vectors, it is possible to check that:

\[ n_s = -i \{(\varepsilon_1 \varepsilon_4)(\varepsilon_3 \varepsilon_2)(s - u) + 2(\varepsilon_1 \varepsilon_4)\{(\varepsilon_3 p_2)\varepsilon_2 \cdot (p_4 - p_1) + (\varepsilon_2 p_3)\varepsilon_3 \cdot (p_1 - p_4)\}\}
\[ n_t = -i \{(\varepsilon_1 \varepsilon_3)(\varepsilon_4 \varepsilon_1)(t - s) + 2(\varepsilon_1 \varepsilon_3)\{(\varepsilon_4 p_4)\varepsilon_4 \cdot (p_3 - p_1) + (\varepsilon_4 p_2)\varepsilon_2 \cdot (p_1 - p_3)\}\}
\[ n_u = -i \{(\varepsilon_1 \varepsilon_4)(\varepsilon_3 \varepsilon_1)(t - s) + 2(\varepsilon_1 \varepsilon_4)\{(\varepsilon_2 p_1)\varepsilon_3 \cdot (p_4 - p_3) + (\varepsilon_3 p_1)\varepsilon_1 \cdot (p_2 - p_4)\}\}

(2.21)

(2.22)

(2.23)

Via direct calculation, exploiting momentum conservation and transversality of the polarization vectors, it is possible to check that:

\[ n_s + n_t + n_u = 0, \]

(2.24)

providing the kinematic counterpart of the Jacobi identity.

At the four particles level, this was actually known since [96] and [97]. Now it is understood to provide the simplest example of a more general property of YM scattering amplitudes termed color-kinematics duality (CK), first identified in [14]. Basically, it amounts to say that the numerator factors in the expression of the amplitude obey the same algebraic relation as the color factors, in spite of the purely group-theoretical nature of the latter. One can also observe that it is possible to perform a redefinition of
the numerator factors $n_i$,

$$
\begin{aligned}
  n'_s &= n_s + s\alpha(p_i, \varepsilon_i), \\
  n'_t &= n_t + t\alpha(p_i, \varepsilon_i), \\
  n'_u &= n_u + u\alpha(p_i, \varepsilon_i),
\end{aligned}
$$

(2.25)

such that the scattering amplitude is invariant:

$$
\mathcal{A}'_4 = \mathcal{A}_4 + (c_s + c_t + c_u)\alpha = \mathcal{A}_4.
$$

(2.26)

In force of (2.18), the possibility of shifting of the kinematic factors $\{n_i\}$ as in (2.25) without affecting the physical amplitude is known as generalized gauge invariance [14]. Both the color-kinematics duality and the generalized gauge are crucial ingredients to the understanding of the double-copy structure of gravitational scattering amplitudes. Therefore, we will devote the next section to a more systematic illustration of these central concepts.

### 2.2.3 Color-kinematics duality

The identity (2.24), whose existence we observed in the simple case of the tree-level scattering amplitude among four gluons, turns out to be a general property of scattering amplitudes in pure YM theory at tree level. This was first understood by Bern, Carrasco and Johansson in [14] and is also known as BCJ duality. In this section we explore the duality in general terms, explaining its content for an arbitrary number of particles at tree level, while we shall postpone to Section 2.2.7 a discussion of its validity at loop level. We will not give any proofs since these are extremely technical, but we will make clear what has been proven and what is still a conjecture.

For a general tree-level scattering amplitude in pure Yang-Mills theory it is always possible to rearrange its expression as a sum over diagrams with only trivalent vertices, that we collectively denote as $\Gamma_3$, in the form:

$$
\mathcal{A}_n^{\text{tree}}(1, \ldots, n) = g^{n-2} \sum_{\sigma \in \Gamma_3} \frac{n_i \alpha_i}{\prod_{i<j} n_i n_j s_{\alpha_i}},
$$

(2.27)

where $c_i$ is a color factor obtained by dressing each vertex of the diagram with a structure constant, $n_i$ is an appropriate kinematic factor containing momenta and polarizations
while the denominator is the product of all the inverse propagators $s_{\alpha_i}$ corresponding to internal lines. As showed in the previous section for a specific case, this representation can be obtained by splitting (in the sense explained at the end of page 19) all the contributions from the contact terms.

As illustrated in [14], the representation of the kinematic factors is not unique. Indeed we can modify the latter with a shift of the form

$$n_i \rightarrow n_i' = n_i + \Delta_i(p_i, \varepsilon_i),$$

and still leave the amplitude invariant if the functions $\Delta_i(p_i, \varepsilon_i)$ are chosen such that:

$$\sum_{i \in \Gamma_3} \frac{\Delta_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0. \quad (2.29)$$

This property was termed *generalized gauge invariance* in [14], since at the field-theoretical level it corresponds to a gauge transformation plus a field redefinition or to the addition of higher derivative (possibly non-local) terms which leave the YM Lagrangian invariant. Bern, Carrasco and Johansson conjectured in [14] that, in the class of amplitudes equivalent under (2.28), there always exists a representation of the kinematic numerators $n_i$ such that, to every Jacobi identity satisfied by the color factors of a triplet of diagrams in the set $\Gamma_3$, it corresponds an analogous identity satisfied by the kinematic numerators:

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \quad (2.30)$$

Moreover, they conjectured the kinematic factors to share the same antisymmetry of the color factors:

$$c_i = -c_j \Rightarrow n_i = -n_j. \quad (2.31)$$

For instance, if we consider two diagrams $i$ and $j$ which are identical, except for the exchange of two external legs attached to the same vertex, owing to the antisymmetry of the structure constants the color factors of the two diagrams are related by $c_i = -c_j$. The conjecture requires also the kinematic factors to share this antisymmetry, which, however, is naturally encoded in the YM Feynman rules (Section 2.1) for the cubic vertex and in each of the three terms resulting from the splitting of the quartic vertex, as it can be
easily checked in the previous sections. Therefore, to satisfy the requirement (2.31), it is sufficient to wisely choose the generalized gauge transformation. This correspondence between the algebraic identities obeyed by the structure constants and analogous identities relating the kinematical factors is known as the color-kinematics (CK) duality.

As already mentioned, the validity of the CK duality for an arbitrary number of external particles at tree level was conjectured in [14]. The existence of duality-satisfying numerators was first exemplified for any \( n \) in [98], while two independent solutions to the problem of finding explicit expressions were proposed in [99] and [100]. What makes this property quite remarkable is that, a priori, there is no apparent connection between the algebra of the gauge group and the kinematic factors. To the present state of knowledge, the reason for such a duality between color and kinematics is not yet understood, with the only indication in this sense provided by [101], where the presence of a kinematic algebra in the self-dual sector of YM theory in \( D = 4 \) was pointed out.

### 2.2.4 Partial amplitudes

As we will discuss in Section 2.4, the existence of the CK duality in YM theory is crucial for the validity of the so-called double-copy relations, connecting gravitational and gauge theory amplitudes, which are the main subject of this Thesis. However, the CK duality is also relevant from another viewpoint, as it highlights a new property of the so-called partial amplitudes, which we now introduce.

Let us consider an \( SU(N) \) YM theory and observe that all the pure gluon amplitudes contain products of structure constants, which one can express in terms of traces of generators of the gauge group. The simplest case involves only one tensor \( f^{abc} \) so that, using

\[
[T^a, T^b] = i f^{abc} T^c, \tag{2.32}
\]

with \( T^a \) the \( SU(N) \) generators in the fundamental representation\(^1\), one can extract

\[
f^{abc} = -2i \left\{ \text{tr}(T^a T^b T^c) - \text{tr}(T^a T^c T^b) \right\}. \tag{2.33}
\]

\(^1\)With normalization \( \text{tr}(T^a T^b) = \frac{i}{2} \delta^{ab} \). Let us mention that typically in the amplitude-related literature a different normalization is used, such that \( \text{tr}(T^a T^b) = \delta^{ab} \).
From the Fierz identity:
\[
(T^a)_{ij}(T^b)_{kl} = \frac{1}{2} \left\{ \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right\},
\] (2.34)

one can derive an analogous expression for the product of two structure constants with one color index contracted:
\[
f^{a_1 a_2 e} f^{a_3 a_4 e} = 4 \left\{ \text{tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - \text{tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) \\
- \text{tr}(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) + \text{tr}(T^{a_2} T^{a_1} T^{a_4} T^{a_3}) \right\}. \] (2.35)

Similar results can be obtained for any products of structure constants, with different index contractions, resulting in single traces or in products of traces of generators. At \( n \) points there are only \( (n - 1)! \) independent single traces, due to the cyclicity of the trace, but this set is not sufficient to describe all the possible configurations of \( n \) gluons. However, it is possible to show \([5]\) that, at tree level\[2\]:
\[
A_{n}^{\text{tree}} = g^{n-2} \sum_{\sigma \in S_{n-1}} A_{n}[1\sigma(2)\sigma(3)...\sigma(n)] \text{tr}(T^{a_1} T^{\sigma(a_2)} T^{\sigma(a_3)} ... T^{\sigma(a_n)}),
\] (2.36)

where \( S_{n-1} \) is the symmetric group of order \( n - 1 \) and \( A_{n}[1...n] \) are called partial or color-ordered amplitudes, owing to the fact that they have a fixed ordering of external particles. The decomposition (2.36) is useful for two reasons: first, partial amplitudes are easier to compute than full amplitudes, since the color-ordered Feynman rules (with definite particle ordering) are simpler than the complete ones. Moreover, they can be proven to be gauge invariant and possess the following interesting properties \([5]\):

- Ciclicity: \( A_{n}[12...n] = A_{n}[2...n1] \),
- Reflection: \( A_{n}[12...n] = (-1)^n A_{n}[n...21] \),
- \( U(1) \) decoupling identity: \( A_{n}[123...n] + A_{n}[213...n] + ... + A_{n}[23...1n] = 0 \),

(2.37) (2.38) (2.39)

together with satisfying the so-called Kleiss-Kuijf relations\([102]\):
\[
A_{n}[1, \{\alpha\}, n, \{\beta\}] = (-1)^{|\beta|} \sum_{\sigma \in \text{OP}(\{\alpha\}, \{\beta^T\})} A_{n}[1, \sigma, n],
\] (2.40)

\[2\text{Loop-level amplitudes require the inclusion of products of traces.}\]
where \( \{ \alpha \} \) and \( \{ \beta \} \) denote two sets of external particle indices, “OP” the ordered permutations of the indices in the set \( \{ \alpha \} \cup \{ \beta^T \} \) and \( \{ \beta^T \} \) a set of indices with the reverse ordering with respect to \( \{ \beta \} \). Together with (2.37)-(2.39), (2.40) can be employed to reduce the number of independent partial amplitudes: from \( (n - 1)! \) (corresponding with the number of independent single traces of \( n \) generators) to \( (n - 2)! \).

The discovery of the CK duality comes into play at this point, since it was realized that the new algebraic relations between the kinematic factors of the cubic diagrams introduced in the previous sections can be exploited to express some of the partial amplitudes in terms of other partial amplitudes and kinematic factors [14]. Then, one can perform a generalized gauge transformation of the type (2.28), setting to zero some of the kinematic factors, resulting in new relations between partial amplitudes. We will not enter in the details of this construction, which are beyond the scope of this Thesis, limiting to state the result: the existence of the CK duality leads to the so-called **BCJ relations**, which further constrain the partial amplitudes and leave only \( (n - 3)! \) independent ones. The original expression of the BCJ relations is the following:

\[
A_n[1, 2, \{ \alpha \}, 3, \{ \beta \}] = \sum_{\sigma_i \in \text{POP}(\{ \alpha \}, \{ \beta \})} A_n(1, 2, 3, \sigma_i) F_i, \tag{2.41}
\]

with “POP” denoting partial-order-preserving permutations: permutations of set \( \{ \alpha \} \cup \{ \beta \} \) preserving the relative ordering inside the set \( \{ \beta \} \), while \( F_i \) stands for complicated kinematic factors containing Lorentz-invariant combinations of the particles momenta, whose explicit definition can be found in [14]. Apart from the exact expressions of such factors, the meaning of (2.41) is that all the partial amplitudes can be expressed in terms of the \( (n - 3)! \) partial amplitudes with particles \( 1, 2 \) and \( 3 \) in the first positions, in this order. In addition, it was further shown [15] that (2.41) contains redundant information, and that it can be derived from the **fundamental BCJ relation**:

\[
\sum_{b=3}^{n} \left( \sum_{c=1}^{n-1} p_2 \cdot p_c \right) A_n[1, 3, \ldots, b - 1, 2, b, \ldots, n] = 0, \tag{2.42}
\]

where \( p_i \) are the gluons momenta. Examples of BCJ relations in the four and five gluons case are the following:

\[
s_{14} A_4[1234] - s_{13} A_4[1243] = 0, \tag{2.43}
\]
\[ s_{12}A_5[21345] - s_{23}A_5[13245] - (s_{23} + s_{24})A[13425] = 0, \]  
\[ (2.44) \]

with as usual \( s_{ij} = 2p_i \cdot p_j \).

There are many proofs of the BCJ relations, both from String Theory \([103-109]\) and from Field Theory \([15, 110, 111]\). In \([110]\) it was also shown that the existence of BCJ relations together with the Kleiss–Kuijf identities give rise to Jacobi-like identities for the kinematic numerators.

### 2.2.5 Five gluons amplitude

Let us move to our last explicit computation in YM theory: the five gluons amplitude. It is interesting to our purposes since it provides an example where the expression of the amplitude derived from Feynman diagrams (after splitting the contact terms) does not satisfy the CK duality. It is then necessary to perform a generalized gauge transformation to bring the amplitude in the form of eq. (2.27). We will also show explicitly that, as observed in \([16]\), this transformation is tantamount to the addition of a non-local five gluons contact term to the YM Lagrangian. The latter contributes non-trivially to the trivalent diagrams \( \Gamma_3 \), although it is identically zero due to the Jacobi identity and therefore has no influence on physical quantities.

The Feynman diagrams contributing to the amplitude contain vertices with either three or four gluons; in the latter case each quartic vertex has to be split into three distinct contributions to the trivalent vertices with fixed color structure needed in order to enforce the color-kinematics duality. One can easily count the distinct tree-level diagrams with only trivalent vertices we can build at tree level: given the number of diagrams for \( n \) particles, call it \( C_n \), in order to obtain a diagram for \( n + 1 \) particles we can attach an additional leg either to one of the \( n \) external legs or to an internal line of each of the \( C_n \) diagrams for \( n \) particles. Denoting with \( E (I) \) the number of external (internal) lines and with \( V \) the number of vertices, then the topology of tree-level diagrams implies:

\[ I = V - 1, \]  
\[ (2.45) \]

which is a specific case of Euler’s formula for planar diagrams in the absence of loops.
In addition, for diagrams with only trivalent vertices, the following relation also holds:

$$3V = E + 2I,$$

(2.46)

as can be understood from the fact that exactly three legs must meet in every vertex and external lines are attached to one vertex only, while internal lines connect two vertices. For $E = n$, (2.45) and (2.46) allow to derive the number of vertices and internal lines, which are given by:

$$V = n - 2, \quad I = n - 3.$$

(2.47)

Since, as mentioned, tree-level diagrams with trivalent vertices and $n + 1$ external legs are built from analogous diagrams with $n$ external legs, attaching an additional leg either to an external or to an internal line one finds:

$$C_{n+1} = (E + I)C_n = (2n - 3)C_n \Rightarrow C_n = (2n - 5)!!$$

(2.48)

Therefore, for $n = 5$, the set $\Gamma_3$ contains $(10 - 5)!! = 15$ diagrams. Contributions to such diagrams come from Feynman diagrams with either only cubic vertices (fig. 2.3a), which are 15, or with one cubic and one quartic vertex (2.3b), which are $\binom{5}{2} = 10$. Since each quartic term contains three different color structures, again we split their expression into three parts and multiply and divide by the appropriate inverse propagator, getting 30 distinct contributions. Therefore, per each color structure we have $30/15=2$ contributions from diagrams with one quartic term (fig. 2.3b).

To check the CK duality we choose three diagrams whose color factors sum up to
zero owing to the Jacobi identity, and compute the contributions to them from the appropriate Feynman diagrams. Without loss of generality, we can consider for instance the diagrams with color structure

\[ c_i := f^{a_1 a_2 b} f^{b c a_4 a_5}, \quad c_j := f^{a_1 a_2 b} f^{b c a_4 a_3}, \quad c_k := f^{a_1 a_2 b} f^{b c a_5 a_3}, \tag{2.49} \]

such that \( c_i + c_j + c_k = 0 \), naming the corresponding diagrams in \( \Gamma_3 \) with \( i, j \) and \( k \).

In Figs. 2.4a, 2.4b and 2.4c we list the Feynman diagrams which contribute to diagrams \( i, j \) and \( k \), respectively, where the vertical bar with \( c_l \) means that only the part of the diagram proportional to \( c_l \), with \( l = i, j, k \), has to be selected. As before, we consider all the particles as ingoing and we label the inverse propagators of the internal lines as \( s_{ij} := (p_i + p_j)^2 = 2p_i \cdot p_j \). We wish rearrange the amplitude in the form:

\[ A_n = g^{n-2} \sum_{i \in \Gamma_3} \frac{n_l c_i}{\prod \alpha_i s_{\alpha_i}}; \tag{2.50} \]
to then check whether $n_i + n_j + n_k = 0$, given that $c_i + c_j + c_k = 0$.

Let us focus, for instance, on the kinematic factor of diagram $i$ (the others can be easily obtained from cyclic permutations of the particle indices). Since each of the three Feynman diagrams in fig. 2.4a contributes to the kinematic numerator $n_i$ in a different way, we analyse them separately, starting from those in fig. 2.5. As clear from fig. 2.4, the Feynman diagrams with only cubic vertices which contribute to the diagrams $i$, $j$ and $k$ in $\Gamma_3$ share a common vertex, containing the external lines 1 and 2 and an internal line carrying a squared momentum $(p_1 + p_2)^2 = s_{12}$. We name $V_{12}^{b,\nu}$ this vertex (fig. 2.6a) contracted with the polarization vectors $\varepsilon_1$ and $\varepsilon_2$, while $b (\nu)$ is the color (Lorentz) index of the line which is not contracted with polarization vectors. The remaining part of the diagram consists in a gluon propagator, carrying squared momentum $s_{12}$, which is common to the three diagrams as well, and a part which is different in the three cubic diagrams. This latter part, which we name $V_{345}^{b,\nu}$, in the case of diagram $i$, is represented in fig. 2.6b (the corresponding terms for $j$ and $k$ are obtained from particle index permutations), where the polarization vectors $\varepsilon_3$, $\varepsilon_4$ and $\varepsilon_5$ are contracted with three of the external lines, while as above $b (\nu)$ is the color (Lorentz) index of the line which is not contracted with polarization vectors. With this factorization, the diagram of fig. 2.5 can be written as:

$$2.5 = V_{12}^{b,\nu} \cdot \frac{-i}{s_{12}} \cdot V_{345}^{b,\nu},$$

(2.51)

where $V_{12}^{b,\nu}$ and $V_{345}^{b,\nu}$ are represented in fig. 2.6a and 2.6b respectively. Their expression is the following:

$$V_{12}^{b,\nu} = g f_{a_1 a_2 b} \{(\varepsilon_1 \varepsilon_2)(p_1 - p_2)^\nu + 2(\varepsilon_1 p_2)\varepsilon_2^\nu - 2(\varepsilon_2 p_1)\varepsilon_1^\nu\},$$

(2.52)
In order to obtain the sought for contribution to the kinematic factors \( n_i \) in eq. (2.50), we must dispose of the coupling constant, of the color factors and of the propagators: we name \( \tilde{V} \) the objects \( V \) defined above, omitting these terms and without color indices (the free color index of the vertices 2.6a and 2.6b are simply contracted with each other by the gluon propagator in Feynman gauge). Employing this prescription in eq. (2.52), we can read the definition of \( \tilde{V}_{12}^{b,\nu} \):\[
V_{12}^{b,\nu} = g f^{a_1 a_2 b} \{ \ldots \}^{\nu} \Rightarrow \tilde{V}_{12}^{\nu} := \{ \ldots \}^{\nu}. \tag{2.54}
\]

In a similar way, we define \( \tilde{V}_{345}^{\nu} \), this time eliminating also the inverse propagator at the denominator:
\[
V_{345}^{b,\nu} = -i \frac{g^2}{s_{45}} f^{ba_3 c} f^{ca_4 a_5} \{ \ldots \}^{\nu} \Rightarrow \tilde{V}_{345}^{\nu} := -i \{ \ldots \}^{\nu}. \tag{2.55}
\]

Moreover, we calculate explicitly only the part concerning diagram \( i \) (with particle ordering 12345), while the contributions to diagrams \( j \) and \( k \) can be calculated easily performing permutations of the particle indices. Therefore, diagrams of the type 2.5 contribute

\[
\begin{align*}
V_{12}^{b,\nu} &= -i \frac{g^2}{s_{45}} f^{ba_3 c} f^{ca_4 a_5} \left\{ \varepsilon_5 p_4 \right\} (\varepsilon_3 p_4) (\varepsilon_4 p_3) (\varepsilon_5 p_5) \\
&\quad - 2(\varepsilon_3 \varepsilon_4) (\varepsilon_5 p_4) (p_3 - p_4 - p_5)^\nu + (\varepsilon_4 \varepsilon_5) (\varepsilon_3 p_4) (p_4 - p_4 - p_5)^\nu \\
&\quad - (\varepsilon_4 \varepsilon_5) (\varepsilon_3 p_5) (p_3 - p_4 - p_5)^\nu + 2(\varepsilon_3 \varepsilon_5) (\varepsilon_4 p_5) (p_3 - p_4 - p_5)^\nu \\
&\quad + 2(\varepsilon_4 \varepsilon_5) (\varepsilon_3 p_4) (p_4 - p_4 - p_5)^\nu + 2(\varepsilon_4 \varepsilon_5) (\varepsilon_4 p_5) (p_4 - p_5)^\nu \\
&\quad - 4\varepsilon_4^{\nu} [(\varepsilon_5 p_4) (\varepsilon_3 p_4) + (\varepsilon_5 p_4) (\varepsilon_3 p_5)] + 4\varepsilon_5^{\nu} [(\varepsilon_3 p_4) (\varepsilon_4 p_5) + (\varepsilon_3 p_5) (\varepsilon_4 p_5)].
\end{align*}
\]
to the sum of kinematic factors as follows:

\[
[n_i + n_j + n_k]_{2.5} = \tilde{V}_{12}^{b,\nu}(-i)\{\tilde{V}_{b,\nu}^{345} + \tilde{V}_{b,\nu}^{453} + \tilde{V}_{b,\nu}^{534}\} = -\tilde{V}_{b,\nu}^{12}(p_3 + p_4 + p_5)^\nu \\
\times \left[ (\varepsilon_4\varepsilon_5)\varepsilon_3 \cdot (p_4 - p_5) + (\varepsilon_5\varepsilon_3)\varepsilon_4 \cdot (p_5 - p_3) + (\varepsilon_3\varepsilon_4)\varepsilon_5 \cdot (p_3 - p_4) \right] \\
+ \varepsilon_3^i(\varepsilon_4\varepsilon_5)(s_{35} - s_{34}) + \varepsilon_4^i(\varepsilon_5\varepsilon_3)(s_{43} - s_{45}) + \varepsilon_5^i(\varepsilon_3\varepsilon_4)(s_{54} - s_{53}) \}.
\]

(2.56)

Using momentum conservation one has \(p_3 + p_4 + p_5 = -p_1 - p_2\) and from the transversality of the polarization vectors and the mass-shell condition, also \(\tilde{V}_{b,\nu}^{12}(p_1 + p_2)^\nu = 0\), so that:

\[
[n_i + n_j + n_k]_{2.5} = -\left\{ [ (\varepsilon_1\varepsilon_2)(\varepsilon_4\varepsilon_5)(\varepsilon_3p_1)(s_{35} - s_{34}) - (\varepsilon_1\varepsilon_2)(\varepsilon_4\varepsilon_5)(\varepsilon_3p_2)(s_{35} - s_{34}) \\
+ 2(\varepsilon_4\varepsilon_5)[(\varepsilon_2\varepsilon_3)(\varepsilon_1p_2) - (\varepsilon_1\varepsilon_3)(\varepsilon_2p_1)](s_{35} - s_{34}) \\
+ [345 \rightarrow 453] + [345 \rightarrow 534] \},
\]

(2.57)

where with the notation \(+[abc \rightarrow cab]\) we mean that the expression also contains the permutations of the first term with index \(a\) replaced with \(c\), \(b\) with \(a\) and \(c\) with \(b\). We notice that every term in \(n_i\) contains either one or two (the integer part of \(n/2\), in the general case) factors of \((\varepsilon_i\varepsilon_j)\), but in the sum of the kinematic numerators from this type of diagrams only terms with two factors survive. This is essential in order for them to be cancelled by diagrams with a contact term, since the Feynman rules of the latter generate only terms with two factors. In order to find the contribution to \(n_i\) from the diagram in fig. 2.7 we must select from the Feynman diagram the part proportional to \(c_i\). Let us write down its full expression, with the appropriate insertions of \(\frac{s_{34}}{s_{ij}}\):

\[
2.7 = ig^2\left\{ \frac{-i}{s_{12}} \right\} \tilde{V}_{b,\nu}^{12}f_{\alpha_1\alpha_2b} \times \left\{ \frac{s_{45}}{s_{45}} f_{\beta a_3c} f^{\alpha a_4 a_5} [\varepsilon_5^\nu(\varepsilon_3\varepsilon_4) - \varepsilon_4^\nu(\varepsilon_3\varepsilon_5)] \\
+ \frac{s_{35}}{s_{35}} f_{\beta a_4 c} f^{\alpha a_5 a_3} [\varepsilon_5^\nu(\varepsilon_4\varepsilon_5) - \varepsilon_5^\nu(\varepsilon_4\varepsilon_3)] \\
+ \frac{s_{34}}{s_{34}} f_{\beta a_5 c} f^{\alpha a_3 a_4} [\varepsilon_4^\nu(\varepsilon_5\varepsilon_3) - \varepsilon_3^\nu(\varepsilon_5\varepsilon_4)] \}.
\]

(2.58)

Since in eq. (2.58) all the color factors of interest are present, we can easily extract from it the kinematic factors \(n_l\), with \(l = i, j, k\), simply suppressing coupling constants, propagators and structure constants after having identified the color factors \(c_i, c_j, c_k\).
Their sum is given by the following expression:

\[
[n_i + n_j + n_k]_{2.7} = s_{45} \{ (\varepsilon_3 \varepsilon_4)[(\varepsilon_1 \varepsilon_2)\varepsilon_5 \cdot (p_1 - p_2) + 2(\varepsilon_2 \varepsilon_5)(\varepsilon_1 p_2) - 2(\varepsilon_1 \varepsilon_5)(\varepsilon_2 p_1)] \\
- (\varepsilon_3 \varepsilon_5)[(\varepsilon_1 \varepsilon_2)\varepsilon_4 \cdot (p_1 - p_2) + 2(\varepsilon_2 \varepsilon_4)(\varepsilon_1 p_2) - 2(\varepsilon_1 \varepsilon_4)(\varepsilon_2 p_1)] \} \\
+ [345 \to 453] + [345 \to 534].
\]  

(2.59)

We can now collect the contribution to the sum of the kinematic factors coming from diagrams of the type 2.5 and 2.7, and observe that:

\[
[n_i + n_j + n_k]_{2.5+2.7} = 0.
\]  

(2.60)

Therefore, any violation of the kinematic Jacobi identity must come from diagrams of the type 2.8. Let us now compute the contribution of this diagram to \(n_i\), therefore extracting only the part with color factor \(c_i\). Again, contributions to \(n_j\) and \(n_k\) can be easily obtained upon permuting of the particle indices \((3, 4, 5)\). We find:

\[
[n_i + n_j + n_k]_{2.8} = 0.
\]  

(2.61)
\[ 2.8 = \frac{g^3}{s_{12}s_{45}}c_is_{12}\{2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_4)(\varepsilon_5p_4) - 2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_5)(\varepsilon_4p_4) \\
+ (\varepsilon_2\varepsilon_3)(\varepsilon_4\varepsilon_5)\varepsilon_1 \cdot (p_4 - p_5) - (\varepsilon_1\varepsilon_3)(\varepsilon_4\varepsilon_5)\varepsilon_2 \cdot (p_4 - p_5) \\
+ 2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_5)(\varepsilon_4p_5) - 2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_5)(\varepsilon_4p_5)\}. \] (2.61)

We notice that with diagrams of the type 2.8 we have multiplied and divided by the same factor \(s_{12}\) the contributions to diagrams \(i, j\) and \(k\) in order to render the correct propagator structure. Therefore the kinematic factors nicely sum keeping \(s_{12}\) factorized, with the result:

\[ [n_i + n_j + n_k]_{2.8} = s_{12}\{2[(\varepsilon_1\varepsilon_4)(\varepsilon_2\varepsilon_5) - (\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_4)]\varepsilon_3 \cdot (p_4 + p_5) \\
+ (\varepsilon_3\varepsilon_5)[(\varepsilon_2\varepsilon_3)\varepsilon_1^\alpha - (\varepsilon_1\varepsilon_3)\varepsilon_2^\alpha](p_4 - p_5)_\alpha \\
+ [345 \to 453] + [345 \to 534] \neq 0. \] (2.62)

To summarize, though we were able to write the amplitude in the form indicated by eq. (2.50), still the CK duality is not yet satisfied and we need to look for a generalized gauge transformation allowing for this property to hold. We shall not evaluate here the factors \(\Delta_i\) appearing in eq. (2.29); rather, in the next section we will show that these can be directly obtained from an appropriate modification of the YM Lagrangian.

### 2.2.6 A non-local Lagrangian for color-kinematics duality at five points

In the previous section we showed how to reorganize the Feynman diagrams expansion of the five gluons tree-level amplitude, in such a way that it can be written as follows:

\[ A_5 = \sum_{i \in \Gamma_3} \frac{c_in_i}{\prod_{\alpha_i} s_{\alpha_i}}, \] (2.63)

We selected three diagrams out of the 15 present in the set \(\Gamma_3\), namely diagrams \(i, j\) and \(k\), whose color factors obey a Jacobi identity: \(c_i + c_j + c_k = 0\). Then, we realized that \(n_i + n_j + n_k \neq 0\), thus seemingly resulting in a violation of the CK duality.

However, as explained, we still have the freedom to perform a generalized gauge transformation, which does not affect the scattering amplitude but still sets \(n_i + n_j + \)
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\( n_k = 0 \). Instead of making the transformation directly on the numerators, we consider
the possibility of adding a 5-gluons contact term to the Yang-Mills Lagrangian, which
vanishes identically due to the Jacobi identity but contributes via Feynman diagrams to
each color structure in order to make the color-kinematics duality evident. This was first
proposed in [16], where the explicit expression of this Lagrangian term was also given:

\[
L_5' = -\frac{g^3}{2} f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \left( \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2} A^{a_3 \mu} + \partial_{[\mu} A^{a_2}_{\nu]} A^{a_3} A^{a_1 \mu} + \partial_{[\mu} A^{a_3}_{\nu]} A^{a_1} A^{a_2 \mu} \right) 1 \Box (A^{a_4 \nu} A^{a_5 \rho}).
\]

Even though in this form it is not clear that \( L_5' \) is identically zero, this can be easily
seen if we relabel the color indices in such a way that only one monomial is present,
multiplied by a sum of color factors:

\[
L_5' = -\frac{g^3}{2} (f^{a_1 a_2 b} f^{b a_3 c} + f^{a_3 a_1 b} f^{b a_2 c} + f^{a_2 a_3 b} f^{b a_1 c}) f^{c a_4 a_5}
\times \left( \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2} A^{a_3 \mu} \right) 1 \Box (A^{a_4 \nu} A^{a_5 \rho}),
\]

where the Jacobi identity is now apparent. It is interesting to notice that such a relevant
symmetry such the CK duality requires non-local contributions in order to be imple-
mented at the Lagrangian level.

In order to extract the Feynman rules from this Lagrangian, we notice that, as usual,
we must sum over the \( n! = 5! = 120 \) ways to contract the \( n = 5 \) fields in the Lagrangian
with the \( n = 5 \) external particles involved in the scattering. Exploiting the total anti-
symmetry of the structure constants, we can organize these 120 contractions into the
15 possible color structures (one per each diagram in \( \Gamma_3 \)), such that each color structure
receives contributions from 8 contractions. These contractions amount to the possible
associations of each external particle \( x_i \) to each field in \( L_5' \), or equivalently to each
color index \( a_i \). If we want to build, for instance, the color factor \( c_i = f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \),
we must contract \( x_3 \) with \( a_3 \) and \( x_1, x_2 \) (\( x_4, x_5 \)) either with \( a_1, a_2 \) (\( a_4, a_5 \)) or with \( a_4, a_5 \)
(\( a_1, a_2 \)) in each order. Therefore, with reference to the order of color indices present in
the Lagrangian of eq. (2.64), we have the freedom to switch \( a_1 \) and \( a_2 \) as well as \( a_4 \) and
\( a_5 \), but also to exchange the pair (12) with the pair (45), overall amounting to a total of
\( 2! \cdot 2! \cdot 2 = 8 \) contractions. Let us notice that, when exchanging (12) with (45), the color
factor stays the same (apart from signs), but \( 1/\Box \) changes value: indeed, when acting
on $A^{a_4} A^{a_5}$ (case “A”) its value is $1/s_{45}$ (to be multiplied by $s_{12}^{34}$), while when acting on $A^{a_1} A^{a_2}$ (case “B”) it is $1/s_{12}$ (to be multiplied by $s_{45}^{34}$).

Here we compute separately the contributions to the kinematic factor of diagram $i$ coming from $\mathcal{L}_5'$, in the two possible cases A and B:

$$n_i^A = \frac{s_{12}}{2} \left\{ (\varepsilon_1 \varepsilon_4) (\varepsilon_2 \varepsilon_5) \varepsilon_3 \cdot (p_1 + p_2) - (\varepsilon_1 \varepsilon_3) (\varepsilon_2 \varepsilon_5) \varepsilon_4 \cdot (p_1 - p_3) \right. \\
+ (\varepsilon_2 \varepsilon_4) (\varepsilon_3 \varepsilon_5) \varepsilon_1 \cdot (p_2 + p_3) + (\varepsilon_1 \varepsilon_5) (\varepsilon_3 \varepsilon_4) \varepsilon_2 \cdot (p_1 + p_3) \\
+ (\varepsilon_1 \varepsilon_5) (\varepsilon_2 \varepsilon_3) \varepsilon_4 \cdot (p_2 - p_3) - (\varepsilon_1 \varepsilon_5) (\varepsilon_2 \varepsilon_4) \varepsilon_3 \cdot (p_1 + p_2) \\
- (\varepsilon_1 \varepsilon_4) (\varepsilon_3 \varepsilon_5) \varepsilon_2 \cdot (p_1 + p_3) - (\varepsilon_2 \varepsilon_5) (\varepsilon_3 \varepsilon_4) \varepsilon_1 \cdot (p_2 + p_3) \\
+ (\varepsilon_1 \varepsilon_5) (\varepsilon_2 \varepsilon_4) \varepsilon_5 \cdot (p_1 - p_3) - (\varepsilon_2 \varepsilon_3) (\varepsilon_1 \varepsilon_4) \varepsilon_5 \cdot (p_2 - p_3) \\
+ (\varepsilon_1 \varepsilon_2) ([\varepsilon_3 \varepsilon_5] \varepsilon_4^\alpha - [\varepsilon_3 \varepsilon_4] \varepsilon_5^\alpha] (p_1 - p_2)_\alpha \right\}, \quad (2.66)$$

$$n_i^B = -\frac{s_{45}}{2} \left\{ (\varepsilon_1 \varepsilon_4) (\varepsilon_2 \varepsilon_5) \varepsilon_3 \cdot (p_4 + p_5) + (\varepsilon_3 \varepsilon_4) (\varepsilon_2 \varepsilon_5) \varepsilon_1 \cdot (p_3 - p_4) \right. \\
+ (\varepsilon_1 \varepsilon_5) (\varepsilon_2 \varepsilon_4) \varepsilon_4 \cdot (p_5 + p_3) + (\varepsilon_1 \varepsilon_3) (\varepsilon_2 \varepsilon_4) \varepsilon_5 \cdot (p_3 + p_4) \\
+ (\varepsilon_2 \varepsilon_4) (\varepsilon_3 \varepsilon_5) \varepsilon_1 \cdot (p_5 - p_3) - (\varepsilon_1 \varepsilon_5) (\varepsilon_2 \varepsilon_4) \varepsilon_3 \cdot (p_4 + p_5) \\
- (\varepsilon_1 \varepsilon_4) (\varepsilon_2 \varepsilon_5) \varepsilon_5 \cdot (p_3 + p_4) - (\varepsilon_2 \varepsilon_5) (\varepsilon_1 \varepsilon_4) \varepsilon_4 \cdot (p_3 + p_5) \\
+ (\varepsilon_1 \varepsilon_5) (\varepsilon_3 \varepsilon_4) \varepsilon_2 \cdot (p_4 - p_3) - (\varepsilon_3 \varepsilon_5) (\varepsilon_1 \varepsilon_4) \varepsilon_2 \cdot (p_5 - p_3) \\
+ (\varepsilon_4 \varepsilon_5) ([\varepsilon_2 \varepsilon_3] \varepsilon_1^\alpha - [\varepsilon_3 \varepsilon_2] \varepsilon_1^\alpha)] (p_4 - p_5)_\alpha \right\}, \quad (2.67)$$

where the $n_{j,k}^{A,B}$ are obtained from cyclic permutations of the particle indices $(3,4,5)$, as usual. We notice that the coefficients $n_{j,k}^B$ contain exactly the same terms as $n_i^B$, except for the $s_{ij}$ terms: since via momentum conservation $s_{34} + s_{25} + s_{45} = s_{12}$ we can sum these terms to the $A$-type contributions, keeping $s_{12}$ factorized. Therefore, if we define $n_i' = n_i + n_i^A + n_i^B$, we end up with:

$$n_i' + n_j' + n_k' = (n_i^A + n_i^B + n_i) + (n_j^A + n_j^B + n_j) + (n_k^A + n_k^B + n_k) = 0, \quad (2.68)$$

which means that the addition of $\mathcal{L}_5'$ to the YM Lagrangian generates Feynman rules which make the CK automatically satisfied up to five points at tree level.

Furthermore, it is possible to consider the addition of another five gluons contact
term to the YM Lagrangian, which again vanishes identically due to the Jacobi identity and whose Feynman rules generate diagrams which automatically satisfy the color-kinematics duality at five points:

\[
L_5'' = -\frac{1}{2} y^3 f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \left( \partial_{(\mu} A_{\nu)}^{a_1} A_{\rho}^{a_2} A_{\sigma}^{a_3 \mu} + \partial_{(\mu} A_{\nu)}^{a_2} A_{\rho}^{a_3} A_{\sigma}^{a_1 \mu} 
+ \partial_{(\mu} A_{\nu)}^{a_3} A_{\rho}^{a_1} A_{\sigma}^{a_2 \mu} \right) \frac{1}{\Box} \left( A^{a_4 \nu} A^{a_5 \rho} \right). \tag{2.69}
\]

Per each value of \( \alpha \), the addition of such a term to the YM Lagrangian implements a generalized gauge transformation which leaves the five-gluons amplitude in the CK dual form, as can be seen via direct computation. The case in which \( \frac{\Lambda}{\mu} = \frac{1}{s_{45}} \) will be referred to as the “C” case, while we shall label with “D” the case in which \( \frac{\Lambda}{\mu} = \frac{1}{s_{12}} \). Simply changing some signs in the expressions of \( n_i^A \) and \( n_i^B \) (in particular, only terms proportional to the difference of two momenta change sign), we find (setting \( \alpha = 1 \)):

\[
n_i^C = \frac{s_{12}}{2} \{ (\epsilon_1 \epsilon_4) (\epsilon_2 \epsilon_5) \epsilon_3 \cdot (p_1 + p_2) + (\epsilon_1 \epsilon_3) (\epsilon_2 \epsilon_5) \epsilon_4 \cdot (p_1 - p_3) 
+ (\epsilon_2 \epsilon_4) (\epsilon_3 \epsilon_5) \epsilon_1 \cdot (p_2 + p_3) + (\epsilon_1 \epsilon_5) (\epsilon_3 \epsilon_4) \epsilon_2 \cdot (p_1 + p_3) 
- (\epsilon_1 \epsilon_5) (\epsilon_2 \epsilon_3) \epsilon_4 \cdot (p_2 - p_3) - (\epsilon_1 \epsilon_5) (\epsilon_2 \epsilon_4) \epsilon_3 \cdot (p_1 + p_2) 
- (\epsilon_1 \epsilon_4) (\epsilon_3 \epsilon_5) \epsilon_2 \cdot (p_1 + p_3) - (\epsilon_2 \epsilon_5) (\epsilon_3 \epsilon_4) \epsilon_1 \cdot (p_2 + p_3) 
- (\epsilon_1 \epsilon_3) (\epsilon_2 \epsilon_4) \epsilon_5 \cdot (p_1 - p_3) + (\epsilon_2 \epsilon_3) (\epsilon_1 \epsilon_4) \epsilon_5 \cdot (p_2 - p_3) 
+ (\epsilon_1 \epsilon_2) [ (\epsilon_3 \epsilon_4) \epsilon_3^a - (\epsilon_3 \epsilon_5) \epsilon_1^a ] (p_1 - p_2) \}, \tag{2.70}
\]

\[
n_i^D = -\frac{s_{45}}{2} \{ (\epsilon_1 \epsilon_4) (\epsilon_2 \epsilon_5) \epsilon_3 \cdot (p_4 + p_5) - (\epsilon_3 \epsilon_4) (\epsilon_2 \epsilon_5) \epsilon_1 \cdot (p_3 - p_4) 
+ (\epsilon_1 \epsilon_5) (\epsilon_2 \epsilon_3) \epsilon_4 \cdot (p_5 + p_3) + (\epsilon_1 \epsilon_3) (\epsilon_2 \epsilon_4) \epsilon_5 \cdot (p_3 + p_4) 
- (\epsilon_2 \epsilon_4) (\epsilon_3 \epsilon_5) \epsilon_1 \cdot (p_5 - p_3) - (\epsilon_1 \epsilon_5) (\epsilon_2 \epsilon_4) \epsilon_3 \cdot (p_4 + p_5) 
- (\epsilon_1 \epsilon_4) (\epsilon_2 \epsilon_3) \epsilon_5 \cdot (p_3 + p_4) - (\epsilon_2 \epsilon_5) (\epsilon_1 \epsilon_4) \epsilon_4 \cdot (p_3 + p_5) 
- (\epsilon_1 \epsilon_5) (\epsilon_3 \epsilon_4) \epsilon_2 \cdot (p_4 - p_3) + (\epsilon_3 \epsilon_5) (\epsilon_1 \epsilon_4) \epsilon_2 \cdot (p_5 - p_3) 
+ (\epsilon_4 \epsilon_5) [ (\epsilon_1 \epsilon_3) \epsilon_2^a - (\epsilon_2 \epsilon_3) \epsilon_1^a ] (p_4 - p_5) \}. \tag{2.71}
\]

Again, the terms contributing to the color structures \( j \) and \( k \) are obtained by cyclic permutations of (3,4,5). Using \( s_{34} + s_{35} + s_{45} = s_{12} \) in the sum over \( D \)-type numerators
we get:

\[(n_i^C + n_i^D) + (n_j^C + n_j^D) + (n_k^C + n_k^D) = 0, \quad (2.72)\]

thus showing that \(L_5''\) generates self-CK dual Feynman rules for five gluon scattering.

### 2.2.7 Color-kinematics duality at loop level

We end our discussion on CK duality in YM amplitudes with an outline of how this property is implemented when quantum corrections are concerned. The idea is that, as in the case of tree-level amplitudes, contributions from contact terms can always be treated is such a way that the amplitude gets rearranged as a sum over diagrams with only trivalent vertices. The expression of a general \(L\)-loops amplitude is then written as:

\[
\mathcal{A}_n^{L\text{-loop}} = iL^2g^{n-2+2L} \sum_{i \in \Gamma_3} \left( \prod_{k=1}^L \frac{d^D l_k}{(2\pi)^D} \right) \frac{1}{S_i} \prod_{a_i} \frac{c_i n_i}{s_{a_i}}, \quad (2.73)
\]

where the notation is the same of Section 2.2.3: \(\Gamma_3\) is the set of diagrams with trivalent vertices, \(c\) are color factors, \(n\) are kinematic numerators and \(1/s_{a_i}\) are scalar-like propagators attached to the internal lines. Moreover, \(l_k\) denote the loop momenta while \(S_i\) is the symmetry factor of the corresponding diagram. The addition of diagrams with ghosts as internal lines does not affect the possibility of deriving such representation of the amplitude: since the ghosts are states in the adjoint representation of the gauge group they behave similarly to the gluons. In particular, they only have trivalent couplings with the gauge field while the corresponding vertex contains exactly one structure constant, like the vertex with three gluons.

The content of the CK duality is totally analogous to the tree-level case: it was proposed in [17] that also at loop level a representation of the kinematic factors in (2.73) exists such that whenever the color factors of a triplet of diagrams \(i, j, k\) satisfy a Jacobi identity \((c_i + c_j + c_k = 0)\), then the same identity is also satisfied by the kinematic factors of the three diagrams \((n_i + n_j + n_k = 0)\). In contrast to the tree-level case, however, the validity of the CK duality at loop level is still a conjecture. Nonetheless, the conjecture has been tested working out explicit numerators in several non-trivial cases, both in pure and Yang-Mills theory, as well as in its supersymmetric counterpart. Striking
examples are:

- 4 points, 4 loops in $\mathcal{N} = 4$ SYM [17, 48],
- 5 points, 2 loops in $\mathcal{N} = 4$ SYM [112],
- 4 points, 2 loops in pure YM (in $D = 4$) [17]
- 4 points, 1 loop in pure YM (in any $D$) [113],
- 4 points, 1 loop in pure YM with matter [114],
- 4 points, 1 loop with less than maximal supersymmetry [115].

It is worth emphasizing that, although most examples are carried out in simplified cases such as maximal supersymmetry or fixed-helicity amplitudes in $D = 4$, CK duality seems to be a general property of every YM theory, in any dimension.

2.3 General Relativity: Lagrangian and Feynman rules

General Relativity is usually formulated as a field theory for the field $g_{\mu\nu}$, playing the role of the dynamical metric of spacetime, which is a Lorentzian manifold endowed with a torsion-free Levi-Civita connection $\Gamma^\lambda_{\mu\nu}(g)$. The Einstein-Hilbert (EH) action is expressed in terms of the curvature as:

$$S_{EH} = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} R, \quad (2.74)$$

where $g$ is the determinant of the metric, $R$ the Ricci scalar and $\kappa^2 = 8\pi G^{(D)}$ (in the usual $c = 1$ units), with $G^{(D)}$ the Newton constant in dimension $D$. We neglected the cosmological constant term, which will not be considered in this Thesis, while we choose the following sign convention for the Riemann tensor:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (2.75)$$

The components of the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$, when solved for the metric tensor, are expressed in terms of the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left\{ \partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right\}. \quad (2.76)$$
Upon discarding a total derivative, the EH Lagrangian can be written in the equivalent form:

\[ L_{\Gamma - \Gamma} = -\frac{1}{2\kappa^2} \sqrt{-g} g^{\mu\nu} \left\{ \Gamma^\beta_{\mu\nu} \Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\beta\nu} \right\} , \]  

(2.77)

with \( \Gamma^\lambda_{\mu\nu} \) given in (2.76), which turns out to simplify the perturbative expansion. In order to study scattering amplitudes on a flat background, we define the graviton as the fluctuation over the flat Minkowski metric \( \eta_{\mu\nu} \):

\[ g_{\mu\nu} := \eta_{\mu\nu} + \tilde{h}_{\mu\nu} := \eta_{\mu\nu} + 2\kappa h_{\mu\nu} , \]  

(2.78)

where \( h_{\mu\nu} \) is the field with the correct mass dimension, being:

\[ [g_{\mu\nu}] = [\eta_{\mu\nu}] = [\tilde{h}_{\mu\nu}] = 0, \quad [\kappa] = -\frac{D - 2}{2}, \quad [h_{\mu\nu}] = \frac{D - 2}{2}. \]  

(2.79)

In order to get a perturbative expansion of the EH Lagrangian in terms of the graviton field, we work out separately the first terms of the expansion of its constituents:

- \( \sqrt{-g} \). First, we use the identity \( \log[\det(A)] = \text{tr}[\log(A)] \) to expand it as follows:

\[ \sqrt{-g} = \sqrt{-\det(\eta + \tilde{h})} = \exp \left\{ \frac{1}{2} \log[-\det(\eta) \det(1 + \eta^{-1}\tilde{h})] \right\} \]

\[ = \exp \left\{ \frac{1}{2} \text{tr}[\log(1 + \eta^{-1}\tilde{h})] \right\} . \]  

(2.80)

Then, we employ the expansions:

\[ \log(1 + x) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} x^k, \quad e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} , \]  

(2.81)

so that, up to cubic order, we find:

\[ \sqrt{-g} = \exp \left\{ \frac{1}{2} \text{tr}[\eta^{-1}\tilde{h} - \frac{1}{2}(\eta^{-1}\tilde{h})^2 + \frac{1}{3}(\eta^{-1}\tilde{h})^3 + o(\tilde{h}^4)] \right\} \]

\[ = 1 + \frac{1}{2} \text{tr}(\eta^{-1}\tilde{h}) - \frac{1}{4} \text{tr}[(\eta^{-1}\tilde{h})^2] + \frac{1}{6} \text{tr}[(\eta^{-1}\tilde{h})^3] + \frac{1}{8} \text{tr}[(\eta^{-1}\tilde{h})^2]^2 \]

\[ - \frac{1}{8} \text{tr}(\eta^{-1}\tilde{h})\text{tr}[(\eta^{-1}\tilde{h})^2] + \frac{1}{48} \text{tr}[(\eta^{-1}\tilde{h})^2]^2 + O(\tilde{h}^4) \]

\[ = 1 + \frac{1}{2} \tilde{h} - \frac{1}{4} \tilde{h}^{\alpha\beta} \tilde{h}_{\alpha\beta} + \frac{1}{8} \tilde{h}^2 + \frac{1}{6} \tilde{h}^{\alpha\beta} \tilde{h}_{\beta\gamma} \tilde{h}_{\gamma\alpha} - \frac{1}{8} \tilde{h} \tilde{h}^{\alpha\beta} \tilde{h}_{\alpha\beta} \]

\[ + \frac{1}{48} \tilde{h}^3 + O(\tilde{h}^4). \]  

(2.82)
• $g^{\mu\nu}$. We write the inverse metric as:

$$g^{-1} = (\eta + \tilde{h})^{-1} = \eta^{-1}(\eta + \tilde{h})^{-1} = \eta^{-1}(1 + \tilde{h} \eta^{-1})^{-1}.$$  

(2.83)

Then, we use:

$$(1 + x)^{-1} = \sum_{k=0}^{+\infty} (-1)^k x^k, \quad g^{-1} = \eta^{-1} \sum_{k=0}^{+\infty} (-1)^k (1 + \tilde{h} \eta)^{-1},$$  

(2.84)

with the result:

$$g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu} + \tilde{h}^{\mu} h^{\nu} - \tilde{h}^{\mu} h^{\nu} + \tilde{h}^{\mu} \tilde{h}^{\nu} + \mathcal{O}(\tilde{h}^4).$$  

(2.85)

• $\Gamma^\beta_{\mu\alpha}$. Directly from eq. (2.76) and (2.78):

$$\Gamma^\beta_{\mu\alpha} = \frac{1}{2} g^{\beta \lambda} (g_{\alpha \lambda, \mu} + g_{\mu \lambda, \alpha} - g_{\alpha \mu, \lambda})$$  

$$= \frac{1}{2} (\partial_\mu \tilde{h}^\beta - \partial_\alpha \tilde{h}^\beta - \partial^\beta \partial_\mu h^{\alpha} - \tilde{h}^{\beta} \partial_\mu h^{\alpha} - \tilde{h}^{\beta} \partial_\alpha h^{\mu} + \tilde{h}^{\beta} \partial_\lambda h^{\mu} + \mathcal{O}(\tilde{h}^3)).$$  

(2.86)

We can now expand the action perturbatively in powers of $\tilde{h}_{\mu\nu}$ or $h_{\mu\nu}$, i.e. schematically:

$$\mathcal{L} = \frac{1}{2\kappa^2} \sum_{n=2}^{+\infty} \mathcal{L}^{(n)} = \frac{1}{2\kappa^2} \sum_{n=2}^{+\infty} \partial^2 \tilde{h}^n = \frac{1}{2} \sum_{n=2}^{+\infty} \kappa^{n-2} \partial^2 h^n.$$  

(2.87)

The first two terms of the expansion (quadratic Lagrangian and cubic interactions) turn out to be:

$$\mathcal{L}^{(2)} = -\frac{1}{2} \partial_\mu h_{\alpha\beta} \partial^{\mu} h^{\alpha\beta} + \partial \cdot h_{\mu} \partial \cdot \tilde{h}^\mu - \partial_\mu h \partial \cdot h^\mu + \frac{1}{2} \partial_\mu \tilde{h} \partial^{\mu} h.$$  

(2.88)

$$\mathcal{L}^{(3)} = \kappa \left\{ (\sqrt{g})^{(1)} g^{\mu\nu} (0) + (\sqrt{g})^{(0)} g^{\mu\nu} (1) \right\} [\Gamma^\mu - \Gamma^\mu_{\mu\nu}] + 2 \partial_\beta hh^{\alpha\beta} \partial \cdot h^{\alpha}$$  

$$- 2h^{\mu\nu} \partial^{\beta} h_{\beta\mu} + h^{\mu\nu} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta}$$  

$$+ 2h^{\mu\nu} \partial^{\beta} h_{\beta\nu} - \partial_\beta hh^{\alpha\beta} \partial_\alpha h + 2h^{\mu\nu} \partial_\alpha h_{\mu\nu} \partial \cdot h^{\alpha} - 4h^{\alpha\beta} \partial_\alpha h_{\beta\mu} \partial^{\beta} h^{\mu}_{\lambda}$$  

(2.89)

Let us stress once more that (2.88) and (2.89), whose calculation already took some effort, provide just the first two terms of the expansion sketched in eq. (2.87), but this actually contains an infinite number of vertices. This makes computations in perturbative quantum gravity extremely difficult at higher points and represents a relevant complication when compared to the case of Yang-Mills theories.
2.3.1 Three-graviton tree-level amplitude

We will now compute the simplest scattering amplitude in General Relativity: the tree-level three-graviton amplitude, that will be useful in the next sections when we will introduce the DC relations; for the moment we shall limit ourselves to state the result.

We recall that in the case of a graviton in $D$ spacetime dimensions the polarization tensors are states in the rank-two, symmetric, traceless representation of $SO(D - 2)$ (therefore, there are $\frac{1}{2}D(D - 3)$ of them). In covariant notation, they are transverse, symmetric, traceless tensors $\varepsilon_{\mu\nu}(p)$ ($p$ is the momentum of the particle) defined, as in the spin-one case, up to an equivalence relation:

$$\varepsilon_{\mu\nu} \sim \varepsilon_{\mu\nu} + cp_{\mu}\Lambda_{\nu} + cp_{\nu}\Lambda_{\mu},$$

(2.90)

where $\Lambda_{\mu}$ is a Lorentz vector and $c \in \mathbb{R}$.

The full expression of the Feynman rules is quite long already for the vertex with three gravitons, but, luckily, we will not need it for our purposes. Indeed, since we are only interested in the scattering amplitude between three gravitons, we can isolate the transverse-traceless (TT) part of $L^{(3)}$: the part of the cubic vertex which does not contain either the trace ($h$) of $h_{\mu\nu}$ or its divergence $\partial \cdot h_{\mu}$. This is sufficient in order to compute the three-graviton amplitude, since all the fields in the vertex are going to be contracted which the physical polarization tensors, which are themselves transverse and traceless. Setting $h = 0$ and $\partial \cdot h_{\mu} = 0$ in eq. (2.89), the expression of the TT vertex turns out to be the following:

$$L^{(3)}_{TT} = \kappa\{h^{\mu\nu}\partial_{\mu}h_{\alpha\beta}\partial_{\nu}h^{\alpha\beta} - 2h^{\mu\nu}\partial^{\beta}h_{\alpha\mu}\partial^{\alpha}h^{\beta\mu} + 2h^{\mu\nu}\partial_{\alpha}h_{\beta\nu}\partial^{\alpha}h^{\beta\mu} - 4h^{\alpha\lambda}\partial_{\alpha}h_{\beta\mu}\partial^{\beta}h^{\mu}_{\lambda}\}.$$

(2.91)

We can now integrate by parts and eliminate terms containing $\Box h_{\mu\nu}$, since they contribute to the amplitude with factors $p^2$ which are zero for massless particles. Therefore the part of $L_{\beta}$ actually contributing to the scattering among three gravitons is:

$$L^{(3)}_{TT} = \kappa\{h^{\mu\nu}\partial_{\mu}h_{\alpha\beta}\partial_{\nu}h^{\alpha\beta} + 2h^{\mu\nu}\partial_{\mu}h_{\alpha\beta}\partial^{\beta}h^{\alpha\nu}\}.$$

(2.92)
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From the TT Lagrangian we can extract the part of the Feynman rules which allows to compute the tree-level amplitude with three gravitons. Taking into account the symmetry of the graviton polarization tensors these are:

\[ V^{\mu\nu\sigma\tau\rho\lambda}_3 = 2i\kappa \left\{ \eta^{\mu\sigma} \eta^{\nu\tau} p_1^\rho p_2^\nu p_3^\rho + \eta^{\mu\lambda} \eta^{\nu\sigma} p_1^\rho p_2^\rho + \eta^{\lambda\sigma} \eta^{\nu\tau} p_3^\rho \right\} + 2\eta^{\mu\sigma} \eta^{\nu\tau} p_2^\rho p_3^\nu + 2\eta^{\nu\sigma} \eta^{\mu\tau} p_1^\rho p_3^\nu + 2\eta^{\lambda\mu} \eta^{\nu\tau} p_3^\rho p_1^\nu + 2\eta^{\lambda\nu} \eta^{\mu\sigma} p_1^\rho p_3^\lambda + 2\eta^{\lambda\nu} \eta^{\mu\sigma} p_3^\rho p_1^\lambda \right\}. \tag{2.93} \]

In order to get the actual amplitude, we contract eq. (2.93) with the physical polarization tensors \( \varepsilon^{\mu}_{\nu}, \varepsilon^{\rho}_{\sigma}, p_i \cdot \varepsilon^{i}_{\mu} = 0, \varepsilon^{i}_{\nu} = 0 \forall i \) and exploit when necessary the mass-shell condition \( p_i^2 = 0 \forall i \) together with momentum conservation \( \sum_i p_i = 0 \). The final result for the three-graviton tree-level scattering amplitude is:

\[ M_3 = \varepsilon^{\mu}_{\nu} \varepsilon^{\rho}_{\sigma} \varepsilon^{\lambda}_{\rho} V^{\mu\nu\sigma\tau\rho\lambda}_3 \]

\[ = 2i\kappa \left\{ \varepsilon^{1}_{\alpha} \varepsilon^{2}_{\beta} \varepsilon^{3}_{\gamma} p_2^\rho p_1^\nu p_3^\rho + \varepsilon^{2}_{\alpha} \varepsilon^{3}_{\beta} \varepsilon^{1}_{\gamma} p_2^\rho p_3^\nu + \varepsilon^{3}_{\alpha} \varepsilon^{1}_{\beta} \varepsilon^{2}_{\gamma} p_1^\rho p_2^\nu + \varepsilon^{1}_{\alpha} \varepsilon^{2}_{\beta} \varepsilon^{3}_{\gamma} p_1^\rho p_3^\nu + \varepsilon^{2}_{\alpha} \varepsilon^{3}_{\beta} \varepsilon^{1}_{\gamma} p_3^\rho p_1^\nu \right\}. \tag{2.94} \]

This amplitude will be exploited in the last part of this chapter when dealing with the double copy, for the moment we only comment on the relative simplicity of the result stemming from the fact that most of the terms in the Lagrangian only have the role to guarantee gauge invariance and do not contribute to physical quantities such as scattering amplitudes.

2.4 The double copy

In this section we will present the main topic of interest for this Thesis: the double-copy (DC) relations [14, 17] connecting gravitational and YM amplitudes. The EH and YM Lagrangians appear quite different in terms of the graviton and gluon fields, respectively. The most striking difference is that the former contains an infinite number of vertices, whereas the latter has only vertices with three and four fields.

The idea of a relation between the scattering amplitudes in the two theories first appeared with the discovery of the Kawai-Lewellen-Tye (KLT) relations [18] in 1985. These are relations between tree-level scattering amplitudes in String Theory, which express the closed string amplitudes in terms of open string ones. In the low-energy limit, String Theory reduces to Field Theory and the KLT relations express gravity tree-level ampli-
Chapter 2. Scattering amplitudes and the double copy

tudes in terms of a sum of products of “left” and “right” YM partial amplitudes (introduced in Section 2.2.4), possibly from two different gauge theories. The lower points KLT relations can be expressed as follows:

\[ M_{\text{tree}}^{4}(1, 2, 3, 4) = -is_{12}(\frac{K}{2})^{2}A_{\text{tree}}^{4}(1, 2, 3, 4)\tilde{A}_{\text{tree}}^{4}(1, 2, 3, 4), \]  
\[ M_{\text{tree}}^{5}(1, 2, 3, 4, 5) = i\left(\frac{K}{2}\right)^{3}s_{12}s_{34}A_{\text{tree}}^{5}(1, 2, 3, 4, 5)\tilde{A}_{\text{tree}}^{5}(2, 1, 4, 3, 5) \]
\[ + i\left(\frac{K}{2}\right)^{3}s_{13}s_{34}A_{\text{tree}}^{5}(1, 3, 2, 4, 5)\tilde{A}_{\text{tree}}^{5}(3, 1, 4, 2, 5), \]  

where \( M_{\text{tree}}^{n} \) are \( n \)-particle tree-level gravitational amplitudes, \( A_{\text{tree}}^{n} \) and \( \tilde{A}_{\text{tree}}^{n} \) are the aforementioned left and right YM partial amplitudes and \( s_{ij} = (p_i + p_j)^2 \), as usual, while the arguments of the amplitudes label the external legs. Such relations hold for any particle state appearing in the closed String Theory and in particular, when the partial amplitudes are from pure YM theory, the gravitational amplitudes contain as external states not only the graviton, but also a two-form and a scalar, as we will discuss in Section 2.4.1. The KLT relations hold for any number \( n \) of external particles, although for high values of \( n \) the explicit expressions tend to be rather cumbersome, therefore we will not write them. Moreover, the validity of the KLT relations is limited to tree-level amplitudes, therefore connecting gravity and YM theories only at the semi-classical level.

The DC relations, equivalent to the KLT ones at tree level, provide an implementation of the more radical idea (still to be explored in all its aspects and meanings) that gravity is in a sense the “square” of a YM theory. These relations strongly rely on the existence of CK duality in YM scattering amplitudes and when compared to the results obtained by KLT they have the advantage that they can be extended also at loop level (with some caveats), while also leading to less involved expressions. At tree level, given a YM amplitude in a CK-dual form,

\[ A_{n} = g^{n-2}\sum\frac{n_{i}c_{i}}{\prod s_{\alpha i}}, \]  

i.e. where \( c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0 \), the basic tenet of the DC paradigm is that it is possible to obtain a gravity amplitude simply substituting the color factors with another copy of kinematic factors (\( \tilde{n}_i \)), possibly from a different YM theory, appropriately
modifying the coupling constant [14]:

$$M_n = i(2\kappa)^{n-2} \sum_{i \in \Gamma_3} \frac{n_i \tilde{n}_i}{\prod_{s_i} s_{s_i}}. \quad (2.98)$$

In what follows we will elaborate on the general meaning and implications of this expression. In the next section we will focus on the explicit case of the product of two YM tree-level scattering amplitudes with three gluons, interpreting the result as an amplitude in an appropriate theory of gravity. Later, we will discuss the DC relations for general amplitudes both at tree level and at loop level.

### 2.4.1 Particle content of the double copy

As we briefly mentioned at the beginning of Section 2.4, the intuitive idea of the DC relations is that an appropriate “square” of scattering amplitudes in YM theory gives rise to amplitudes in a theory of gravity. As a first step let us try to figure out the particle content of this putative theory.

When multiplying two YM amplitudes in the spirit of eq. (2.98), the product of kinematic factors \( \{ n \} \) gives rise, among other terms, to products of spin-one-polarization vectors. Let us focus on a single particle and, since we are dealing with massless quanta, let us choose a frame where its momentum is

$$p_\mu = E(1, 0, ..., 0, 1), \quad (2.99)$$

and its little group (LG), i.e. the subgroup of the Poincaré group which leaves this vector unaltered, is \( ISO(D - 2) \). Particle states with finite spin are defined as reducible, yet undecomposable representations of the LG such that, in order to avoid continuous spin representations, an equivalence relation between polarization tensors is assumed such to trivialize the action of the commuting generators of \( ISO(D - 2) \). (See e.g. [116].) For helicity \( h = 1 \) two polarizations \( \varepsilon_\mu \) and \( \varepsilon'_\mu \) belonging to the same equivalence class differ by an on-shell gauge transformation

$$\varepsilon'_\mu(p) = \varepsilon'_\mu(p) + \alpha(p)p_\mu, \quad (2.100)$$

with \( \alpha(p) \) an arbitrary function. In the standard frame defined by (2.99) we can choose a
Chapter 2. Scattering amplitudes and the double copy

representative which satisfies \( \varepsilon^\lambda_0 = \varepsilon^\lambda_D = 0 \ \forall \lambda \) (\( \lambda \) is the polarization), so that the (outer) product \( \varepsilon_{\mu\nu} := \varepsilon^\lambda_{\mu} \varepsilon^{\lambda'}_{\nu} \) of two such vectors is a \( D \times D \) matrix with zeros on the first and on the last row and column:

\[
\varepsilon_{\mu\nu} = \varepsilon^\lambda_{\mu} \varepsilon^{\lambda'}_{\nu} = \\
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & \varepsilon_{11} & \ldots & \varepsilon_{1,D-2} & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \varepsilon_{D-2,1} & \ldots & \varepsilon_{D-2,D-2} & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix} = \\
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & \varepsilon_{ij} & \ldots & \varepsilon_{ij} & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad (2.101)
\]

where the latin indices are LG indices (\( i = 1, \ldots, D - 2 \)).

The result of the outer product is a reducible \( \text{SO}(D - 2) \) rank-two tensor; in order to extract its actual particle content we must decompose it into irreducible representations. In the language of Young tableaux this is easily done:

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\end{array}
\end{array}, \quad (2.102)
\end{array}
\]

where the three tableaux on the right hand side of (2.102) correspond to:

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\end{array}
\end{array} \big|_T \leftrightarrow \text{symmetric, traceless tensor with } \frac{D(D-3)}{2} \text{ polarizations: graviton } h_{\mu\nu},
\)

- \( \begin{array}{c}
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\begin{array}{c}
\end{array}
\end{array} \leftrightarrow \text{antisymmetric tensor with } \frac{(D-2)(D-3)}{2} \text{ physical polarizations: two-form } B_{\mu\nu},
\)

- \( \begin{array}{c}
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\begin{array}{c}
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\end{array} \leftrightarrow \text{trace part, corresponding to a scalar degree of freedom: scalar field } \varphi.
\)

The sum of the physical degrees of freedom carried by the single representations, of course, matches the number of degrees of freedom of the product of two spin-one polarizations:

\[
(D - 2) \times (D - 2) = (D - 2)^2 = \frac{D(D-3)}{2} + \frac{(D-2)(D-3)}{2} + 1. \quad (2.103)
\]

The explicit decomposition of the tensor \( \varepsilon_{\mu\nu} \) into irreducible representations requires a trace only on the LG indices. In order to make it covariant we need an auxiliary vector. Indeed, we can define a matrix which is a \( (D - 2) \times (D - 2) \) identity on the LG indices,
completed with zeros to a covariant $D$-dimensional matrix as follows:

$$\mathbf{1}_{\mu \nu}^{LG} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1}_{(D-2) \times (D-2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \eta_{\mu \nu} + \frac{1}{2} p(\mu q^\nu), \quad (2.104)$$

where $q_\mu$ is an auxiliary vector which, in the frame where $p_\mu = E(1, 0, \ldots, 0, 1)$ takes the form $q_\mu = E^{-1}(1, 0, \ldots, 0, -1)$. (With this normalization $p \cdot q = 2$.) With the LG identity defined by (2.104), we can work out the decomposition of the product of two spin-one polarizations:

$$\varepsilon_{\mu \nu} = \varepsilon^\lambda_{\mu \nu} \tilde{\varepsilon}^{\lambda'}_{\mu \nu} = \frac{1}{2} \varepsilon^\lambda_{(\mu} \tilde{\varepsilon}^{\lambda'}_{\nu)} - \frac{\varepsilon^\lambda \cdot \tilde{\varepsilon}^{\lambda'} \eta_{\mu \nu} + \frac{1}{2} p(\mu q^\nu)}{D - 2} + \frac{1}{2} \varepsilon^\lambda_{[\mu} \tilde{\varepsilon}^{\lambda'}_{\nu]} \eta_{\mu \nu} + \frac{1}{2} p(\mu q^\nu), \quad (2.105)$$

As we shall see, in order to implement this decomposition at the field theoretical level, we shall be forced to perform a non-local projection as to extract a gauge-invariant scalar mode. The same decomposition was also used in [11]. Neglecting the contributions containing the auxiliary vector $q$, amounting to an on-shell gauge transformation, we obtain alternative (but equivalent) decomposition:

$$\varepsilon_{\mu \nu} = \varepsilon^\lambda_{\mu \nu} \tilde{\varepsilon}^{\lambda'}_{\mu \nu} = \frac{1}{2} \varepsilon^\lambda_{(\mu} \tilde{\varepsilon}^{\lambda'}_{\nu)} - \frac{\varepsilon^\lambda \cdot \tilde{\varepsilon}^{\lambda'} \eta_{\mu \nu} + \frac{1}{2} p(\mu q^\nu)}{D - 2} + \frac{1}{2} \varepsilon^\lambda_{[\mu} \tilde{\varepsilon}^{\lambda'}_{\nu]} \eta_{\mu \nu}, \quad (2.106)$$

in which the scalar “polarization” is the part proportional to $\eta_{\mu \nu}$ in the last term, as we would expect in a covariant theory. Moreover, we notice that now the graviton, defined as:

$$G_{\mu \nu} = \frac{1}{2} \varepsilon^\lambda_{(\mu} \tilde{\varepsilon}^{\lambda'}_{\nu)} - \frac{\varepsilon^\lambda \cdot \tilde{\varepsilon}^{\lambda'} \eta_{\mu \nu}}{D - 2}, \quad (2.107)$$

is in de Donder gauge: $p \cdot G_\nu = \frac{1}{2} \eta_{\mu \nu}$. This confirms the fact that the difference from the decomposition in eq. (2.105), in which the graviton is transverse and traceless, and the one in eq. (2.106), is simply an on-shell gauge transformation (i.e. with $p^2 = 0$). This is also discussed in [117] from the vantage point of String Theory, in which case the antisymmetric component is absent.
Although this Thesis will not deal with supersymmetry, it is worth mentioning that in the case of supersymmetric theories the particle content of the DC is richer and can be varied accordingly to the choice of the two copies of super-YM theories which participate to the DC. In the case of no supersymmetry, the possibility of choosing two different YM theories reduces to the choice of two different gauge groups; however, since the only information on the gauge group in pure YM amplitudes comes from the structure constants, which are removed when performing the DC, the choice of the gauge group does not affect the result\(^3\). On the contrary, in the case of supersymmetric theories, there is the possibility to choose two super-Yang-Mills (SYM) theories with different amount of supersymmetry, giving rise to the various supergravities (SUGRA). In particular, the relation between the number of supersymmetry of the various theories (see e.g. [118]) is:

\[
[N_L \text{ SYM}] \otimes [N_R \text{ SYM}] \rightarrow [N = N_L + N_R \text{ SUGRA}],
\]

(2.108)

### 2.4.2 The double copy for three-particle amplitudes

We now want to make the content of the DC explicit in the simplest possible case: the tree-level amplitude with three particles. In this case, the prescription of eq. (2.98) is simply to eliminate the coupling constant and the structure constants from the two YM amplitudes and to multiply them, since at the three-particle level there are neither propagators nor any CK duality to satisfy. We label the particles of the first gauge theory with indices \(i = 1, 2, 3\) and the particles of the second with barred indices \(\bar{i} = \bar{1}, \bar{2}, \bar{3}\). We stress that \(p_i = p_{\bar{i}} \ \forall i\), while in general \(\varepsilon_i\) and \(\varepsilon_{\bar{i}}\) carry different polarizations, where we use a notation such that \(\varepsilon_{\bar{i}} := \tilde{\varepsilon}\).

Suppressing from the amplitude in eq. (2.10) the factors of \(g\) and \(f^{abc}\), we get:

\[
\mathcal{A}_3(1, 2, 3)\tilde{\mathcal{A}}_3(\bar{1}, \bar{2}, \bar{3}) \sim 4\{(\varepsilon_1\varepsilon_2)(\varepsilon_3 p_2) + (\varepsilon_2\varepsilon_3)(\varepsilon_1 p_3) + (\varepsilon_3\varepsilon_1)(\varepsilon_2 p_1)\}
\]

\[
\times \{(\tilde{\varepsilon}_1\tilde{\varepsilon}_2)(\tilde{\varepsilon}_3 p_2) + (\tilde{\varepsilon}_2\tilde{\varepsilon}_3)(\tilde{\varepsilon}_1 p_3) + (\tilde{\varepsilon}_3\tilde{\varepsilon}_1)(\tilde{\varepsilon}_2 p_1)\}
\]

\[
= 4\{(\varepsilon_1\varepsilon_2)(\varepsilon_3 p_2)(\tilde{\varepsilon}_1\tilde{\varepsilon}_2)(\tilde{\varepsilon}_3 p_2) + (\varepsilon_1\varepsilon_2)(\varepsilon_3 p_2)(\tilde{\varepsilon}_2\tilde{\varepsilon}_3)(\tilde{\varepsilon}_1 p_3) + (\varepsilon_1\varepsilon_2)(\varepsilon_3 p_2)(\tilde{\varepsilon}_3\tilde{\varepsilon}_1)(\tilde{\varepsilon}_2 p_1)\}
\]

\(^3\)In this sense, the DC relations display the somewhat unsatisfactory feature of "total color blindness", as one can apparently build the same gravitational amplitudes from YM theories with different color structures.
Chapter 2. Scattering amplitudes and the double copy

\[ + (\varepsilon_2\varepsilon_3)(\varepsilon_1p_3)(\tilde{\varepsilon}_1\tilde{\varepsilon}_2)(\tilde{\varepsilon}_3p_2) + (\varepsilon_2\varepsilon_3)(\varepsilon_1p_3)(\tilde{\varepsilon}_2\tilde{\varepsilon}_3)(\tilde{\varepsilon}_1p_2) + (\varepsilon_2\varepsilon_3)(\varepsilon_1p_3)(\tilde{\varepsilon}_3\tilde{\varepsilon}_1)(\tilde{\varepsilon}_2p_1) \]

\[ + (\varepsilon_3\varepsilon_1)(\varepsilon_2p_1)(\tilde{\varepsilon}_1\tilde{\varepsilon}_2)(\tilde{\varepsilon}_3p_2) + (\varepsilon_3\varepsilon_1)(\varepsilon_2p_1)(\tilde{\varepsilon}_2\tilde{\varepsilon}_3)(\tilde{\varepsilon}_1p_2) + (\varepsilon_3\varepsilon_1)(\varepsilon_2p_1)(\tilde{\varepsilon}_3\tilde{\varepsilon}_1)(\tilde{\varepsilon}_2p_1) \]

\[ = 4\{[(\varepsilon_0^{\alpha\beta}\varepsilon_2^{\theta\sigma}\varepsilon_3^{\mu\nu}p_2^\nu p_2^\sigma) + (123 \rightarrow 231) + (123 \rightarrow 312)] \]

\[ + [(\varepsilon_0^{\alpha\beta}\varepsilon_2^{\theta\sigma}\varepsilon_3^{\mu\nu}p_3^\nu p_3^\sigma) + (123 \rightarrow 231) + (123 \rightarrow 312)] \}

\[ (2.109) \]

As in the previous section, we can define for each particle \( i = 1, 2, 3 \) the product of two polarization vectors \( \varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \varepsilon_i^\nu \) to be split in a symmetric traceless part (graviton \( G_{\mu\nu} \)), an antisymmetric part (two-form \( B_{\mu\nu} \)) and the trace part (scalar). This product of YM amplitudes in principle decomposes in all the possible amplitudes that can consistently mix \( h_{\mu\nu} \), \( B_{\mu\nu} \) and \( \varphi \) and we wish to compute all these contributions. To this end we define:

- \[ \{1\}_{123} = \varepsilon_0^{\alpha\beta}\varepsilon_2^{\theta\sigma}\varepsilon_3^{\mu\nu}p_2^\nu p_2^\sigma \] (2.110)

- \[ \{2\}_{123} = \varepsilon_0^{\alpha\beta}\varepsilon_2^{\theta\sigma}\varepsilon_3^{\mu\nu}p_3^\nu p_3^\sigma + \varepsilon_0^{\alpha\beta}\varepsilon_2^{\theta\sigma}\varepsilon_3^{\mu\nu}p_3^\nu p_3^\sigma \] (2.111)

Our strategy will be to split the polarizations in eq. (2.110) and (2.111), computing their separate contributions to all the possible amplitudes with gravitons (polarizations \( G_{\mu\nu} \)), two-forms (polarizations \( A_{\mu\nu} \)) and scalars (\( S \)), which are collected in table 2.1. Then, as prescribed in eq. (2.109), we add analogous terms but with the two other cyclic permutations of particle indices. The list of amplitudes of table 2.1 distinguishes different particle-orders: in the decomposition of \( \varepsilon_{\mu\nu} \) into \( G_{\mu\nu} \), \( A_{\mu\nu} \) and \( S \) all the possible orderings are met, and they produce different results in (2.110) and (2.111).

<table>
<thead>
<tr>
<th>GGG</th>
<th>GGA</th>
<th>GGS</th>
<th>GAG</th>
<th>GSG</th>
<th>GAA</th>
<th>GAS</th>
<th>GSA</th>
<th>GSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
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<td>AAS</td>
<td>AGA</td>
<td>ASA</td>
<td>AGG</td>
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<td>ASS</td>
</tr>
<tr>
<td>SSS</td>
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<td>SSA</td>
<td>SGS</td>
<td>SAS</td>
<td>SGG</td>
<td>SGA</td>
<td>SAG</td>
<td>SAA</td>
</tr>
</tbody>
</table>

Table 2.1: Possible permutations of external particles in the DC amplitude \( A_3, \tilde{A}_3 \)

separately all the contributions of the type \( 1 \) and \( 2 \) to the amplitudes of table 2.1:

- **AAA, single-A terms:**
  - \( 1 = 0 \) (G, S, \( p_2^\nu p_2^\sigma \) are symm, A is antisymmm)
  - \( 2 = 0 \) (\{ \mu \} \leftrightarrow \{ \nu \}, A or AAA antisymmm)
Now, summing over particle-index even permutations, and taking into account the factor of four in front of the curly bracket in eq. (2.109), we can compute all the non-vanishing amplitudes (taking into account momentum conservation and the properties of G and A):

\[ M_3(GGG) = 4 \left\{ G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_2^\mu p_2^\nu + G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_3^\mu p_3^\nu + G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_1^\mu p_1^\nu \right\} 
+ 2 G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_1^\lambda p_2^\nu + 2 G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_2^\lambda p_3^\mu + 2 G_{\alpha \beta}^{\mu \nu} G_{\alpha \beta}^{\mu \nu} p_3^\lambda p_1^\mu \right\} \cdot p_2^\mu p_3^\nu p_1^\lambda \] \quad (2.112)

\[ M_3(GGS) = (GGS)_{123} + (GSG)_{231} + (SGG)_{312} \]

\[ = 4 \varepsilon^\lambda \varepsilon^\nu \left\{ \frac{2}{D-2} G_{\mu \nu}^{\mu \nu} p_1^\lambda p_2^\mu p_3^\nu + \frac{2}{D-2} G_{\mu \nu}^{\mu \nu} p_2^\mu p_3^\nu p_1^\lambda \right\} = 0. \quad (2.113) \]
\[ = 4(\epsilon^\lambda \cdot \epsilon^{\lambda'})^2 \left\{ \frac{D}{(D-2)^2} G_{\mu \nu}^1 p_3^\mu p_3^\nu + \frac{2}{(D-2)^2} G_{\mu \rho}^2 p_3^\mu p_3^\rho \right\} = \frac{4(\epsilon^\lambda \cdot \epsilon^{\lambda'})^2}{D - 2} G_{\mu \rho}^1 p_3^\mu p_3^\rho. \]  

(2.114)

\[ \mathcal{M}_3(GAA) = (GAA)_{123} + (AAG)_{231} + (AGA)_{312} \]

(2.115)

\[ = 4 \left\{ A_3^{\lambda\beta} A_3^\alpha p_3^\beta p_3^\alpha + 2G_{\mu \nu}^1 A_2^\lambda A_3^{\mu \lambda} p_3^\mu p_3^\nu + 2A_2^{\mu \nu} A_3^\lambda G_{\rho \lambda}^1 p_3^\mu p_3^\rho + 2A_3^{\mu \nu} G_{\rho \lambda}^1 A_2^{\mu \rho} p_3^\nu p_3^\rho \right\}. \]

(2.116)

\[ \mathcal{M}_3(SAA) = (SAA)_{123} + (AAS)_{231} + (ASA)_{312} \]

(2.116)

\[ = 4\varepsilon^\lambda \cdot \varepsilon^{\lambda'} \left\{ \frac{2}{D - 2} A^2_{\mu \nu} A_3^\lambda p_3^\mu p_3^\nu + \frac{2}{(D-2)} A_3^{\mu \nu} A_2^\lambda p_3^\mu p_3^\nu \right\} = \frac{16\varepsilon^\lambda \cdot \varepsilon^{\lambda'}}{D - 2} A^2_{\mu \nu} A_3^\mu p_3^\mu p_3^\nu. \]

As a first comment, let us observe that the scattering amplitude between three gravitons obtained from the product of two YM amplitudes between three gluons exactly matches the result of eq. (2.94), when multiplied by \( \kappa/2 \). In the next section we are going to comment about all the other amplitudes.

### 2.4.3 Three-particle amplitudes in \( \mathcal{N} = 0 \) Supergravity

Although in this Thesis we will not deal with it, a comment about String Theory is in order. In fact, while the DC relations can be shown to hold from QFT arguments, at tree level they are equivalent to the older KLT relations [18], which are derived from String Theory. These are originally relations between scattering amplitudes of open strings (whose low-energy limit contains a spin-one gauge theory) and scattering amplitudes of closed strings (whose low-energy limit contains a graviton). Therefore, it is natural to expect the theory described by the DC to bear some relations to the low-energy limit of the bosonic (in the case of no supersymmetry) closed sector of String Theory, which is exactly a theory of gravity coupled to a two-form field, known as the Kalb-Ramond field, and a scalar field, known as the dilaton. In the DC literature this theory is sometimes referred to as the \( \mathcal{N} = 0 \) Supergravity (see e.g. [119] or [11]), and its action is:

\[ S_{\mathcal{N}=0} = \int d^Dx \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{6} e^{-\frac{2\varphi}{H_{\mu \nu}}} H_{\mu \lambda \nu} H^{\mu \nu \lambda} - \frac{1}{2(D-2)} \partial_{\mu} \varphi \partial^{\mu} \varphi \right\}, \]  

(2.117)

where \( R \) is the Ricci scalar, \( \varphi \) is the dilaton and \( H_{\mu \nu \lambda} \) is the field strength of the two-form field \( B_{\mu \nu} \) \( (H = dB) \). All the contractions of greek indices are performed by the dynamical metric field \( g_{\mu \nu} \). We observe that the scalar normalization is clearly non-canonical:
we should rescale the dilaton by a factor of $\sqrt{D-2}$ in order to achieve the usual $1/2$ factor in front of the kinetic term. However, we will keep the normalization as it is in order to simplify the comparison with the results of the DC: indeed, if we look at the decomposition in eq. (2.105), we see that also in that case the normalization of the scalar “polarization” displays a factor of $\sqrt{D-2}$. Of course if we want to get the physical amplitude we must rescale it by an appropriate factor.

In order to compute the three-particles scattering amplitudes we can expand the fields about the IR vacuum expectation values $\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}$, $\langle B_{\mu\nu} \rangle = 0$, $\langle \phi \rangle = 0$. We neglect the pure gravitational term, since it has already been computed in Section 2.3, while expanding $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ we extract the cubic interactions from the action in eq. (2.117), where now the greek indices are Lorentz indices, contracted with $\eta_{\mu\nu}$:

$$S_{N=0}^{(3)} = S_{EH}^{(3)} + \kappa \int d^Dx \left\{ -\frac{1}{2(D-2)}(h\eta^{\mu\nu} - 2h^{\mu\nu})\partial_\mu \phi \partial_\nu \phi \\ + \frac{2}{D-2} \phi (\partial_\mu B_{\nu\lambda} \partial^\mu B^{\nu\lambda} + 2\partial_\mu B_{\nu\lambda} \partial^\nu B^{\lambda\mu}) \\ - \frac{1}{6} (h\eta^{\mu\alpha} \eta^{\nu\beta} \eta^{\lambda\gamma} - 2h^{\mu\alpha} \eta^{\nu\beta} \eta^{\lambda\gamma} - 2\eta^{\mu\alpha} h^{\nu\beta} \eta^{\lambda\gamma} - 2\eta^{\mu\alpha} \eta^{\nu\beta} h^{\lambda\gamma}) \\ \times (\partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\nu\mu} + \partial_\nu B_{\lambda\mu})(\partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}) \right\}. \quad (2.118)$$

From the cubic action we can compute all the non-vanishing tree-level scattering amplitudes (taking into account only the TT part of the vertices):

- \[ S_{TT}^{(3)}[h] = \kappa \int d^Dx \left\{ h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} h_{\alpha\beta} + 2h^{\mu\nu} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} \right\}, \text{ from which:} \]

$$M_3(GGG) = 2i\kappa \left\{ G_{\alpha\beta}^{G_1} G_{\nu\lambda}^{G_2} G_{\mu\sigma}^{G_3} p_\mu p_\nu p_\sigma + G_{\alpha\beta}^{G_2} G_{\nu\lambda}^{G_3} G_{\mu\sigma}^{G_1} p_\mu p_\nu p_\sigma + G_{\alpha\beta}^{G_3} G_{\nu\lambda}^{G_1} G_{\mu\sigma}^{G_2} p_\mu p_\nu p_\sigma + 2G^{\mu\nu} G_{\mu\sigma}^{G_1} G_{\nu\lambda}^{G_2} G_{\mu\nu}^{G_3} p_\mu p_\nu p_\sigma + 2G^{\mu\nu} G_{\mu\sigma}^{G_3} G_{\nu\lambda}^{G_1} G_{\mu\nu}^{G_2} p_\mu p_\nu p_\sigma \right\}. \quad (2.119)$$

- \[ S_{TT}^{(3)}[\phi, h] = \frac{\kappa}{D-2} \int d^Dx h_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi, \text{ from which:} \]

$$M_3(GSS) = \frac{2i\kappa}{D-2} G_{\mu\nu}^{G_1} G_{\nu\lambda}^{G_2} G_{\mu\sigma}^{G_3} p_\mu p_\nu p_\sigma. \quad (2.120)$$
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\[ S_{\mathcal{A}}^{(3)}(\phi, B) = \frac{2\kappa}{D-2} \int d^Dx \phi_B(x) \left( \partial_\mu B_{\nu\lambda} \partial^\mu B^{\nu\lambda} + 2 \partial_\mu B_{\nu\lambda} \partial^\nu B^{\lambda\mu} \right), \]

from which:

\[ \mathcal{M}_3(SAA) = \frac{8i\kappa}{D-2} A_{\mu\nu} A_{\alpha}^{\mu} p_{\beta}^{\nu} p_{\gamma}^{\alpha}. \] (2.121)

\[ \mathcal{M}_3(GAA) = 2i\kappa (A_{\alpha\beta} A_{\lambda}^{\alpha\beta} G_{\mu\nu} p_{2\rho}^{\mu} p_{2\sigma}^{\nu} + 2G_{\mu\nu} A_{\lambda}^{\mu\lambda} A_{\rho}^{\nu\lambda} p_{3\rho}^{\mu} p_{3\sigma}^{\nu} + 2G_{\mu\nu} A_{\lambda}^{\mu\lambda} A_{\rho}^{\nu\lambda} p_{3\rho}^{\mu} p_{3\sigma}^{\nu} + 2A_{\mu\nu} G_{\lambda}^{\nu\lambda} A_{\rho}^{\mu\lambda} p_{3\rho}^{\mu} p_{3\sigma}^{\nu}). \] (2.122)

When comparing these amplitudes with the ones calculated in Section 2.4.2 we notice that, when the latter are multiplied by \( \kappa/2 \) and \( \varepsilon^\lambda \cdot \tilde{\varepsilon}^{\lambda'} = 1 \), they are all equal, including the amplitudes which are identically zero. Let us stress that, while the cubic self-interacting gravitons vertex is fixed by the gauge invariance of the free theory, it is not at all obvious why the vertices involving the scalar fields should comply with the couplings of the \( \mathcal{N} = 0 \) Supergravity. In this sense it is a non-trivial result to observe that \( \mathcal{N} = 0 \) Supergravity is, at least at the level of cubic vertices, the result of the DC.

2.4.4 The double copy at tree level

Having discussed an explicit example, which helps to clarify how is it possible to build gravity amplitudes from gauge theory ones, we now want to clarify the content of the DC relations more in general. In this section, we deal with tree-level amplitudes, for which the DC relations were conjectured in [14]. As already mentioned, the idea is that when a YM tree-level amplitude is written in a CK-dual form:

\[ A_n = g^{n-2} \sum_{i \in \Gamma_3} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}}, \] (2.123)

with \( c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0 \), the replacement of the color factors \( \{ c \} \) with a new set ok kinematic factors coming from a different gauge theory \( \{ \tilde{n} \} \) leads to an amplitude in a theory of gravity:

\[ \mathcal{M}_n = i \left( \frac{\kappa}{2} \right)^{n-2} \sum_{i \in \Gamma_3} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}. \] (2.124)
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Initially it was believed that also the second YM amplitude, with kinematic factors \( \tilde{n} \), should be in a CK-dual form. However, if we perform on one of the two amplitudes (say the second, \( \tilde{A} \), with kinematic factors \( \tilde{n} \)) a generalized gauge transformation (see Section 2.2.3):

\[
\tilde{n}_i \rightarrow \tilde{n}_i' = \tilde{n}_i + \Delta_i : \tilde{A}_n' = \tilde{A}_n \iff \sum_{i \in \Gamma_3} \Delta_i \prod_{\alpha_i} s_{\alpha_i} = 0.
\] (2.125)

The last identity, which is the condition for the amplitude to be invariant, must hold for any gauge group, therefore its validity cannot depend on the specific values of the color factors (and hence of the structure constants), but only on their algebraic properties [16]. Then, if the first amplitude is in a CK-dual form, the kinematic factors \( \{n\} \) satisfy the same algebraic identity of the color factors, therefore the following identity must hold:

\[
\sum_{i \in \Gamma_3} \frac{\Delta_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0.
\] (2.126)

This indicates that it is sufficient that one of the two YM amplitudes is in a CK dual form, whereas the other does not have to:

\[
\sum_{i \in \Gamma_3} \frac{n_i \tilde{n}_i'}{\prod_{\alpha_i} s_{\alpha_i}} = \sum_{i \in \Gamma_3} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}} + \sum_{i \in \Gamma_3} \frac{n_i \Delta_i}{\prod_{\alpha_i} s_{\alpha_i}}.
\] (2.127)

This allows a greater freedom in the representation of YM amplitudes, which is useful when the two YM theories are different and CK-dual representations are not known for one of the two. Also, we observe that at tree level, due to the vertex structure of \( \mathcal{N} = 0 \) Supergravity, if the external states are all gravitons then the two-form and the dilaton are not allowed to propagate in internal lines and the result of the DC is pure Einstein gravity. It is therefore simple, at least at tree level, to select the pure gravity sector among the products of the DC. The situation is more complicated at loop level, as we will comment in the following section.

At tree level, the validity of the DC relations was proven inductively in [16], with the technique of complex shifts [13] and strongly relying upon the existence of the CK duality in YM amplitudes.
2.4.5 The double copy at loop level

At loop level, the existence of DC relations, as well as the CK duality, is still a conjecture (first proposed in [17]). The idea is in many ways similar to what happens at tree level: again one exploits the possibility to “blow up” four-gluon vertices into three-gluon ones, so that the amplitude can be written as:

\[ \mathcal{A}_{n}^{L-\text{loop}} = i^{L} g^{n-2+2L} \sum_{i \in \Gamma_{3}} \int \prod_{j=1}^{L} \left( \frac{dL_{j}}{(2\pi)^{D}} \right) \frac{1}{S_{i}} \frac{c_{i}n_{i}}{\prod_{\alpha_{i}, \alpha_{i}}}, \tag{2.128} \]

where \( S_{i} \) is the symmetry factor\(^4\) of the diagram and \( l_{j} \) are the loop momenta. If it is possible to find a CK-dual representation for this amplitude, then the DC proceeds as at tree level, \textit{i.e.} the amplitude:

\[ \mathcal{M}_{n}^{L-\text{loop}} = i^{L+1} \left( \frac{\kappa}{2} \right)^{n-2+2L} \sum_{i \in \Gamma_{3}} \int \prod_{j=1}^{L} \left( \frac{dL_{j}}{(2\pi)^{D}} \right) \frac{1}{S_{i}} \frac{n_{i} \tilde{n}_{i}}{\prod_{\alpha_{i}, \tilde{\alpha}_{i}}}, \tag{2.129} \]

is an amplitude in a theory of gravity, with \( \{ \tilde{n} \} \) a second copy of kinematic factors. At the loop-level, however, even in the case of purely graviton amplitudes the two-form and the scalar are allowed to propagate as internal states in the loops. Therefore, in this case, if one is interested in a scattering amplitude in pure gravity, it is necessary to employ projectors which select only gravitons in the internal lines [17]. Alternatively, in [61] ghosts states have been introduced in order to cancel the contributions of the dilaton and the two-form from loop amplitudes, obtaining pure gravity.

An important point to stress is that, as evident from eqs. (2.128) and (2.129), the DC relations at loop level relate the formal loop integrands, rather than the amplitudes (which are the result of the loop integral). This is crucial to account for the very different UV behavior of YM theory and gravity. For instance, in \( D = 4 \) the former is a renormalizable theory while the latter is not. From the point of view of the DC, this different behaviour in the UV is “explained” by the fact that the second copy of kinematic factors \( \{ \tilde{n} \} \) increases the powers of momenta in the numerator, thus enhancing the degree of divergence of the theory. This corresponds, from a Lagrangian perspective, to gravity having two-derivative interactions, as opposed to the one-derivative cubic vertex of YM.

\(^4\)Number of ways of interchanging internal or external lines without changing the topology of the diagram. The expression of a Feynman diagram is given by the product of the Feynman rules divided by the symmetry factor, in order to prevent overcounting.
The non-existence of a proof for the DC relations at loop level is strictly related to the lack of a proof for the validity of CK duality at loop level. Indeed, by means of the unitarity cuts [120] and assuming the existence of a CK-dual representation of YM amplitudes, formula (2.129) can be easily justified. The unitarity cuts are a powerful tool which allows to relate loop amplitudes to tree-level amplitudes, on account of the unitarity of the $S$–matrix of the theory. Indeed, if we write

$$S = 1 + iT,$$  

(2.130)

with $T$ known as the transfer matrix (it contains the relevant information for the scattering), and impose unitarity we get:

$$SS^\dagger = 1 = 1 + iT - iT^\dagger + TT^\dagger \iff TT^\dagger = -i(T - T^\dagger) = 2\text{Im}(T).$$  

(2.131)

Thus, upon expanding $T$ in powers of the coupling constant, one realizes that being $SS^\dagger = 1$ a quadratic constraint, it relates amplitudes at different order in perturbation theory, such as for instance one-loop amplitudes and tree-level ones. The explicit relations can be found inserting a complete set if intermediate states with the result:

$$2\text{Im}\mathcal{A}(i \to f) = i \sum_X \int d\Pi_X (2\pi)^D\delta^D(p_i - p_X)\mathcal{A}(i \to X)\mathcal{A}^*(f \to X),$$  

(2.132)

where $\mathcal{A}(i \to f)$ is the amplitude of interest, $X$ denotes the intermediate states, $d\Pi_X$ the Lorentz-invariant measure for the phase space of the particle in state $X$ and $\mathcal{A}(i \to X)$, $\mathcal{A}(f \to X)$ the two amplitudes which are obtained by cutting the propagators of $\mathcal{A}(i \to f)$, i.e. putting on shell the momenta of the propagators without violating momentum conservation, with the insertion of $X$ as intermediate states. In [7] and [6] the idea of generalized unitarity was introduced: a technique which allows, relying on appropriate unitarity cuts, to entirely build loop amplitudes from tree-level ones, apart from rational terms. Rational terms are rational functions in the loop amplitudes which does not possess branch cuts, being therefore undetectable using unitarity cuts (indeed, we see from (2.131), that the unitarity method reconstructs the imaginary part of the loop amplitude). Such terms are known to be absent in certain theories (such as SYM), but they are present for example in the case of pure YM theory.
In order to justify the DC formula at loop level [17], we assume the existence of a CK dual representation for loop amplitudes in YM theory and we apply the unitarity method. For the sake of concreteness, we consider a three-loops example [4], in which a cubic diagram contributing to a gravity amplitude is built from tree-level diagrams by means of generalized unitarity (fig. 2.9). In the figure, the uncut propagators are named $1/p_i^2$

\[
\begin{align*}
\text{(Figure 2.9: Unitarity cuts on loop amplitudes [4]).}
\end{align*}
\]

\[
(i = 1, \ldots, 5), \quad n \text{ is the numerator of the three-loop diagram and } n_a, n_b, n_c \text{ are the numerators of the tree-level diagrams in terms of which the loop diagram is built with the unitarity method. If we consider the corresponding gauge theory diagrams, the assumption that the kinematic factors in } n \text{ are in a CK-dual form implies that also } n_a, n_b, n_c \text{ satisfy the duality for the individual tree-level diagrams. Then, we can “square” the tree-level YM amplitudes individually, obtaining tree-level gravity amplitudes (since the DC relations are proven to hold at tree level, as we mentioned): this guarantees that the “square” of the YM loop amplitude also result in a gravity loop amplitude. Despite them being not cut-constructible, formula (2.129) also correctly reproduces the rational terms in the amplitude: this is due to the fact that rational terms are obtained calculating the loop integral in higher dimensions, using the extra-dimensional momenta as regulators. However, the DC relations are valid in arbitrary spacetime dimensions: since formula (2.129) reproduces the correct cuts in any } D, \text{ it must also reproduce correctly the rational terms.}
\]

More recently, the difficulty in finding CK-dual representations of YM amplitudes at loop level has led to the development of generalized DC constructions [54, 55], which compensate the violation of BCJ relations in YM amplitudes when building gravity amplitudes in cases where CK-dual kinematic factors are have not been found (though the CK duality is believed to hold).
2.4.6 The zeroth copy

It was also noticed in [121] that, starting from eq. (2.73) another replacement is possible, which goes in opposite direction with respect to the double copy. Indeed, we can eliminate the kinematic factors \( \{n\} \) in favour of another copy of color factors \( \{\tilde{c}\} \), coming from a YM theory with a possibly different gauge group. Inserting an appropriate coupling constant \( y \), the new amplitude reads:

\[
\mathcal{T}_n^{L-\text{loop}} = i^L y^{n-2+2L} \sum_{i \in \Gamma_3} \int \left( \prod_{k=1}^L \frac{d^D l_k}{(2\pi)^D} \right) \frac{1}{S_t} \prod_{\alpha_i} c_i \tilde{c}_i,
\]

(2.133)

with the usual notation.

This is a scattering amplitude in a theory of scalar fields \( \Phi^{aa'} \), transforming in the adjoint representation of two (possibly different) Lie algebras. The Lagrangian of this theory, which correctly reproduces the scattering amplitudes of eq. (2.133), is:

\[
\mathcal{L}_\Phi = -\frac{1}{2} \partial_\mu \Phi^{aa'} \partial^\mu \Phi^{aa'} - \frac{y}{3} f^{abc} f^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'},
\]

(2.134)

where unprimed indices belong to one of the two Lie algebras and primed indices to the other one.
Since the discovery of the DC relations, many endeavors have been made to extend their scope beyond the realm of perturbative scattering amplitudes, pursuing the goal of better understanding their origin and their breadth. In this chapter, we will review the main approaches pursued in the attempt to generalize such relations, at the (semi-)classical level.

First, we will discuss the definition of the “product” between two YM fields given in [58], whose main advantage is to correctly reproduce the linearized diffeomorphisms and two-form gauge, starting from the abelianized version of the YM gauge symmetry. With this definition, in [59] the issue of reproducing the linearized equations of motion for the fields of $N = 0$ Supergravity was also addressed, both in presence and in absence of sources for the fields. We shall elaborate the results of [59] from a slightly different perspective, proposing in particular an alternative derivation of the equations of motion.

The second approach that we review was first illustrated in [16]. It amounts to the construction of a YM Lagrangian which, order by order in the number of external gluons, produces Feynman rules which automatically encode the CK duality for tree-level scattering amplitudes, by means of the introduction of appropriate auxiliary fields. From such Lagrangian, at fixed gauge, a “square” of the YM vertices is defined, building a gravity Lagrangian which reproduces the correct scattering amplitudes at tree level, in agreement with the DC relations.

Last, for the sake of completeness, we will review some results obtained in the ap-
plication of the DC approach to classical solutions in YM theories and gravity. Since
this latter lies a bit outside the scope of this Thesis we shall limit ourselves to illustrate
the first work on the topic [19], which already contains some of the key features of the
classical DC.

3.1 Relating Yang-Mills and gravitational symmetries

In this section we will review the off-shell definition of the product between YM
fields introduced in [58], showing how it allows to reproduce the linearized local symmetries
of a theory containing a graviton, a two form and a scalar field from the abelianized YM
symmetries, while also leading to a definition of the gravitational curvature tensors built
out of two spin-one abelianized field strength.

Let us stress here that, while [58] deals with the supersymmetric versions of YM the-
tory and gravity, in this Thesis we will be mainly concerned with the non-supersymmetric
case only. Barring this aspect, the results of [58] provide the basis on which our work is
built upon.

Last, following [59], we review the possibility of reproducing the linearized equations
of motion for the gravitational fields, both in presence and in absence of sources.

3.1.1 Product of two Yang-Mills fields

Inspired both by String Theory, where the massless states of the closed string arise from
tensor products of the massless states of the open string [122], and by the DC relations,
it is natural to formulate the idea that gravity fields may be defined as “products” of YM
fields. However, it was only in [58] that a precise definition of the product was given, in
the following schematic form:

\[
H_{\mu\nu}(x) := [A^a_{\mu} \circ \Phi^{-1}_{aa'} \circ \tilde{A}^{a'}_{\nu}](x) := [A_{\mu} \star \tilde{A}_{\nu}](x), \tag{3.1}
\]

where \( H_{\mu\nu} \) contains, as we will explain, the gravity fields, \( \circ \) is a convolution product
and \( A^a_{\mu}, \tilde{A}^{a'}_{\nu} \) are two YM fields, which we will also denote as the left (L) and the right (R)
factors respectively, transforming in the adjoint representation of two (possibly differ-
tent) gauge groups \( G_L, G_R \). Furthermore, the field \( \Phi^{-1}_{aa'} \) is a scalar field with two color
indices belonging to the groups \( G_L \) and \( G_R \), transforming in the adjoint representation.
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of both. Several comments are in order, to explain the rationale behind this definition.

First, we want to motivate the introduction of the convolution product $\circ$ and to recall some of its basic properties. Both in the case of String Theory and of the DC constructions, the result that gravity states are actual products of spin-one states arises in momentum space (as in the decomposition of polarizations of Section 2.4.1), therefore it is somewhat natural to think of a convolution in coordinate space when attempting to implement this product at the level of off-shell fields. We remind that the convolution product is defined as:

$$[f \circ g](x) := \int d^Dy f(y)g(x - y),$$  \hspace{1cm} (3.2)

and satisfies the following properties:

- $(f \circ g) \circ h = f \circ (g \circ h)$ (associativity), \hspace{1cm} (3.3)
- $f \circ g = g \circ f$ (commutativity), \hspace{1cm} (3.4)
- $(af + bg) \circ h = a(f \circ h) + b(f \circ g)$, $a, b \in \mathbb{R}$ (linearity), \hspace{1cm} (3.5)
- $\partial_{\mu}(f \circ g) = (\partial_{\mu}f) \circ g = f \circ (\partial_{\mu}g)$ (differentiation rule), \hspace{1cm} (3.6)
- $\mathcal{F}\{f \circ g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$ (convolution theorem), \hspace{1cm} (3.7)

where, as already mentioned, $\mathcal{F}\{f\}$ denotes the Fourier transform of $f$.

Secondly, we want to explain and justify the presence of the scalar field $\Phi^{-1}_{aa'}$. Its role is to saturate the color indices of the YM fields, so as to define a gravity field $H_{\mu\nu}$ which does not contain color indices. This scalar field will be always present in the product of two fields, or more generally functions, with color indices, in the sense of eq. (3.1). Therefore, apart from cases in which we will need its transformation properties to be manifest, we will not write it explicitly and rather denote the operation “$\circ \Phi^{-1}_{aa'} \circ$” with the symbol “$\ast$”, as in eq. (3.1). Moreover, following [59], the field was introduced via its inverse, since it enters as the convolution inverse of the biadjoint scalar field $\Phi_{aa'}$ introduced in Section 2.4.6, in the sense that:

$$[\Phi^{-1}_{aa'} \circ \Phi_{bb'}](x) = \delta_{ab}\delta_{a'b'}\delta^{(D)}(x),$$  \hspace{1cm} (3.8)

with $\delta^{(D)}(x)$ a $D$-dimensional Dirac delta function. This has been motivated in two ways
in [59]. On the one hand, as explained in Section 2.4.6, the biadjoint scalar field is in a sense an output of the DC: the scattering amplitudes for a theory with cubic interactions among biadjoint scalars is indeed the zeroth copy of two YM theories [121]. Therefore, the actual biadjoint scalar should appear on the side of the “results” of the DC and thus, when we write it together with the YM fields its inverse should appear. On the other hand, one has to account for the mass dimensions of the fields involved in eq. (3.2). Indeed, both the gravity field $H_{\mu\nu}$ (which, as we will see shortly, contains the graviton, the two-form and the scalar) and the YM fields should have mass dimension $D - 2$, compatibly with the mass dimensions introduced by the convolutions. This is because the measure $d^D x$ of the integrals performed for the convolution product brings a dimension $[d^D x] = -D$, therefore, for instance, the convolution of two fields of dimension $D - 2$ has dimension $2 \frac{D-2}{2} - D = -2$. Thus, if the biadjoint field had dimension $D - 2$, as expected from a scalar field, it would produce a dimension for $H_{\mu\nu}$ such that $[H_{\mu\nu}] = -\frac{D+6}{2}$, which is not correct. However, if we define $\Phi^{-1}_{aa'}$ as the convolution inverse of a scalar field with mass dimension one, we have $[\Phi^{-1}_{aa'}] = 3D + 2$ (since $[\delta(D)(x)] = D$) and, correctly, $[H_{\mu\nu}] = \frac{D-2}{2}$ (from eq. (3.1)).

3.1.2 Field content

We would now like to covariantly extract, from eq. (3.1), the actual physical fields. In Section 2.4.1 we achieved a similar goal, but for on-shell polarization vectors; in this section we are going to perform a completely off-shell decomposition.

Being the graviton and the two-form respectively symmetric and antisymmetric rank-two tensors, the most natural idea might be to define such fields as the symmetric and antisymmetric part of $H_{\mu\nu}$, respectively. This indeed reproduces the correct linearized gauge transformations of a graviton and of a two-form field, as we will illustrate in Section 3.1.3. However, this naive identification would miss two relevant aspects, as we are now going to discuss.

Let us recall that the physical states of massless particles lie in irreducible representations of $SO(D - 2)$, whereas the corresponding covariant fields are representations of $GL(D, \mathbb{R})$. The representations of both groups can be described by means of Young tableaux: in these terms, we can employ the same tableaux to describe a particle and the corresponding covariant field, provided that in the first case (particle) the tableau is
interpreted as a tableau for $SO(D - 2)$, while in the second as a tableau for $GL(D, \mathbb{R})$.

For instance, the physical states of a graviton are described by a symmetric, traceless tensor $h_{ij}$ ($i, j = 1, ..., D - 2$), while the covariant field describing a graviton is a symmetric tensor $h_{\mu\nu}$ ($\mu, \nu = 0, ..., D - 1$), with no constraints on the trace. Both cases are represented by the same Young tableau \[ \begin{array}{c} 
\end{array} \], but interpreted in different ways, as explained. The relevant point for us is that in the covariant construction of the graviton field there is some degree of arbitrariness in the definition of its trace, and this leads to possible mixings with the scalar field of the $\mathcal{N} = 0$ Supergravity multiplet.

Indeed, while the definition of the graviton as the symmetric part of $H_{\mu\nu}$,

$$h_{\mu\nu} := H^S_{\mu\nu} := \frac{1}{2} H_{(\mu\nu)},$$

(3.9)
correctly reproduces the linearized gauge transformation of a spin-two field, as we will show in Section 3.1.3 still, since we are now off shell, we can freely modify its trace provided that we do not modify its transformation rule. Therefore, the graviton can be equivalently defined as

$$h_{\mu\nu} := H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi,$$

(3.10)

where $\gamma \in \mathbb{R}$ is an arbitrary parameter, while $\varphi$ is the (gauge invariant) scalar field. For the above reasons, (3.10) is as good a definition of the covariant field describing the graviton for every real value of $\gamma$.\footnote{In the following we will sometimes pause to comment on what appears to be the most suitable value for $\gamma$ in different situations.}

Whatever choices one eventually makes, they will have an impact on the very definition of the scalar field $\varphi$ in terms of $H_{\mu\nu}$. In order to better frame the issue at stake, it is useful to look back once again to Section 2.4.1, from which it is clear that, the scalar degree of freedom is a singlet of $SO(D - 2)$, which arises from the trace of the product (in momentum space) $\varepsilon_i \bar{\varepsilon}_j$, with $\varepsilon_i$, $\bar{\varepsilon}_j$ physical spin-one polarizations and $i = 1, ..., D - 2$. On the other hand, the obvious candidate to play the role of the physical scalar, that is the trace $H := H^\alpha_\alpha$ of $H_{\mu\nu}$, However, is not off-shell gauge-invariant, thus leading to the need to covariantly compensate its gauge transformation. In particular, since the
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symmetric part of \( H_{\mu\nu} \) is subject to the gauge transformation (see Section 3.1.3):

\[
\delta H_{\mu\nu}^S = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{3.11}
\]

with \( \xi_\mu \) a Lorentz vector, the trace of \( H_{\mu\nu} \) transforms as:

\[
\delta H = 2 \partial \cdot \xi, \tag{3.12}
\]

Hence, in order to give a gauge-invariant definition of the scalar field, we can subtract from \( H \) a term which compensates the variation (3.12). Such term must be unphysical, since \( H \) already contains the physical scalar degree of freedom, and covariant, thus leading to the following definition:

\[
\varphi := H - \frac{\partial \cdot \partial \cdot H}{\Box}, \tag{3.13}
\]

where a non-local projector enters, in the spirit of [123]. We will shortly comment more about the presence of \( \frac{1}{\Box} \), but first let us observe that (3.13) is a good, gauge-invariant off-shell definition of the scalar field\(^2\). Indeed, we can first fix \( \partial \cdot \partial \cdot H = 0 \) with a gauge parameter \( \xi_\mu \) such that \( \Box \xi_\mu \neq 0 \), being left with the possibility of performing residual gauge transformations with \( \Box \xi_\mu = 0 \) and \( \partial \cdot \xi = 0 \), which preserve the gauge-fixing \( \partial \cdot \partial \cdot H = 0 \).

Let us also stress that the definition (3.13) of \( \varphi \) suggests a special value for the parameter \( \gamma \) introduced in (3.10). Indeed, when the field \( H_{\mu\nu} \) is gauge fixed to be transverse with respect to both indices (\( \partial^\alpha H_{\alpha\mu} = \partial^\alpha H_{\mu\alpha} = 0 \)), so that the scalar reduces to \( \varphi = H_{\alpha\alpha} \), then the graviton turns out to be in the harmonic (or de Donder) gauge if \( \gamma = \frac{1}{D-2} \). Indeed, from (3.10):

\[
\begin{cases}
\partial \cdot h_\mu = -\gamma \partial_\mu H, \\
h = H - \gamma D H,
\end{cases} \Rightarrow \partial \cdot h_\mu = \frac{1}{2} \partial_\mu h \leftrightarrow -\gamma = \frac{1}{2} (1 - \gamma D), \tag{3.14}
\]

which is solved precisely by \( \gamma = \frac{1}{D-2} \). Since the transverse gauge for \( H_{\mu\nu} \) is equivalent to the Lorenz gauge condition for both the YM fields involved in the product

\(^2\)This is true at the linearized level; when interactions are included, as in Section 4.4, the definition of the scalar has to be reconsidered.
(\partial \cdot A = \partial \cdot \tilde{A} = 0), we can state that when \( \gamma = \frac{1}{D-2} \) the Lorenz gauge on the YM side implies the de Donder gauge for the graviton. Furthermore, this value of \( \gamma \) makes the definition of the graviton similar to the one given in Section 2.4.1 for the on-shell graviton polarization tensor.

Now, let us comment about the introduction of the inverse d’Alembertian operator in eq. (3.13). First of all, we give a precise definition of what we mean by this inversion, listing some of its properties. We can introduce a Green’s function for the d’Alembert operator \( \Box = \eta^{\mu\nu} \partial_{\mu} \partial_{\nu} \),

\[
\Box_x G(x - y) = \delta^{(D)}(x - y), \tag{3.15}
\]

which allows to find the solution to equations of the type \( \Box f = j \) in the following form

\[
f(x) = [G \circ j](x) + g(x), \tag{3.16}
\]

with \( g(x) \) satisfying \( \Box g = 0 \). The function \( G \) defines the inverse d’Alembertian operator, that we shall interpret as

\[
\frac{1}{\Box}(j)(x) := [G \circ j](x) = \frac{d^D p}{(2\pi)^D} \frac{\mathcal{F}\{j\}(p)}{p^2} e^{-ip \cdot x}, \tag{3.17}
\]

where \( \mathcal{F}\{j\}(p) \) is the Fourier transform of \( j(x) \). In what follows, we will exploit mainly the two following properties of \( \frac{1}{\Box} \) (see [59]):

- \( \frac{1}{\Box}(\Box f) = \Box(\frac{1}{\Box} f) = f \) when \( \Box f \neq 0 \), \tag{3.18}
- \( \frac{1}{\Box}(f \circ g) = (\frac{1}{\Box} f) \circ g = f \circ (\frac{1}{\Box} g) \). \tag{3.19}

At the same time, the very definition of \( \frac{1}{\Box} f \) relies on the fact that the function \( f \) is not in the kernel of the \( \Box \) operator, i.e. \( \Box f \neq 0 \).

Furthermore, let us stress that the presence of non-localities was expected, since the scalar is defined projecting out covariantly the spin-zero part from a rank-two tensor, and such projections require the introduction of non-localities [123], which sometimes are “hidden” by the introduction of auxiliary fields. Let us now show explicitly how this
happens, starting from the spin-one case described by Maxwell’s equations

\[ \Box A_\mu - \partial_\mu \partial \cdot A = 0, \quad (3.20) \]

which are invariant under the gauge transformation

\[ \delta A_\mu = \partial_\mu \Lambda, \quad (3.21) \]

with \( \Lambda \) an arbitrary scalar function. If we covariantly split the field into its transverse and longitudinal component, by putting:

\[ A_\mu = A^T_\mu + \partial_\mu \chi, \quad (3.22) \]

with \( \partial \cdot A^T = 0 \), and we plug it in eq. (3.20), we can observe that the Maxwell’s equations only rule the dynamics of the \( D - 1 \) transverse components \( A^T_\mu \), with the equation of motion:

\[ \Box A^T_\mu = 0, \quad (3.23) \]

while being

\[ \Box (\partial_\mu \chi) - \partial_\mu \partial^\alpha (\partial_\alpha \chi) = 0 \quad (3.24) \]

identically, the longitudinal component \( \chi \) is actually unconstrained by the equations of motion of \( A_\mu \). This is due to the fact that the longitudinal mode is non-physical, and can be eliminated altogether by means of a gauge transformation \( \delta A_\mu = \partial_\mu \Lambda \), simply choosing \( \Lambda = -\chi \). These are very well known facts, but we found useful to recall them in order to make the two following remarks. The first is that, although it is true that the fields describing free massless particles satisfy wave equations of the type \( \Box = 0 \), this is not actually true for all their components: in the Maxwell case, it is not true that the equations imply \( \Box \chi = 0 \). The second remark is that, in the decomposition (3.22), we introduced two auxiliary fields \( A^T_\mu \) and \( \chi \) in order to disentangle the transverse and the longitudinal components of \( A_\mu \). However, if we insist on a description based solely on the field \( A_\mu \), it is possible to derive an expression for such auxiliary fields terms of \( A_\mu \).
but this necessarily requires the introduction of the inverse d’Alembertian operator:

\[ \partial \cdot A = \Box \chi \Rightarrow \chi = \frac{\partial \cdot A}{\Box}. \]  

(3.25)

If we now apply the same arguments to \( H_{\mu\nu}^S \), we can show that eq. (3.13) is a perfectly sensible covariant definition for the scalar field. Indeed, we can decompose the field as follows:

\[ H_{\mu\nu}^S = H_{\mu\nu}^{ST} + \partial_{\mu} \pi_{\nu}^T + \partial_{\nu} \pi_{\mu}^T + \partial_{\mu} \partial_{\nu} \lambda, \]

(3.26)

with \( \partial \cdot H_{\mu\nu}^{ST} = 0 = \partial \cdot \pi^T \) and \( \lambda \) a scalar. Now, the covariant and gauge invariant equations of motion for \( H_{\mu\nu}^S \) (that we will derive in Section 4.1.1), cannot constrain the components \( \pi_{\mu}^T \) and \( \lambda \), which are pure gauge just like the longitudinal modes in the spin-one case. Actually, the only components which must satisfy \( \Box = 0 \) are the fully transverse ones \( (H_{\mu\nu}^{S, T})^3 \). Indeed, from (3.26), it is also possible to check that the trace of \( H_{\mu\nu}^S \) and the trace of \( H_{\mu\nu}^{S, T} \) are different, which is the reason why we could not define the scalar simply as \( \varphi := H \). The difference can be expressed as follows (we omit the index \( S \) since the contraction with \( \eta^{\mu\nu} \) when extracting the trace automatically selects the symmetric part of \( H_{\mu\nu} \)):

\[ H = H^T + \Box \lambda. \]

(3.27)

Then, we can work out an expression for \( \lambda \) in terms of \( H_{\mu\nu} \), in a way which is analogous to eq. (3.25) (omitting \( S \) for reasons similar to above: \( \partial^\mu \partial_\nu \) is symmetric):

\[ \partial \cdot \partial \cdot H = \Box^2 \lambda. \]

(3.28)

Therefore, we can conclude that a covariant expression for the trace of the transverse part of \( H_{\mu\nu}^S \), which comprises the physical modes, can be written entirely in terms of \( H_{\mu\nu}^S \) itself, only upon introducing the inverse d’Alembertian operator:

\[ H^T = H - \frac{\partial \cdot \partial \cdot H}{\Box}. \]

(3.29)

At any rate, let us anticipate that, as we shall discuss in Chapter 4, the theory which

\footnote{Including the trace of \( H_{\mu\nu}^{S, T} \), which contains the propagating scalar.}
we are going to build is completely local in terms of $h_{\mu\nu}$, $B_{\mu\nu}$ and $\varphi$.

To summarize, the physical field content of our DC theory built upon the definition (3.1) is the following:

$$A_\mu \ast \tilde{A}_\nu = H_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu} + \gamma \eta_{\mu\nu} \varphi,$$

(3.30)

where

$$\begin{cases}
h_{\mu\nu} := H_{\mu\nu}^S - \gamma \eta_{\mu\nu} \varphi, \\
B_{\mu\nu} := H_{\mu\nu}^A, \\
\varphi := H - \partial \cdot \partial = H. 
\end{cases}$$

(3.31)

Additional support to these identifications will be given in the next section, where we shall discuss the gauge transformations of these fields.

### 3.1.3 Linearized symmetries

The aim of [58] was to reproduce the linearized symmetries of gravity, starting from the linearized version of YM theory, i.e. the $g \to 0$ limit of a YM theory, where $g$ is the coupling constant. In this limit the theory is equivalent to a set of free spin-one fields $A^a_\mu$, with local transformation rule (see eq. (2.3)):

$$\delta_0 A^a_\mu = \lim_{g \to 0} \left( \partial_\mu \varepsilon^a - gf_{abc} A^c_\mu \right) = \partial_\mu \varepsilon^a,$$

(3.32)

with $\varepsilon^a = \varepsilon^a(x)$ a set of local parameters and where the subscript $0$ in $\delta_0$ refers to the fact that this transformation holds at the linearized level. Also the field strength is “abelianized” in the limit $g \to 0$, indeed from eq. (2.5):

$$\left( F^{\alpha}_{\mu\nu} \right)^{(0)} = \lim_{g \to 0} \left( \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - gf_{abc} A^b_\mu A^c_\nu \right) = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu.$$

(3.33)

In the remainder of this chapter we will often employ eq. (3.32) and (3.33), but in order to avoid cumbersome formulas we will omit the “0”, since we will only deal with the linearized local transformation and field strength.

We also take into account the global transformation rule of the YM fields under the
gauge group, which we write with a set of global (constant) parameters $\vartheta^a$:

$$\delta_\vartheta A^a_\mu = f^{abc} A^b_\mu \vartheta^c,$$  \hspace{1cm} (3.34)

since the YM connection transforms in the adjoint representation, as already mentioned. Therefore, keeping into account both the local and the global transformation, under the gauge group:

$$\begin{align*}
\delta A^a_\mu &= \partial_\mu \varepsilon^a + f^{abc} A^b_\mu \vartheta^c, \\
\delta \tilde{A}^a_\mu &= \partial_\mu \tilde{\varepsilon}^a + f^{a'b'c'} \tilde{A}^{b'}_\mu \tilde{\vartheta}^{c'}.
\end{align*}$$  \hspace{1cm} (3.35)

The field $\Phi_{-1}^{a\alpha}$ is chosen to transform only under global transformations, and as already explained it transforms under the adjoint representations of the two groups $G_L$ and $G_R$:

$$\delta \Phi_{-1}^{a\alpha} = f^{abc} \Phi_{-1}^{b\beta} \vartheta^c + f^{a'b'c'} \Phi_{-1}^{ab'} \tilde{\vartheta}^{c'}.$$  \hspace{1cm} (3.36)

Given these transformations for the fields, exploiting the properties of the convolution (in particular eq. (3.6)) and the antisymmetry of the structure constants we find:

$$\begin{align*}
\delta H^{S}_{\mu\nu} &= (\partial_\mu \varepsilon^a + f^{abc} A^b_\mu \vartheta^c) \circ \Phi_{-1}^{a\alpha} \circ \tilde{A}^{\alpha'}_{\nu} \\
&\quad + A^a_\mu \circ (f^{abc} \Phi_{-1}^{b\beta} \vartheta^c + f^{a'b'c'} \Phi_{-1}^{ab'} \tilde{\vartheta}^{c'}) \circ \tilde{A}^a_{\nu} + A^a_\mu \circ \Phi_{-1}^{a\alpha} \circ (\partial_\nu \tilde{\varepsilon}^{\alpha'} + f^{a'b'c'} \tilde{A}^{b'}_{\nu} \tilde{\vartheta}^{c'}) \\
&= \partial_\mu \alpha_\nu + \partial_\nu \tilde{\alpha}_\mu,
\end{align*}$$  \hspace{1cm} (3.37)

where we have defined:

$$\begin{align*}
\alpha_\mu &:= \varepsilon^a \circ \Phi_{-1}^{a\alpha} \circ \tilde{A}^a_{\mu} = \varepsilon^a \star \tilde{A}^a_{\mu}, \\
\tilde{\alpha}_\mu &:= A^a_\mu \circ \Phi_{-1}^{a\alpha} \circ \tilde{\varepsilon}^{\alpha'} = A^a_\mu \star \tilde{\varepsilon}^{\alpha'},
\end{align*}$$  \hspace{1cm} (3.38)

which will be the gauge parameters of the DC theory\(^4\). Splitting the result of eq. (3.37) into its symmetric ($H^{S}_{\mu\nu} := \frac{1}{2} H_{(\mu\nu)}$) and antisymmetric ($H^{A}_{\mu\nu} := \frac{1}{2} H_{[\mu\nu]}$) part:

$$\begin{align*}
\delta H^{S}_{\mu\nu} &= \frac{1}{2} \partial_\mu (\alpha_\nu + \tilde{\alpha}_\nu) + \frac{1}{2} \partial_\nu (\alpha_\mu + \tilde{\alpha}_\mu) := \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\
\delta H^{A}_{\mu\nu} &= \frac{1}{2} \partial_\mu (\alpha_\nu - \tilde{\alpha}_\nu) - \frac{1}{2} \partial_\nu (\alpha_\mu - \tilde{\alpha}_\mu) := \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu,
\end{align*}$$  \hspace{1cm} (3.39)
where we have introduced new gauge parameters, built from the ones of eq. (3.38):

\[
\begin{align*}
\xi_\mu &:= \frac{1}{2}(\alpha_\mu + \tilde{\alpha}_\mu), \\
\Lambda_\mu &:= \frac{1}{2}(\alpha_\mu - \tilde{\alpha}_\mu).
\end{align*}
\] (3.40)

The key observation of [58] is that the two equations in the system (3.39) correspond respectively to the linearized version of the gauge transformation of a graviton and to the gauge transformation of a two-form field. However, as we discussed at length in the previous section, the introduction of a gauge invariant scalar field, together with the fact that an off-shell graviton has no constraint on its trace, introduces an ambiguity in the definition of the graviton. From eq. (3.39) we can check that, as anticipated, the definition (3.13) of \( \varphi \) is gauge-invariant, therefore we can complete the definition of the fields in eq. (3.31) by including their gauge transformations:

\[
\begin{align*}
&h_{\mu\nu} := H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi \to \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\
&B_{\mu\nu} := H^A_{\mu\nu} \to \delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \\
&\varphi := H - \frac{\partial \varphi}{\Box} \to \delta \varphi = 0.
\end{align*}
\] (3.41)

### 3.1.4 Field strength and Riemann tensor

In [58], a linearized field strength for \( H_{\mu\nu} \) was also defined, built from the “abelianized” field strengths (defined in eq. (2.5)) of the two YM fields:

\[
R^*_{\mu\nu\rho\sigma} := -\frac{1}{2} F^a_{\mu\nu} \circ \Phi^{-1} \circ \tilde{F}^a_{\rho\sigma} = -\frac{1}{2} F_{\mu\nu} \star \tilde{F}_{\rho\sigma}.
\] (3.42)

\( R^*_{\mu\nu\rho\sigma} \) is clearly gauge invariant, since it is built from two gauge invariant field strength tensors. If we expand (3.42) in terms of \( H_{\mu\nu} = H^S_{\mu\nu} + H^A_{\mu\nu} = H^S_{\mu\nu} + B_{\mu\nu} \) we get the expression:

\[
\begin{align*}
R_{\mu\nu\rho\sigma} &= \frac{1}{2} \left\{ \partial_\rho \partial_\sigma H^S_{\mu\nu} + \partial_\mu \partial_\sigma H^S_{\rho\sigma} - \partial_\mu \partial_\nu H^S_{\rho\sigma} - \partial_\nu \partial_\rho H^S_{\mu\sigma} \right\} \\
&+ \frac{1}{2} \left\{ \partial_\rho \partial_\sigma B_{\mu\nu} + \partial_\mu \partial_\sigma B_{\rho\sigma} - \partial_\mu \partial_\nu B_{\rho\sigma} - \partial_\nu \partial_\rho B_{\mu\sigma} \right\},
\end{align*}
\] (3.43)

In the first curly bracket, we can recognize the linearized Riemann tensor of a manifold with Levi-Civita connection and a metric \( g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu} \). The second curly bracket, instead, contains the field strength of the two-form, amenable to be identified as a con-
tribution to the Riemann tensor coming from the torsion part of the connection. Indeed a Kalb-Ramond two-form minimally coupled to gravity contributes to the geometry of the spacetime manifold with an effective torsion, with totally antisymmetric torsion tensor given by the field strength of the two-form field. We consider the case in which the torsion tensor is given by the field strength \( H_{\mu\alpha\beta} \) of the Kalb-Ramond two-form \( B_{\mu\nu} \):

\[
-T_{\mu\alpha\beta} = H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} + \partial_\alpha B_{\mu\beta} + \partial_\beta B_{\mu\alpha}.
\] (3.45)

Even in the presence of torsion, the definition of the Riemann tensor in terms of the connection is unaltered, with the only caveat that the connection is no longer symmetric in its lower indices:

\[
\hat{R}^\rho_{\sigma\mu\nu} = \partial_\mu \hat{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \hat{\Gamma}^\rho_{\mu\sigma} + \hat{\Gamma}^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \hat{\Gamma}^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma},
\] (3.46)

as in eq. (2.75). For our purposes, it is useful to split the connection into a torsion-free part \( \hat{\Gamma} \), symmetric in its lower indices, and a pure torsion part:

\[
\Gamma^\mu_{\alpha\beta} = \hat{\Gamma}^\mu_{\alpha\beta} + \frac{1}{2} T^\mu_{\alpha\beta}.
\] (3.47)

Now, \( \hat{\Gamma}^\mu_{\alpha\beta} \) is the usual Levi-Civita connection, determined by the metric as in Section 2.3, and exploiting the full antisymmetry of the torsion we can also split the Riemann tensor in a purely metric part \( \hat{R}(\hat{\Gamma}) \) and a part which contains the torsion:

\[
R^\rho_{\sigma\mu\nu} = \hat{R}^\rho_{\sigma\mu\nu} + \frac{1}{2} D_\mu T^\rho_{\nu\sigma} + \frac{1}{4} T^\rho_{\mu\lambda} T^\lambda_{\nu\sigma} + \frac{1}{2} T^\rho_{\lambda\sigma} T^\lambda_{\mu\nu},
\] (3.48)

where \( D_\mu \) is the covariant derivative built with the Levi-Civita connection \( \hat{\Gamma} \). Expanding \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and linearizing eq. (3.48) on a flat background, we obtain:

\[
R_{\rho\sigma\mu\nu} = \frac{1}{2} \left\{ \partial_\mu \partial_\sigma h_{\rho\nu} + \partial_\rho \partial_\sigma h_{\mu\nu} - \partial_\mu \partial_\rho h_{\sigma\nu} - \partial_\nu \partial_\sigma h_{\rho\mu} \right\} + \frac{1}{2} \left\{ \partial_\mu \partial_\sigma B_{\rho\nu} + \partial_\rho \partial_\sigma B_{\mu\nu} - \partial_\mu \partial_\rho B_{\sigma\nu} - \partial_\nu \partial_\sigma B_{\rho\mu} \right\},
\] (3.49)

5We recall that the torsion tensor is defined as the antisymmetric part of the connection in its lower indices (when expressed in a coordinate basis):

\[
T^\mu_{\alpha\beta} := \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} = \Gamma^\mu_{[\alpha\beta]},
\] (3.44)

where \( T \) is the torsion tensor and \( \Gamma \) is the connection.
which exactly matches eq. (3.43) if we identify \( H^S_{\mu\nu} = h_{\mu\nu} \). Therefore, we can interpret the field strength of \( H_{\mu\nu} \) given in eq. (3.42) as the linearized Riemann tensor of a manifold with metric \( g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu} \) and connection with torsion \( T_{\mu\nu\lambda} = -H_{\mu\nu\lambda} \). This seems to indicate that the correct definition of the graviton is \( h_{\mu\nu} = H^S_{\mu\nu} \) (therefore, \( \gamma = 0 \) in eq. (3.31)) and (3.30), since the above reasoning shows that we can interpret \( H^S_{\mu\nu} \) as the metric fluctuation over flat spacetime.

Also, we observe that from eq. (3.48) we can calculate the Ricci scalar:

\[
R = \hat{R} - \frac{3}{4} T^\lambda_{\mu\nu} T_{\lambda\mu\nu},
\]

from which we can build a Einstein-Hilbert like action (recalling the identification (3.45)):

\[
S = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} \left\{ \hat{R} - \frac{3}{4} H^\lambda_{\mu\nu} H_{\lambda\mu\nu} \right\},
\]

which, apart from the normalization of the two-form (which can be made canonical with a field redefinition), is exactly the action of a two-form field with field strength \( H_{\mu\nu\lambda} \) minimally coupled to gravity.

### 3.1.5 Free equations of motion

In the present and in the following section we address the question of reproducing the linearized equations of motion of the \( \mathcal{N} = 0 \) Supergravity, given the linearized YM equations and the definition (3.1). Actually, we proceed as follows:

- We shall assume the free equations of motion for the fields in the \( \mathcal{N} = 0 \) Supergravity multiplet,

- We shall rewrite them in terms of the component gauge fields in order to check whether they are automatically satisfied when the gauge fields are on shell or, if not, what are the additional conditions to be imposed on \( A^a_{\mu} \) and \( \tilde{A}^a'_{\mu} \).

This section deals with the free equations of motion (no sources), while in the following we will study the sourced equations. The same equations have been explored in [59], although in a slightly different manner with respect to the one that we follow here. From now on we will not write \( \Phi^{-1}_{\mu a' \mu} \) explicitly anymore in the convolutions, rather we
shall systematically adopt the notation introduced in Section 3.1.1:

\[
[X \star Y] := \left[ X^a \circ \Phi_{aa'}^{-1} \circ Y^{a'} \right](x) \tag{3.52}
\]

We begin our analysis with the linearized Einstein-Hilbert equations, which in absence of sources read:

\[
R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = 0, \tag{3.53}
\]

with \( R_{\mu\nu} \) and \( R \) being the linearized Ricci tensor and Ricci scalar, respectively. We express the linearized Ricci tensor \( R_{\mu\nu} \) in terms of YM fields, exploiting the definition of the graviton given in eq. (3.31) and keeping \( \gamma \) arbitrary for the time being.

Since \( R_{\mu\nu} \) is gauge invariant at the linearized level, we try as much as possible to implement its DC expression in terms of YM field strengths, although, as we shall see, other types of contributions appear. Moreover, when ambiguities appear due to the fact that derivatives can act equivalently on both fields involved in the \( \star \)-product, we choose to keep the expression symmetric in the left (\( A \)) and right (\( \tilde{A} \)) fields. Thus, exploiting the definitions:

\[
h_{\mu\nu} = H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi, \tag{3.54}
\]

\[
H^S_{\mu\nu} = \frac{1}{2} A_{(\mu} \star \tilde{A}_{\nu)}, \tag{3.55}
\]

\[
\varphi = \left( A^\alpha \star \tilde{A}_\alpha - \frac{(\partial \cdot A) \star (\partial \cdot \tilde{A})}{\Box} \right), \tag{3.56}
\]

it is possible to derive the following form for the linearized Ricci tensor:

\[
R_{\mu\nu} = \frac{1}{2} \left\{ \partial_\mu \partial_\nu h_{\alpha\beta} + \partial_{(\alpha} \partial_{\nu)} h_{\mu\beta} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h \right\}
\]

\[
= \frac{1}{4} \left\{ (\partial_\mu A_{\nu} + \partial_\nu A_{\mu}) \star (\partial \cdot \tilde{A}) + (\partial \cdot A) \star (\partial_\mu \tilde{A}_{\nu} + \partial_\nu \tilde{A}_{\mu}) \\
- \Box (A_{\mu} \star \tilde{A}_{\nu} + A_{\nu} \star \tilde{A}_{\mu}) - 2 \partial_\mu \partial_\nu A_{\alpha} \star \tilde{A}^\alpha + \frac{1}{2} \gamma (D - 2)(\partial_{(\mu} A_{\nu)} + \eta_{\mu\nu} \Box) \varphi \\
= \frac{1}{4} \left\{ (-\Box A_{\mu} + \partial_\mu \partial \cdot A) \star \tilde{A}_\nu + A_{\nu} \star (-\Box \tilde{A}_\mu + \partial_\mu \partial \cdot \tilde{A}) \\
+ A^\alpha \star \partial_\nu (\partial_\alpha A_{\mu} - \partial_\mu A_{\alpha}) + \partial_\nu (\partial_\alpha A_{\mu} - \partial_\mu A_{\alpha}) \star \tilde{A}^\alpha + \frac{1}{2} \gamma (D - 2)(\partial_{(\mu} A_{\nu)} + \eta_{\mu\nu} \Box) \varphi \\
= \frac{1}{4} \left\{ - \partial^\alpha F_{\alpha\mu} \star \tilde{A}_{\nu} - A_{\nu} \star \partial^\alpha \tilde{F}_{\alpha\mu} + \partial_\nu F_{\alpha\mu} \star \tilde{A}^\alpha + A^\alpha \star \partial_\nu F_{\alpha\mu} \right\}
\]
\[ + \frac{1}{2} \gamma [(D - 2) \partial_\mu \partial_\nu + \eta_{\mu\nu} \Box] \varphi \]

\[ = \frac{1}{4} \{ F_{\alpha\mu} \tilde{F}^\alpha_{\nu} + F^\alpha_{\nu} \tilde{F}_{\alpha\mu} \} + \frac{1}{2} \gamma [(D - 2) \partial_\mu \partial_\nu + \eta_{\mu\nu} \Box] \varphi. \tag{3.57} \]

Then, the Ricci scalar is:

\[ R = -\frac{1}{2} F_{\alpha\beta} \tilde{F}^{\alpha\beta} + (D - 1) \gamma \Box \varphi. \tag{3.58} \]

Furthermore, we observe that:

\[ F_{\alpha\beta} \tilde{F}^{\alpha\beta} = 2 \Box (A^\alpha \star \tilde{A}_\alpha) - 2 (\partial \cdot A) \star (\partial \cdot \tilde{A}) = 2 \Box \varphi, \tag{3.59} \]

hence the Ricci scalar can be equivalently written as:

\[ R = \{ \gamma(D - 1) - 1 \} \Box \varphi = \frac{1}{2} \{ \gamma(D - 1) - 1 \} F_{\alpha\beta} \tilde{F}^{\alpha\beta}. \tag{3.60} \]

As a next step we wish to explore to what extent assuming that the linearized YM equations of motion hold\(^6\),

\[ \partial^\alpha F^a_{\alpha\mu} = \Box A^a_\mu - \partial_\mu \partial \cdot A^a \approx 0, \tag{3.61} \]

imply that the gravitational ones hold as well i.e. that the Ricci tensor vanishes.

First, let us observe that the free equations of motion for the scalar field are easily satisfied:

\[ \Box \varphi = \Box A^a \star \tilde{A}_a - \partial \cdot A \star \partial \cdot \tilde{A} = (\Box A_a - \partial_\alpha \partial \cdot A) \star \tilde{A}_a = \partial^\mu F_{\mu\alpha} \star \tilde{A}^\alpha \approx 0, \tag{3.62} \]

thus implying that \( R \approx 0 \), in force of (3.58). Therefore, the linearized version of (3.53) reduces to:

\[ R_{\mu
u} - \frac{1}{2} \eta_{\mu\nu} R \approx \frac{1}{4} \{ \partial_\nu F_{\alpha\mu} \star \tilde{A}^\alpha + A^a \star \partial_\nu \tilde{F}_{\alpha\mu} \} + \frac{1}{2} \gamma (D - 2) \partial_\mu \partial_\nu \varphi \]

\[ = \frac{1}{2} \gamma (D - 2) \partial_\mu \partial_\nu A_a \star \tilde{A}^a \]

\[ + \frac{1}{4} \partial_\nu \left( A_\mu - \frac{\partial_\mu \partial \cdot A}{\Box} \right) \star \partial \cdot \tilde{A} + \frac{1}{4} \partial \cdot A \star \left( \tilde{A}_\mu - \frac{\partial_\mu \partial \cdot \tilde{A}}{\Box} \right). \tag{3.63} \]

\(^6\)With the symbol \( \approx \) we denote an equality which only holds when the YM equations of motion are satisfied.
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If we want (3.63) to vanish, we can choose \( \gamma = \frac{1}{D-2} \) to eliminate one term, while the only option to eliminate the last line is to fix the Lorenz gauge condition for both the YM fields: \( \partial \cdot A = \partial \cdot \tilde{A} = 0 \). Therefore, when the gauge fields are on shell and they satisfy the Lorenz gauge condition, the linearized Einstein equations of motion are satisfied too:

\[
R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \approx 0.
\]

(3.64)

Similarly, we can recover the free e.o.m. for the two-form field:

\[
\Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} \\
= \frac{1}{2} \left\{ \Box (A_\mu \ast \tilde{A}_\nu - A_\nu \ast \tilde{A}_\mu) - \partial_\mu \partial^\rho A_\rho \ast \tilde{A}_\nu \\
+ \partial_\mu \partial^\rho A_\nu \ast \tilde{A}_\rho + \partial_\nu \partial^\rho A_\rho \ast \tilde{A}_\mu - \partial_\nu \partial^\rho A_\mu \ast \tilde{A}_\rho \right\} \\
= \frac{1}{2} \left\{ (\Box A_\mu - \partial_\mu \partial \cdot A) \ast \tilde{A}_\nu - A_\nu \ast (\Box \tilde{A}_\mu - \partial_\mu \partial \cdot \tilde{A}) + \partial_\nu \partial \cdot A \partial_\rho \tilde{A}_\mu - A_\mu \ast \partial_\nu \partial \cdot \tilde{A} \right\}.
\]

(3.65)

Again, the result is that when the YM fields are on shell and in Lorenz gauge, the equations of motion for \( B_{\mu\nu} \) are satisfied: \( \Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} \approx 0 \).

To summarize, with the product of eq. (3.1) and the definition of the gravity fields given in Section 3.1.2, the linearized equations of motion for the fields of \( \mathcal{N} = 0 \) Supergravity,

\[
\begin{cases}
R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = 0, \\
\Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} = 0, \\
\Box \phi = 0,
\end{cases}
\]

(3.66)

are indeed satisfied when the YM fields are on shell \( (\partial^\mu F_{\mu\nu} = \partial^\mu \tilde{F}_{\mu\nu} = 0) \) and in Lorenz gauge \( (\partial \cdot A = \partial \cdot \tilde{A} = 0) \), if, in addition, in the definition of the graviton given in eq. (3.31) one chooses \( \gamma = \frac{1}{D-2} \).

### 3.1.6 Sourced equations of motion

In this section we consider the addition of sources for the YM fields and for the (inverse of the) biadjoint scalar field. We refer to ideas presented in [59], but we proceed from a
slightly different perspective. Indeed, we introduce a new product (in the spirit of (3.1)), which we shall denote with “⊕” and we shall employ in order to build gravitational sources from the YM ones\(^7\). The definition is as follows:

\[
J_{\mu\nu} := \left[ j^{a\mu} \circ [j^{(\Phi)}]^{-1} \circ \tilde{j}^{a\nu} \right] (x) := \left[ j^{a\mu} \oplus \tilde{j}^{a\nu} \right],
\]

where \(J_{\mu\nu}\) is the “DC source”, \(j^{a\mu}, \tilde{j}^{a\nu}\) are two YM currents and \([j^{(\Phi)}]^{-1}\) is the convolution inverse of the source for the biadjoint scalar field \(\Phi^{aa'}\). In terms of these sources, the linearized equations of motion for YM fields read:

\[
\begin{aligned}
\partial^{\mu} F_{a\mu\nu} &= \Box A^{a}_{\mu} - \partial_{\mu} \partial \cdot A^{a}_{\nu} = j^{a\nu}, \\
\partial^{\mu} \tilde{F}^{a'}_{a\mu\nu} &= \Box \tilde{A}^{a'}_{\mu} - \partial_{\mu} \partial \cdot \tilde{A}^{a'}_{\nu} = \tilde{j}^{a'}_{\nu},
\end{aligned}
\]

while the equations of motion for \(\Phi^{-1}_{aa'}\) deserve a separate discussion. Indeed, we can start from the biadjoint scalar, which in presence of a source satisfies the following equation:

\[
\Box \Phi^{aa'} = j^{\phi}_{aa'},
\]

whose solution, focusing on the inhomogeneous part, is (in Fourier transform):

\[
\mathcal{F}\{\Phi^{aa'}\}(k) = -\frac{\mathcal{F}\{j^{\phi}_{aa'}\}(k)}{k^2}.
\]

Since \(\Phi^{-1}_{aa'}\) is defined to be the convolution inverse of the biadjoint scalar, their Fourier transforms satisfy the following equation:

\[
\mathcal{F}\{\Phi^{-1}_{aa'}\}(k) = -\frac{1}{\mathcal{F}\{\Phi^{aa'}\}(k)}.
\]

Therefore, we can derive the equations of motion for \(\Phi^{-1}_{aa'}\) from (3.70) and (3.71):

\[
\mathcal{F}\{\Phi^{-1}_{aa'}\}(k) = -\frac{k^2}{\mathcal{F}\{j^{\phi}_{aa'}\}(k)},
\]

or, in coordinate space:

\[
\Phi^{-1}_{aa'} = \Box [j^{\phi}_{aa'}]^{-1}.
\]

\(^7\)We are grateful to Silvia Nagy for private communications in connection to the definition of this product.
Let us comment more on the introduction of the $\star$-product in eq. (3.67). The underlying idea is that, in the DC spirit, as the gravity fields are built from YM fields, the sources for these fields should be built from the YM sources. Indeed, without introducing $\star$, we would get convolutions of YM currents with YM fields, which should be identified with the sources for the gravity fields. On the other hand, it is more natural to think that the gravitational sources be determined only by the YM currents, $J^\text{grav} \sim j \otimes \tilde{j}$, consistently with what expected from the DC of classical solutions, as we will discuss in Section 3.3. Moreover, the use of the product $\star$ also for sources, which would appear maybe more natural, would also be problematic because of the mass dimension of the currents: $[j \star \tilde{j}] = \frac{D+6}{2}$, since $[j] = \frac{D+2}{2}$. The introduction of $\otimes$ solves this issue, since $[(j^\Phi)^{-1}] = \frac{3D-2}{2}$, therefore $[j \otimes \tilde{j}] = \frac{D+2}{2}$, which is the correct mass dimension expected from a source.

To summarize, we have defined two kinds of products: “$\star$” and “$\otimes$”. Their aim is similar, in the sense that they are both used to perform the DC of two quantities in YM theory. However, for the reasons which we detailed, we shall from now on employ “$\star$” to multiply the YM fields and “$\otimes$” to multiply the YM currents. More explicitly:

$$X \star Y = \left[ X^a \circ \varphi_{aa'} \circ Y^{a'} \right](x) \quad \text{if } X, Y \text{ are fields,}$$

$$j^X \otimes j^Y = \left[ j^X_a \circ (j^{aa'}_a)^{-1} \circ j^Y_a \right](x) \quad \text{if } j^X, j^Y \text{ are currents},$$

where $(j_{aa'}^{(\Phi)})^{-1}$ is the convolution inverse of the source $j_{aa'}^\Phi$ of the biadjoint scalar. From the last equation in the system (3.68), we can read the relation

$$X \star Y = \Box X \otimes Y,$$  

which we will use extensively in what follows.

We proceed employing the expressions for the equations of motion derived in eqs (3.57),(3.58),(3.65), then substituting the eqs. (3.68). First of all, we consider the scalar field, since this will turn out to be useful also in the case of gravity. As in eq. (3.62):

$$\Box \varphi = \partial^\mu F_{\mu \alpha} \star \bar{A}^\alpha = \partial^\mu F_{\mu \alpha} \otimes \bar{A}^\alpha,$$  

therefore, again if the YM fields are in Lorenz gauge, and denoting with “$\approx$” equalities
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which hold on shell one has:

\[ \Box \varphi \approx j^\alpha \otimes \tilde{j}_\alpha. \]  

(3.78)

From now on we shall always assume the gauge fields to satisfy the Lorenz gauge condition and \( \gamma = \frac{1}{D-2} \), since in many occasions they both proved to be necessary conditions to the goal of reproducing the equations of motion for the fields in the gravity theory.

Then, from eqs. (3.57) and (3.58), we derive:

\[ R_{\mu\nu} \approx -\frac{1}{4} \left\{ j_\mu \otimes \tilde{j}_\nu + j_\nu \otimes \tilde{j}_\mu \right\} + \frac{1}{2(D-2)} \eta_{\mu\nu} j^\alpha \otimes \tilde{j}_\alpha, \]  

(3.79)

\[ R = -\frac{1}{2} \left\{ \partial^\alpha F_{\alpha\beta} \ast \tilde{A}^\beta + \tilde{A}^\beta \ast \partial^\alpha \tilde{F}_{\alpha\beta} \right\} + \frac{D-1}{D-2} \Box \varphi \approx \frac{1}{D-2} j^\alpha \otimes \tilde{j}_\alpha, \]  

(3.80)

from which we obtain the sourced EH equations in the form

\[ R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \approx -\frac{1}{4} j_{(\mu} \otimes \tilde{j}_{\nu)}, \]  

(3.81)

where in particular we have a DC form, in the sense of the \( \otimes \)-product, for the energy-momentum tensor of matter. Last we consider the two-form for which, in the same setup, one obtains:

\[ \Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} \approx \frac{1}{2} j_{[\mu} \otimes \tilde{j}_{\nu]}. \]  

(3.82)

To summarize, with the choice \( \gamma = \frac{1}{D-2} \), when the YM fields are on shell and in the Lorenz gauge, the sourced, linearized equations of \( \mathcal{N} = 0 \) Supergravity:

\[
\begin{cases}
R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = -\frac{1}{4} j_{(\mu} \otimes \tilde{j}_{\nu)} := -T_{\mu\nu}, \\
\Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} = \frac{1}{2} j_{[\mu} \otimes \tilde{j}_{\nu]} := j^{(B)}_{\mu\nu}, \\
\Box \varphi = j^\alpha \otimes \tilde{j}_\alpha := j^{(\varphi)}.
\end{cases}
\]  

(3.83)

3.2 BDHK Lagrangian and the double copy

In this section we review the proposal for a Lagrangian implementation of the DC relations made in [16], trying to highlight its essential features. The program of [16] was to build a Lagrangian for pure YM theory with only cubic vertices, whose Feynman rules would automatically obey the CK duality, to then define an appropriate “square” of such
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a Lagrangian, whose tree-level scattering amplitudes would reproduce the results of the DC relations.

3.2.1 Color-kinematics duality in YM Lagrangian

As we explicitly showed in Section 2.2.2, the CK duality is a property of the tree-level four gluons scattering amplitude which can be derived directly from the Feynman rules of YM theory, provided that the vertices are rearranged in a cubic (trivalent) form. In [16] Bern, Dennen, Huang and Kiermaier (BDHK) showed how to derive the CK duality for the four-point amplitude directly from a Lagrangian, suitably arranged to contain only cubic interactions by means of the introduction of an appropriate additional field.

The BDHK Lagrangian is the following:

\[
\mathcal{L}_{\text{BDHK}} = \frac{1}{2} A^a_{\mu} \Box A^a_{\mu} + \frac{1}{2} (\partial \cdot A^a)^2 - B^{a\rho\mu\nu} \Box B^a_{\rho\mu\nu} - g f^{abc} (\partial_\mu A^b_{\nu} + \partial_\rho B^a_{\rho\mu\nu}) A^b_{\mu} A^c_{\nu},
\]

(3.84)

where \( B^a_{\rho\mu\nu} \) denotes new fields, which only propagate as internal lines in the Feynman diagrams when computing gluon amplitudes. Given the form of its interaction in the Lagrangian of eq. (3.84), it can be taken antisymmetric in its last two indices:

\[
B^a_{\rho\mu\nu} = -B^a_{\rho\nu\mu}.
\]

Equivalence to the YM Lagrangian can be proven upon formally integrating out the auxiliary field, i.e. substituting in the Lagrangian the equations of motion for \( B^a_{\rho\mu\nu} \):

\[
\frac{\delta \mathcal{L}_{\text{BDHK}}}{\delta B^a_{\rho\mu\nu}} = 0 \Rightarrow B^a_{\rho\mu\nu} = \frac{g}{2} f^{abc} \partial_\rho (A^b_{\mu} A^c_{\nu}),
\]

(3.85)

\[
\mathcal{L}_{\text{BDHK}}(A, B(A)) = \frac{1}{2} A^{a\mu} \Box A^a_{\mu} + \frac{1}{2} (\partial \cdot A^a)^2 - g f^{abc} \partial_\mu A^b_{\nu} A^b_{\mu} A^c_{\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_{\mu} A^{c\mu} A^{d\nu} A^{e\nu},
\]

(3.86)

where in the intermediate steps algebraic manipulations of \( \frac{1}{2} \) were assumed to be well defined.

The main advantage of (3.84) with respect to the conventional YM Lagrangian is that it only contains cubic vertices. Since the amplitudes obtained from \( \mathcal{L}_{YM} \) automatically satisfy the CK duality up to four points, the same must true for the amplitudes obtained from \( \mathcal{L}_{BDHK} \) when the external states are gluons, but with a simplification: in the YM...
theory we were forced to split the four-gluons vertex into three-point contributions “by hand”, dividing and multiplying by appropriate inverse propagators, while starting from
(3.84) this operation is automatically implemented when computing Feynman diagrams. As an example, let us compute the contribution to the s-channel in the tree-level four-gluon scattering. First, let us derive the Feynman rules from (3.84), representing the field
$B^a_{\rho\mu\nu}$ with a straight continuous line and the gluon with a curly line, as usual:

\[
\frac{1}{2p^2} \left( \delta_{ab} \eta_\mu \eta_\nu \eta_\rho \eta_\sigma \right),
\]

\[
g f^{a_1 a_2 a_3} \left\{ \eta^{\mu_1 \mu_2} (p_1 - p_2)^{\mu_3} + \eta^{\mu_2 \mu_3} (p_2 - p_3)^{\mu_1} + \eta^{\mu_3 \mu_4} (p_3 - p_1)^{\mu_2} \right\},
\]

\[
g f^{a_1 a_2 a_3} \left( \eta_{\beta \mu_1} \eta_{\gamma \mu_2} - \eta_{\beta \mu_2} \eta_{\gamma \mu_1} \right).
\]

In Feynman gauge (eq. (2.8)), the contribution from the exchange of an auxiliary field to the s-channel in four gluons scattering is:

\[
g f^{a_1 a_2 a_3} \left( \eta_{\beta \mu_1} \eta_{\gamma \mu_2} - \eta_{\beta \mu_2} \eta_{\gamma \mu_1} \right),
\]

The result of this diagram is exactly the same as the part proportional to $c_s$ in eq. (2.19), i.e. the part of the contact term diagram with color structure $c_s$. It is important to stress the role of the kinetic term of the field $B^a_{\rho\mu\nu}$ and of the derivative in the corresponding interaction vertex: in the given example they provide an apparently irrelevant factors $s_s = 1$, which is actually instrumental in reproducing the “splitting” of the contact term performed in Section 2.2.2. It is in this sense that $B^a_{\rho\mu\nu}$ is “auxiliary”, although its equa-
tions of motion are not algebraic and it actually propagates in the Feynman diagrams. Indeed, in order to recover the YM scattering amplitudes we must limit ourselves to consider, among the possible amplitudes produced by (3.84), only the ones in which the external states are all gluons, and $B_{\mu_\rho_\nu}$ only propagates in internal lines.

Moreover, as we discussed in Section 2.2.6, in [16] it was shown how to introduce a five gluons (non-local) contact term in the YM Lagrangian, which on the one hand is identically zero due to Jacobi identity, but on the other hand it contributes to the Feynman diagrams so as to render the CK duality apparent up to five points. Similarly to (3.84), it was also shown how to introduce additional “auxiliary” fields in order to make all the interactions only cubic. While the procedure can be extended to in principle arbitrarily higher points, explicit results so far have been found only up to six points at tree level [124].

### 3.2.2 Double copy for BDHK Lagrangian

In [16] a possible Lagrangian realization of the DC relations is also discussed. The idea is to perform, at the Lagrangian level, the same steps which guide the construction of gravitational scattering amplitudes by means of the DC relations. We now review this approach, providing some comments about it in Section 3.2.3.

Let us review, step by step, the DC construction of scattering amplitudes, while discussing its Lagrangian implementation as proposed in [16]. For the moment, we limit ourselves to discuss the analysis up to the four-point scattering amplitudes.

- The first step required to obtain gravitational amplitudes with the DC is to rearrange the expressions of YM amplitudes, writing them as a sum over diagrams with cubic vertices whose kinematic numerators respect the CK duality. At the four-point level, this is reproduced by the BDHK Lagrangian (3.84): as we discussed in the previous section it contains only cubic vertices, whose Feynman rules automatically encode the CK duality.

- The DC of scattering amplitudes is naturally performed in momentum space, with the color factors replaced by another copy of kinematic factors. Therefore, let us

---

8In the sense that, like $B_{\mu_\nu_\rho}$ in (3.84), they have a kinetic term but they contribute to the scattering amplitudes only propagating in the internal lines of the diagrams, like ghosts in gauge theories quantized à la Faddeev-Popov.
consider the BDHK Lagrangian written in momentum space, and introduce a prescription to remove the color indices and the structures constants. The idea of [16] is to consider a key feature of the CK duality the fact that the kinematic numerators of the cubic diagrams in YM theory, when in the CK-dual form, share the same antisymmetry as the structure constants. Therefore, when suppressing the color indices and the structure constants from the YM Lagrangian in momentum space, they proposed to make this antisymmetry explicit in the vertex. As an example of this procedure, let us consider the YM cubic vertex:

\[ S_{YM}^1 = \int d^D x f^{abc} \partial_\mu A^a_\mu A^b_\nu A^c_\nu, \]  

(3.87)

and let us write it in momentum space\(^9\)

\[ S_{YM}^1 = -i \int d^3 k_1 d^3 k_2 d^3 k_3 \delta^{(D)}(k_1 + k_2 + k_3) f^{abc} k^{1}_\mu(k_1) A^a_\mu(k_2) A^b_\nu(k_3) A^c_\nu(k_3). \]  

(3.88)

Then, the prescription tells to remove color indices and structure constants, while still endowing the remaining part of the vertex with the same antisymmetry of the constants \(f^{abc}\). To this end, we explicitly write a sum over permutations with sign of the momentum indices 1, 2 and 3, since in (3.88) there is a one-to-one correspondence between the momentum indices and the color indices. The result is the following\(^{10}\):

\[ [S_{YM}^1]_{\text{no color}} = -i \int d^3 k_1 d^3 k_2 d^3 k_3 \delta^{(D)}(k_1 + k_2 + k_3) \sum_{\sigma \in S_3} (-1)^{\text{sgn(\sigma)}} k^{1}_\mu(k_1) A^{(1)}_\sigma(k_2) A^{(2)}_\sigma(k_3) A^{(3)}_\sigma. \]  

(3.89)

In the kinetic term, where no structure constants are present, the prescription is to simply suppress the color indices. The result, for the full BDHK Lagrangian in \textit{Feynman gauge} (this is crucial, as we will discuss), is:

\[ S_{BDHK}^{YM} \rightarrow \frac{1}{2} \int d^3 k_1 d^3 k_2 \delta^{(D)}(k_1 + k_2) k_1^2 [A^1_\mu A^2_\mu - 2 B^{\rho \mu \nu} B^2_{\rho \mu \nu}] \]

\(^9\)We introduce the notation: \(d^3 k = \frac{d^D k}{(2\pi)^D}\).

\(^{10}\)In order to avoid cumbersome formulas, from now on we adopt the notation \(X^i = X(k_i)\), where \(X\) is any field.
\[-i \int dk_1 dk_2 dk_3 \delta^{(D)} \left( \sum_{i=1}^{3} k_i \right) \sum_{\sigma \in S_3} (-1)^{sgn(\sigma)} \left[ \left( k^{\sigma(1)}_\mu A^{\sigma(1)}_\nu + k^{\rho(1)}_\sigma B^{\sigma(1)}_{\rho\mu\nu} \right) A^{\mu}_\sigma A^{\nu}_\sigma \right]. \]

\[(3.90)\]

- Now that the color factors have been suppressed, with the vertices of (3.90) properly antisymmetrized, let us insert a new copy of kinematic factors. To this end, it is necessary to adopt two different prescriptions for the kinetic term and for the vertices. Indeed, in the DC of scattering amplitudes, while passing from YM to gravity the denominators of the YM diagrams, containing propagators, are left unaltered, while their numerators are multiplied by another copy of kinematic numerators. Therefore, we insert a new copy of YM fields (with the “tilde”, since they come from the second copy of YM theory), but leaving the momentum structure of the kinetic term unaltered, so as to obtain a two-derivative kinetic term also for the gravitational theory:

\[ \frac{1}{2} \int dk_1 dk_2 \delta^{(D)}(k_1 + k_2) k_1^2 \left[ A^1_\mu A^2_\mu - 2 B^1_{\rho\mu\nu} B^2_{\rho\mu\nu} \right] \]

\[ - \frac{1}{2} \int dk_1 dk_2 \delta^{(D)}(k_1 + k_2) k_1^2 \left[ A^1_\mu A^2_\mu - 2 B^1_{\rho\mu\nu} B^2_{\rho\mu\nu} \right] \left[ \tilde{A}_\alpha^1 \tilde{A}_\beta^2 - 2 \tilde{B}_{\gamma\alpha\beta} \tilde{B}_{\gamma\alpha\beta}^2 \right]. \]

\[(3.91)\]

On the contrary, in the interaction vertices, we insert a new copy of the fields and of the derivatives (or, equivalently, of the momenta) acting on them. For the time being, we work up to an overall constant. The result is the following:

\[ \int dk_1 dk_2 dk_3 \delta^{(D)} \left( \sum_{i=1}^{3} k_i \right) \sum_{\sigma \in S_3} (-1)^{sgn(\sigma)} \left[ \left( k^{\sigma(1)}_\mu A^{\sigma(1)}_\nu + k^{\rho(1)}_\sigma B^{\sigma(1)}_{\rho\mu\nu} \right) A^{\mu}_\sigma A^{\nu}_\sigma \right] \]

\[ \rightarrow \int dk_1 dk_2 dk_3 \delta^{(D)} \left( \sum_{i=1}^{3} k_i \right) \sum_{\sigma \in S_3} (-1)^{sgn(\sigma)} \left[ \left( k^{\sigma(1)}_\mu A^{\sigma(1)}_\nu + k^{\rho(1)}_\sigma B^{\sigma(1)}_{\rho\mu\nu} \right) A^{\mu}_\sigma A^{\nu}_\sigma \right] \]

\[ \times \sum_{\tau \in S_3} \left[ \left( k^{\tau(1)}_{\alpha} \tilde{A}^{\tau(1)}_\beta + k^{\tau(1)}_{\gamma} \tilde{B}^{\tau(1)}_{\gamma\alpha\beta} \right) \tilde{A}^{\mu}_\tau \tilde{A}^{\nu}_\tau \right]. \]

\[(3.92)\]

- Finally, in the DC the states in the gravitational theory are built from products of spin-one state, which do not carry color indices. The proposal of Bern and collaborators is therefore to define a physical “gravitational field” \( H_{\mu\nu} \) with the identification \( A_\mu(k) \tilde{A}_\nu(k) \rightarrow H_{\mu\nu}(k) \), where \( A_\mu(k) \) and \( \tilde{A}_\nu(k) \) are two YM fields, coming from the two YM theories which are involved in the DC, with their color
indices simply suppressed and carrying momentum $k_\mu$. As mentioned, the symbol “tilde” over the fields will be used to denote quantities in the second copy of YM theory. The physical states of the field $H_{\mu\nu}$ arise from the product of spin-one states, therefore they include a graviton, a two-form and a scalar, as discussed in Section 2.4.1. The other products of YM fields which appear in (3.91) and (3.92) contain at least one copy of the field $B^a_{\rho\mu\nu}$, and from these products new gravitational fields are defined, which will have, in the theory of gravity, a role similar to the one of $B^a_{\rho\mu\nu}$ in YM theory, as we shall discuss. Such fields are defined as

$$A^\mu \tilde{B}^{\gamma\alpha\beta} := g^{\mu\gamma\alpha\beta}, B^\rho_{\mu\nu} \tilde{A}^\alpha := \tilde{g}^{\rho\mu\nu\alpha} \text{ and } B^\rho_{\mu\nu} \tilde{B}^{\gamma\alpha\beta} := f^{\rho\mu\nu\gamma\alpha\beta}. $$

Let us now study the kinetic part (3.91) and the vertices (3.92) separately, so as to inspect the DC Lagrangian which we just built following [16]. First, we consider (3.91), which can be easily written back in coordinate space, with the following result:

$$S_{\text{DC kin}} = \frac{1}{2} \int d^D x \left[ H^{\mu\alpha} \Box H_{\mu\alpha} - 2 g^{\mu\gamma\alpha\beta} \Box g_{\mu\gamma\alpha\beta} - 2 \tilde{g}^{\rho\mu\nu\alpha} \Box \tilde{g}_{\rho\mu\nu\alpha} + 4 f^{\mu\nu\gamma\alpha\beta} \Box f_{\mu\nu\gamma\alpha\beta} \right]. $$

The role of the new fields will be clear as soon as we study the interaction part of this double-copy Lagrangian. Before doing this, let us write the various propagators of the fields, contracting independently the indices from the left and the right gauge theories:

$$\langle H_{\mu\alpha}(k_1) H_{\nu\beta}(k_2) \rangle = \frac{i}{k_1^2} (\eta_{\mu\nu} \eta_{\alpha\beta}) \delta^{(D)}(k_1 + k_2), $$

$$\langle g_{\mu\alpha\beta\gamma}(k_1) g_{\rho\delta\zeta\eta}(k_2) \rangle = -\frac{i}{2k_1^2} (\eta_{\mu\rho} \eta_{\delta\gamma} \eta_{\beta\zeta} \eta_{\zeta\eta}) \delta^{(D)}(k_1 + k_2), $$

$$\langle \tilde{g}_{\mu\rho\alpha\beta}(k_1) \tilde{g}_{\sigma\tau\lambda\beta}(k_2) \rangle = -\frac{i}{2k_1^2} (\eta_{\mu\sigma} \eta_{\rho\tau} \eta_{\delta\rho} \eta_{\alpha\beta}) \delta^{(D)}(k_1 + k_2), $$

$$\langle f_{\mu\nu\rho\alpha\beta\gamma}(k_1) f_{\sigma\tau\lambda\delta\zeta\eta}(k_2) \rangle = \frac{i}{4k_1^2} (\eta_{\mu\sigma} \eta_{\nu\tau} \eta_{\rho\lambda} \eta_{\alpha\delta} \eta_{\beta\zeta} \eta_{\gamma\eta}) \delta^{(D)}(k_1 + k_2). $$

The interaction Lagrangian of eq. (3.92) contains $12^2 = 144$ terms, that can be conveniently grouped as follows: there are 36 vertices of the type $HHH$, 36 $gHH$ vertices, 36 $\tilde{g}HH$ vertices, 24 $g\tilde{g}H$ vertices and 12 $fHH$ vertices. They can be derived writing explicitly all the terms in the product of two sums over permutations in (3.92). In [16] the interaction vertices are not displayed because of their lengthy expressions; however, we realized that there is quite a bit of redundancy in the above counting, as it is possible to collect groups of six terms which generate identical Feynman rules. Therefore, in
coordinate space the vertices can be rewritten as:

\[
\mathcal{L}^{HHH} = H^\mu_\nu \partial_\mu \partial_\nu H^{\alpha\beta} H_{\alpha\beta} - H^\mu_\nu \partial_\mu \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\alpha\beta} + H^\mu_\nu \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\beta} - H^\mu_\nu \partial_\beta H^{\alpha\beta} \partial_\alpha H_{\alpha\beta} - H^\mu_\nu H^{\alpha\beta} \partial_\alpha \partial_\beta H_{\alpha\beta}, \tag{3.98}
\]

\[
\mathcal{L}^{gHH} = \partial_\mu \partial_\gamma g_{\gamma\alpha\beta} H^{\mu\alpha} H^{\nu\beta} + \partial_\gamma g_{\gamma\alpha\beta} \partial_\mu H^{\mu\alpha} H^{\nu\beta} + \partial_\gamma g_{\gamma\alpha\beta} \partial_\mu H^{\mu\alpha} H^{\nu\beta} - \partial_\gamma g_{\gamma\alpha\beta} \partial_\mu H^{\mu\alpha} H^{\nu\beta}, \tag{3.99}
\]

\[
\mathcal{L}^{\tilde{g}HH} = \partial_\alpha \partial_\rho g_{\rho\mu\nu} H^{\mu\alpha} H^{\nu\beta} + \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^{\nu\beta} + \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^{\nu\beta} - \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^{\nu\beta}, \tag{3.100}
\]

\[
\mathcal{L}^{g\tilde{g}H} = \partial_\alpha \partial_\rho g_{\rho\mu\nu} \partial_\gamma g^{\gamma\alpha\beta} H^\mu_\beta + \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^\nu_\beta + \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^\nu_\beta - \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^\nu_\beta, \tag{3.101}
\]

\[
\mathcal{L}^{fHH} = \partial_\rho \partial_\gamma f_{\rho\mu\nu} \partial_\gamma g^{\gamma\alpha\beta} H^\mu_\beta - \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^\nu_\beta - \partial_\gamma g_{\rho\mu\nu} \partial_\alpha H^{\mu\alpha} H^\nu_\beta. \tag{3.102}
\]

Such a Lagrangian reproduces, by construction, the results of the DC in the case of three and four particles scattering amplitudes at tree level. Indeed, the interactions are built exactly in such a way that, if the left and right YM fields contract independently, the vertices are products of two YM vertices which encode the CK duality, without the structure constants but taking into account their antisymmetry. This is crucial: the YM Feynman rules always inherit from the structure constants the antisymmetry under the exchange of color indices (see Section 2.1), therefore when suppressing the structure constants the antisymmetry must be enforced “manually”. Furthermore, since in the DC of the quadratic Lagrangian only the fields were squared, leaving the derivative part unaltered, the propagators are \( \sim \frac{1}{p^2} \) as in the YM theory, as required by the DC prescription in order to leave the denominators of the amplitude unaltered. Finally, being all the vertices cubic, the “doubling” of the interactions in the Lagrangian exactly corresponds to the “doubling” of the kinematic factors in the YM amplitudes, diagram by diagram.

We can represent schematically how the gravity vertices are built, drawing diagrams where curly lines are gluons, straight lines represent the field \( B^{\mu\nu}_{\rho\nu} \), double wavy lines the field \( H_{\mu\nu} \), wavy + straight lines either \( g_{\gamma\alpha\beta} \) or \( \tilde{g}_{\mu\nu\alpha} \) (according to the order) and double straight lines the field \( f_{\rho\mu\nu\gamma\alpha\beta} \). With this notation, the DC of the BDHK Lagrangian is represented in fig. 3.1. As an example, we can compute the three particles tree-level scattering amplitude between gravitons, two-forms and scalars, and compare the result with what we obtained in Section 2.4.2. Integrating by parts and requiring transversality
of the fields (inherited by the transversality of gluons polarization vectors), we can recast (3.98) into the form:

$$L_{TT}^{HHH} = 2 H^\mu_\alpha \partial_\mu H^{\alpha \beta} H_\beta + 2 H^\mu_\alpha \partial_\mu H^{\alpha \beta} \partial_\beta H_\alpha + 2 H^\mu_\alpha \partial_\mu H^{\alpha \beta} \partial_\alpha H_\mu.$$  (3.103)

which, with the insertion of a factor $\frac{\kappa}{2}$ ($\kappa$ is the gravitational coupling constant), gives exactly the amplitude of eq. (2.109).

If we adopt the same prescription for the coupling constant for all the cubic vertices, which is the natural thing to do if one thinks of the DC construction of gravitational scattering amplitudes, we can write the complete result for the Lagrangian proposed in [16] in order to reproduce the DC relations up to the four-point scattering amplitudes. Upon integration by parts of some terms in eqs. (3.98)-(3.102), it is possible to write the result as follows:

$$L_{BDHK}^{DC} = \left\{ \frac{1}{2} H^{\mu \alpha} \Box H_\mu - g^{\mu \gamma \alpha \beta} \Box g_{\mu \gamma \alpha \beta} - \tilde{g}^{\rho \mu \alpha} \Box \tilde{g}_{\rho \mu \alpha} + 2 f^{\rho \mu \gamma \alpha \beta} \Box f_{\rho \mu \gamma \alpha \beta} \right\}
+ \kappa \left\{ H^{\mu \nu} \partial_\nu H^{\alpha \beta} H_\alpha + H^{\mu \nu} \partial_\nu H^{\alpha \beta} \partial_\beta H_\alpha + H^{\mu \nu} \partial_\nu H^{\alpha \beta} \partial_\alpha H_\mu \right\}
+ \frac{1}{2} H_{\alpha \beta} \partial_\nu H^{\mu \nu} (\partial_\mu H^{\alpha \beta} + \partial^\alpha H_\mu) + \frac{1}{2} H^{\mu \nu} H_{\mu \beta} \partial_\beta H^{\alpha \nu}
+ \frac{\kappa}{2} \left\{ \partial^\gamma g_{\mu \nu | \alpha \beta} \left( 2 \partial_\nu H^{\mu \alpha} H^{\nu \beta} + H^{\mu \alpha} \partial_\nu H^{\nu \beta} - \partial^\mu H^{\nu \alpha} H_\nu^{\beta} - \partial_\nu g_{\mu \nu | \alpha \beta} H_\nu^{\beta} \right)
+ \partial^\rho g_{\mu \nu | \alpha \beta} \left( 2 \partial_\beta H^{\mu \alpha} H^{\nu \beta} + H^{\mu \alpha} \partial_\beta H^{\nu \beta} - \partial^\alpha H^{\mu \beta} H_\beta^{\nu} - \partial_\beta g_{\mu \nu | \alpha \beta} H_\beta^{\nu} \right) - \partial^\rho g_{\mu \nu | \alpha \beta} \partial_\gamma g^{\mu \gamma | \alpha \beta} H_\rho^{\nu \beta} + \frac{1}{2} f_{\rho \mu \nu | \alpha \beta} H^{\rho \mu} H^{\nu \beta} \right\}.$$  (3.104)

As in the BDHK theory, integrating out the fields $g_{\mu \gamma \alpha \beta}$, $\tilde{g}_{\mu \nu | \beta \alpha}$ and $f_{\rho \mu \nu | \alpha \beta}$ might in principle result in a non-local gravity Lagrangian in force of their non-trivial kinetic operators. However, it is possible to show that this does not happen in this case, owing to the particular form of the interaction vertices in eq. (3.104) and assuming that all intermediate formal manipulations are well-defined.
Let us prove it, first for the fields $g_{\mu\gamma\alpha\beta}$ and $\tilde{g}_{\mu\nu\rho\alpha}$ (the logic is the same for these two fields) and then for $f_{\rho\mu\nu\gamma\alpha\beta}$. Let us write, schematically, the part of the action containing $g_{\mu\gamma\alpha\beta}$:

$$S_g = \int d^D x \left[ -g^{\mu\gamma\alpha\beta} \Box g_{\mu\gamma\alpha\beta} + \frac{\kappa}{2} \partial^\gamma g_{\mu\gamma\alpha\beta} X^\mu_{\alpha\beta} \right], \quad (3.105)$$

where with $X^\gamma_{g\alpha\beta}$ we denote the remaining part of the vertices which contain $\partial^\gamma g_{\mu\gamma\alpha\beta}$. The key fact is that in all its interactions the field $g_{\mu\gamma\alpha\beta}$ appears with a derivative contracted with its second index, which allowed us to write (3.105) in this form. If we then vary the action and we assume intermediate formal manipulations to be well defined, we can compute the equations of motion for the field under inspection, which are the following:

$$g_{\mu\gamma\alpha\beta} = -\frac{\kappa}{4} \partial_{\gamma} X^g_{\mu\alpha\beta}. \quad (3.106)$$

Last, replacing (3.106) into (3.105) we can “integrate out” $g_{\mu\gamma\alpha\beta}$, with a resulting action:

$$S_g = \int d^D x \left[ -\frac{\kappa}{4} \partial^\gamma X^\mu_{g\alpha\beta} \Box X^g_{\mu\alpha\beta} - \frac{\kappa^2}{8} \partial^\gamma \partial^{\alpha} X^g_{\mu\alpha\beta} X^g_{\mu\alpha\beta} \right]$$

$$= -\frac{\kappa^2}{16} \int d^D x X^\mu_{g\alpha\beta} X^g_{\mu\alpha\beta}. \quad (3.107)$$

From (3.107) it is evident that, as long as $X^\gamma_{g\alpha\beta}$ is a local quantity, integrating out $g_{\mu\gamma\alpha\beta}$ produces a local action even though its equations of motion (3.106) present a non-locality. The same argument applies to $\tilde{g}_{\mu\nu\rho\alpha}$, whose action can be written as:

$$S_{\tilde{g}} = \int d^D x \left[ -\tilde{g}^{\rho\mu\nu\alpha} \Box \tilde{g}_{\rho\mu\nu\alpha} + \frac{\kappa}{2} \partial^\rho \tilde{g}_{\rho\mu\nu\alpha} X^\mu_{\tilde{g}\alpha} \right], \quad (3.108)$$

The equations of motion for $\tilde{g}_{\mu\nu\rho\alpha}$ are the following:

$$\tilde{g}_{\mu\nu\rho\alpha} = -\frac{\kappa^2}{4} \partial_{\rho} X_{\tilde{g}\mu\nu\alpha}, \quad (3.109)$$

by means of which one can integrate out $\tilde{g}_{\mu\nu\rho\alpha}$, with the result:

$$S_{\tilde{g}} = -\frac{\kappa^2}{16} \int d^D x X^\mu_{\tilde{g}\alpha} X^g_{\mu\alpha}. \quad (3.110)$$
Similarly, we can write the action as for $f_{\rho\mu\nu\gamma\alpha\beta}$:

$$S_f = \int d^D x \left[ + 2 f^{\rho\mu\nu\gamma\alpha\beta} \Box f_{\rho\mu\nu\gamma\alpha\beta} + \frac{\kappa}{2} \partial^\rho \partial^\gamma f_{\rho\mu\nu\gamma\alpha\beta} X_f^{\mu\nu\alpha\beta} \right],$$

with the only difference with respect to the previous cases being the fact that there are two derivatives acting on $f_{\rho\mu\nu\gamma\alpha\beta}$ in the interaction term. Indeed, from (3.111), the equations of motion for $f_{\rho\mu\nu\gamma\alpha\beta}$ are the following:

$$f_{\rho\mu\nu\gamma\alpha\beta} = -\frac{\kappa}{8} \Box \partial_\rho \partial_\gamma X_f^{\mu\nu\alpha\beta}.$$

Integrating out $f_{\rho\mu\nu\gamma\alpha\beta}$ from (3.111) by means of (3.112) yields the following result:

$$S_f = \int d^D x \left[ \frac{\kappa}{8} \Box \partial_\rho \partial_\gamma X_f^{\mu\nu\alpha\beta} \right]$$

$$= -\frac{\kappa^2}{32} \int X_f^{\mu\nu\alpha\beta} \Box X_f^{\mu\nu\alpha\beta}.$$

Again, the action is local if $X_f^{\mu\nu\alpha\beta}$ itself is local.

In the above discussion we have introduced three quantities, namely $X^{\gamma\alpha\beta}_g$, $X^{\mu\nu\rho}_\tilde{g}$ and $X_f^{\mu\nu\alpha\beta}$, implicitly defined by eqs. (3.105), (3.108) and (3.111), respectively. They can be straightforwardly derived from (3.104), with the following results:

- $X^{\gamma\alpha\beta}_g = 2 \partial_\nu H^{\mu[\alpha|} H^{\nu|\beta]} + H^{\mu[\alpha|} \partial_\nu H^{\nu|\beta]} - \partial^\mu H^{\nu[\alpha} H^{\nu|\beta]} - \partial_\nu \tilde{g}^{\rho[\mu|\nu|\alpha} H^{\rho|\beta]},$ (3.114)
- $X^{\mu\nu\rho}_\tilde{g} = 2 \partial_\nu H^{\mu[\alpha|} H^{\nu|\beta]} + H^{\mu[\alpha|} \partial_\nu H^{\nu|\beta]} - \partial^\alpha H^{\nu[\mu|} H^{\nu|\beta]} - \partial_\nu \tilde{g}^{\gamma[\alpha|} H^{\gamma|\beta]},$ (3.115)
- $X_f^{\mu\nu\alpha\beta} = H^{\mu[\alpha|} H^{\nu|\beta]}.$ (3.116)

As a last step, we wish to display explicitly the action which results integrating out the “auxiliary” fields, in order to obtain an action for the only physical field $H_{\mu\nu}$. However, comparing (3.106) and (3.109) with (3.114) and (3.115) it is evident that the equations of motion of $g_{\mu\gamma\alpha\beta}$ and $\tilde{g}_{\mu\nu\rho}$ are coupled:

$$g^{\mu\gamma\alpha\beta} = -\frac{\kappa}{4} \Box \partial_\gamma \left( 2 \partial_\nu H^{\mu[\alpha|} H^{\nu|\beta]} + H^{\mu[\alpha|} \partial_\nu H^{\nu|\beta]} - \partial^\mu H^{\nu[\alpha} H^{\nu|\beta]} - \partial_\nu \tilde{g}^{\rho[\mu|\nu|\alpha} H^{\rho|\beta]}, \right),$$

(3.117)

$$\tilde{g}_{\mu\nu\rho} = -\frac{\kappa}{4} \Box \partial_\rho \left( 2 \partial_\beta H^{\mu[\alpha|} H^{\nu|\beta]} + H^{\mu[\alpha|} \partial_\beta H^{\nu|\beta]} - \partial^\alpha H^{\nu[\mu|} H^{\nu|\beta]} - \partial_\nu \tilde{g}^{\gamma[\alpha|} H^{\gamma|\beta]}, \right).$$

(3.118)
As a result, while $X_{\mu}^{\nu\alpha\beta}$ is completely determined in terms of $H_{\mu\nu}$, $X_{\gamma}^{\nu\alpha\beta}$ also depends on $\tilde{g}_{\mu\nu\alpha}$ and $\tilde{X}_{\mu}^{\nu\alpha\beta}$ on $g_{\mu\nu\alpha\beta}$. To settle this, we can solve the equations of motion (3.117) and (3.118) with a perturbative expansion in terms of powers of $H_{\mu\nu}$ (or, equivalently, in terms of powers of $\kappa$). The non-locality in the equations of motion of $g_{\mu\nu\alpha\beta}$ and $\tilde{g}_{\mu\nu\alpha}$ is canceled as above: in (3.114) and (3.115) the two fields appear with one derivative contracted with the derivative factored out in the equations of motion (3.117) and (3.118), therefore formally eliminating $\Box$. The final expression of the action is, therefore, completely local, but can be determined only perturbatively.

Since this DC action allows, by construction, to compute gravitational scattering amplitudes up to the four-point ones, we can compute the quartic vertices which result integrating out $g_{\mu\nu\alpha\beta}$, $\tilde{g}_{\mu\nu\alpha}$ and $f_{\mu\nu\rho\alpha\beta}$. To this end, we can safely neglect the aforementioned mixing between $g_{\mu\nu\alpha\beta}$ and $\tilde{g}_{\mu\nu\alpha}$: all the vertices produced by the above mechanism contain two powers of the quantities $X$ previously introduced, and all of them are at least quadratic in $H_{\mu\nu}$. Therefore, a non-zero value for $g_{\mu\nu\alpha\beta}$ and $\tilde{g}_{\mu\nu\alpha}$ in (3.114) and (3.115) would produce, from the respective equations of motion (3.117) and (3.118), terms at least cubic in $H_{\mu\nu}$, which would result in vertices with at least five fields.

Hence, the result of this discussion is that the quartic couplings of $H_{\mu\nu}$ in the DC Lagrangian proposed in [16] are the following:

$$\mathcal{L}^{DC}_{(4)} = -\frac{\kappa^2}{16} \left\{ \frac{1}{2} \left[ H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} \right] \Box \left[ H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} \right] + \right. \left[ 2 \partial_{\nu} H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} + H_{\mu}^{[\alpha]} \partial_{\nu} H_{\nu}^{[\beta]} - \partial_{\nu} H_{\nu}^{[\alpha]} H_{\nu}^{[\beta]} \right] \times \left[ 2 \partial_{\nu} H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} + H_{\mu}^{[\alpha]} \partial_{\nu} H_{\nu}^{[\beta]} - \partial_{\nu} H_{\nu}^{[\alpha]} H_{\nu}^{[\beta]} \right] + \right. \left[ 2 \partial_{\nu} H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} + H_{\mu}^{[\alpha]} \partial_{\nu} H_{\nu}^{[\beta]} - \partial_{\nu} H_{\nu}^{[\alpha]} H_{\nu}^{[\beta]} \right] \times \left[ 2 \partial_{\nu} H_{\mu}^{[\alpha]} H_{\nu}^{[\beta]} + H_{\mu}^{[\alpha]} \partial_{\nu} H_{\nu}^{[\beta]} - \partial_{\nu} H_{\nu}^{[\alpha]} H_{\nu}^{[\beta]} \right] \left\} . \tag{3.119} \right.$$
order to bring these additional contributions (such as the one discussed in Section 2.2.6) into a local form with only cubic vertices. Once the YM Lagrangian is recast in this form, with manifest CK duality, it is possible to obtain a gravity Lagrangian with a procedure analogous to the one discussed in this section for the BDHK Lagrangian.

### 3.2.3 Comments on the BDHK approach

The method proposed in [16] is, as of today, the only approach available in the literature which actually implements the DC relations at the Lagrangian level, which makes it of extreme interest to our purposes. For this reason, we would like to add a few more comments on its main features.

First, let us observe that two different prescriptions were employed in the DC procedure discussed in the previous section: as we already stressed, in the kinetic terms only the fields were “double-copied”, leaving the derivatives structure unaltered. On the contrary, in the interaction part of the Lagrangian both the derivatives and the fields were “double-copied”, as expected from the DC procedure on scattering amplitudes. However, this procedure was successful only because in [16] the YM Lagrangian was chosen to be in Feynman gauge, where the quadratic term is simply $A^2$, with no derivatives contracted with the gauge fields. Had one chosen a different gauge or no gauge-fixing at all, a prescription should have been specified on how to treat the derivatives contracted with the fields in the quadratic part of the Lagrangian. Second, in this discussion the problem of symmetries and geometry, which is the main focus of this Thesis, was not addressed at all: it is not clear how the off-shell symmetries of gravity should emerge from YM theory and indeed the gauge invariance of the resulting Lagrangian is not discussed at all, the reason being again that a gauge choice has been made before performing the “square”. Third, no prescription is given on how to treat the color indices of the YM fields in the DC: they are simply eliminated from the fields, together with the structure constants, and the identification $H_{\mu\nu}(k) = A_\mu(k)A_\nu(k)$ (and similarly for the auxiliary fields) is not really mathematically clarified, precisely because the color indices are simply "removed", even though $A^a_\mu$ in principle contains a multiplet of spin-one fields. Last, the authors of [16] observed that they have found no indication that the procedure necessary to build a Lagrangian with CK duality might eventually terminate, nor did they find any pattern or symmetry principle behind their construction, which we ascribe once again to a lack of geometrical insight on the DC relations.
3.3 The double copy of classical solutions

Another interesting direction in which double-copy relations were studied and developed is the realm of classical solutions. The first work on the subject [19] explored the possibility to relate the Kerr-Schild class of solutions to Einstein’s equations [125] to classical solutions in YM theory. The program was then extended to consider different and possibly more complicated solutions, including the Taub-NUT spacetime [126, 127] as an example of double Kerr-Schild metric [22], Kerr-Schild metrics over curved backgrounds [20], perturbative solutions [119], accelerating particles and radiative solutions [24, 26–29, 128], GR and YM solutions over maximally symmetric spacetimes [21].

In the next few sections we shall review the basic, yet quite relevant case of the Kerr-Schild DC [19]. Indeed, while the DC relations only hold perturbatively, extending the discussion to classical solutions opens up the possibility to relate exact, nonperturbative solutions in gravity and YM theory, and this is exactly what was first done in [19].

3.3.1 Kerr-Schild ansatz in General Relativity

The Einstein-Hilbert equations are a complicated set of non-linear partial differential equations, whose exact solutions are known only in a few cases. Among them, a particularly interesting class was identified in [125]: the Kerr-Schild (KS) solutions. In this section, we review the properties of KS solutions which turn out to be useful in the corresponding double-copy construction.

The starting point is the KS ansatz for the metric:

\[ g_{\mu\nu} = \eta_{\mu\nu} + \varphi k_\mu k_\nu, \]  \hspace{1cm} (3.120)

where \( \eta_{\mu\nu} \) is the Minkowski metric, \( \varphi \) is a scalar field and \( k_\mu \) a (co-)vector field. Upon further requiring that \( k_\mu \) be null both w.r.t. Minkowski metric and to \( g_{\mu\nu} \)

\[ \eta^{\mu\nu} k_\mu k_\nu = g^{\mu\nu} k_\mu k_\nu = 0, \]  \hspace{1cm} (3.121)

Together with the condition

\[ k^\mu \partial_\mu k_\nu = 0, \]  \hspace{1cm} (3.122)

it is possible to prove that the metric (3.120) possesses a number of interesting properties:

1. \( g^{\mu\nu} = \eta^{\mu\nu} - \varphi k^\mu k^\nu. \)
Proof: let us try the ansatz $g^{\mu\nu} = \eta^{\mu\nu} + X^\mu{}\nu$, with symmetric $X^\mu{}\nu$. Raising and lowering the indices of $X^\mu{}\nu$ and $k^\mu$, with $\eta_{\mu\nu}$ we find

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu = \delta^\mu_\nu + X^\mu{}\nu + \varphi k^{\mu} k_\nu + \varphi X^\mu {}\alpha k_\alpha k_\nu,$$

(3.123)

and therefore:

$$X^\mu_\nu + \varphi k^{\mu} k_\nu + \varphi X^\mu {}\alpha k_\alpha k_\nu = 0.$$  (3.124)

Since $\eta^{\mu\nu} k_\mu k_\nu = g^{\mu\nu} k_\mu k_\nu = 0$, then also $X^{\mu\nu} k_\mu k_\nu = 0$; thus, by contracting (3.124) with $k_\mu$, we find:

$$k_\mu X^\mu_\nu + \varphi k^{\mu} k^{\mu} k_\nu + \varphi X^\mu {}\alpha k_\alpha k_\nu = k_\mu X^\mu_\nu = 0.$$  (3.125)

Substituting back into (3.124) yields $X^\mu_\nu = -\varphi k^\mu$. Since $g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\mu - \varphi k^{\mu} k_\mu k_\nu$, it turns out that the vector index of $k$ can be raised/lowered equivalently with $\eta_{\mu\nu}$ or with $g_{\mu\nu}$.

2. $\sqrt{-g} = 1$.

Proof: let us write the determinant of the metric as in Section 2.3:

$$\sqrt{-g} = \exp \left\{ \frac{1}{2} \text{tr} \left[ \log (\delta^\mu_\nu + \eta^{\mu\alpha} \varphi k_\alpha k_\nu) \right] \right\},$$  (3.126)

Given that

$$\text{tr} (\eta^{\mu\alpha} \varphi k_\alpha k_\nu) = \varphi \eta^{\mu\nu} k_\mu k_\nu = 0,$$

(3.127)

and similarly all its powers vanish, one finds $\sqrt{-g} = e^0 = 1$.

3. $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} \left\{ \partial_\mu (\varphi k^\alpha k_\nu) + \partial_\nu (\varphi k^\alpha k_\mu) - \partial^\alpha (\varphi k_\mu k_\nu) + \varphi k^{\alpha} k_\mu k_\nu k \cdot \partial \varphi \right\}$.

Proof: exploiting (3.121) and (3.122), we can write

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu})$$

$$= \frac{1}{2} (\eta^{\alpha\lambda} - \varphi k^{\alpha\lambda}) \left[ \partial_\mu (\varphi k^\lambda k_\nu) + \partial_\nu (\varphi k^\lambda k_\mu) - \partial_\lambda (\varphi k_\mu k_\nu) \right]$$

$$= \frac{1}{2} \left\{ \partial_\mu (\varphi k^\alpha k_\nu) + \partial_\nu (\varphi k^\alpha k_\mu) - \partial^\alpha (\varphi k_\mu k_\nu) + \varphi k^{\alpha} k_\mu k_\nu k \cdot \partial \varphi \right\}. \quad (3.128)$$

If we then use

$$k^\lambda \partial_\mu (\varphi k_\lambda k_\nu) = \partial_\mu \varphi k^\lambda k_\nu + \varphi \partial_\mu k_\alpha k^\lambda + \varphi k_\nu k^\lambda \partial_\mu k_\lambda,$$

(3.129)

$$k^\lambda \partial_\mu k_\lambda = \frac{1}{2} \partial_\mu (k^2) = 0,$$

(3.130)

the proof follows.
4. $\Gamma_{\mu\nu}^\alpha k^\mu k^\nu = 0$.

Proof: one has to contract independently all the terms contained in the expression for $\Gamma_{\mu\nu}^\lambda$ found at point 3.

$$\partial_\mu(\varphi k^\alpha k^\nu)k^\mu k^\nu = \partial_\mu \varphi k^\alpha k^\mu k^\nu + \varphi k^\cdot \partial(k^\alpha k^\nu)k^\nu = 0, \quad (3.131)$$

$$\partial_\nu(\varphi k^\alpha k^\mu)k^\mu k^\nu = \partial_\nu \varphi k^\alpha k^\mu k^\nu + \varphi k^\cdot \partial(k^\alpha k^\mu)k^\mu = 0, \quad (3.132)$$

$$\partial^\alpha (\varphi k^\mu k^\nu)k^\mu k^\nu = \partial^\alpha \varphi(k^2)^2 + 2 \varphi k^\cdot \partial^\alpha k^\mu k^\nu = 0, \quad (3.133)$$

$$\varphi k^\alpha k^\mu k^\nu \cdot \partial k^\mu k^\nu = \varphi k^\alpha (k^2)^2 k^\cdot \partial \varphi = 0. \quad (3.134)$$

5. $\Gamma_{\mu\nu}^\alpha k^\mu k^\nu = 0$.

Proof: as above, we simply evaluate all the contributions:

$$\partial_\mu(\varphi k^\alpha k^\nu)k^\mu k^\nu = \partial_\mu \varphi k^\alpha k^\mu k^\nu + \varphi k^\cdot \partial(k^\alpha k^\nu)k^\nu = 0, \quad (3.135)$$

$$\partial_\nu(\varphi k^\alpha k^\mu)k^\mu k^\nu = \partial_\nu \varphi k^\alpha k^\mu k^\nu + \varphi k^\cdot \partial(k^\alpha k^\mu)k^\mu = 0, \quad (3.136)$$

$$\partial^\alpha (\varphi k^\mu k^\nu)k^\mu k^\nu = k^\cdot \partial k^\mu k^\nu + \varphi k^\cdot k^\nu(k^\mu k^\nu) = 0, \quad (3.137)$$

$$\varphi k^\alpha k^\mu k^\nu \cdot \partial k^\mu k^\nu = \varphi k^\alpha (k^2)^2 k^\cdot \partial \varphi = 0. \quad (3.138)$$

6. $(D_\mu k^\nu)k^\nu = (\partial_\mu k^\nu)k^\nu = 0$.

Proof: $(D_\mu k^\nu)k^\nu = (\partial_\mu k^\nu)k^\nu - \Gamma^\alpha_{\mu\nu} k^\alpha k^\nu = 0$. the proof then follows from the relation $k^\lambda \partial_\mu k^\lambda = \frac{1}{2} \partial_\mu (k^2) = 0$.

7. $(D_\mu k^\nu)k^\mu = (\partial_\mu k^\nu)k^\mu = 0$.

Proof: $(D_\mu k^\nu)k^\mu = (\partial_\mu k^\nu)k^\mu - \Gamma^\alpha_{\mu\nu} k^\alpha k^\nu = 0$. Thus, the null vector $k$ is geodesic both respect to $g_{\mu\nu}$ and to $\eta_{\mu\nu}$.

8. $R^\lambda_\nu = \frac{1}{2} \left\{ \partial_\alpha \partial^\lambda (\varphi k^\alpha k^\nu) + \partial_\alpha \partial_\nu (\varphi k^\alpha k^\lambda) - \partial^\alpha \partial_\alpha (\varphi k^\lambda k^\nu) \right\}$,

where $R^\lambda_\nu = g^{\lambda\mu} R_{\mu\nu}$, but with the convention that $\partial^\lambda = \eta^{\lambda\mu} \partial_\mu$. The mixed convention on how the indices are raised is useful since it allows to build a form of the Ricci tensor which is linear in the (non-perturbative) graviton field defined as $h_{\mu\nu} = \varphi k^\mu k^\nu$.

Proof: we simplify the Ricci tensor using the above properties.

$$R^\lambda_\nu = g^{\lambda\mu} \left\{ \partial_\mu \partial_\nu \ln(\sqrt{-g}) - \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} \Gamma^\alpha_{\mu\nu}) + \Gamma^\beta_{\mu\alpha} \Gamma_{\beta\nu}^\lambda \right\}$$

$$= (g^{\lambda\mu}) \left\{ - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\beta_{\mu\alpha} \Gamma_{\beta\nu}^\lambda \right\}, \quad (3.139)$$
Then, we compute separately the two contributions, keeping in mind the convention $\partial^\lambda = \eta^{\lambda\mu} \partial_\mu$:

\[
g^{\lambda\mu}\partial_\alpha \Gamma^\alpha_{\mu\nu} = \frac{1}{2}(\eta^{\lambda\mu} - \varphi k^\mu \lambda^\lambda)\left\{ \partial_\alpha \partial_\mu (\varphi k^\alpha k_\nu) + \partial_\alpha \partial_\nu (\varphi k^\alpha k_\mu) - \partial^\kappa \partial_\alpha (\varphi k^\mu k_\nu) + k_\mu k_\nu (\partial k \cdot \partial k) \right\}
\]

\[
= \frac{1}{2}\left\{ \partial_\alpha \partial^\lambda (\varphi k^\alpha k_\nu) + \partial_\alpha \partial_\nu (\varphi k^\alpha k^\lambda - \partial^\mu \partial_\alpha (\varphi k^\mu k_\nu)) + k^\lambda k_\nu (\partial k \cdot \partial k)^2 
+ \varphi k^\alpha k^\beta \partial_\alpha \partial_\beta \varphi + \varphi k \cdot \partial \varphi \partial \cdot k \right\} - \frac{1}{2} \varphi \left\{ k^\lambda k^\mu [\partial_\alpha \partial_\mu (\varphi k^\alpha k_\nu) + \partial_\alpha \partial_\nu (\varphi k^\alpha k_\mu) - \partial^\alpha \partial_\alpha (\varphi k^\mu k_\nu)] \right\},
\]

\[
(3.140)
\]

\[
\Gamma^\beta_{\mu\alpha\Gamma} = \frac{1}{2}\Gamma^\alpha_{\beta\mu} \left\{ \partial_\mu (\varphi k^\alpha k_\nu) + \partial_\nu (\varphi k^\alpha k_\mu) - \partial^\alpha (\varphi k^\mu k_\nu) + \varphi k^\beta k_\alpha k_\mu k \cdot \partial \varphi \right\} = 0 \text{ contracted with } \Gamma
\]

\[
= \frac{1}{4} \left\{ \partial_\mu (\varphi k^\beta k_\alpha) \partial_\nu (\varphi k^\gamma k_\nu) + \partial_\mu (\varphi k^\beta k_\nu) \partial_\nu (\varphi k^\gamma k_\alpha) 
- \partial_\mu (\varphi k^\beta k_\alpha) \partial^\gamma (\varphi k^\gamma k_\nu) + \partial_\mu (\varphi k^\beta k_\nu) \partial^\gamma (\varphi k^\gamma k_\alpha) 
+ \partial_\alpha (\varphi k^\beta k_\mu) \partial_\beta (\varphi k^\nu k_\nu) - \partial_\alpha (\varphi k^\beta k_\nu) \partial_\beta (\varphi k^\mu k_\nu) 
- \partial^\beta (\varphi k^\nu k_\nu) \partial_\alpha (\varphi k^\beta k_\mu) - \partial^\beta (\varphi k^\mu k_\nu) \partial_\alpha (\varphi k^\beta k_\nu) 
+ \partial^\beta (\varphi k^\nu k_\nu) \partial_\alpha (\varphi k^\beta k_\mu) + \varphi k^\alpha k^\beta k_\nu k \cdot \partial \varphi [\partial_\mu (\varphi k^\gamma k_\beta) + \partial_\mu (\varphi k_\beta k_\nu)] \right\}
\]

\[
= \frac{1}{2} \partial^\gamma (\varphi k^\beta k_\nu) \partial^\alpha (\varphi k^\nu k_\alpha) + \frac{1}{4} k_\mu k \cdot \partial \varphi [\varphi k^\beta k_\mu] \Gamma_{\mu\nu} - \frac{1}{4} \partial_\alpha [(\varphi k^\beta)^2] = 0
\]

\[
(3.141)
\]

In the last equality we integrated by parts, exploiting the fact that total derivative terms always contain $k^2 = 0$. Then, contracting the last expression with the inverse metric tensor:

\[
g^{\lambda\mu}\Gamma^\beta_{\mu\alpha\Gamma} = \frac{1}{2} \partial^\gamma (\varphi k^\lambda k_\beta) \partial^\delta (\varphi k^\nu k_\alpha) - \frac{1}{2} \varphi k^\lambda \underbrace{k^\mu \partial^\alpha (\varphi k_\mu k_\beta)}_{k^2 = k^\mu \partial_\mu k_\beta} \partial^\beta (\varphi k^\nu k_\alpha).
\]

\[
(3.142)
\]

Therefore, from eq. (3.139):

\[
R^\lambda_{\mu\nu} = \frac{1}{2} \left\{ \partial_\alpha \partial^\alpha (\varphi k^\beta k_\nu) + \partial_\alpha \partial_\nu (\varphi k^\beta k^\lambda) - \partial^\mu \partial_\alpha (\varphi k^\mu k_\nu) \right\},
\]

\[
(3.143)
\]

since it is possible to show that other contributions cancel out integrating by parts and exploiting the above identities.

For what the KS DC is concerned, eq. (3.143) is extremely important: it shows that, for the particular class of solutions to Einstein-Hilbert equations which admit a metric in the KS form, the equations can be recast in a linear form, owing to the properties of the
vector field $k_\mu$.

### 3.3.2 Vacuum, stationary Kerr-Schild solutions and the double copy

The authors of [19] considered in particular stationary solutions, where the components of the metric (3.120) are time independent: $\partial_0 \varphi = \partial_0 k_\mu = 0$. Moreover, they observed that redefinitions of the type $k_\mu \rightarrow \alpha(x)k_\mu$, $\varphi \rightarrow \varphi(\alpha(x))^2$ leave $g_{\mu\nu}$ unaltered, and exploited this freedom to set $k_0 = 1$, absorbing all of its dynamics in the function $\varphi$.

Under these conditions, the vacuum Einstein-Hilbert equations read, in components:

\begin{align}
R^0_0 &= -\frac{1}{2} \Box \varphi = -\frac{1}{2} \nabla^2 \varphi = 0, \quad (3.144) \\
R^i_0 &= \frac{1}{2} \{ \partial^i \partial_j (\varphi k^j) - \partial_j \partial^i (\varphi k^j) \} = \frac{1}{2} \partial_j [\partial^i (\varphi k^j) - \partial^j (\varphi k^i)] = 0, \quad (3.145) \\
R^i_j &= \frac{1}{2} \{ \partial^i \partial_j (\varphi k^j k^l) + \partial_j \partial^j (\varphi k^i k^l) - \partial_l \partial^j (\varphi k^i k^j) \} = 0. \quad (3.146)
\end{align}

The key prescription of [19] is to define the single copy of this KS solution to be a YM field of the form:

\[ A^a_\mu = \varphi^a k_\mu, \quad (3.147) \]

where the $\varphi^a$'s are a set of scalars, which are assumed to solve the same equations as the scalar $\varphi$ defined in eq. (3.120). The YM field strength of this gauge field turns out to be linear, since the dependence on the color and the Lorentz indices is factorized:

\begin{align}
F^{ab}_\mu A^b_\nu &= f_{abc} A^c_\mu A^b_\nu = 0 \Rightarrow F^{a b}_\mu \nu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \quad (3.148)
\end{align}

due to the contraction of the symmetric product $\varphi^a \varphi^b$ with the structure constants. Then, it is straightforward to check that the Einstein-Hilbert equations for the metric in the KS form imply the abelian Yang-Mills equations of motion for the gauge field defined in eq. (3.147):

\begin{align}
\partial^\mu F^{a}_\mu \nu &= -\nabla^2 \varphi^a, \quad (3.149) \\
\partial^\mu F^a_\mu &= -\partial_j [\partial^i (\varphi^a k^j) - \partial^j (\varphi^a k^i)], \quad (3.150)
\end{align}
since eq. (3.149) ((3.150)) is the same as eq. (3.144) ((3.145)). The fact that the YM equations are in the abelian form (where the covariant derivative on the principal bundle is replaced with a partial derivative) reflects the linear structure of the Einstein-Hilbert equations in KS coordinates.

As far as the general features of this approach to the DC are concerned, two remarks are in order. First, the KS DC relates YM solutions to GR solutions, but the two-form and the scalar field typically present in the DC multiplet are absent. This is due to the fact that the graviton defined by this approach is $h_{\mu\nu} = \phi k_{\mu} k_{\nu}$, which is automatically symmetric and traceless ($k^2 = 0$). In this sense, the KS ansatz naturally selects only the purely gravitational sector of the DC. Moreover, it is worth stressing that in this approach the DC is defined in coordinate space, whereas in the case of scattering amplitudes it is properly defined in momentum space.

3.3.3 The double copy for the Schwarzschild black-hole

As an application of the general construction presented in the previous section, let us review a concrete example of the KS DC, namely the Schwarzschild black-hole. By Birkhoff’s theorem it is the most general spherically symmetric solution to the Einstein equations in vacuum. In Schwarzschild coordinates the line element reads:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2.$$  \hspace{1cm} (3.151)

For the sake of simplicity we consider the case of $D = 4$, in which $F(r) = 1 - \frac{2GM}{r}$ is a scalar function and $d\Omega^2$ is the measure on the two-sphere. Even though the Schwarzschild solution is a vacuum solution, it is possible to source it with a static point-like mass $M$ in the origin of the Schwarzschild coordinates:

$$T_{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(x),$$  \hspace{1cm} (3.152)

with $v^\mu = (1, 0, 0, 0)$.

In order to illustrate the physical meaning of the KS DC in this case, first of all we need to perform a coordinate transformation so as to recast the metric in the KS form.
to this end, starting from eq. (3.151) we define $du := dt + \frac{dr}{S(r)}$:

$$ds^2 = -F(r)du^2 + 2drdu - \frac{dr^2}{F(r)} + \frac{dr^2}{F(r)} + r^2d\Omega^2 = -F(r)du^2 + 2drdu + r^2d\Omega^2. \quad (3.153)$$

Then, introducing $T := u - r$:

$$ds^2 = -F(r)(dT + dr)^2 + 2drdT + 2dr^2 + r^2d\Omega^2$$

$$= \eta_{\mu\nu}dx^\mu dx^\nu + (1 - F(r))(dT^2 + 2\frac{x_i}{r}dx^idT + \frac{x_i x_j}{r^2}dx^i dx^j)$$

$$= \eta_{\mu\nu}dx^\mu dx^\nu + \varphi k_\mu k_\nu dx^\mu dx^\nu, \quad (3.154)$$

where $\varphi = 1 - F(r)$ and $k_\mu = (1, \frac{x_i}{r})$, and where one can explicitly check that:

- $k^2 = 0$: indeed, $k^2 = 1 - \frac{x_i x_j}{r^2} = 0$.
- $k \cdot \partial(k_\mu) = 0$: observe $\partial_\mu(\frac{1}{r}) = -\frac{x_i}{r^2}$ and $\partial_\mu(\frac{x_i}{r}) = -\frac{x_i x_j}{r^2} + \frac{\delta_{ij}}{r}$. Clearly $k \cdot \partial(k_0) = 0$, while $k \cdot \partial(k_i) = \frac{x_j}{r}(-\frac{x_i x_j}{r^2} + \frac{\delta_{ij}}{r}) = 0$.

This proves that a KS form is possible for every metric in the form (3.151).

In the specific case of the Schwarzschild black-hole $1 - F(r) = \frac{2GM}{r}$. Therefore, if we redefine the scalar $\varphi$ factoring the gravitational coupling constant ($\kappa^2 = 8\pi G$ in our conventions) and the mass, i.e. $\varphi = \frac{1}{4\pi}$, while at the same time defining the graviton field $h_{\mu\nu}$ in $g_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, one eventually finds:

$$h_{\mu\nu} = \frac{\kappa}{2} M \varphi k_\mu k_\nu = \frac{\kappa}{2} \frac{M}{4\pi r} k_\mu k_\nu. \quad (3.155)$$

In order to build the corresponding gauge theory solution, we follow the prescriptions of [19] to extract the single copy via the substitutions:

$$\frac{\kappa}{2} \rightarrow g, \quad M \varphi \rightarrow \varphi^a, \quad k_\mu k_\nu \rightarrow k_\mu. \quad (3.156)$$

The resulting Yang-Mills field is then:

$$A^a_\mu = \frac{g c^a}{4\pi r} k_\mu = \frac{g c^a}{4\pi r} \left(1, \frac{x_i}{r}\right). \quad (3.157)$$

The goal of gaining a better physical interpretation of this gauge field, we perform a
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gauge transformation $A^a_\mu \rightarrow A^a_\mu + \partial_\mu \chi^a$, with a time-independent scalar $\chi$ in order to leave the zeroth component of $A^a_\mu$ unaltered. It is then possible to completely eliminate the spatial components of $A^a_\mu$ choosing

$$\partial_i \chi^a = -A^a_i \Rightarrow \chi^a = \frac{-gc^a}{4\pi} \int \frac{x_i dx_i}{r^2} = \frac{-gc^a}{4\pi} \int \frac{dr}{r} = \frac{-gc^a}{4\pi} \ln \left( \frac{r}{r_0} \right).$$ (3.158)

In this gauge, the single copy of a Schwarzschild black-hole turns out to be a Coulomb potential:

$$A^a_\mu = \left( \frac{gc^a}{4\pi r}, 0 \right).$$ (3.159)

We can also compute the “single-copy” of the source exploiting the abelian Yang-Mills equations to extract $j^a_\nu = \partial^\mu F^a_{\mu\nu}$:

- $\partial^\mu F^a_{\mu 0} = \Box A^a_0 - \partial_0 \partial \cdot A^a = \nabla^2 A^a_0 = \frac{gc^a}{4\pi} \nabla^2 \frac{1}{r} = -gc^a \delta^{(3)}(\vec{x})$, (3.160)

- $\partial^\mu F^a_{\mu i} = \Box A^a_i - \partial_i \partial \cdot A^a = 0$. (3.161)

The resulting current is:

$$\partial^\mu F^a_{\mu\nu} = -gc^a v_\nu \delta^{(3)}(\vec{x}) = j^a_\nu.$$ (3.162)

Therefore, $T_{\mu\nu}$ is a DC of $j^a_\nu$ with the same set of substitution rules given for the fields, except that $k_\mu k_\nu \rightarrow k_\mu$ is replaced by $v_\mu v_\nu \rightarrow v_\mu$. The physical interpretation of $j^a_\mu$ is clearly that of a static color charge source located at the origin.

This result was obtained in $D = 4$ but it is easily generalizable to any spacetime dimension $D$ just by substituting $1/r \rightarrow 1/r^{D-3}$ and accordingly modifying the angular measure.

3.3.4 The double copy for (A)dS spacetime

As an additional explicit example of the KS DC, we review the construction for the (A)dS spacetime [22]. Since the cosmological constant term can be considered as an effective energy density, we can apply the KS DC construction in the presence of sources. (A)dS spacetime admits coordinates in which the line element reads as in eq. (3.151), although
with a different value of the function $F(r)$ that takes the form:

$$
F(r) = \begin{cases} 
1 + \frac{r^2}{l^2} & \text{AdS}, \\
1 - \frac{r^2}{l^2} & \text{dS},
\end{cases}
$$

(3.163)

with $\frac{1}{l^2} = \frac{|\Lambda|}{3}$ in $D = 4$. With the same coordinate transformations employed in the previous section (eqs. (3.153) and (3.154)), the (A)dS metric can be recast in the form:

$$
ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \mp \frac{r^2}{l^2} (dT + dr)^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \varphi k_\mu k_\nu dx^\mu dx^\nu,
$$

(3.164)

with $\varphi = r^2/l^2$ and, as in the case of the Schwarzschild black-hole, $k_\mu = (1, k_i/r)$. In this case, the role of the gravitational charge is played by the cosmological constant, or equivalently by the factor $1/l^2 = \frac{\Lambda}{3}$. The cosmological constant is related to the vacuum energy density via $\Lambda = 8\pi G \rho_{\text{vac}} = \kappa^2 \rho_{\text{vac}}$, therefore in this case to keep the correct mass dimensions we make the substitution $\rho_{\text{vac}} \to \rho_{\text{YM}}^a$, with $\rho_{\text{YM}}^a$ a uniform color charge density. All in all, with the graviton defined as before in $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, the result is:

$$
h_{\mu\nu} = \frac{1}{6} \kappa r^2 \rho_{\text{vac}} k_\mu k_\nu = \frac{\kappa}{2} \varphi \rho_{\text{vac}} k_\mu k_\nu.
$$

(3.165)

Thus, performing the substitutions:

$$
\frac{\kappa}{2} \to g, \quad \rho_{\text{vac}} \varphi \to \varphi^a = \rho_{\text{YM}}^a \varphi, \quad k_\mu k_\nu \to k_\mu,
$$

(3.166)

the single-copy of (A)dS spacetime is:

$$
A^a_\mu = \mp g\rho_{\text{YM}}^a r^2 k_\mu = \mp g\rho_{\text{YM}}^a r^2 \left(1, \frac{x_i}{r}\right),
$$

(3.167)

As in the case of the single copy of the Schwarzschild black-hole, the spatial part of this gauge field actually is pure gauge and as such it can be eliminated via a transformation $A^a_\mu \to A^a_\mu + \partial_\mu \chi^a$, with

$$
\partial_i \chi^a = - A^a_i \Rightarrow \chi^a = \pm g\rho_{\text{YM}}^a \int r x_i dx_i = \pm g\rho_{\text{YM}}^a \int r^2 dr = \pm g\rho_{\text{YM}}^a \frac{r^3}{3}.
$$

(3.168)
The resulting field is an electrostatic field with potential:

\[ A_\mu^a = \pm g \rho_{YM}^a r^2 k_\mu = \pm g \rho_{YM}^a r^2 (1, 0). \tag{3.169} \]

In this case we expect an extended source for the gauge field configuration, since it is the single copy of a GR solution with non-vanishing energy-momentum tensor everywhere in spacetime. The current can be extracted from the linearized YM equations,

\[ \partial^\mu F_{\mu 0}^a = \Box A_0^a - \partial_0 \partial \cdot A^a = \nabla^2 A_0^a = \mp 6 g \rho_{YM}^a \quad (\nabla^2 (r^2) = 6), \tag{3.170} \]
\[ \partial^\mu F_{\mu i}^a = \Box A_i^a - \partial_i \partial \cdot A^a = 0, \tag{3.171} \]

and turns out to be:

\[ \partial^\mu F_{\mu \nu}^a = \mp 6 g \rho_{YM}^a v_\nu = j_\nu^a, \tag{3.172} \]

with \( v_\mu = (1, 0, 0, 0) \). This current has the physical interpretation of a uniform distribution of (color) charge, with density \( \rho^a = \mp 6 g \rho_{YM}^a \): it is the YM correspondent of a cosmological constant term, which is a uniform energy density completely filling the spacetime.
Chapter 4

Lagrangian formulation

In this chapter we collect the original results that we obtained so far in the attempt to understand the DC relations from the Lagrangian perspective. We shall exploit the product of two YM fields introduced in [58] and discussed in Section 3.1 in order to define the graviton, the two-form and the scalar in terms of YM fields, to then build a suitable quadratic Lagrangian for the corresponding DC system. Then we shall review the Noether procedure, which is a perturbative algorithm allowing to build interacting gauge theories starting from a quadratic action with free gauge invariance, while also providing some examples of concrete implementations of this method. We shall exploit the Noether procedure to build the cubic vertices for our DC quadratic Lagrangian, proving their consistency with the amplitude results of Section 2.4.2, 2.4.3 and comparing with the off-shell formulation of [16], here discussed in Section 3.2.2. Last, we shall discuss the deformation of the linear gauge transformation of $H_{\mu\nu}$ which results from the Noether procedure, introducing the necessary non-linear deformations to the definitions of the graviton and of the scalar field.

4.1 Quadratic Lagrangian for the double-copy

In this section we build a quadratic Lagrangian which is the sum of the quadratic Lagrangians of the gravity fields $h_{\mu\nu}$, $B_{\mu\nu}$, $\varphi$, based on the definition of the field $H_{\mu\nu}$ in terms of YM fields given in [58]. The final result has a clear interpretation as the “square”, in a generalized sense, of two YM quadratic Lagrangians. In addition, we discuss the equations of motion and the gauge-fixing which is needed to extract the physical degrees of freedom propagated by our quadratic Lagrangian.
Our starting point is the definition (3.1) of the DC field $H_{\mu\nu}$, together with its gauge transformation (3.37), that we reproduce here for the reader’s convenience:

\[ H_{\mu\nu} : = A_\mu \star \tilde{A}_\nu, \quad (4.1) \]

\[ \delta H_{\mu\nu} = \partial_\mu \alpha_\nu + \partial_\nu \tilde{\alpha}_\mu. \quad (4.2) \]

In terms of this field, we defined in (3.31) the graviton ($h_{\mu\nu}$), the two-form ($B_{\mu\nu}$) and the scalar ($\phi$):

\[
\begin{cases}
    h_{\mu\nu} := H_{\mu\nu}^S - \gamma \eta_{\mu\nu} \phi, \\
    B_{\mu\nu} := H_{\mu\nu}^A, \\
    \phi := H - \frac{\partial \partial \cdot}{\Box},
\end{cases}
\]

where we recall that

\[ H_{\mu\nu}^S := \frac{1}{2} H_{(\mu\nu)} = \frac{1}{2} (H_{\mu\nu} + H_{\nu\mu}), \]

\[ H_{\mu\nu}^A := \frac{1}{2} H_{[\mu\nu]} = \frac{1}{2} (H_{\mu\nu} - H_{\nu\mu}). \]

In order to build an action for $H_{\mu\nu}$ we first consider the most general quadratic Lagrangian with two derivatives; taking into account that $H_{\mu\nu} \neq H_{\nu\mu}$ we have, up to total derivatives:

\[
\mathcal{L}_0^{(H)} = a_1 \partial_\mu H_{\alpha\beta} \partial^\mu H^{\alpha\beta} + a_2 \partial_\mu H_{\alpha\beta} \partial^\mu H^\alpha \partial^\beta + a_3 \partial^\alpha H_{\alpha\beta} \partial_\gamma H^{\gamma\beta} + a_4 \partial^\alpha H_{\alpha\beta} \partial_\gamma \partial^\beta H^{\gamma\alpha}
\]

\[
+ a_5 \partial^\alpha H_{\beta\alpha} \partial_\gamma H^{\beta\gamma} + a_6 \partial^\alpha H_{\alpha\beta} \partial^\beta H + a_7 \partial_\alpha H \partial^\alpha H.
\]

Let us compute the gauge variation of each term separately to then impose gauge invariance of the full expression. Denoting with $\delta_i$ the variation of the term with coefficient $a_i$ in (4.6) and allowing for integrations by parts we obtain

\[
\begin{align*}
    \delta_1 &= 2 \partial_\mu H_{\alpha\beta} \partial^\mu \partial^\alpha \alpha^\beta + 2 \partial_\mu H_{\alpha\beta} \partial^\mu \partial^\beta \partial^\alpha \tilde{\alpha}^\alpha = 2 \Box \partial^\alpha H_{\alpha\beta} \alpha^\beta + 2 \partial \partial H_{\alpha\beta} \partial^\alpha \tilde{\alpha}^\alpha, \\
    \delta_2 &= 2 \partial_\mu H_{\alpha\beta} \partial^\mu \partial^\alpha \alpha^\alpha + 2 \partial_\mu H_{\alpha\beta} \partial^\mu \partial^\beta \partial^\alpha \tilde{\alpha}^\beta = 2 \partial \partial H_{\alpha\beta} \alpha^\alpha + 2 \partial \partial H_{\alpha\beta} \tilde{\alpha}^\beta, \\
    \delta_3 &= 2 \partial^\alpha (\partial_\alpha \alpha^\beta + \partial_\beta \tilde{\alpha}^\alpha) \partial_\gamma H^{\gamma\beta} = 2 \Box \partial_\gamma H^{\gamma\beta} \alpha^\beta + 2 \partial^\alpha \partial_\beta \partial_\gamma H^{\beta\gamma} \tilde{\alpha}^\alpha, \\
    \delta_4 &= \Box \alpha^\beta \partial_\gamma H^{\beta\gamma} + \partial^\alpha \partial_\beta \partial_\gamma H^{\beta\gamma} + \partial_\beta \partial_\gamma \alpha \partial_\alpha H^{\alpha\beta} + \Box \tilde{\alpha}^\beta \partial^\alpha H_{\alpha\beta} \\
    &= \Box \partial^\beta H^{\alpha\beta} (\alpha + \tilde{\alpha})^\alpha + \partial^\alpha \partial_\beta \partial_\gamma H^{\beta\gamma} (\alpha + \tilde{\alpha})^\alpha.
\end{align*}
\]
\[ \delta_5 = 2 \partial^\alpha (\partial_\beta \alpha + \partial_\alpha \hat{\alpha}) \partial_\gamma H^{\beta \gamma} = 2 \partial^\alpha \partial_\beta \partial_\gamma H^{\beta \gamma} \alpha + 2 \Box \partial_\gamma H^{\beta \gamma} \hat{\alpha}, \quad (4.11) \]

\[ \delta_6 = \partial^\alpha (\partial_\alpha \alpha + \partial_\alpha \hat{\alpha}) \partial_\beta H + \partial^\alpha H_{\alpha \beta} \partial^\beta (\partial \cdot \alpha + \partial \cdot \hat{\alpha}) \\
= \partial^\beta \Box H (\alpha + \hat{\alpha})_\beta + \partial^\gamma \partial_\alpha \partial_\beta H^{\alpha \beta} (\alpha + \hat{\alpha})_\gamma, \quad (4.12) \]

\[ \delta_7 = 2 \partial^\beta H \partial_\beta (\partial \cdot \alpha + \partial \cdot \hat{\alpha}) = 2 \partial^\beta \Box H (\alpha + \hat{\alpha})_\beta. \quad (4.13) \]

Collecting separately the terms proportional to \( \alpha \) and \( \hat{\alpha} \) and imposing \( \delta L_0^{(H)} = 0 \) we derive the conditions:

\[
\begin{align*}
2a_1 + 2a_3 &= 0, \\
2a_1 + 2a_5 &= 0, \\
2a_2 + a_4 &= 0, \\
2a_3 + a_4 + a_6 &= 0, \\
2a_5 + a_4 + a_6 &= 0, \\
a_6 + 2a_7 &= 0.
\end{align*}
\]

(4.14)

The system admits a two-parameters family of solutions \((a := a_1, b := a_2)\):

\[
a_3 = -a, \quad a_4 = -2b, \quad a_5 = -a, \quad a_6 = 2(a + b), \quad a_7 = -(a + b),
\]

leading to the Lagrangians:

\[
L_0^{(H)}(a, b) = a \partial_\mu H_{\alpha \beta} \partial^\mu H^{\alpha \beta} + b \partial_\mu H_{\alpha \beta} \partial^\mu H^{\beta \alpha} - a \partial^\alpha H_{\alpha \beta} \partial_\gamma H^{\gamma \beta} - 2b \partial^\alpha H_{\alpha \beta} \partial_\gamma H^{\gamma \beta} \\
- a \partial^\alpha H_{\beta \alpha} \partial_\gamma H^{\beta \gamma} + 2(a + b) \partial^\alpha H_{\alpha \beta} \partial_\beta H - (a + b) \partial_\alpha H \partial^\alpha H.
\]

(4.16)

In terms of the gravitational multiplet \( H_{\mu \nu} = h_{\mu \nu} + B_{\mu \nu} + \gamma \eta_{\mu \nu} \varphi \) one has:

\[
L_0^{(H)}(a, b) = (a + b) \left[ \partial_\mu h_{\alpha \beta} \partial^\mu h^{\alpha \beta} - 2 \partial \cdot h^\alpha \partial \cdot h_{\alpha} + 2 \partial \cdot h^\alpha \partial_\alpha h - \partial^\alpha h \partial_\alpha h \right] + \\
+ \gamma (a + b) \partial^\mu \varphi \left[ 2(D - 2) \partial \cdot h_{\mu} - 2(D - 2) \partial_\mu h - \gamma (D^2 - 3D + 2) \partial_\mu \varphi \right] + \\
+ (a - b) \left[ \partial_\mu B_{\alpha \beta} \partial^\mu B^{\alpha \beta} - 2 \partial^\alpha B_{\alpha \beta} \partial_\beta \gamma \beta \right],
\]

(4.17)

where one has to keep in mind the relations connecting the field in the scalar sector of
the theory:

\[
\begin{aligned}
&h = H - \gamma D \varphi, \\
&\partial \partial \cdot h = \partial \partial \cdot H - \gamma \varphi, \\
\Rightarrow h - \partial \partial \cdot h = [1 - \gamma (D - 1)] \varphi, \\
\end{aligned}
\]  

which we shall use many times in this chapter. In particular, integrating by parts the terms with a mixing between \(h_{\mu \nu}\) and \(\varphi\) in (4.17) one finds

\[
2(D - 2) \partial^\mu \varphi \partial \cdot h_{\mu} - 2(D - 2) \partial^\mu \varphi \partial_{\mu} h = 2(D - 2) \varphi (\Box h - \partial \partial \cdot h) \\
= 2(D - 2) [1 - \gamma (D - 1)] \varphi \Box \varphi, 
\]

so that, upon substituting in (4.17), one eventually finds

\[
\mathcal{L}^{(H)}_0 = (a + b) \left[ \partial_{\mu} h_{\alpha \beta} \partial^\mu h^{\alpha \beta} - 2 \partial \cdot h^\alpha \partial \cdot h_\alpha + 2 \partial \cdot h^\alpha \partial_\alpha h - \partial^\mu h_{\alpha} \partial_\alpha h \right] \\
- \gamma (a + b) \left[ -2(D - 2) [1 - \gamma (D - 1)] + \gamma (D^2 - 3D + 2) \right] \partial_\mu \varphi \partial^\mu \varphi' \\
+ (a - b) \left[ \partial_\mu B_{\alpha \beta} \partial^\mu B^{\alpha \beta} - 2 \partial^\alpha B_{\alpha \beta} \partial_\gamma B^{\gamma \beta} \right]. 
\]

Up to overall normalizations, (4.20) is the sum of the quadratic Lagrangians for a graviton, a two-form and a scalar field. We can exploit the parameters \(a\) and \(b\) in order to fix the canonical normalization for the graviton and the two-form:

\[
\begin{aligned}
&a + b = -\frac{1}{2}, \quad \Rightarrow a = -\frac{1}{2}, b = 0. \\
&a - b = -\frac{1}{2}, \\
\end{aligned}
\]

The result is, denoting with \(\mathcal{L}^{EH}_0\) the quadratic part in the expansion of the Einstein-Hilbert Lagrangian and with \(\mathcal{L}^B_0\) the two-form quadratic Lagrangian (both with correct normalizations):

\[
\mathcal{L}^{(H)}_0 = \mathcal{L}^{EH}_0 + \mathcal{L}^B_0 + \frac{1}{2} \gamma (D - 2) \left[ 2 - \gamma (D - 1) \right] \partial_\mu \varphi \partial^\mu \varphi, 
\]  

where the normalization of the scalar field depends on the value of \(\gamma\). In particular:

- When \(\gamma = 0\) the scalar is absent,
- When \(\gamma = \frac{1}{D - 2}\), the normalization is \(\frac{D - 3}{2(D - 2)}\), which has the wrong sign for \(D > 3\) and anyway does not have a clear interpretation,
• More generally, the normalization can be either positive or negative, according to
the value of $\gamma$.

Since at the level of gauge symmetries all the values of $\gamma$ are equivalent, as we noticed in
Section 3.1, it may be unexpected that for some values of $\gamma$ the scalar is a ghost. We also
observed that, for some applications, two values of $\gamma$ seem to be the most plausible ones:
$\gamma = 0$ when dealing with the field strength of $H_{\mu\nu}$ in Section 3.1.4 and $\gamma = \frac{1}{D-2}$ when
dealing with the equations of motion in Section 3.1.5 and 3.1.6. In particular, $\gamma = \frac{1}{D-2}$
seems to be promising also because it resonates with the decomposition of polarization
vectors of Section 2.4.1. However, the normalization of the scalar in eq. (4.22) apparently
favors other choices for $\gamma$.

In order to clarify this issue let us notice that, given our definition of the scalar field,
there is another term which might be included in the Lagrangian of (4.6):

$$
\delta_8 = -2a_8 \partial^\mu \partial^\nu H_{\alpha\beta} \partial^\gamma H_{\alpha\beta} - 2a_8 \partial^\mu \partial^\nu H \partial_{\alpha\beta},
$$

(4.23)

leading to the most general quadratic Lagrangian where one allows non-local terms in
pure gauge contributions. Including (4.23) in (4.6), the system (4.14) changes to

$$
\begin{align*}
2a_1 + 2a_3 &= 0, \\
2a_1 + 2a_5 &= 0, \\
2a_2 + a_4 &= 0, \\
2a_3 + a_4 + a_6 - 2a_8 &= 0, \\
2a_5 + a_4 + a_6 - 2a_8 &= 0, \\
a_6 + 2a_7 &= 0,
\end{align*}
$$

(4.24)

that admits a three parameters family of solutions ($a := a_1, b := a_2, c := a_8$):

$$
a_3 = -a, \ a_4 = -2b, \ a_5 = -a, \ a_6 = 2(a + b + c), \ a_7 = -(a + b + c).
$$

(4.25)

The resulting Lagrangian is:

$$
\mathcal{L}^{(H)}_0 = a \partial_{\mu} H_{\alpha\beta} \partial^\mu H^{\alpha\beta} + b \partial_{\mu} H_{\alpha\beta} \partial^\mu H^{\gamma\beta} - a \partial^\alpha H_{\alpha\beta} \partial_{\gamma} H^{\gamma\beta} - 2b \partial^\alpha H_{\alpha\beta} \partial_{\gamma} H^{\gamma\beta} - a \partial^\alpha H_{\beta\delta} \partial_{\beta} H_{\delta\gamma} + 2(a + b) \partial^\alpha H_{\alpha\beta} \partial_{\beta} H - (a + b) \partial_{\alpha} H \partial^\alpha H
$$
Moreover, using the definition \( H_{\mu\nu} = A_{\mu} \star \tilde{A}_\nu \) and integrating by parts

\[
L_0^{(H)} = \frac{1}{2} H_{\alpha\beta} \Box H^{\alpha\beta} - \partial^\alpha \partial_\gamma H_{\alpha\beta} \partial_\gamma H^{\alpha\beta} - \frac{1}{2} \Box H_{\alpha\beta} \partial_\gamma H^{\alpha\beta} + \frac{1}{2} \Box (\partial \cdot H)^2
\]

(4.30)

Moreover, using the definition \( H_{\mu\nu} = A_{\mu} \star \tilde{A}_\nu \) and integrating by parts

\[
L_0^{(H)} = \frac{1}{2} H_{\alpha\beta} \{ \Box H^{\alpha\beta} - \partial_\gamma \partial_\gamma H^{\alpha\beta} - \partial_\gamma \partial_\gamma H^{\alpha\beta} + \partial_\gamma \partial_\gamma \partial \cdot H \}
\]

(4.27)

 tantamount to the possibility of fixing the normalization of the scalar independently of \( h_{\mu\nu} \) and \( B_{\mu\nu} \). However, the scalar kinetic term receives different contributions for different values of \( \gamma \), as evident from (4.22). Thus, keeping \( c \) arbitrary but fixing \( a = -\frac{1}{2}, b = 0 \) (canonical normalization for \( h_{\mu\nu} \) and \( B_{\mu\nu} \)), eq. (4.26) can be written as:

\[
L_0^{(H)} = -\frac{1}{2} \partial_\mu H_{\alpha\beta} \partial^\mu H^{\alpha\beta} + \frac{1}{2} \partial^\mu H_{\alpha\beta} \partial_\gamma H^{\gamma\beta} + \frac{1}{2} \partial^\mu H_{\beta\alpha} \partial_\gamma H^{\beta\gamma} - (1 - 2c) \partial^\mu H_{\alpha\beta} \partial_\beta H
\]

(4.28)

An interesting possibility emerges from this Lagrangian: the choice \( c = \frac{1}{2} \) eliminates two of its terms,

\[
L_0^{(H)} = \frac{1}{2} H_{\alpha\beta} \Box H^{\alpha\beta} - \frac{1}{2} H_{\alpha\beta} \partial^\alpha H^{\gamma\beta} \partial_\gamma H^{\alpha\beta} - \frac{1}{2} H_{\alpha\beta} \partial^\beta \partial_\gamma H^{\alpha\gamma} + \frac{1}{2} \Box (\partial \cdot H)^2
\]

(4.29)

with the appealing feature that the term in the curly brackets can be seen as the product of two YM kinetic operators, divided by \( \Box \):

\[
L_{YM} = A^{\alpha}_a \{ \Box \eta_{\alpha\mu} - \partial_\alpha \partial_\mu \} A_{\alpha}^a, \quad L_{YM} = \tilde{A}_{\alpha}^a \{ \Box \eta_{\beta\nu} - \partial_\beta \partial_\nu \} \tilde{A}_{\alpha}^a,
\]

\[
\{ \Box \eta_{\alpha\mu} \eta_{\beta\nu} - \partial_\alpha \partial_\mu \eta_{\beta\nu} - \partial_\beta \partial_\nu \eta_{\alpha\mu} + \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \} = \{ \Box \eta_{\alpha\mu} - \partial_\alpha \partial_\mu \} \frac{1}{4} \{ \Box \eta_{\beta\nu} - \partial_\beta \partial_\nu \}
\]

(4.30)
\[ \times \frac{1}{\Box} (\partial^\mu \partial^\nu A^\alpha \star \tilde{A}^\beta - \partial^\mu \partial^\beta A^\alpha \star \tilde{A}^\nu - \partial^\alpha \partial^\nu A^\mu \star \tilde{A}^\beta + \partial^\alpha \partial^\beta A^\mu \star \tilde{A}^\nu), \] (4.31)

therefore, even more suggestively in terms of the (abelian) field strengths:

\[ \mathcal{L}^{(H)}_0 = \frac{1}{8} (F_{\mu\alpha} \star \tilde{F}_{\nu\beta}) \frac{1}{\Box} (F_{\mu\alpha} \star \tilde{F}_{\nu\beta}). \] (4.32)

Amongst all our options, the Lagrangian (4.32) has the cleanest interpretation as the “square” of two Yang-Mills quadratic Lagrangians; while the non-local factor also accounts for the restoration of the correct dimension for a quadratic Lagrangian. Moreover, recalling from Section 3.1.4 the definition of the curvature tensor for \( H_{\mu\nu} \):

\[ R^s_{\mu\nu\rho\sigma} := -\frac{1}{2} F_{\mu\nu} \star \tilde{F}_{\rho\sigma}, \] (4.33)

from eq. (4.32) we conclude that this Lagrangian can be written as:

\[ \mathcal{L}^{(H)}_0 = \frac{1}{2} R^s_{\mu\nu\rho\sigma} \frac{1}{\Box} R^s_{\mu\nu\rho\sigma}. \] (4.34)

This form of the quadratic Lagrangian is strongly reminiscent of the spin-two representative in the class of quadratic Lagrangians found in [129] for Maxwell-like higher spin fields. In the case of [129] the gauge potential was chosen to be totally symmetric, and the corresponding equations of motion where shown to propagate a massless graviton together with a massless scalar, but of course no two-form was present in the spectrum. Differently, in (4.34) the generalized curvature \( R^s_{\mu\nu\rho\sigma} \) also includes a torsion which, as we explained in Section 3.1.4, is exactly due to the presence of the two-form field\(^1\).

Therefore, we see that the choice \( c = \frac{1}{2} \) in eq. (4.28) leads to a Lagrangian with a clear geometric interpretation (eq. (4.34)) which, when expressed in terms of gauge fields, can be seen in a sense as the “square” of two linearized YM Lagrangians in the sense of eq. (4.32).

\(^1\)Interestingly, the geometric Lagrangians of [129] have an interpretation as sectors of the tensionless Lagrangian of free Open String Field Theory, after integration over the auxiliary fields present in the corresponding BRST construction [130, 131], while also relating to a Lagrangian description of massless higher spins allowing for the propagation of reducible spectra of particles, more akin to the spirit of the DC construction under scrutiny in this work [129, 132].
geometric interpretation, turns out to be promising also for other reasons. Let us go back to the expansion \( H_{\mu \nu} = h_{\mu \nu} + B_{\mu \nu} + \gamma \eta_{\mu \nu} \phi \) in eq. (4.29), in order to study the normalization of the scalar for different values of \( \gamma \):

\[
\mathcal{L}_0^{(H)} = \mathcal{L}_0^{EH} + \mathcal{L}_0^B - \frac{1}{2} \{(D - 2)(D - 1)\gamma^2 - 2\gamma(D - 2) + 1\}\partial_\mu \phi \partial^\mu \phi. \tag{4.35}
\]

Where the normalization \( N \) of the scalar kinetic term is given by:

\[
N = -\frac{1}{2} \{(D - 2)(D - 1)\gamma^2 - 2\gamma(D - 2) + 1\}. \tag{4.36}
\]

The discriminant of the polynomial is \( \Delta = -4(D - 2) \), while the coefficient of \( \gamma^2 \) is also positive when \( D > 2 \), we conclude that the polynomial is always positive and therefore the sign of \( N \) is always correct. Moreover, one can notice that

- \( \gamma = 0 \): \( N = -\frac{1}{2} \). In this case, the normalization of the scalar is canonical.
- \( \gamma = \frac{1}{D-2} \): \( N = -\frac{1}{2(D-2)} \). In this case the normalization is not canonical, but it corresponds to the normalization of the scalar field in the action of \( N = 0 \) Supergravity, given in eq. (2.117).

### 4.1.1 Equations of motion

From the Lagrangian (4.29), it is possible to derive the equations of motion for the field \( H_{\mu \nu} \):

\[
\mathcal{E}_{\mu \nu} := \frac{\delta \mathcal{L}_H}{\delta H_{\mu \nu}} = \Box H_{\mu \nu} - \partial_\mu \partial^\alpha H_{\alpha \nu} - \partial_\nu \partial^\alpha H_{\mu \alpha} + \partial_\mu \partial_\nu \frac{\partial \cdot \partial H}{\Box} = 0, \tag{4.37}
\]

that we now scrutinize, in order to understand whether and how they contain the equations of motion for all the fields in the gravitational multiplet. To this end we can study separately symmetric and antisymmetric parts of the equations of motion.

First, let us consider the antisymmetric part which, upon identifying \( B_{\mu \nu} = \frac{1}{2} H_{[\mu \nu]} \), gives indeed:

\[
\frac{1}{2} \mathcal{E}_{[\mu \nu]} = \Box B_{\mu \nu} - \partial_\mu \partial^\alpha B_{\alpha \nu} + \partial_\nu \partial^\alpha B_{\mu \alpha} = 0, \tag{4.38}
\]

i.e. the equations of motion for a two-form gauge field. Recalling that, from the definition
(4.3) of the fields in the gravitational multiplet, we can use
\[ \frac{\partial \cdot \partial \cdot H}{\Box} = H - \varphi = h + (\gamma D - 1) \varphi, \] (4.39)
the symmetric part of (4.37) gives instead
\[ \frac{1}{2} \mathcal{E}_{(\mu \nu)} = \Box h_{\mu \nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\mu + \partial_\mu \partial_\nu h + \gamma \eta_{\mu \nu} \Box \varphi + [\gamma (D - 2) - 1] \partial_\mu \partial_\nu \varphi = 0. \] (4.40)
We consider the two values of $\gamma$ which proved to be the most relevant for our discussion, with the results:

- $\gamma = 0$: we can read the equations of motion in the two equivalent ways
  \[ \frac{1}{2} \mathcal{E}_{(\mu \nu)} = \Box h_{\mu \nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\mu + \gamma \eta_{\mu \nu} \Box \varphi \] (4.41)
  \[ = \Box h_{\mu \nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\mu + \gamma \eta_{\mu \nu} \frac{\partial \cdot \partial \cdot h}{\Box}, \] (4.42)
where from (4.41) to (4.42) we used $\varphi = h - \frac{\partial \cdot \partial \cdot h}{\Box}$, which hold when $\gamma = 0$. The first part of (4.41) is equivalent to $R^\text{lin}_{\mu \nu}(h)$ (apart from an overall constant), i.e. the linearized Ricci tensor in terms of a metric fluctuation $h_{\mu \nu}$. However, the presence of the term $\partial_\mu \partial_\nu \varphi$ does not allow the interpretation of (4.40) with $\gamma = 0$ as the linearized equations of motion for a graviton. We can conclude that, even though the symmetric part of $\mathcal{E}_{\mu \nu}$ correctly propagates the degrees of freedom of a spin-two and of a spin-zero field (as shown in [60] for equations in the form (4.42) and in Section 4.1.2), when $\gamma = 0$ the decomposition (4.3) into graviton and scalar does not reproduce the linearized Einstein-Hilbert equations of motion.

- $\gamma = \frac{1}{D-2}$: one of the terms containing $\varphi$ in eq. (4.40) cancels out, for the other we can use eq. (4.18) with the given value of $\gamma$ to write $\gamma \Box \varphi = \Box h - \partial \cdot \partial \cdot h = R^\text{lin}_{(\mu \nu)}(h)$, i.e. the linearized Ricci scalar in terms of a metric fluctuation $h_{\mu \nu}$. Then, eq. (4.40) can be written as
  \[ \Box h_{\mu \nu} - \partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu - \eta_{\mu \nu} (\Box h - \partial \cdot \partial \cdot h) = 0, \] (4.43)
which is equivalent to
\[ R^\text{lin}_{\mu \nu}(h) - \frac{1}{2} \eta_{\mu \nu} R^\text{lin}(h) = 0. \] (4.44)
Last, we consider the trace of (4.37), which can be expressed as follows:

\[ E = \eta^{\mu\nu} \mathcal{E}_{\mu\nu} = \Box H - \partial \cdot \partial \cdot H = \Box \varphi. \]  \hspace{1cm} (4.45)

The last equation exactly corresponds to the equations of motion of a free scalar field, therefore we can state that, when \( \gamma = \frac{1}{\sqrt{2}} \) in the decomposition of \( H_{\mu\nu} \), the equation \( \mathcal{E}_{\mu\nu} = 0 \) actually contains the free, linearized equations of motion for the fields \( h_{\mu\nu}, B_{\mu\nu} \) and \( \varphi \).

### 4.1.2 Gauge-fixing and degrees of freedom

The quadratic Lagrangian (4.29) possesses a large gauge invariance, which, as discussed in Section 3.1, originates from the spin-one gauge invariance and is given by (4.2):

\[ \delta H_{\mu\nu} = \partial_{\mu} \alpha_{\nu} + \partial_{\nu} \tilde{\alpha}_{\mu}, \]  \hspace{1cm} (4.46)

In this section we would like to illustrate two possible gauge fixings, aiming to highlight from a different angle the physical meaning of \( H_{\mu\nu} \). The first is to proceed from the point of view of gravity, splitting:

\[ \delta H_{\mu\nu} = \partial_{\mu} \alpha_{\nu} + \partial_{\nu} \tilde{\alpha}_{\mu} = \frac{1}{2} \partial_{\mu} (\alpha_{\nu} + \tilde{\alpha}_{\nu}) + \frac{1}{2} \partial_{\nu} (\alpha_{\mu} + \tilde{\alpha}_{\mu}) + \frac{1}{2} \partial_{\mu} (\alpha_{\nu} - \tilde{\alpha}_{\nu}) - \frac{1}{2} \partial_{\nu} (\alpha_{\mu} - \tilde{\alpha}_{\mu}) \]

\[ = \partial_{\mu} (\xi_{\nu}) + \partial_{\nu} (\Lambda_{\mu}) = \delta H_{\mu\nu}^{S} + \delta H_{\mu\nu}^{A}, \]  \hspace{1cm} (4.47)

where, \( \xi_{\mu} \) and \( \Lambda_{\mu} \) are the gauge parameters for \( H_{\mu\nu}^{S} \) and \( H_{\mu\nu}^{A} \), respectively. Since we are describing massless particles we want to reduce the equations of motion to \( \Box H_{\mu\nu} = 0 \), and this can be achieved with the covariant gauge-fixing

\[ \partial \gamma H_{\gamma\mu} = \partial \gamma H_{\mu\gamma} = 0. \]  \hspace{1cm} (4.48)

In order to see that (4.48) is indeed a possible gauge-fixing, it is convenient to consider separately the semi-sum and the semi-difference of eq. (4.48):

\[
\begin{cases}
S_{\mu} := \frac{1}{2} \partial \gamma (H_{\gamma\mu} + H_{\mu\gamma}) \to \delta S_{\mu} = \Box \xi_{\mu} + \partial_{\mu} \partial \cdot \xi, \\
A_{\mu} := \frac{1}{2} \partial \gamma (H_{\gamma\mu} - H_{\mu\gamma}) \to \delta A_{\mu} = \Box \Lambda_{\mu} - \partial_{\mu} \partial \cdot \Lambda.
\end{cases}
\]  \hspace{1cm} (4.49)
In order to set $S_{\mu} = 0$, we most solve the differential equation

$$\Box \xi_{\mu} + \partial_{\mu} \partial \cdot \xi = S_{\mu},$$  \hspace{1cm} (4.50)

which can be solved since the differential operator $\Box + \partial_{\mu} \partial \cdot$ has no zero modes and thus, in order to eliminate both longitudinal and transverse parts of $S_{\mu}$, one has to solve for ordinary wave equations with sources. This eliminates $D$ degrees of freedom of $H_{\mu\nu}$, corresponding to the $D$ components of $\xi_{\mu}$ needed in order to fix in order to set $S_{\mu} = 0$.

For the two-form $H_{\mu\nu}^A$, on the other hand, the differential operator entering $\delta H_{\mu\nu}^A$ has zero modes, but, as usual for $p$-forms, the gauge transformation of a two-form field, gauge transformation itself is reducible. Indeed, a transformation with parameter $\Lambda_{\mu}' = \Lambda_{\mu} + \partial_{\mu} \chi$, is the same as one without the term in $\chi$:

$$\delta H_{\mu\nu}^A = \partial_{\mu} \Lambda_{\nu}' - \partial_{\nu} \Lambda_{\mu}' = \partial_{\mu} (\Lambda_{\mu} + \partial_{\mu} \chi) - \partial_{\nu} (\Lambda_{\mu} + \partial_{\mu} \chi) = \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu}. \hspace{1cm} (4.51)$$

One can then exploit this “gauge-for-gauge” symmetry\(^2\) and choose $\chi$ such that $\partial \cdot \Lambda = 0$, so that in order to set $A_{\mu} = 0$ one must solve once again the sourced wave equation:

$$\Box \Lambda_{\mu} = A_{\mu} \Rightarrow \Lambda_{\mu} = \frac{1}{\Box} A_{\mu}, \hspace{1cm} (4.52)$$

while also taking into account that $\partial \cdot A \equiv 0$. Altogether, at this stage we have eliminated $D + D - 1 = 2D - 1$ degrees of freedom from $H_{\mu\nu}$. The gauge (4.48) corresponds to the de-Donder gauge for the graviton ($\partial \cdot h_{\mu} = \frac{1}{2} \partial_{\mu} h$) when the latter is defined as $h_{\mu\nu} = H_{\mu\nu}^S - \frac{1}{D-2} \eta_{\mu\nu} \varphi$, and to the Lorenz gauge for the two-form field ($\partial^\alpha B_{\alpha\beta} = 0$). Moreover, it reduces the definition of the scalar to $\varphi = H$, as expected from the discussion of Section 3.1.2. The equations of motion (4.37) in their turn simplify to

$$\Box H_{\mu\nu} = 0, \hspace{1cm} (4.53)$$

thus confirming that the theory describes massless particles. If we choose a reference frame where $p^\mu = (E, \vec{0}, E)$ (from now on latin indices denote little group indices: $i = 1, \ldots, D - 2$), the gauge fixing conditions $\partial^\gamma H_{\gamma\mu} = \partial^\gamma H_{\mu\gamma} = 0$ expressed in Fourier

\(^2\)Such symmetry also implies that only $D - 1$ degrees of freedom of $\Lambda_{\mu}$ can be exploited to the aim of performing the gauge-fixing.
Chapter 4. Lagrangian formulation

space can be written as follows:

\[
\begin{cases}
  H_{00} + H_{D-1,0} = 0, & (a) \\
  H_{0i} + H_{D-1,i} = 0, & (b) \\
  H_{0,D-1} + H_{D-1,D-1} = 0, & (c) \\
  H_{00} + H_{0,D-1} = 0, & (d) \\
  H_{0i} + H_{i,D-1} = 0, & (e) \\
  H_{D-1,0} + H_{D-1,D-1} = 0. & (f)
\end{cases}
\]  

(4.54)

These $2(D - 2) + 4 = 2D$ equations are not all independent: indeed, the last one is related to the others via \((f) = (a) + (c) - (d)\). The number of actually independent equations is therefore $2D - 1$, matching the degrees of freedom counting performed before. In this reference frame, taking into account the gauge-fixing conditions (4.54), we can write the independent (at this level) components of $H_{\mu\nu}$ as:

\[
H_{\mu\nu} = \begin{pmatrix}
  a_0 & a_1 & \ldots & a_{D-1} & -a_0 \\
  b_1 & & & & -b_1 \\
  & \ddots & & & \vdots \\
  b_{D-1} & H_{ij} & \cdots & & -b_{D-1} \\
  -a_0 & -a_1 & \ldots & -a_{D-2} & a_0
\end{pmatrix},
\]

where $H_{ij}$ is a $(D - 2) \times (D - 2)$ matrix. As in the case of the Maxwell theory or of linearized gravity, we can now exploit the equations of motion and the residual gauge invariance in order to eliminate further components of $H_{\mu\nu}$. To this end, we observe that eq. 4.50 is unaltered if we make a gauge transformation with parameter $\xi_\mu$ satisfying \(\Box \xi_\mu = 0\) and $\partial \cdot \xi = 0$. With such a $\xi$ we can remove $D - 1$ additional components of $H_{\mu\nu}$:

- $\delta H_{00}^S = 2ip\xi_0 \to \xi_0 = \frac{a_0}{2ip}$ sets $H_{00}^S = H_{00} = 0$. From the system (4.54):
  \[
  H_{00} = 0 \Rightarrow H_{D-1,0} = 0 \Rightarrow H_{D-1,D-1} = 0 \Rightarrow H_{0,D-1} = 0. 
  \]  

(4.55)

- $\delta H_{0i}^S = ip\xi_i \to \xi_i = \frac{a_i + b_i}{2ip}$ sets $H_{0i}^S = 0$.

(4.56)

- $\xi_{D-1}$ is fixed by $\partial \cdot \xi = 0$. 

\[\xi_{D-1} = \text{fixed} \quad \partial \cdot \xi = 0.\]
Therefore, at this stage we are removing further $D - 1$ degrees of freedom from $H_{\mu\nu}$, corresponding to the independent components of a vector $\xi_\mu$ satisfying $\partial \cdot \xi = 0$.

Then, we can exploit the residual gauge freedom with a parameter $\Lambda_\mu$ satisfying $\Box \Lambda_\mu = 0$ and $\partial \cdot \Lambda = 0$, after fixing its own residual gauge-for-gauge freedom, which leaves it with only $D - 2$ independent components, just those that are needed in order to satisfy the condition

$$\delta H_{0i}^S = ip\Lambda_i \to \xi_i = \frac{a_i - b_i}{2ip} \text{ sets } H_{0i}^A = 0,$$

which completes the elimination of the components of $H_{\mu\nu}$ with at least one index equal to 0 or $D - 1$, resulting in the fact that $H_{\mu\nu}$ propagates the same number of degrees of freedom as the $SO(D-2)$-reducible tensor $H_{ij}$. Indeed, since $H_{\mu\nu}$ is a non-symmetric matrix with $D^2$ components, we obtain the result that the number of physical degrees of freedom is given by:

$$\# \text{physical d.o.f.} = D^2 - 2(D-1) - (2D - 3) = D^2 - 4D + 4 = (D - 2)^2$$

$$= \underbrace{\frac{2}{\text{graviton}}} + \underbrace{\frac{2}{\text{two-form}}} + \frac{1}{\text{scalar}}.$$ (4.58)

At this stage, the actual decomposition of $H_{ij}$ into three irreducible representations can be performed as in Section 2.4.1, with the result that the Lagrangian (4.29), together with the gauge invariance of (4.46), actually describes the physical degrees of freedom of a graviton, a two-form and a scalar.

From an alternative perspective, one can focus on the fact that both the field $H_{\mu\nu}$ is defined in terms of Yang-Mills fields as in (3.1)

$$H_{\mu\nu} = A_\mu \star \tilde{A}_\nu.$$ (4.59)

Therefore, we can fix the Lorenz gauge for the YM fields

$$\partial \cdot A^a = 0 = \partial \cdot \tilde{A}^a.$$ (4.60)
with the result that, as in (4.48)

\[ \partial^{\gamma} H_{\gamma \mu} = \partial^{\gamma} H_{\mu \gamma} = 0. \]  

(4.61)

Such a gauge-fixing reduces the equations of motion to

\[ \Box A_{\mu} = 0 \]

on the Yang-Mills side and \( \Box H_{\mu \nu} = 0 \) on the gravity side. Still, it is possible to exploit the residual gauge invariance: as a result, the two Yang-Mills fields have only \( D - 2 \) physical components, reflecting into \( (D - 2)^2 \) physical degrees of freedom of \( H_{\mu \nu} \). Then, the decomposition into graviton, two-form and scalar directly follows, exactly as in Section 2.4.1.

### 4.2 Troubles with the naive inclusion of double-copy interactions

Our construction so far led to a free DC Lagrangian which can be suggestively expressed in terms of YM fields as

\[ L_0 = F^{\mu \nu} \ast \tilde{F}^{\alpha \beta} \frac{1}{\Box} F_{\mu \nu} \ast \tilde{F}_{\alpha \beta}. \]  

(4.62)

When extending the theory to the interacting level, a natural idea might be to insert in (4.62) the full YM field strength,

\[ F_{\mu \nu}^{a} = \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} - g f^{abc} A_{\mu}^{b} A_{\nu}^{c}, \]  

(4.63)

so as to obtain DC gravitational interactions in the most direct possible fashion. However, this solution is problematic from various perspectives, and eventually does not seem to lead to a sensible result at least if one attempts its most straightforward implementation.

First, the number of derivatives in the interaction vertices would not match the two-derivative interactions of gravity. Indeed, since the non-abelian part of (4.63) does not contain derivatives, putative interactions built in this way would not match, by construction, the gravitational ones.

Second, the definition of the product (3.1) itself does not allow a clean interpretation...
for products between more than two YM fields. Indeed, a “cubic vertex” derived from (4.62) with the inclusion of the non-abelian part of (4.63) would have the schematic form

\[ S_1 = \int d^D x [(A_1 A_2) \star (\tilde{A}_1 \tilde{A}_2)](x) [A_3 \star \tilde{A}_3](x). \]  

(4.64)

One can write (4.64) in momentum space, obtaining

\[ S_1 = \int dp_1 dp_2 \delta^{(D)}(p_1 + p_2) F\{A_1 A_2\}(p_1) F\{\tilde{A}_1 \tilde{A}_2\}(p_1) A_3(p_2) \tilde{A}_3(p_2), \]  

(4.65)

where we used the convolution theorem. As evident, the presence of more than one field in the factors of the \(\star\)-product leads to a vertex containing the Fourier transform of the product between two fields, such as \(F\{A_1 A_2\}(p_1)\) in (4.65). This can be expressed as the convolution (in momentum space) between the two fields

\[ F\{A_1 A_2\}(p_1) = \int dq A_1(q) A_2(p_1 - q). \]  

(4.66)

As a result, the vertex (4.65) can be written as

\[ S_1 = \int dp_1 dp_2 dq dk \delta^{(D)}(p_1 + p_2) A_1(q) A_2(p_1 - q) \tilde{A}_1(k) \tilde{A}_2(p_1 - k) A_3(p_2) \tilde{A}_3(p_2), \]  

(4.67)

from which it is clear that only the fields \(A_3\) and \(\tilde{A}_3\) come in pair with the same momentum, allowing the identification \(A_3(p_2) \tilde{A}_3(p_2) \rightarrow H(p_2)\), with \(H\) a new field. On the contrary, an actual vertex of the type \(H^3\) is built from \(\star\)-products involving only one field on each side of the \(\star\):

\[ \int d^D x (A_1 \star \tilde{A}_1)(A_2 \star \tilde{A}_2)(A_3 \star \tilde{A}_3). \]  

(4.68)

Finally, one can notice the most apparent of the difficulties: how can it be possible to obtain a non-polynomial theory like \(N = 0\) Supergravity by squaring theories with at most quartic interactions?\(^\text{3}\)

\(^\text{3}\)For the sake of simplicity we neglect color indices, derivatives and structure constants. Also, we drop the biadjoint scalar in the \(\star\)-product, since it does not play any role in this discussion, with the result that here \(\star\) actually coincides with a convolution.

\(^\text{4}\)Using the notation \(dk = \frac{d^D k}{(2\pi)^D}\) already introduced in Section 3.2.2 and denoting once again with \(F\{f\}(p)\) the Fourier transform of a function \(f(x)\).

\(^\text{5}\)In principle, however, one may try to make contact with polynomial formulations of gravity like, e.g., the Palatini formulation. See also [133].
For all these reasons it appears that the extension of the double copy at the interacting Lagrangian level, if at all possible, involves additional subtleties that defy an immediate intuitive understanding and require to go through a less straightforward route.

In the next sections we shall choose an alternative, systematic approach, leading to a successful construction of cubic interactions for the double-copy field $H_{\mu\nu}$, in principle amenable to be extended to all orders.

4.3 The Noether procedure

In the previous section we built a quadratic Lagrangian for the field $H_{\mu\nu}$ and highlighted its relation to the YM Lagrangian together with its particle content. One guiding principle has been to ensure invariance under the abelian gauge transformation of $H_{\mu\nu}$, inherited from those of its YM constituents.

Now we want to add interactions among these massless particles. This is typically accomplished by adding to the Lagrangian polynomials in the fields and their derivatives of degree higher than two. In general, a given deformation of the free theory by the inclusion of higher-order polynomials would break its free gauge-invariance, so the difficult part of the construction of interactions is to select those vertices that do not break the free gauge symmetry, while possibly deforming it into a non-abelian one. A systematic algorithm which allows the extension of a quadratic action with gauge invariance to the interacting level is the Noether procedure, which is defined perturbatively, order by order in the number of fields (or, equivalently, in powers of the coupling constant).

In order to outline how the Noether procedure works in general, let us consider the action $S[\varphi]$ of the putative complete theory which we would like to build and let us expand it perturbatively in the number of fields, here collectively denoted with $\varphi$

$$S[\varphi] = S_0[\varphi] + gS_1[\varphi] + g^2S_2[\varphi] + ..., \quad (4.69)$$

where $g$ is the coupling constant of the theory while $S_n[\varphi]$ includes all the vertices of the complete action with $n - 2$ fields, so that for instance $S_0$ is the free theory. Cor-
respondingly, we expand the gauge variation $\delta \varphi$ of the field $\varphi$ in the complete theory perturbatively in the number of fields, or equivalently in the power of the coupling constant $g$:

$$\delta \varphi = \delta_0 \varphi + g \delta_1 \varphi + g^2 \delta_2 \varphi + \ldots.$$  \hspace{1cm} (4.70)

The requirement of gauge invariance of the action, interpreted order by order in perturbation theory, leads to the following system of equations:

$$\delta S = 0 \Rightarrow \begin{cases} 
\delta_0 S_0 = 0, \\
\delta_0 S_1 + \delta_1 S_0 = 0, \\
\delta_0 S_2 + \delta_1 S_1 + \delta_2 S_0 = 0, \\
\vdots 
\end{cases}$$  \hspace{1cm} (4.71)

which is known as the Noether system. In order to find a solution to the system, we can observe that

$$\delta_n S_0 = \frac{\delta S_0}{\delta \varphi} \delta_n \varphi \approx 0,$$  \hspace{1cm} (4.72)

where, with "$\approx 0$", we denote quantities that vanish when the free equations of motion are satisfied. This observation allows solve the system order by order, starting from the lowest non-trivial one:

$$\delta_0 S_1 + \delta_1 S_0 = 0 \Rightarrow \delta_0 S_1 \approx 0.$$  \hspace{1cm} (4.73)

Therefore, in order to build the cubic vertex, one can make an ansatz for $S_1$ with undetermined coefficients, to be fixed in order to obtain a quantity whose variation under the free gauge symmetry is zero when the free equations of motion are satisfied. Once such a $S_1$ is obtained, we collect the terms proportional to the equations of motion in its variation and compute the first correction to the gauge transformation of the fields exploiting once more the Noether equation

$$\delta_1 S_0 = \frac{\delta S_0}{\delta \varphi} \delta_1 \varphi = -\delta_0 S_1.$$  \hspace{1cm} (4.74)

This procedure can be carried on at every order: at the $n$-th step one knows all the
vertices up to $S_{n-1}$, and the equation to be solved is

$$\delta_0 S_n + \ldots + \delta_{n-1} S_1 \approx 0,$$

which allows to determine $S_n$. Next, collecting all the terms proportional to the free equations of motion in $\delta_0 S_n$ and solving

$$\delta_n S_0 = -\delta_0 S_n - \ldots - \delta_{n-1} S_1,$$

it is possible to find $n$-th correction to the gauge variation $\delta_n \varphi$.

In this section, for pedagogical purposes, we provide two examples of application of the Noether procedure to YM theory and General Relativity. In the first case, the procedure terminates with the quartic vertices, and we shall indeed build the complete theory. In the case of gravity, the Einstein-Hilbert action expanded on flat background is non-polynomial in the fluctuation $h_{\mu\nu}$, while here we will only derive the expression for its cubic vertex, together with the first correction to the free gauge transformation of the graviton field.

4.3.1 Noether procedure for Yang-Mills theory

As a first example of how the Noether procedure works in practice, we are going to build the full interacting Yang-Mills theory deforming the quadratic Lagrangian for a set of $N$ massless, spin-one fields $A^a_\mu$ ($a = 1, \ldots, N$), with free gauge invariance $\delta_0 A^a_\mu = \partial_\mu \epsilon^a$. The free Lagrangian is the sum of $N$ Maxwell Lagrangians:

$$L_0 = \frac{1}{2} A^a_\mu \left( \Box A^{a\mu} - \partial^\mu \partial \cdot A^a \right),$$

where the sum over the internal indices is left implicit, so that the free equations of motion correspond to Maxwell’s equations for each field:

$$\mathcal{E}^a_\mu = \Box A^a_\mu - \partial_\mu \partial \cdot A^a \approx 0.$$

As a next step, we make an ansatz for a vertex with three fields trying to keep at a minimum the number of derivatives involved. Since we must contract all the Lorentz indices to obtain a scalar the number of derivatives must be odd, the simplest possibility
thus being

$$L_1 = f_{abc} A^a_\mu (\partial^\mu A^b_\nu) A^{c\nu}, \quad (4.79)$$

with $f_{abc}$ a set of (at this stage) arbitrary real numbers. As a first step, we compute the free gauge variation of (4.79), isolating terms proportional to the equations of motion via the identity $\Box A^a_\mu = E^a_\mu + \partial_\mu \partial \cdot A^a$. The variation is as follows:

$$\delta_0 L_1 = f_{abc} \partial_\mu \varepsilon^a \partial^\mu \varepsilon^b A^c_\nu + f_{abc} A^a_\mu \partial^\mu \partial^\nu \varepsilon^b A^c_\nu + f_{abc} A^a_\mu \partial^\mu A^{bc} \partial_\nu \varepsilon^c$$

$$= f_{abc} \{-\varepsilon^a [A^{c\nu} \Box A^b_\nu + \partial^\mu A^{bc} \partial_\mu A^c_\nu] + \varepsilon^b [A^c_\nu \partial^\nu \partial \cdot A^a + A^a_\nu \partial^\mu \partial \cdot A^c + \partial \cdot A^a \partial \cdot A^c]$$

$$- \varepsilon^c [A^a_\mu \partial^\mu \partial \cdot A^b + \partial_\mu A^b_\nu \partial^\nu A^{a\mu}]\}$$

$$= \varepsilon^a \{ - f_{abc} [A^{c\nu} \Box A^b_\nu + \partial^\mu A^{bc} \partial_\mu A^c_\nu] - f_{cba} [A^c_\nu \partial^\mu \partial \cdot A^b + \partial_\mu A^b_\nu \partial^\nu A^{c\mu}]$$

$$+ f_{bac} [A^c_\mu \partial^\mu \partial \cdot A^b + \partial_\mu A^b_\nu \partial^\nu A^{c\mu}] - \partial \cdot A^b \partial \cdot A^c + \partial_\mu A^b_\nu \partial^\nu A^{c\mu}]$$

$$\approx \varepsilon^a \{ - f_{abc} - f_{cba} + f_{bac} \} A^c_\mu \partial^\mu \partial \cdot A^b + f_{bac} \partial \cdot A^b \partial \cdot A^c$$

$$- f_{abc} \partial_\mu A^b_\nu \partial^\mu A^{c\nu} + (f_{bac} - f_{cba}) \partial_\mu A^b_\nu \partial^\nu A^{c\mu}\}, \quad (4.80)$$

where we have integrated by parts in order to isolate the gauge parameter and, to obtain the last expression, we have exploited the equations of motion (4.78). Moreover, we have collected separately the different tensorial structures in the last two lines: all those terms must vanish independently, upon imposing conditions on their coefficients, since must require that $\delta_0 L_1 \approx 0$. To this end, we make the following observations:

- There is only one term of the form $\partial \cdot A^b \partial \cdot A^c$: since it is symmetric in its indices, the only way it can drop out of the result is by imposing $f_{bac} = -f_{cab}$.

- Similarly, there is only one term of the form $\partial_\mu A^b_\nu \partial^\mu A^{c\nu}$: in its turn, it is symmetric under the exchange of its indices $(b, c)$, therefore it can be eliminated only by requiring its coefficient to be antisymmetric under the same exchange: $f_{abc} = -f_{acb}$.

- The relation $f_{bac} = -f_{cab}$ obtained in the first point also implies that another terms vanishes due to its symmetry:

$$f_{bac} = -f_{cab} A^b_\nu \partial^\nu A^{c\mu} = -f_{cba} A^b_\nu \partial^\nu A^{c\mu}. \quad (4.81)$$

Being this the only term of this type, the only way to cancel it is, as above, to
require $f_{cbo} = - f_{bca}$.

- The arguments of the previous points imply that the constants $f_{abc}$ are completely antisymmetric in their indices. Moreover, since there is no other possible choice to cancel the above terms, we can conclude that the solution is unique. As a last step, we can check that the last term remaining in (4.80) also cancel due to the antisymmetry of $f_{abc}$:

$$- f_{abc} - f_{cbo} + f_{bca} + f_{cab} = 0. \quad (4.82)$$

Therefore we can conclude that the complete antisymmetry of the $f_{abc}$'s is a necessary and sufficient condition for $\delta_0 L_1$ to vanish on shell.

Having obtained $\delta_0 L_1 \approx 0$, we collect the terms proportional to the equations of motion $E^a_{\mu}$ in the expression (4.80) of $\delta_0 L_1$ in order to derive the correction to the gauge transformation:

$$\delta_0 S_1 + \delta_1 S_0 = \int \left\{ \frac{\delta L_0}{\delta A^a_{\mu}} \delta_1 A^a_{\mu} + \frac{\delta L_1}{\delta A^a_{\mu}} \delta_0 A^a_{\mu} \right\} = \int \left\{ E^{a\mu} \delta_1 A^a_{\mu} + \frac{\delta L_1}{\delta A^a_{\mu}} \partial_\mu e^a \right\}$$

$$= \int \left\{ E^{a\mu} \delta_1 A^a_{\mu} + f_{abc} \varepsilon^b A^c_{\mu} \right\}, \quad (4.83)$$

from which, exploiting the antisymmetry of the coefficients $f_{abc}$ one finds

$$\delta_1 A^a_{\mu} = f_{abc} \varepsilon^b A^c_{\mu}. \quad (4.84)$$

This is the correct result for a Yang-Mills theory, therefore we don’t expect further corrections to the gauge transformation, although we ought to be able to recover this information from the procedure.

Still, at this stage $\delta_1 L_1 \neq 0$ (as we will show in (4.88)), therefore we must add a new term with four fields to compensate this piece of gauge variation. To this end, we must solve the following equation in the Noether system:

$$\delta_2 L_0 + \delta_1 L_1 + \delta_0 L_2 = 0. \quad (4.85)$$

Since $\delta_1 L_1$ contains only one derivative, we choose $L_2$ with no derivatives, so that $\delta_0 L_2$
has the same number of derivatives as $\delta_1 L_1$. Our ansatz for the quartic vertex is therefore:

$$L_2 = K_{abcd} A^a_{\mu} A^b_{\nu} A^c_{\mu} A^d_{\nu}, \quad (4.86)$$

with $K_{abcd}$ encoding arbitrary real coefficients, at this stage symmetric under the exchanges $a \leftrightarrow c, b \leftrightarrow d$ and $(ab) \leftrightarrow (cd)$. Making use of these symmetries we obtain:

$$\begin{align*}
\delta_0 L_2 &= 2K_{abcd} \partial_\mu \varepsilon^a A^b_{\nu} A^c_{\mu} A^d_{\nu} + 2K_{abcd} A^a_{\mu} \partial_\nu \varepsilon^b A^c_{\mu} A^d_{\nu} \\
&\quad - 2K_{abcd} \partial_\nu A^b_{\mu} A^c_{\mu} A^d_{\nu} + A^b_\nu \partial_\nu A^c_{\mu} A^d_{\nu} + A^b_\mu \partial_\nu A^c_{\nu} A^d_{\nu} + A^b_\mu \partial_\nu A^c_{\mu} \partial_\nu A^d_{\nu} \\
&= -4\varepsilon^a \left( K_{abcd} (A^b \cdot A^d)(\partial_\nu A^c) + 2K_{abcd} A^b_{\mu} A^c \cdot \partial A^d_{\nu} \right). \quad (4.87)
\end{align*}$$

The other variation to be taken into account is:

$$\begin{align*}
\delta_1 L_1 &= -f_{abc} f_{ade} \varepsilon^d A^c_{\mu} A^d_{\nu} A^a_{\mu} - f_{abc} f_{cde} A^a_{\mu} \partial_\mu (\varepsilon^d A^c_{\nu}) A^c_{\nu} - f_{abc} f_{ede} A^a_{\mu} \partial_\mu A^b_{\nu} \varepsilon^d A^c_{\nu} \\
&= -\varepsilon^a \left( -f_{ced} f_{eab} A^b \cdot A^d \partial_\nu A^c + f_{ced} f_{eab} - f_{ced} f_{eab} A^b A^c \cdot \partial A^d_{\nu} \right). \quad (4.88)
\end{align*}$$

Since there are no terms proportional to the equations of motion, we can directly look for a solution with $\delta_2 A^\mu_\mu = 0$, imposing $\delta_1 L_1 + \delta_0 L_2 = 0$. Requiring independent tensorial structures to independently cancel and exploiting the symmetries of $f_{abc}$ and $K_{abcd}$, this leads to the system:

$$\begin{align*}
4K_{abcd} &= -f_{eab} f_{ecd}, \\
8K_{abcd} &= 4K_{abcd} + 4K_{cbad} = -f_{edb} f_{eac} - f_{eab} f_{ecd} - f_{eab} f_{ecd},
\end{align*} \quad (4.89)$$

which is solved by:

$$\begin{align*}
K_{abcd} &= -\frac{1}{4} f_{eab} f_{ecd}, \\
f_{eab} f_{ecd} + f_{eac} f_{edb} + f_{ead} f_{ebc} &= 0. \quad (4.90)
\end{align*}$$

We observe that the solution obtained for $K_{abcd}$ has less symmetry than originally expected, but if we go bottom up and exploit only the symmetries of the solution, everything is consistent either way. The second expression in (4.89) is just the Jacobi identity, i.e. the condition under which the $f_{abc}$’s are the structure constants of a Lie Group.
One can also check that these results solve the entire system: $\delta_2 A^a_\mu = 0, \delta_1 S_2 = 0$, therefore nothing else must be added. Thus, the solution of the Noether system is

$$\begin{align*}
\mathcal{L} & = \frac{1}{2} A^a_\mu \left( \Box A^{a\mu} - \partial^\mu \partial^a \right) + f_{abc} A^a_\mu (\partial^\mu A^b_\nu) A^{c\nu} - \frac{1}{4} f_{eab} f_{ecd} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu}, \\
\delta A^a_\mu & = \partial_\mu \epsilon^a + f_{abc} \epsilon^b A^c_\mu.
\end{align*}$$

(4.91)

If we define $F^a_{\mu\nu} := \partial_{[\mu} A_{\nu]} - f_{abc} A^b_\mu A^c_\nu$, the Lagrangian can be written as:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu},$$

(4.92)

with $\delta F^a_{\mu\nu} = -f_{abc} \epsilon^b F^c_{\mu\nu}$, which is the usual form of a Yang-Mills theory.

### 4.3.2 Noether procedure for graviton cubic vertex

We now move to a more complicated case, which will also serve as a “warm-up” for the computation of the cubic vertex for the field $H_{\mu\nu}$ in the next section. We want to build the cubic self-interaction vertex for a massless spin-two field, starting from the massless Fierz-Pauli Lagrangian by means of the Noether procedure. This approach reproduces the perturbative expansion of the Einstein-Hilbert action, modulo boundary contributions.

A massless spin-two particle can be described in a covariant fashion with a symmetric tensor $h_{\mu\nu}$, together with an equivalence relation $h_{\mu\nu} \sim h_{\mu\nu} + \partial_{(\mu} \xi_{\nu)}$, where $\xi_\mu$ is an arbitrary vector which serves as a gauge parameter. An action principle can be formulated in terms of the massless Fierz-Pauli Lagrangian, which corresponds to the most general quadratic Lagrangian with two derivatives which is compatible with the gauge invariance of a massless spin-two field, and coincides with the quadratic order of the expansion of the Einstein-Hilbert Lagrangian in terms of $h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}$, up to total derivatives:

$$\mathcal{L}_0 = \frac{1}{2} h^{\mu\nu} \left\{ \square h_{\mu\nu} - 2 \partial_\mu \partial_\nu h + \partial_\mu \partial_\nu h - \eta_{\mu\nu} (\square h - \partial_\mu \partial_\nu h) \right\},$$

(4.93)
The free equations of motion for $h_{\mu\nu}$ are then:

$$\Box h_{\mu\nu} - \partial_{(\mu} \partial \cdot h_{\nu)} + \partial_{\mu} \partial_{\nu} h - \eta_{\mu\nu} (\Box h - \partial \cdot \partial \cdot h) = 0.$$  \hfill (4.94)

The trace of eq. (4.94) implies

$$\Box h - \partial \cdot \partial \cdot h = 0.$$  \hfill (4.95)

Therefore, the linearized equations of motion for $h_{\mu\nu}$ can be equivalently written as:

$$R_{\mu\nu} := \Box h_{\mu\nu} - \partial_{(\mu} D_{\nu)} = 0,$$

$$D_{\mu} := \partial \cdot h_{\mu} - \frac{1}{2} \eta_{\mu\nu} R,$$  \hfill (4.96, 4.97)

where we wrote the linearized Ricci tensor $R_{\mu\nu}$ in terms of the de-Donder tensor $D_{\mu}$. In terms of these variables, the quadratic Lagrangian can be rewritten as:

$$L_0 = \frac{1}{2} h^{\mu\nu} \left\{ R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \right\},$$

whose gauge invariance relies on the Bianchi identity of the linearized Einstein tensor:

$$\delta_0 S_0 = \int \delta S_0 \delta_0 h_{\mu\nu} = 2 \int (R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R) \partial^\nu \xi^\mu = -2 \int \xi^\nu \partial^\mu (R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R) = 0$$

due to $\partial \cdot R = \frac{1}{2} \partial_{\mu} R$.  \hfill (4.99)

For later convenience, we also state two useful relations:

$$\delta_0 D_{\mu} = \Box \xi_{\mu}, \quad \delta_0 h = 2 \partial \cdot \xi.$$  \hfill (4.100)

We can now move to building the cubic vertex $S_1$ for the graviton self-interaction. To this end, we make a few observations which will simplify the calculation:

- As a first step, we build the TT (transverse, traceless) part of the cubic Lagrangian, i.e. the part which doesn’t contain traces ($h$) and divergences ($\partial \cdot h_{\mu}$, or equivalently $D_{\mu}$) of the graviton. We make an ansatz and discard all terms containing $F_{\mu\nu}$, $h$ and $D_{\mu}$: these terms will enter the procedure at different stages.

- When two derivatives are contracted with each other we can make use of the
following identity (where we neglect total derivative contributions):

\[
\varphi_1 \partial^\mu \varphi_2 \partial_\mu \varphi_3 = \frac{1}{2} (\Box \varphi_1 \varphi_2 \varphi_3 - \varphi_1 \Box \varphi_2 \varphi_3 - \varphi_1 \varphi_2 \Box \varphi_3). \tag{4.101}
\]

This generates two types of contributions: \( \Box h_{\mu \nu} = F_{\mu \nu} + \partial_\mu \partial_\nu \) (or its trace) and \( \Box \xi_\mu = \delta \mathcal{D}_\mu \) which, looking at the TT-level, can both be neglected. At the next step we include terms containing \( \mathcal{D}_\mu \) or \( h \). proportional to the equations of motion \( F_{\mu \nu} \).

• In order to avoid ambiguities related to the freedom of integrating by parts, we employ the so-called cyclic ansatz: in the TT Lagrangian, every derivative acting on a field must be contracted with an index of the previous one. Since our ansatz for \( S_1 \) contains two derivatives and three fields, every term which is not cyclic can be brought in this form via integration by parts upon discarding non-TT terms and terms proportional to the equations of motion.

Given these prescriptions, the most general TT Lagrangian with three fields and two derivatives has the form:

\[
\mathcal{L}^{TT}_1 = ah^{\mu \nu} \partial_\mu \partial_\nu h_{\alpha \beta} h^{\alpha \beta} + bh^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\alpha h_{\beta \nu}, \tag{4.102}
\]

with \( a, b \) real coefficients. We can compute the variation of \( \mathcal{L}^{TT}_1 \) under a free gauge transformation, exploiting the above prescriptions:

• \( \delta_0 (h^{\mu \nu} \partial_\mu \partial_\nu h_{\alpha \beta} h^{\alpha \beta}) = 2 (\partial^\mu \xi_\nu \partial_\mu \partial_\nu h_{\alpha \beta} + h^{\mu \nu} \partial_\mu \partial_\nu \partial_\alpha \xi_\beta) h^{\alpha \beta} + 2h^{\mu \nu} \partial_\mu h_{\alpha \beta} \partial^\alpha \xi_\beta \)

\[
= \Box h_{\alpha \beta} \partial_\mu \partial_\nu h^{\alpha \beta} - \Box \xi_\nu \partial_\mu h_{\alpha \beta} \partial_\nu h^{\alpha \beta} - \partial_\nu \Box h_{\alpha \beta} h^{\alpha \beta} \xi_\nu \partial_\mu h^{\alpha \beta} \partial^\alpha \partial_\beta h^{\alpha \beta} 
- \partial_\nu h^{\mu \nu} \partial_\mu \partial_\nu h_{\alpha \beta} \partial_\alpha \xi_\beta, \tag{4.103}
\]

• \( \delta_0 (h^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\alpha h_{\beta \nu}) = \partial^\mu (\xi_\nu) \partial_\mu h^{\alpha \beta} \partial_\alpha h_{\beta \nu} + h^{\mu \nu} \partial_\mu \partial_\nu h^{\alpha \beta} \partial_\alpha \xi_\beta + h^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\alpha \partial_\beta h_{\beta \nu} \)

\[
= \frac{1}{2} \partial_\alpha \Box h_{\beta \nu} h^{\alpha \beta} \xi_\nu - \frac{1}{2} \Box h^{\alpha \beta} \partial_\alpha h_{\beta \nu} - \frac{1}{2} \Box h^{\alpha \beta} \partial_\alpha h_{\beta \nu} \xi_\nu + h^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\alpha \partial_\beta h_{\beta \nu} 
+ \frac{1}{2} \Box h^{\mu \nu} h_{\beta \nu} \partial_\mu \xi_\beta \partial_\alpha h_{\beta \nu} - \frac{1}{2} \partial_\mu \partial_\nu \xi_\beta h^{\mu \nu} h_{\beta \nu} - \partial_\mu \partial_\nu \xi_\beta h^{\mu \nu} h_{\beta \nu} 
- \partial_\mu \partial_\nu \xi_\beta \partial_\mu \partial_\nu h^{\alpha \beta} h_{\alpha \beta} \partial_\alpha \xi_\beta, \tag{4.104}
\]

where the underlined terms provide the TT part of the variation: requiring them to mutually cancel fixes the coefficients \( a \) and \( b \) to satisfy \( a = b/2 \). In what follows, we shall work modulo an overall constant and set \( a = 1/2, b = 1 \) and only at the end we add
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the appropriate normalization and coupling constant. We shall exploit $\Box x_\mu = \delta D_\mu$ and work with $D_\mu$ instead of $\partial \cdot h_\mu = D_\mu + \frac{1}{2} \partial_\mu h$; every time we encounter $\Box h_{\mu\nu}$ we use eq. (4.96), replacing it with $R_{\mu\nu} + \partial_\mu D_\nu$.

For the full computations one can consult appendix A while here we only discuss the results. Compensating the several non-TT terms which are generated by $\delta_0 L_{TT}^1$, we can build the full vertex $L_1$, which satisfies $\delta_0 L_1 \approx 0$:

$$L_1 = \frac{1}{2} h^{\mu\nu} \partial_\mu \partial_\nu h_{\alpha\beta} + h^{\mu\nu} \partial_\mu h^{\alpha\beta} \partial_\alpha h_{\beta\nu} - \frac{1}{4} \partial \cdot D h_{\alpha\beta} h^{\alpha\beta} - \frac{1}{2} h^{\alpha\beta} \partial_\alpha h D_\beta. \quad (4.105)$$

Up to boundary terms this result matches with the order $h^3$ expansion of the EH action derived in Section 2.3. Then, we collect all the terms proportional to the equations of motion in order to derive the correction to the gauge transformation $\delta_1 h_{\mu\nu}$. A key observation is in order at this point: terms in $\delta_0 L_{TT}^1$ which contain the equations of motion $E_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} R$, (4.106)

and can be expressed as a $E_{\alpha\beta} \delta_0 (F(h^2)_{\alpha\beta})$, with $F(h^2)$ a function quadratic in $h_{\mu\nu}$, can be absorbed via a field redefinition. Indeed, since $\delta_0 E_{\alpha\beta} = 0$, the corresponding term in $L_1$ is $E_{\alpha\beta} (F(h^2)_{\alpha\beta})$, and if we truncate the Lagrangian at cubic order:

$$L_0 + L_1 = \frac{1}{2} h_{\alpha\beta} E^{\alpha\beta} + E^{\alpha\beta} F(h^2)_{\alpha\beta} = E^{\alpha\beta} \{ h_{\alpha\beta} + F(h^2)_{\alpha\beta} \} = E^{\alpha\beta} h'_{\alpha\beta}, \quad (4.107)$$

with $h'_{\alpha\beta} = h_{\alpha\beta} + F(h^2)_{\alpha\beta}$ the field redefinition which removes the vertex. This shows that cubic vertices proportional to the free equations of motion are actually “fake” interactions, and would produce identically zero scattering amplitudes. Of course, such field redefinitions will affect the higher order vertices $L_n$ with $n > 2$, but since in this section we only deal with the cubic interaction of the graviton, we can safely perform the field redefinition without worrying about its consequences on the Lagrangian of the complete theory. With this in mind, we move to the analysis of the part of $\delta_0 L_1$ which vanishes on shell. Integrating by parts in order to isolate $x_\mu$ (without derivatives acting
on it) and replacing:

\[
\begin{align*}
R_{\alpha\beta} &\rightarrow E_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta} R, \\
R &\rightarrow -\frac{2}{D-2} E,
\end{align*}
\]  

(4.108)

we obtain:

\[
\delta_0 L_1 = E_{\alpha\beta}\left\{\partial_\nu h^{\alpha\beta}\xi^\nu + \frac{1}{2} h^{(\alpha\nu}(\partial^\beta\xi^\nu - \partial_\nu\xi^{\beta)}\right\} + \frac{1}{2} E_{\alpha\beta} h^{(\alpha\beta} \partial \cdot \xi - \frac{1}{4} E_{\alpha\beta} \partial^\alpha h\xi^\beta
\]

\[
- \frac{1}{2} E_{\alpha\beta} \partial^\alpha (h^{\beta\nu} \xi^\nu) + \frac{1}{4} E_{\alpha\beta} h\partial^\alpha \xi^\beta - \frac{1}{2(D-2)} E h \partial \cdot \xi
\]

\[
= E_{\alpha\beta}\left\{\partial_\nu h^{\alpha\beta}\xi^\nu + \frac{1}{2} h^{\nu(\alpha}(\partial^{\beta)}\xi^\nu - \partial_\nu\xi^{\beta)}\right\} + E_{\alpha\beta}\left\{\delta_0 (h^{\alpha\beta} h) - \frac{1}{8(D-2)} \eta^{\alpha\beta} \delta_0 (h^2)\right\},
\]

(4.109)

In addition we can observe that:

\[
\frac{1}{2} h^{(\alpha\nu}(\partial^\beta)\xi^\nu - \partial_\nu\xi^{\beta)} = h^{\nu(\alpha}(\partial^{\beta)\xi^\nu - \frac{1}{2} \delta_0 (h^{\nu\alpha} h^\beta). 
\]

(4.110)

Therefore, (4.109) can be rewritten as:

\[
\delta_0 L_1 = E_{\alpha\beta}\left\{\partial_\nu h^{\alpha\beta}\xi^\nu + h^{\nu(\alpha}(\partial^{\beta)}\xi^\nu\right\} + \frac{1}{4} E_{\alpha\beta} \delta_0 \left\{h^{\alpha\beta} h - 2h^{\alpha\nu} h^\beta - \frac{1}{2(D-2)} \eta^{\alpha\beta} h^2\right\}.
\]

(4.111)

The second part, as previously discussed, can be reabsorbed by the free Lagrangian with the following field redefinition:

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + \delta h_{\mu\nu} = h_{\mu\nu} + \frac{1}{4}\left\{h^{\alpha\beta} h - 2h^{\alpha\nu} h^\beta - \frac{1}{2(D-2)} \eta^{\alpha\beta} h^2\right\}.
\]

(4.112)

Therefore, the first part defines the actual correction to the gauge transformation,

\[
\delta_1 h_{\alpha\beta} = \xi^\nu \partial_\nu h_{\alpha\beta} + \partial_\rho (\xi^\nu h_{\beta\nu}),
\]

(4.113)

which corresponds to the Lie derivative of a rank-two symmetric tensor. Finally, by comparison with the results of Section 2.3, we can fix the overall constant of the vertex by comparison with the perturbative expansion of the Einstein-Hilbert action, discussed
in Section 2.3. The result is:

\[
\mathcal{L}_1 = \kappa \left\{ h^{\mu\nu} \partial_\mu \partial_\nu h_{\alpha\beta} h^{\alpha\beta} + 2 h^{\mu\nu} \partial_\mu h^{\alpha\beta} \partial_\alpha h_{\beta\nu} - \frac{1}{2} \partial \cdot D h_{\alpha\beta} h^{\alpha\beta} - h^{\alpha\beta} \partial_\alpha h \partial_\beta \right\},
\]

(4.114)

\[
\delta h_{\mu\nu} = 2 \kappa \left\{ \xi_{\nu} \partial_\nu h_{\alpha\beta} + \partial_{(\alpha} \xi_{\beta)} h_{\gamma\nu} \right\},
\]

(4.115)

granting for the graviton and the gauge parameter to have the correct mass dimensions:

\[
[h_{\mu\nu}] = \frac{D - 2}{2}, \quad [\xi_\mu] = \frac{D - 4}{2}, \quad [\kappa] = -\frac{D - 2}{2}.
\]

(4.116)

### 4.4 Cubic vertex for the double-copy Lagrangian

In this section we shall build the cubic vertices for the field $H_{\mu\nu}$ as a deformation of the DC quadratic Lagrangian (4.29), by means of the Noether procedure. Also, we shall derive and discuss the corresponding deformation of the gauge transformation. First, in order to make this section self-consistent, let us summarize the key pieces of information about the free theory. The Lagrangian is

\[
\mathcal{L}_0 = \frac{1}{2} H_{\alpha\beta} \left\{ \Box H^{\alpha\beta} - \partial^\alpha \partial_\gamma H^{\gamma\beta} - \partial^\beta \partial_\gamma H^{\alpha\gamma} + \partial^\alpha \partial^\beta \frac{\partial \cdot \partial}{\Box} H \right\},
\]

(4.117)

invariant under the free gauge transformation:

\[
\delta_0 H_{\mu\nu} = \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu.
\]

(4.118)

The free equations of motion are

\[
\mathcal{E}_{\alpha\beta} = \Box H_{\alpha\beta} - \partial_\alpha \partial^\gamma H_{\gamma\beta} - \partial_\beta \partial^\gamma H_{\alpha\gamma} + \partial_\alpha \partial_\beta \frac{\partial \cdot \partial}{\Box} H = \Box H_{\alpha\beta} - \partial_\alpha D_\beta - \partial_\beta \tilde{D}_\alpha \approx 0,
\]

(4.119)

where we have defined:

\[
\begin{align*}
D_\alpha & := \partial^\gamma H_{\gamma\alpha} - \frac{1}{2} \partial_\alpha \frac{\partial \partial \Box H}{\Box}, \\
\tilde{D}_\alpha & := \partial^\gamma H_{\alpha\gamma} - \frac{1}{2} \partial_\alpha \frac{\partial \partial \Box H}{\Box}.
\end{align*}
\]

(4.120)

As in the case of gravity, we shall first build the TT part of the vertex, thus ignoring
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terms which contain $H, \partial^\alpha H_{\alpha \beta}$ and $\partial^3 H_{\alpha \beta}$, such as $D_\mu$ and $\hat{D}_\mu$. Moreover, also in this case we shall use the cyclic ansatz for the TT vertex. The most general local TT cubic Lagrangian in the cyclic ansatz for the field $H_{\mu \nu}$ contains ten terms$^7$:

$$
L_{1}^{TT} = a_1 H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + a_2 H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} H^{\alpha \beta}
+ a_3 H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\alpha H_{\beta \nu} + a_4 H^{\mu \nu} \partial_\nu H^{\alpha \beta} \partial_\alpha H_{\beta \mu}
+ a_5 H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\beta H_{\alpha \nu}
+ a_6 H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\beta H_{\alpha \mu}
+ a_7 H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\nu H_{\alpha \beta}
+ a_8 H^{\mu \nu} \partial_\nu H^{\alpha \beta} \partial_\alpha H_{\beta \mu}
+ a_9 H^{\mu \nu} \partial_\nu H^{\alpha \beta} \partial_\beta H_{\alpha \mu}
+ a_{10} H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\beta H_{\nu \alpha}.
$$

(4.121)

As a first step, we require the gauge invariance at the TT order, ignoring for the moment in the variations all the non-TT contributions. Integrating by parts and neglecting such terms, we can isolate the gauge parameters $\alpha_\mu$ and $\tilde{\alpha}_\mu$, obtaining the ten variations

- $\delta_1^{TT} = -\alpha_\beta \partial_\mu H^{\mu \nu} \partial_\nu H_{\alpha \beta} - \tilde{\alpha}_\alpha \partial_\beta H^{\mu \nu} \partial_\nu H_{\alpha \beta} - \alpha^\beta \partial^\alpha H^{\mu \nu} \partial_\nu H_{\alpha \beta}$

(4.122)

- $\delta_2^{TT} = -\alpha_\beta \partial_\mu H^{\mu \nu} \partial_\nu H_{\alpha \beta} - \tilde{\alpha}_\alpha \partial_\beta H^{\mu \nu} \partial_\nu H_{\alpha \beta} - \alpha^\beta \partial^\alpha H^{\mu \nu} \partial_\nu H_{\alpha \beta}$

(4.123)

- $\delta_3^{TT} = -\tilde{\alpha}_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.124)

- $\delta_4^{TT} = -\alpha_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.125)

- $\delta_5^{TT} = \alpha_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.126)

- $\delta_6^{TT} = \alpha_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.127)

- $\delta_7^{TT} = \tilde{\alpha}_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.128)

- $\delta_8^{TT} = \tilde{\alpha}_\beta \partial_\mu H_{\alpha \beta} \partial^\alpha H^{\mu \nu} + \tilde{\alpha}_\beta \partial_\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta} + \alpha_\nu \partial^\alpha H^{\mu \nu} \partial_\mu H_{\alpha \beta}$

(4.129)

$^6$Even if the quadratic Lagrangian is non-local, we do not include non-localities in our TT ansatz, as they only lie in pure-gauge, unphysical sectors of the theory and as such cannot have a direct influence on the physical degrees of freedom, contained in the TT components of $H_{\mu \nu}$.

$^7$Keep in mind that $H_{\mu \nu} \neq H_{\nu \mu}$. 
\[ \begin{align*}
\delta_9^{\text{TT}} & = -\tilde{\alpha}^\mu \partial_\mu \partial_\nu H_{\alpha\beta} \partial H^{\nu\alpha} + \alpha^\nu \partial^\beta H^{\mu\nu} \partial_\nu \partial_\mu H_{\alpha\beta} + \alpha_\mu \partial_\mu \partial H^{\nu\alpha} \partial_\nu H_{\alpha\beta} \\
& + \tilde{\alpha}^\mu \partial H^{\mu\nu} \partial_\nu \partial_\mu H_{\alpha\beta},
\end{align*} \]

\hspace{1cm} (4.130)

\[ \begin{align*}
\delta_{10}^{\text{TT}} & = -\tilde{\alpha}^\mu \partial_\mu \partial_\nu H_{\alpha\beta} \partial H^{\nu\alpha} + \alpha^\nu \partial^\beta H^{\mu\nu} \partial_\nu \partial_\mu H_{\alpha\beta} + \alpha_\mu \partial_\mu \partial H^{\nu\alpha} \partial_\nu H_{\alpha\beta} \\
& + \tilde{\alpha}^\alpha \partial H^{\mu\nu} \partial_\nu \partial_\mu H_{\alpha\beta}.
\end{align*} \]

\hspace{1cm} (4.131)

Requiring that the TT part of the total variation is zero leads the following system:

\[
\begin{aligned}
-2a_1 + a_3 + a_5 + a_7 + a_8 &= 0, \\
-2a_1 + a_3 + a_6 + a_8 + a_{10} &= 0, \\
-2a_2 + a_4 + a_6 + a_9 + a_{10} &= 0, \\
-2a_2 + a_3 + a_4 + a_7 + a_9 &= 0, \\
a_3 - a_4 &= 0, \\
a_3 - a_7 &= 0, \\
a_4 - a_6 &= 0, \\
a_5 - a_8 &= 0, \\
a_6 - a_{10} &= 0, \\
a_7 - a_9 &= 0, \\
a_9 - a_{10} &= 0,
\end{aligned}
\]

\hspace{1cm} (4.132)

which admits a two parameters family of solutions:

\[
a_5 = a_8 = a, \quad a_3 = a_4 = a_6 = a_7 = a_9 = a_{10} = b, \quad a_1 = a + b, \quad a_2 = 2b.
\]

\hspace{1cm} (4.133)

Therefore, in contrast to the case of the graviton for which the same analysis at the TT level fixes the coefficients up to an overall constant, in this case we still have the freedom to change a relative coefficient arbitrarily, to the point of possibly “switch off” some interaction terms from the vertex. Indeed, using (4.133) we find the following form for the cubic TT Lagrangian

\[
L^{\text{TT}}_1(a, b) = a \left[ H^{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} \partial H^{\alpha\beta} + H^{\mu\nu} \partial_\mu \partial_\nu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H^{\mu\nu} \partial_\mu \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta} \right] \\
+ b \left[ H^{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} \partial H^{\alpha\beta} + 2H^{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} H^{\beta\alpha} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\alpha H_{\beta\nu} \\
+ H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\alpha H_{\beta\mu} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\alpha H_{\nu\beta} \right].
\]
Although, as mentioned, all possible values of \(a\) and \(b\) are possible at this level, there is a unique choice of the parameters which reproduces the results of the DC for three particles amplitudes at tree level. Indeed, in the case \(a = 1\), \(b = 0\), the TT Lagrangian is simply:

\[
\mathcal{L}^{TT}_1(a = 1, b = 0) = H_{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} H^{\alpha\beta} + H_{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H_{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta},
\]

from which it is possible to compute the three particles scattering amplitudes, assigning to the field \(H_{\mu\nu}\) asymmetric transverse polarizations \(\varepsilon_{\mu\nu}\), to be decomposed as in Section 2.4.1. The result of the amplitude is the following:

\[
\mathcal{A}_3 = \varepsilon_{\alpha\beta} \varepsilon_{\mu\nu} p_2 \varepsilon_{\mu\nu} p_2 + \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} p_2 \varepsilon_{\mu\nu} p_2 + \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} \varepsilon_{\mu\nu} p_2 \varepsilon_{\mu\nu} p_2 + \text{perms. of } 1,2,3,
\]

which exactly matches the result of Section 2.4.2 up to the overall constant, that can be fixed only moving to the quartic step of the Noether procedure.

Let us comment more on this choice of the parameters in (4.134). If we split \(H_{\mu\nu}\) into its symmetric and antisymmetric parts, we can explicitly study the effect of the parameters \(a\) and \(b\), so as to understand what are the minimal requirements which lead to the choice \(b = 0\). To this end, we write \(H_{\mu\nu} = H^S_{\mu\nu} + H^A_{\mu\nu}\) in (4.134) and we recall that \(H^S_{\mu\nu}\) contains both the graviton and the scalar fields, while \(H^A_{\mu\nu}\) coincides with the two-form field. The result is the following

\[
\mathcal{L}^{TT}_1(a, b) = (a + 3b) \left\{ H^S_{\mu\nu} \partial_\mu \partial_\nu H^S_{\alpha\beta} H^S_{\alpha\beta} + 2H^S_{\mu\nu} \partial_\mu H^S_{\alpha\beta} \partial^\alpha H^S_{\beta\nu} \right\}
+ (a - b) \left\{ H^S_{\mu\nu} \partial_\mu \partial_\nu H^A_{\alpha\beta} H^A_{\alpha\beta} + 2H^A_{\mu\nu} \partial_\mu H^S_{\alpha\beta} \partial^\alpha H^S_{\beta\nu} 
- 2H^S_{\mu\nu} \partial_\mu H^A_{\alpha\beta} \partial^\alpha H^S_{\beta\nu} - 2H^A_{\mu\nu} \partial_\mu H^A_{\alpha\beta} \partial^\alpha H^S_{\beta\nu} \right\}.
\]

Since our goal is to build a DC Lagrangian, we wish to fix the coefficients \(a\) and \(b\) in order for (4.137) to reproduce the results of Section 2.4.2. To this end, let us notice that in (4.137) all the vertices containing the two-form field \((B^A_{\mu\nu} \equiv H^A_{\mu\nu})\) appear with a factor \((a - b)\), while all the other vertices with a factor \((a + 3b)\). Therefore, in order to fix \(a\) and
we can limit ourselves to consider the amplitudes with three gravitons ($M_3(GGG)$) and with one graviton and two two-forms ($M_3(GAA)$). From the only requirement that these two amplitudes match the ones in Section 2.4.2, we derive the conditions

\[
\begin{align*}
3a + 3b &= \kappa, \\
3a - 3b &= \kappa,
\end{align*}
\]

whose unique solution is $a = \kappa$, $b = 0$. Therefore, we have shown that the choice of the coefficients $a$ and $b$ which allows to reproduce the results of the DC does not involve the analysis of amplitudes which contain the scalar field. This is particularly important for what we will discuss in Section 4.4.1. For the remainder of this section let us consider, unless otherwise specified, $a = 1$, dropping the gravitational coupling constant for the sake of simplicity.

Let us also notice that the choice $b = 0$ selects of the only terms in the Lagrangian of eq. (4.121) which possess an enhanced Lorentz symmetry: indeed, the vertex of eq. (4.135) is invariant under two independent Lorentz transformations acting, respectively, on the left and on the right indices of $H_{\mu\nu}$. This is due to the fact that L-type (R-type) indices are always contracted only with other L-type (R-type) indices or with derivatives. Moreover, a closer inspection of the quadratic Lagrangian (4.117), shows that the same factorization into L and R indices also holds for the free theory. This is the natural consistency feature of the DC construction, where the gravity amplitudes are obtained from the product of two factors (the left and the right ones) which are separately Lorentz invariant, each containing cubic vertices with exactly one derivative. The origin of such a twofold Lorentz-invariance $O(D - 1, 1)_L \otimes O(D - 1, 1)_R$ can be traced back to String Theory, and indeed a field-theoretical linear realization of this symmetry in the low-energy effective action of the closed string was discussed already in [134–136], while now being studied in the context of Double Field Theory [137]. In [138], inspired by the KLT relations, a pure gravity action whose vertices respect this symmetry was built up to the quintic interactions and in [139] it was shown that it is possible to choose a field basis for the full EH action in which this symmetry is manifest. Also, as in the Double Field Theory and in [139], both the quadratic action (4.117) and the TT cubic vertex (4.135), are invariant under a $Z_2$ symmetry which exchanges the L and the R indices of the field $H_{\alpha\beta}$, together with the indices of the derivatives.
The emergence of this additional symmetry, together with the fact that this choice reproduces the results of the DC relations at the cubic level, gives a particular significance to the case $b = 0$ ($a = 1$ still is an arbitrary choice at the cubic level), which we call the double-copy part of the vertex. Therefore, we shall first focus on this special case in the following section, while in Section 4.4.3 we shall discuss the completion of the cubic vertex also including the case $b \neq 0$.

### 4.4.1 Double-copy part of the cubic vertex

As in the case of gravity, once we have fixed (partially, in this case) the coefficients of the ansatz for the TT Lagrangian with the TT part of its gauge variation, we proceed collecting all the terms in $\delta_0 L_{TT}^1$, and looking for possible counterterms so as to build the double-copy cubic vertex $L_1$. For the details of the computation we refer to appendix B.1, while here we simply state the result:

$$L_1 = H^{\mu\nu} \partial_\mu H_{\alpha\beta} H^{\alpha\beta} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta} - \frac{1}{2} \partial^\alpha \partial^\beta \partial^\cdot D H_{\alpha\beta} - \tilde{D}^\alpha \partial^\beta \partial^\cdot D H_{\alpha\beta}. \quad (4.139)$$

We remark that $L_1$ shares the same $O(D-1,1)_L \otimes O(D-1,1)_R \otimes \mathbb{Z}_2$ symmetry as its TT part (eq. (4.135)). As a last step of the Noether procedure, we collect all the terms proportional to the equations of motion generated along the calculations of appendix B.1:

$$\delta_0 L_1 = \frac{1}{2} \left\{ \epsilon^{\alpha\beta} \partial_\mu H_{\alpha\beta} \partial^\cdot D H_{\alpha\beta} \right\} + \partial_\beta \epsilon_{\alpha\beta} H_{\alpha\beta} \partial^\cdot D H_{\alpha\beta} - \epsilon^{\alpha\beta} \partial_\beta H_{\alpha\beta} \partial^\cdot D H_{\alpha\beta} + \epsilon^{\alpha\beta} \partial_\mu H_{\alpha\beta} \partial^\cdot D H_{\alpha\beta}. \quad (4.140)$$
Similarly to the case of the graviton vertex we notice that
\[
H_{\alpha\nu}(\partial_\beta \alpha^\nu - \partial^\nu \alpha_\beta) + H_{\mu\beta}(\partial_\alpha \delta^\mu - \partial^\mu \delta_\alpha)
\]
\[
= H_{\alpha\nu}\partial_\beta(\alpha^\nu + \tilde{\alpha}^\nu) + H_{\nu\beta}\partial_\alpha(\alpha^\nu + \tilde{\alpha}^\nu) - \delta_0(H^{\alpha\nu}H_{\nu}^\beta),
\]  
(4.141)
therefore, with an appropriate field redefinition we eliminate the terms in the vertex which are proportional to the equations of motion:
\[
H_{\mu\nu} \rightarrow H_{\mu\nu} - H_{\mu\alpha}H_{\alpha\nu} + \frac{1}{2}H_{\mu\nu}\frac{\partial \cdot \partial \cdot H}{\Box}.
\]  
(4.142)

As mentioned in Section 3.1.2, we should not worry about the presence of the possibly singular term \(\frac{\partial \cdot \partial \cdot H}{\Box}\), since it only affects a pure gauge sector of the theory and indeed it simply corresponds to the difference \(H - \varphi\), which has no reason to be singular. By means of this redefinition, we can recast the variation of \(L_1\) in the form:
\[
\delta L_1 = \mathcal{E}^{\alpha\beta}\left\{\partial_\mu H_{\alpha\beta}(\alpha^\mu + \tilde{\alpha}^\mu) + H_{\alpha\nu}\partial_\beta(\alpha^\nu + \tilde{\alpha}^\nu) + H_{\nu\beta}\partial_\alpha(\alpha^\nu + \tilde{\alpha}^\nu)\right\},
\]  
(4.143)
therefore from \(\delta_0 S_1 = -\delta_1 S_0\) we can read the correction to the gauge transformation:
\[
\delta_1 H_{\alpha\beta} = 2\{\xi^\nu \partial_\nu H_{\alpha\beta} + H_{\alpha\nu}\partial_\beta \xi^\nu + H_{\nu\beta}\partial_\alpha \xi^\nu\},
\]  
(4.144)
\[
\xi_\mu := \frac{1}{2}(\alpha_\mu + \tilde{\alpha}_\mu).
\]  
(4.145)

Finally, we can add the appropriate coupling constant, in order to match the three particles scattering amplitudes of Section 2.4.2 one should multiply (4.139) and (4.144) by \(\kappa\).

Having led the cubic step of the Noether procedure to completion, we can now compare this result with the expectation to reproduce the vertices of the \(\mathcal{N} = 0\) Supergravity. Then, in order to make full contact with the structure of \(\mathcal{N} = 0\) Supergravity, in Section 4.4.2, we shall discuss the correction to the gauge transformation of \(H_{\mu\nu}\) showing how to connect with the known gauge transformations of the fields in the gravitational multiplet at this order.

We already observed that the double-copy vertex correctly reproduces the results of the DC of tree-level scattering amplitudes at the three particles level. The same amplitudes are also derived from the \(\mathcal{N} = 0\) Supergravity, as discussed in Section 2.4.3. Then,
it is natural to ask whether it is possible to show explicitly that the Lagrangian of eq. (4.139) matches the Lagrangian of the $\mathcal{N} = 0$ Supergravity, if expanded perturbatively at cubic order. In order to check this, we write $H_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu} + \gamma \eta_{\mu\nu} \varphi$, and try to make contact with the vertices derived in Section 2.4.3. The computations are, once again, rather convoluted, and require several field redefinitions which eliminate vertices proportional to the equations of motion of the various fields. Therefore, we collect them in appendix B.2, and we simply state the result here.

After some algebra, we find that the Lagrangian of eq. (4.139) can be shown to correctly match the perturbative expansion of the $\mathcal{N} = 0$ Supergravity action (2.117) at the cubic order, including the non-TT parts, but only in the case $\gamma = \frac{1}{D-2}$. This is an important result, since it is an off-shell constructive proof (actually the first proof, to our knowledge) that $\mathcal{N} = 0$ Supergravity actually matches the result of the DC at this order, with all the field redefinitions and the field basis which lead to this result explicitly stated. The special role played by the value $\gamma = \frac{1}{D-2}$ already emerged at the quadratic level, since it is only for this value of $\gamma$ that the normalization of the scalar field, given in eq. (4.36), matches the normalization of the same field in eq. (2.117).

Let us also mention an additional relevant feature of the $\mathcal{N} = 0$ Supergravity action, that our Lagrangian (4.139) correctly reproduces. In the Lagrangian of $\mathcal{N} = 0$ Supergravity (2.117) the coupling between the scalar field, the two-form and the graviton takes a very specific form involving the product

$$\exp\left[ -\frac{4\kappa}{D-2} \varphi \right] H_{\mu\nu\lambda} H_{\alpha\beta\gamma} g^{\mu\alpha} g^{\nu\beta} g^{\lambda\gamma}. \quad (4.146)$$

If we imagine to perform the Noether procedure separately for $h_{\mu\nu}$, $B_{\mu\nu}$ and $\varphi$, the scalar couplings, in general, would be very much unconstrained: given a scalar Lagrangian density, the multiplication by an arbitrary function $F(\varphi)$ (not of $\partial_{\mu} \varphi$) would give rise to another allowed Lagrangian density, to the extent that it would be perfectly consistent to have no cubic coupling at all with the two-form. However, implementing the Noether procedure on the DC field $H_{\mu\nu}$, we reproduced at the cubic level exactly the coupling between $\varphi$ and the two-form coming from the expansion of the exponential (4.146) to first order in $\varphi$. It would be interesting to check whether, extending the procedure to the quartic interactions and beyond, also the following terms in the expansion of the
exponential are correctly reproduced. Let us also stress that, given the specific choice of the parameters made in the cubic TT Lagrangian of eq. (4.134) in order to reproduce the results of the DC, it might seem that we have actually chosen the coupling of the scalar so as to reproduce (4.146). However, as we also stressed in Section 4.4, the value of the arbitrary parameters in (4.134) was actually fixed without invoking amplitudes involving the scalar, but only looking at the sector containing the graviton and the two-form. Therefore, the fact that the couplings of the scalar in our DC Lagrangian (4.139) can be viewed as a prediction of our construction.

Another relevant comparison is with the cubic vertex of the DC Lagrangian obtained in [16], which was explicitly built to reproduce the results of the DC of scattering amplitudes. Following [16], in Section 3.2.2 we defined a field $H_{\mu\nu}$, which in momentum space is the product of two YM fields, deprived of their color indices. This field contains the gravity fields, and we explicitly wrote its self-interacting cubic vertex, which we can compare with (4.139). To this end, we display here the vertex for $H_{\mu\nu}$ derived in Section 3.2.2, and integrate it by parts in order to make contact as much as possible with eq. (4.139):

$$L_{HHH} = H_{\mu
u} \partial_\mu H^{\alpha\beta} H_{\alpha\beta} - H_{\mu
u} \partial_\mu H^{\alpha\beta} \partial_\nu H_{\alpha\beta} + H_{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu}$$

$$- H_{\mu\nu} \partial_\beta H^{\alpha\beta} \partial_\alpha H_{\mu\beta} + H_{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta} - H_{\mu\nu} H^{\alpha\beta} \partial_\alpha \partial_\beta H_{\mu\nu}$$

$$= H_{\mu\nu} (\partial_\mu H^{\alpha\beta} - \partial_\nu H_{\alpha\beta}) + 2 H_{\mu\nu} H^{\alpha\beta} \partial_\beta H_{\alpha\nu}$$

$$+ 2 H_{\mu\nu} \partial_\beta H^{\alpha\beta} \partial_\alpha H_{\mu\beta} - H_{\mu\nu} \partial^\gamma H_{\alpha\beta} \partial_\gamma H_{\mu\beta}.$$  \hspace{1cm} (4.147)

Then, integrating by parts eq. (4.139) (with the appropriate coupling constant), the cubic vertex resulting from the Noether procedure can be written as

$$\mathcal{L}_1 = \frac{\kappa}{2} \left\{ H_{\mu\nu} (\partial_\mu H^{\alpha\beta} - \partial_\nu H_{\alpha\beta}) + 2 H_{\mu\nu} H^{\alpha\beta} \partial_\beta H_{\alpha\nu}$$

$$+ \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta} \right\} + \kappa H_{\alpha\beta} (\mathcal{D}^\alpha \mathcal{D}^\beta - \mathcal{D}^\alpha \mathcal{D} \partial_\mu H^{\mu\beta} - \mathcal{D}^\beta \partial_\mu H^{\mu\alpha}).$$ \hspace{1cm} (4.148)

We conclude that our results match indeed the cubic vertices obtained in [16], when the latter are multiplied by $\kappa/2$ (in agreement with the prescription of eq. (2.98) when $n = 3$). However, our cubic Lagrangian also includes non-TT contributions which ensure the gauge invariance of the theory at cubic order, whereas in [16] the issue of gauge invariance was not addressed at all.
4.4.2 Interpretation of $\delta_1 H_{\mu\nu}$

We now turn to the study of the correction to the free gauge transformation $\delta_1 H_{\mu\nu}$, which was determined in eq. (4.144). We wish to understand how such a transformation for the field $H_{\mu\nu}$ reflects into the transformations of the gravitational multiplet. In order to accomplish this, we first study the symmetric and antisymmetric parts of the transformation separately, since in Section 3.1.2 this procedure directly led to the correct (linearized) gauge transformations of the graviton and the two-form (which, for the sake of simplicity, we write here without a factor of $2\kappa$). The result is:

$$
\begin{align*}
\delta_1 H^S_{\mu\nu} &= \xi \cdot \partial H^S_{\mu\nu} + \partial_\mu \xi^\alpha H^S_{\alpha\nu} + \partial_\nu \xi^\alpha H^S_{\mu\alpha}, \\
\delta_1 H^A_{\mu\nu} &= \xi \cdot \partial H^A_{\mu\nu} + \partial_\mu \xi^\alpha H^A_{\alpha\nu} - \partial_\nu \xi^\alpha H^A_{\mu\alpha}.
\end{align*}
$$

As for what concerns the symmetric part, if we add it to the free gauge transformation, the total gauge variation is:

$$
\delta H^S_{\mu\nu} = \delta_0 H^S_{\mu\nu} + \delta_1 H^S_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi \cdot \partial H^S_{\mu\nu} + \partial_\mu \xi^\alpha H^S_{\alpha\nu} + \partial_\nu \xi^\alpha H^S_{\mu\alpha},
$$

which corresponds to the Lie derivative of a metric tensor written as $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$.

This matches with the analysis of Section 3.1, where we concluded that, at the linearized level, everything is compatible with the fact that $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$ is a metric tensor.

Similarly, the antisymmetric part of eq. (4.149) corresponds to the Lie derivative of a two-form.

---

8The Lie derivative of a metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$ is

$$
\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\nu} \partial_\alpha \xi^\alpha + g_{\alpha\nu} \partial_\mu \xi^\alpha = \xi^\alpha \partial_\alpha (\eta_{\mu\nu} + H^S_{\mu\nu}) + (\eta_{\mu\nu} + H^S_{\mu\nu}) \partial_\alpha \xi^\alpha + (\eta_{\alpha\nu} + H^S_{\alpha\nu}) \partial_\mu \xi^\alpha = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi \cdot \partial H^S_{\mu\nu} + \partial_\mu \xi^\alpha H^S_{\alpha\nu} + \partial_\nu \xi^\alpha H^S_{\mu\alpha} = \delta H^S_{\mu\nu}.
$$

9The Lie derivative of a two-form $B = \frac{1}{2}B_{\mu\nu}dx^\mu dx^\nu$ is $\mathcal{L}_\xi B = d(i_\xi B) + i_\xi (d B)$. Therefore, the Lie derivative of $B$ reads:

$$
\mathcal{L}_\xi B_{\mu\nu} = \xi \cdot \partial B_{\mu\nu} + \partial_\mu \xi^\alpha B_{\alpha\nu} - \partial_\nu \xi^\alpha B_{\alpha\mu}.
$$
Last, we study the transformation of the scalar field dictated by eq. (4.144). We recall that our definition of the scalar is:

$$\varphi = H - \frac{\partial \cdot \partial \cdot H}{\Box},$$  

(4.153)

which was designed in order to have a gauge-invariant scalar in the free theory. In order to compute $\delta_1 \varphi$, we first compute separately the two variations:

$$\delta_1 H = \xi \cdot \partial H + (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) H_{\mu\nu} = \xi \cdot \partial H + (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) H^S_{\mu\nu},$$  

(4.154)

$$\delta_1 \frac{\partial \cdot \partial \cdot H}{\Box} = \frac{1}{\Box} \left( \partial^\mu \partial^\nu \xi^\alpha \partial_\alpha H^S_{\mu\nu} + 2(\partial^\nu \xi^\alpha) \partial_\alpha \partial \cdot H^S + \xi^\alpha \partial_\alpha \partial \cdot H^S \\
+ 2\partial^\mu \partial^\nu \xi^\alpha \partial_\alpha H^S_{\alpha\nu} + 2\Box \xi^\alpha \partial \cdot H^S_{\alpha} + \Box (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) H^S_{\mu\nu} \right).$$  

(4.155)

Therefore, the correction to the transformation of $\varphi$ reads

$$\delta_1 \varphi = \delta_1 H - \delta_1 \frac{\partial \cdot \partial \cdot H}{\Box}$$

$$= \xi \cdot \partial H + (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) H^S_{\mu\nu} - \frac{1}{\Box} \left( \partial^\mu \partial^\nu \xi^\alpha \partial_\alpha H^S_{\mu\nu} + 2(\partial^\nu \xi^\alpha) \partial_\alpha \partial \cdot H^S + \xi^\alpha \partial_\alpha \partial \cdot H^S \\
+ 2\partial^\mu \partial^\nu \xi^\alpha \partial_\alpha H^S_{\alpha\nu} + 2\Box \xi^\alpha \partial \cdot H^S_{\alpha} + \Box (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) H^S_{\mu\nu} \right).$$  

(4.156)

As evident from (4.156) $\delta_1 \varphi \neq \mathcal{L}_\xi \varphi$, since the Lie derivative of a scalar is $\mathcal{L}_\xi \varphi = \xi \cdot \partial \varphi$. However, the transformation of a scalar field under diffeomorphisms in a theory of gravity must be its Lie derivative, in order to preserve the diffeomorphism invariance of the full theory. Thus, we face an apparent paradox.

As a solution, we propose to rethink the definition of the scalar $\varphi$. Indeed, the definition (4.153) was designed somewhat “ad hoc” for the linearized theory, in the sense that the term $\frac{\partial \cdot \partial \cdot H}{\Box}$ was introduced simply to compensate the free gauge variation of $H$, but there is no reason to think that the same term should, in general, cancel the unwanted part of the transformation of $H$ when also the nonlinear part of the transformation is included. Since indeed this does not happen, we propose to think of the scalar in the complete theory as a non-linear function of $H^S_{\mu\nu}$, which one can reconstruct perturba-
tively. Therefore, we can expand:

\[ \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots, \tag{4.157} \]

with \( \psi^{(n)} \) containing \( n \) powers of the field \( H_{\mu\nu} \) and \( \psi^{(1)} = \varphi \) as defined in (4.153). Then, we require the correct transformation rule under diffeomorphisms for the scalar in the complete theory, and read this order by order:

\[ \delta \psi = \xi \cdot \partial \psi \Rightarrow \begin{cases} 
\delta_0 \psi^{(1)} = \xi \cdot \partial \psi^{(0)} = 0, \\
\delta_1 \psi^{(1)} + \delta_0 \psi^{(2)} = \xi \cdot \partial \psi^{(1)}, \\
\delta_2 \psi^{(1)} + \delta_1 \psi^{(2)} + \delta_0 \psi^{(3)} = \xi \cdot \partial \psi^{(2)}, \\
\ldots 
\end{cases} \tag{4.158} \]

The first line of the system is satisfied, by construction, by \( \varphi \). We can now exploit the second line in order to determine \( \psi^{(2)} \) (then we stop, since at the cubic step of the Noether procedure we do not know \( \delta_2 \)):

\[ \delta_0 \psi^{(2)} = -H^S_{\mu\nu} \delta_0(H^S_{\mu\nu}) - \xi \cdot \partial H \square + \delta_1 \left( \frac{\partial \cdot \partial \cdot H}{\square} \right). \tag{4.159} \]

In order to extract the value of \( \psi^{(2)} \) from such expression, we formally integrate by parts the inverse d’Alembertian using the relation

\[ f(x)g(x) = \frac{1}{\square} \left( f(x)g(x) \right) = \frac{1}{\square} \left( \square f \right) g + \left( \square g \right) f + 2 \partial^\mu f \partial_\mu g, \tag{4.160} \]

so as to derive

\[ -H^S_{\mu\nu} \delta_0(H^S_{\mu\nu}) = -\frac{1}{2} \delta_0(H^S_{\mu\nu}H^S_{\mu\nu}) - \frac{1}{\square} \delta_0 \left\{ H^S_{\mu\nu} \square H^S_{\mu\nu} + \partial_\alpha H^S_{\mu\nu} \partial^\alpha H^S_{\mu\nu} \right\}, \tag{4.161} \]

and

\[ -\xi \cdot \partial \frac{\partial \cdot \partial \cdot H^S}{\square} = -\frac{1}{\square} \left( \xi^\alpha \partial_\alpha \partial \cdot H^S + \square \xi^\alpha \partial_\alpha \partial \cdot H^S + 2 \partial^\alpha \partial^\beta \partial \cdot H^S \partial_\alpha \right). \tag{4.162} \]

This allows to sum the first two terms on the r.h.s. of eq. (4.159) with the last one. After
some algebra, recalling that $\delta_0 H^S_{\mu\nu} = \partial(\mu \xi_\nu)$, the result can be expressed as:

$$
\delta_0 \psi^{(2)} = \frac{1}{\Box} \left( - H^S_{\mu\nu} \Box H^S_{\mu\nu} + \frac{1}{2} \partial^\alpha H^S_{\mu\nu} \partial_\alpha H^S_{\mu\nu} - \frac{3}{4} \partial^\alpha H^S_{\mu\nu} \partial_\alpha H^S_{\mu\nu} + 2 H^S_{\mu\nu} \partial_\mu \partial \cdot H^S_{\nu} \\
- H^S_{\mu\nu} \partial_\mu \partial_\nu - \partial \cdot H^S_{\alpha} \partial_\alpha - \partial^\alpha \partial \cdot H^S_{\alpha} - \partial \cdot \partial \cdot H^S_{\alpha} \right) \quad (4.163)
$$

Interestingly, this expression is reminiscent of the second-order term in the perturbative expansion of the Ricci scalar of a manifold with Levi-Civita connection, in terms of a metric perturbation over flat background $H^S_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Indeed, if we expand the Ricci scalar with respect to such perturbation, the first two orders are:

$$
R^{(1)} = \partial \cdot \partial \cdot H^S - \Box H^S, \quad (4.164)
$$

$$
R^{(2)} = H^S_{\mu\nu} \Box H^S_{\mu\nu} - \frac{1}{2} \partial^\nu H^S_{\mu\nu} \partial_\alpha H^S_{\mu\nu} + \frac{3}{4} \partial^\alpha H^S_{\mu\nu} \partial_\alpha H^S_{\mu\nu} - 2 H^S_{\mu\nu} \partial_\mu \partial \cdot H^S_{\nu} \\
+ H^S_{\mu\nu} \partial_\mu \partial_\nu - \partial \cdot H^S_{\alpha} \partial_\alpha + \partial^\alpha \partial \cdot H^S_{\alpha} - \frac{1}{4} \partial^\alpha H^S \partial_\alpha H^S. \quad (4.165)
$$

From the first equation, it is tantalizing that our definition of the scalar field can be read as:

$$
\varphi = - \frac{R^{(1)}}{\Box}. \quad (4.166)
$$

Since we are looking for the full scalar $\psi$, which must be a scalar under diffeomorphisms, we are naturally led to a geometrical guess for the general solution, in the form

$$
\chi := - \frac{R}{\Box}, \quad (4.167)
$$

where $R$ is the Ricci scalar of the manifold, expressed in terms of the metric $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$, while $\Box$ is the Laplace-Beltrami operator for a scalar field, i.e. the covariant version of the usual wave operator in flat spacetime:

$$
\hat{\Box} := g^{\mu\nu} D_\mu \partial_\nu = g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma^\alpha_{\mu\nu} \partial_\alpha), \quad (4.168)
$$

with $D_\mu$ the covariant derivative on the spacetime manifold. We can expand also $\Box$ in
powers of $H_{\mu\nu}^S$, with the result:

\begin{equation}
\hat{\Box}^{(0)} = \eta^{\mu\nu}\partial_\mu \partial_\nu = \Box,
\end{equation}

\begin{equation}
\hat{\Box}^{(1)} = -H_S^{\mu\nu}\partial_\mu \partial_\nu - \partial \cdot H_S^\alpha \partial_\alpha + \frac{1}{2} \partial^\alpha H_S^\alpha \partial_\alpha.
\end{equation}

Then, we can compare the result of eq. (4.163) with the perturbative expansion of eq. (4.167):

\begin{equation}
\chi = -R \hat{\Box} = -R^{(1)} + R^{(2)} + \ldots = -\frac{1}{\Box(1 + \hat{\Box}^{(1)})} \left( R^{(1)} + R^{(2)} + \ldots \right).
\end{equation}

Therefore, from eqs. (4.165) and (4.170), the order two term in this expansion is:

\begin{equation}
\chi^{(2)} = -\frac{1}{\Box} \left( H_S^{\mu\nu} \Box H_S^{\mu\nu} - \frac{1}{2} \partial^\rho H_S^{\mu\nu} \partial_\rho H_S^{\mu\nu} + \frac{3}{4} \partial^\alpha H_S^{\mu\nu} \partial_\alpha H_S^{\mu\nu} - 2 H_S^{\mu\nu} \partial_\mu \partial_\nu \right)
\end{equation}

Since $\delta_0 H_S = \delta_0 \partial \cdot \partial H_S$, one can easily check that $\delta_0 \chi^{(2)} = \delta_0 \psi^{(2)}$, therefore at this order we can safely state that the non-linear correction to the dilaton is compatible with the fact that the scalar field in the complete theory is:

\begin{equation}
\psi = \frac{R}{\Box}.
\end{equation}

Now that we have defined a “full scalar” with the correct transformation properties under diffeomorphisms, we can wonder whether the definition of the graviton as:

\begin{equation}
h_{\mu\nu} := H_S^{\mu\nu} - \gamma \eta_{\mu\nu} \varphi,
\end{equation}

is still a good definition for every value of $\gamma \in \mathbb{R}$, i.e. if it displays the expected trans-
formation rules. At the order $\delta_1$, the expected transformation for a graviton is:

$$\delta_1 h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} = \xi \cdot \partial h_{\mu\nu} + h_{\mu\alpha} \partial_\nu \xi^\alpha + h_{\alpha\nu} \partial_\mu \xi^\alpha.$$  \hfill (4.175)

However, while this is indeed the transformation of $H^S_{\mu\nu}$, the cumbersome expression (4.156) of $\delta_1 \varphi$ spoils our expectations. To solve this issue the natural attitude, following the interpretation of the scalar, is to assume that also the definition of the graviton in terms of $H_{\mu\nu}$ gets corrections in the complete theory. Thus, we are led to propose the following ansatz

$$h_{\mu\nu} = H^S_{\mu\nu} - \gamma X_{\mu\nu} \psi,$$  \hfill (4.176)

where $\psi$ is the scalar of the complete theory, as discussed above, and $X_{\mu\nu}$ is a symmetric rank-two tensor, whose expression is to be determined. As in the case of $\psi$, we define $X_{\mu\nu}$ perturbatively in powers of $H^S_{\mu\nu}$, with $X_{\mu\nu}^{(0)} = \eta_{\mu\nu}$ (then, $\delta_0 X_{\mu\nu}^{(0)} = 0 \ \forall n$). We require from eq. (4.176) to reproduce the correct transformation properties of the graviton, but since we only worked out the cubic order of the Noether procedure, all we can ask from $X_{\mu\nu}$ is that:

$$\delta_1 h_{\mu\nu} = \xi \cdot \partial (H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \psi^{(0)}) + \gamma \delta_0 X_{\mu\nu}^{(1)} \psi^{(0)} - \gamma X_{\mu\nu}^{(0)} \delta_1 \psi^{(0)} - \gamma \delta_1 \psi^{(0)} + \gamma X_{\mu\nu}^{(0)} \delta_0 \psi^{(0)},$$  \hfill (4.177)

where $h_{\mu\nu}^{(1)}$ is defined as the order-one contribution in the expansion of eq. (4.176):

$$h_{\mu\nu}^{(1)} := H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi.$$  \hfill (4.178)

We can exploit the second equation in the system (4.158), rearranging eq. (4.177) in the form:

$$\delta_1 h_{\mu\nu} = \xi \cdot \partial (H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \psi^{(0)}) + (H^S_{\mu\alpha} - \gamma \eta_{\mu\alpha} \psi^{(0)}) \partial_\nu \xi^\alpha + (H^S_{\alpha\nu} - \gamma \eta_{\alpha\nu} \psi^{(0)}) \partial_\mu \xi^\alpha +$$
$$\gamma \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \delta_0 X_{\mu\nu}^{(1)} \psi^{(0)} = \xi \cdot \partial h_{\mu\nu}^{(1)} + h_{\mu\alpha}^{(1)} \partial_\nu \xi^\alpha + h_{\alpha\nu}^{(1)} \partial_\mu \xi^\alpha,$$  \hfill (4.179)

which is satisfied if:

$$\delta_0 X_{\mu\nu}^{(1)} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$$  \hfill (4.180)
Chapter 4. Lagrangian formulation

Hence, we can conclude that:

\[ X_{\mu\nu} = X^{(0)}_{\mu\nu} + X^{(1)}_{\mu\nu} + \mathcal{O}(H^2) = \eta_{\mu\nu} + H^S_{\mu\nu} + \Delta_{\mu\nu} + \mathcal{O}(H^2), \tag{4.181} \]

with \( \delta_0 \Delta_{\mu\nu} = 0 \). At this order we cannot exclude the presence of a non-vanishing \( \Delta_{\mu\nu} \), but we can argue that, in order for \( \Delta_{\mu\nu} \) to be gauge invariant, it must be proportional to \( \eta_{\mu\nu} \varphi \). Therefore, at this order, the definition of the graviton as

\[ h_{\mu\nu} = H^S_{\mu\nu} - \gamma \left[ (\eta_{\mu\nu} + H^S_{\mu\nu}) \varphi + \eta_{\mu\nu} \psi^{(1)} \right], \tag{4.182} \]

possesses the correct transformation property under diffeomorphisms for every value of \( \gamma \). However, such a non-linear redefinition of the fields might generate new cubic vertices (we do not care about the quartic ones at this level) when employed in the quadratic action. This aspect may not be a problem in general, but since in the previous section we showed that the double-copy vertex \( \mathcal{L}_1 \) for the self-interaction of \( H_{\mu\nu} \) contains exactly the same (off-shell) vertices of the \( \mathcal{N} = 0 \) Supergravity, we do not want this nice result to be spoiled by the modified definition of the fields in the gravitational multiplet.

If we include the quadratic corrections, the decomposition (4.182) of \( H_{\mu\nu} \) can be written as:

\[ H_{\mu\nu} = h^{(1)}_{\mu\nu} + B_{\mu\nu} + \gamma \eta_{\mu\nu} \varphi + \gamma \delta S_{\mu\nu}, \tag{4.183} \]

with \( h^{(1)}_{\mu\nu} = H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi \) and \( \delta S_{\mu\nu} = \left( X^{(1)}_{\mu\nu} \varphi + \eta_{\mu\nu} \psi^{(2)} \right) \) symmetric. If we replace (4.183) in the quadratic Lagrangian of eq. (4.117), we get a new cubic vertex:

\[
\begin{align*}
\delta \mathcal{L}^{(0)}_H & = \gamma \delta S^{\alpha\beta} \left( \Box \eta_{\alpha\mu} \eta_{\beta\nu} - \partial_{\alpha} \partial_{\mu} \eta_{\beta\nu} - \partial_{\beta} \partial_{\mu} \eta_{\alpha\nu} + \frac{\partial_{\alpha} \partial_{\beta} \partial_{\mu} \partial_{\nu}}{\Box} \right) (h_{\mu\nu}^{(1)} + B_{\mu\nu} + \gamma \eta_{\mu\nu} \varphi) \\
& = \gamma \delta S^{\mu\nu} \left( \Box h_{\mu\nu}^{(1)} - \partial_{\mu} \partial \cdot h_{\nu}^{(1)} - \partial_{\nu} \partial \cdot h_{\mu}^{(1)} + \partial_{\mu} \partial_{\nu} h_{\nu}^{(1)} \right) + \gamma \delta S \Box \varphi \\
& - \gamma \delta S^{\mu\nu} \partial_{\mu} \partial_{\nu} \left( h_{\mu\nu}^{(1)} - \frac{\partial \cdot \partial \cdot h_{\mu\nu}^{(1)}}{\Box} \right) + \gamma \varphi. \tag{4.184} \end{align*}
\]

The second line of eq. (4.184) contains vertices proportional to the linearized equations of motion of the graviton (\( R_{\mu\nu} = 0 \)) and of the scalar field (\( \Box \varphi = 0 \)), respectively, therefore they can be trivially absorbed in the quadratic Lagrangian. The last line, however, contains a term which is in general non-vanishing. At the linearized level, with
\( h^{(1)}_{\mu\nu} = H^S_{\mu\nu} - \gamma \eta_{\mu\nu} \varphi \), as we already mentioned the following identity holds:

\[
\frac{\partial \cdot \partial \cdot h^{(1)}}{\Box} = [1 - \gamma (D - 1)] \varphi.
\] (4.185)

Therefore, from eq. (4.184) we can read that the quadratic correction to the definition of the gravity fields (in particular \( h_{\mu\nu} \) and \( \varphi \), in this case) manifestly does not generate new vertices if \( \gamma = 0 \), or:

\[
1 - \gamma (D - 1) + \gamma = 0 \Rightarrow \gamma = \frac{1}{D - 2},
\] (4.186)

thus confirming the fact that among all possible values for \( \gamma \) the two choices \( \gamma = 0 \) and \( \gamma = \frac{1}{D - 2} \) are somewhat special, as they allow to display in a more explicit fashion a number of relevant features of the construction.

### 4.4.3 Full cubic vertex for \( H_{\mu\nu} \)

In the course of our construction of the cubic vertex for the DC Lagrangian in Section 4.4, we found an arbitrariness in the coefficients of the cubic TT vertex. However, owing to the particular features of one of the solutions, in Section 4.4.1 we discussed only what we called the double-copy part of the vertex, hence neglecting some terms of \( \mathcal{L}^{TT} \) which are not otherwise forbidden by gauge invariance at the cubic level. Indeed, the full TT cubic Lagrangian can be written as in (4.134) that we report here for additional clarity:

\[
\mathcal{L}^{TT}_1 = a \left[ H^{\mu\nu} \partial_\mu H_{\alpha\beta} H^{\alpha\beta} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta} \right] \\
+ b \left[ H^{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} H^{\alpha\beta} + 2 H^{\mu\nu} \partial_\mu \partial_\nu H_{\alpha\beta} H^{\beta\alpha} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\alpha H_{\beta\nu} \\
+ H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\beta\mu} + H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\beta H_{\alpha\mu} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\alpha H_{\nu\beta} \\
+ H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\nu\alpha} + H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\beta H_{\mu\alpha} \right],
\] (4.187)

with arbitrary \( a \) and \( b \). In Section 4.4.1 we only discussed the case \( b = 0 \) (\( a = 1 \) is not actually a limitation, since the cubic level does not fix the overall constant, as already stressed). Now, we explore the completion of the full TT vertex (4.187). However, since calculations are very close to the ones of appendix B.1 (only the position of some indices changes), we shall limit ourselves to state the result.
Chapter 4. Lagrangian formulation

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The full off-shell cubic vertex $L_1$, satisfying $\delta_0 L_1 \approx 0$, is:

$$L_1 = a \left\{ H^{\mu\nu} \partial_{\mu} \partial_{\nu} H_{\alpha\beta} H^{\alpha\beta} + H^{\mu\nu} \partial_{\mu} H^{\alpha\beta} \partial_{\beta} H_{\alpha\nu} + H^{\mu\nu} \partial_{\nu} H^{\alpha\beta} \partial_{\alpha} H_{\mu\beta} \right. $$

$$- \frac{1}{2} \partial \cdot D H_{\alpha\beta} H^{\alpha\beta} - D^{\alpha} \partial_{\alpha} \partial \cdot D H_{\alpha\beta} \right\}$$

$$+ b \left\{ H^{\mu\nu} \partial_{\mu} \partial_{\nu} H_{\alpha\beta} H^{\alpha\beta} + 2 H^{\mu\nu} \partial_{\mu} \partial_{\nu} H_{\alpha\beta} H_{\beta\alpha} + H^{\mu\nu} \partial_{\mu} H^{\alpha\beta} \partial_{\alpha} H_{\beta\nu} + H^{\mu\nu} \partial_{\nu} H^{\alpha\beta} \partial_{\beta} H_{\alpha\mu} + H^{\mu\nu} \partial_{\mu} H^{\alpha\beta} \partial_{\beta} H_{\nu\alpha} \right. $$

$$- \frac{1}{2} \partial \cdot D H_{\alpha\beta} H^{\alpha\beta} - \partial \cdot D H_{\alpha\beta} H^{\beta\alpha} - D^{\alpha} \partial_{\alpha} \partial \cdot D H_{\alpha\beta} \right\}$$

$$- D^{\alpha} \partial_{\alpha} \partial \cdot D H_{\beta\alpha} - 2 D^{\beta} \partial_{\alpha} \partial \cdot D H_{\beta\alpha} - 2 D^{\alpha} \partial_{\alpha} \partial \cdot D H_{\beta\alpha} \} . \quad (4.188)$$

Then, collecting terms proportional to the free equations of motion, the complete gauge variation of the vertex can be written as follows

$$\delta_0 L_1 = \mathcal{E}^{\alpha\beta} \left\{ (a + b) \left[ \partial_{\mu} H_{\alpha\beta}(\alpha^\mu + \bar{\alpha}^\mu) + H_{\alpha\nu}(\partial_{\beta} \alpha^\nu - \partial^\nu \alpha_{\beta}) + H_{\mu\beta}(\partial_{\alpha} \bar{\alpha}^\mu - \partial^\mu \bar{\alpha}_{\alpha}) \right. $$

$$+ \delta_0(H_{\alpha\beta} \partial \cdot D) \right\} + 2 b \left[ \partial_{\mu} H_{\beta\alpha}(\alpha^\mu + \bar{\alpha}^\mu) + H_{\nu\alpha}(\partial_{\beta} \alpha^\nu - \partial^\nu \alpha_{\beta}) \right. $$

$$+ H_{\beta\mu}(\partial_{\alpha} \bar{\alpha}^\mu - \partial^\mu \bar{\alpha}_{\alpha}) + \delta_0(H_{\beta\alpha} \partial \cdot D) \} . \quad (4.189)$$

As in Section 4.4.1, it is now possible to perform a field redefinition:

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + (a + b) \left\{ - H_{\mu\alpha} H^\alpha_{\nu} + \frac{1}{2} H_{\mu\nu} \partial \cdot \partial \cdot H \right\}$$

$$+ 2 b \left\{ - H_{\nu\alpha} H^\alpha_{\mu} + \frac{1}{2} H_{\nu\mu} \partial \cdot \partial \cdot H \right\} , \quad (4.190)$$

in order to eliminated vertices proportional to the equations of motion. Implementing this redefinition, the correction to the gauge transformation reads:

$$\delta_1 H_{\alpha\beta} = 2(a + b) \left[ \xi^\nu \partial_{\nu} H_{\alpha\beta} + H_{\alpha\nu} \partial_{\beta} \xi^\nu + H_{\nu\beta} \partial_{\alpha} \xi^\nu \right]$$

$$+ 4 b \left[ \xi^\nu \partial_{\nu} H_{\beta\alpha} + 2 H_{\nu\alpha} \partial_{\beta} \xi^\nu + 2 H_{\beta\nu} \partial_{\alpha} \xi^\nu \right] , \quad (4.191)$$

again with $\xi_{\mu} = \frac{1}{2}(\alpha + \bar{\alpha})_{\mu}$. It is possible to observe that the choice $a = b$ in eq. (4.188) decouples the two-form because of the symmetries under exchange of indices, whereas for general $b \neq a$ this Lagrangian produces the same type of amplitudes of the double-copy vertex, but with different relative coefficients between them. We expect
the extension of the Noether procedure to the level of quartic vertices to teach us more about the allowed values of $a$ and $b$; this happens indeed both in the case of gravity and in the case of YM theory, where consistency with the quartic vertices require the structure constants (i.e. the couplings of the cubic vertex) to satisfy the Jacobi identity.
Let us summarize the results presented in this Thesis, so as to look at them from a broader perspective.

We analyzed the product between two YM fields given in [58], finding useful to revisit the definitions of the graviton and of the scalar field also at the linearized level. Our conclusion, given the discussion of Section 3.1.2 on the scalar, the results on the linearized equations of motion of Section 4.1.1 and the idea of reproducing the $N = 0$ Supergravity off-shell cubic vertices, is that the more appropriate decomposition of $H_{\mu\nu}$ at the linearized level is:

$$H_{\mu\nu} := A_\mu \star \tilde{A}_\nu := h_{\mu\nu} + B_{\mu\nu} + \frac{1}{D-2} \eta_{\mu\nu} \varphi,$$

with

$$\begin{cases} 
  h_{\mu\nu} := H_{\mu\nu}^S - \frac{1}{D-2} \eta_{\mu\nu} \varphi, \\
  B_{\mu\nu} := H_{\mu\nu}^A, \\
  \varphi := H - \frac{\partial \partial H}{\Box}. 
\end{cases}$$

In particular, the scalar-graviton mixing is chosen so as to reproduce the $N = 0$ Supergravity action in our "canonical" basis. Given these definitions, we were able to build a quadratic Lagrangian for the field $H_{\mu\nu}$ that can be written in several forms:

$$L_0^{(H)} = \frac{1}{2} H^{\alpha\beta} \left\{ \Box \eta_{\alpha\mu} \eta_{\beta\nu} - \partial_\alpha \partial_\mu \eta_{\beta\nu} - \partial_\beta \partial_\nu \eta_{\alpha\mu} + \frac{\partial_\alpha \partial_\beta \partial_\mu \partial_\nu}{\Box} \right\} H_{\mu\nu}$$

$$= \frac{1}{8} (F_{\mu\alpha} \star \tilde{F}_{\nu\beta}) \left\{ (F^\alpha \star \tilde{F}^\nu) \right\}$$
\[ R^*_\mu\nu\rho\sigma = -\frac{1}{2} F^*_\mu\nu \star \tilde{F}^*_{\rho\sigma} = \frac{1}{2} \left\{ \partial_\nu \partial_\rho H_{\mu\sigma} + \partial_\mu \partial_\sigma H_{\nu\rho} - \partial_\nu \partial_\sigma H_{\mu\rho} - \partial_\mu \partial_\rho H_{\nu\sigma} \right\} . \] (5.7)

The Lagrangian (5.3) can be split as a sum of quadratic Lagrangians for a graviton, a two-form field and a scalar, thus making its particle content explicit. In addition, it possesses two notable features. First, the form (5.4) makes it explicit that it can be interpreted as the “square”, in a sense which was specified, of two YM Lagrangians. Second, when expressed in terms of the linearized field strength \( R^*_\mu\nu\rho\sigma \) for \( H^\pm_{\mu\nu} \) (5.5), our Lagrangian is analogous to the “geometric” Maxwell-like Lagrangian derived in [60] from tensionless strings. In this direction, a suggestive outlook for the future is to extend \( R^*_\mu\nu\rho\sigma \) at the non-linear level, looking for a “geometric” Lagrangian for the \( \mathcal{N} = 0 \) Supergravity.

The presence of non-localities in our Lagrangian formulation of the DC might be a source of perplexity. However, as we commented extensively in our analysis, all these non localities arise purely from the need to covariantly project the scalar degree of freedom out of the double-copy field \( H^\pm_{\mu\nu} \), while also ensuring its gauge invariance. Indeed, in the other covariant approaches that we reviewed in this Thesis, aiming at an off-shell interpretation of the double copy, non localities are always present, either as a tool to ensure the CK duality to be manifest in YM amplitudes [16], or, when sources are included, in the linear equations of motion of [59].

Then, we extended our quadratic Lagrangian to the interacting level, with the insertion of cubic vertices for \( H^\pm_{\mu\nu} \). Among the vertices allowed by the Noether procedure it is possible to select a sector which reproduces the amplitude results of the DC at the cubic level, while also displaying an enhanced Lorentz symmetry of the form \( O(D - 1, 1)_L \otimes O(D - 1, 1)_R \otimes \mathbb{Z}_2 \). We showed that these vertices are equivalent, up to field redefinitions, to the cubic terms in the perturbative expansion of the \( \mathcal{N} = 0 \) Supergravity Lagrangian. Furthermore, we compared our vertices with the ones found in [16] finding agreement between the two proposals, up to longitudinal terms at the cubic level. Therefore, following the discussion of Section 4.4.1, one can say that our results provide a completion of what found in [16]. In this respect, the obvious exten-
sion of our work would be to carry on the Noether procedure (at least) to the following order, with the inclusion of quartic vertices. Also upon comparing with [16], quartic vertices might tell us something about the role of the CK duality in the DC relations, since the first non-trivial example of the former manifests itself for the four-point amplitude. Moreover, our work suggests the possibility of rewriting the full action of the $\mathcal{N} = 0$ Supergravity in terms of $H_{\mu\nu}$, which contains the complete multiplet, thus in particular making the $O(D - 1, 1)_L \otimes O(D - 1, 1)_R \otimes \mathbb{Z}_2$ symmetry explicit at the action level.

The other main output of the Noether procedure at the cubic level, namely the first correction to the gauge transformation of $H_{\mu\nu}$, was also analyzed. The non-linear part of the YM transformation does not contain derivatives, differently from the gravitational one, therefore one could expect their connection to be harder to spot (or even to define in principle) than the one between cubic vertices, where there is a natural match between the number of derivatives on the two sides. Nevertheless, if the Lagrangian we started to build is to be thought of as the “square” of two YM Lagrangians, a link between the two gauge transformation is somewhat expected also at the non-linear level, and we maintain that an investigation in this direction could provide deeper insights on the geometric origin of the DC relations.

The main lesson that one gets from the first non-linear corrections to the gauge transformation is the need to consistently deform the definitions of the scalar and of the graviton, by means of the insertion of terms which are quadratic in $H_{\mu\nu}$. Amusingly, we found glimpses of geometry behind this observation, since the redefinition of the scalar field is compatible, to this order, with the convolution of the Green’s function of the Laplace-Beltrami operator with the Ricci scalar of a manifold with metric $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$ and Levi-Civita connection. The insertion of quartic vertices would provide a validation or a refutation of this result. A hidden geometrical structure in the definition of the scalar field in terms of $H_{\mu\nu}$ might also justify the structure of its couplings to the other fields in the $\mathcal{N} = 0$ Supergravity multiplet, which is (somewhat unexpectedly) also reproduced by the Noether procedure.

Apart from these direct extensions of this Thesis, further and more far-reaching directions can be envisaged as applications or continuations of our work:
As we stressed several times, although we focused on the DC of pure YM theory, the CK duality and the DC relations also hold for the supersymmetric extensions of YM theory and Einstein Gravity. Our work can be viewed as an extension of [58], since we exploited the product between YM fields and the analysis of the linearized gauge symmetries given in that work. However, since in [58] also the supersymmetric case is considered, a natural extension of our results is to apply the same machinery to supersymmetric theories.

As we discussed, our quadratic Lagrangian, together with the appropriate sector of the cubic vertices, is physically equivalent to the $\mathcal{N}=0$ Supergravity one, up to the cubic interactions. This theory has been considered, although with different goals, also from the Double Field Theory (DFT) perspective [135–137]. The latter description and ours share the idea of using a field basis such that the three fields of the theory are viewed as components of a unique tensor, and the comparison of our results with the DFT ones might teach something more on the geometry of the theory, and possibly on the DC relations\(^1\).

In recent years, renewed attention was paid to a rather old subject: the study of asymptotic symmetries. It was recently shown how these symmetries can, both in spin-one gauge theories and in General Relativity, be linked to Weinberg’s soft theorems and to the memory effect (see [57] for a review and [140] for the extension to spin greater than two). The recent derivation of a unified soft theorem valid for amplitudes with gravitons, two-form fields and scalars [141], in which the DC construction for scattering amplitudes was also employed, calls for the study of its asymptotic symmetry origin. Our work should be relevant in this sense since the study of asymptotic symmetries requires a Lagrangian formulation, or equivalently the knowledge of the off-shell gauge symmetry.

Among the various extensions of the DC paradigm, we briefly reviewed the classical DC in Section 3.3. We consider it a particularly promising direction to the

\[ \delta_1 H_{\alpha\beta} = (\alpha^\mu + \tilde{\alpha}^\mu) \partial_\mu H_{\alpha\beta} + H_{\alpha\nu}(\partial_\beta \alpha^\nu - \partial^\nu \alpha_{\beta}) + H_{\mu\beta}(\partial_\alpha \tilde{\alpha}^\mu - \partial^\mu \tilde{\alpha}_\alpha), \]  

(5.8)

where in particular the differential structure is same as the generalized Lie derivative encountered in the context of the DFT. Let us also observe, however, that a similar possibility exists in the case of pure gravity as well, at least to cubic order in the Noether procedure.

\(^1\)Fore instance, one can easily write the correction to the free gauge transformation of $H_{\mu\nu}$ (4.144) in a different basis as follows

where in particular the differential structure is same as the generalized Lie derivative encountered in the context of the DFT. Let us also observe, however, that a similar possibility exists in the case of pure gravity as well, at least to cubic order in the Noether procedure.
goal of getting a better understanding of the DC, in view of the fact that it relates exact solutions in YM theory and gravity. Since classical solutions are naturally derived from a Lagrangian (via the Euler-Lagrange equations) and, at least when the Kerr-Schild DC is concerned, the considered solutions are in a sense “linear” (as detailed in Section 3.3), we hope that with our Lagrangian it can be possible to discern the origin of the Kerr-Schild DC. In particular, one of its most puzzling properties is that in this case the DC arises in coordinate space, while the DC of scattering amplitudes is naturally identified in momentum space.

- One of the natural motivations for the study of the DC relations, at least from the phenomenological perspective, is that they provide tremendous simplifications in gravity calculations. In particular, an ongoing problem is the study of binary black-hole systems, for future applications to LIGO and VIRGO. In this sense, the DC was shown to work for radiative solutions in the post-Minkowskian regime [142], both at leading order [26, 27, 128] and, recently, at next-to-leading order [29], non-trivially generalizing the CK duality to the perturbative expansion of classical solutions. The analysis of the post-Newtonian regime [143, 144] was addressed only at leading order [25], but one may hope that DC constructions, in particular with the support of a Lagrangian formulation, could help in the future to increase the precision of analytic calculations of interest for the gravitational waves observatories.

- In view of a recent work [94] generalizing the DC construction for scattering amplitudes to the three-point amplitudes on a curved background and of the extension of the classical DC to non-flat backgrounds [20, 21], it is conceivable that the DC relations might be applied, in the future, to the AdS/CFT correspondence [145].
Appendices
Appendix A

Noether procedure for gravity

A.1 Completion of the cubic vertex

We write here the full computations which led to the derivation of the graviton cubic vertex $L_1$ in Section 4.3.2. We recall that we started from a TT Lagrangian in the cyclic ansatz and we derived a constraint on its coefficient, with the result:

$$L_{1}^{TT} = \frac{1}{2} h^{\mu \nu} \partial_{\mu} \partial_{\nu} h_{\alpha \beta} h^{\alpha \beta} + h^{\mu \nu} \partial_{\mu} h^{\alpha \beta} \partial_{\alpha} h_{\beta \nu}, \quad (A.1)$$

up to an overall coefficient (which we fix only at the end, by comparison with the perturbative expansion of the EH Lagrangian). After a free gauge transformation $\delta_0 h_{\mu \nu} = \partial_{(\mu} \xi_{\nu)}$, the variation of this Lagrangian is:

$$\delta_0 L_{1}^{TT} = \frac{1}{2} \mathcal{F}_{\alpha \beta} \partial_{\nu} h^{\alpha \beta} \xi^{\nu} + \frac{1}{2} \partial_{\alpha} D_{\beta} \partial_{\mu} h^{\alpha \beta} h_{\alpha \beta} \xi^{\nu} - \frac{1}{2} \partial_{\alpha} D^{\nu} \partial_{\nu} h^{\alpha \beta} \partial_{\alpha} h_{\beta \mu} - \frac{1}{2} \partial_{\alpha} F_{\alpha \beta} h^{\alpha \beta} \xi^{\nu}$$

$$- \frac{1}{2} \partial_{\alpha} \partial_{\beta} D_{\mu} h^{\alpha \beta} \xi^{\nu} - h^{\mu \nu} \partial_{\mu} \partial_{\nu} \xi_{\beta} D^{\beta} - \frac{1}{2} h^{\mu \nu} \partial_{\mu} \partial_{\nu} \xi_{\beta} \partial_{\mu} h + \frac{1}{2} \partial_{\alpha} F_{\alpha \beta} h^{\alpha \beta} \xi^{\nu}$$

$$+ \frac{1}{2} \partial_{\alpha} \partial_{\nu} D_{\beta} h^{\alpha \beta} \xi^{\nu} + \frac{1}{2} \partial_{\alpha} \partial_{\beta} D_{\mu} h^{\alpha \beta} \xi^{\nu} - \frac{1}{2} \delta_0 D^{\mu} h^{\alpha \beta} \partial_{\alpha} h_{\beta \mu} - \frac{1}{2} \partial_{\alpha} F_{\alpha \beta} h^{\alpha \beta} \xi^{\nu}$$

$$- \frac{1}{2} \partial_{\alpha} D^{\beta} \partial_{\mu} h^{\beta \nu} \xi^{\nu} - \frac{1}{2} \partial_{\beta} D^{\alpha} \partial_{\alpha} h^{\beta \nu} \xi^{\nu} + \frac{1}{2} \mathcal{F}^{\mu \nu} h^{\beta \nu} \partial_{\mu} h + \frac{1}{2} \partial_{\alpha} D^\mu \partial_{\nu} h^{\mu \nu} \partial_{\mu} \xi_{\beta}$$

$$+ \frac{1}{2} \partial_{\alpha} \partial_{\nu} D^{\mu} h^{\alpha \beta} \partial_{\mu} \xi_{\beta} - \frac{1}{2} \partial_{\beta} D^{\mu} h^{\mu \nu} \partial_{\mu} \xi_{\beta} - \frac{1}{2} \partial_{\alpha} \partial_{\beta} D_{\mu} h^{\mu \nu} \partial_{\mu} \xi_{\beta}$$

$$- \frac{1}{2} \partial_{\alpha} \partial_{\beta} D^{\mu} h^{\mu \nu} \partial_{\mu} \xi_{\beta} - h^{\mu \nu} \partial_{\mu} \xi_{\alpha} \partial_{\alpha} D_{\nu} - \frac{1}{2} h^{\mu \nu} \partial_{\mu} \xi_{\alpha} \partial_{\alpha} \partial_{\beta} h - \mathcal{F}^{\mu \nu} \partial_{\mu} h^{\alpha \beta} \partial_{\alpha} \xi_{\beta}$$
\[ -\frac{1}{2} \partial^\mu h \partial_\mu h^{\alpha\beta} \partial_\alpha \xi_\beta. \]  

(A.2)

In the previous expression, some terms were underlined in different ways: we collect terms with equal underlining and treat separately these terms.

- **____**: terms proportional to the equations of motion. They will have the same underlining also in what follows since they have to be collected all together at the end of this step of the procedure, for the moment we neglect them since we first want to build \( \mathcal{L}_1 \) such that \( \delta_0 \mathcal{L}_1 \approx 0 \).

\[ \sim \quad -\frac{1}{2} \delta_0 D^\nu \partial_\alpha h_{\alpha\beta} h^{\beta\gamma} - D^\nu \partial_\mu h^{\alpha\beta} \partial_\alpha \xi_\beta = -\frac{1}{4} \delta_0 D^\nu \partial_\nu h_{\alpha\beta} h^{\alpha\beta} - \frac{1}{2} D^\nu \partial_\nu h^{\alpha\beta} \delta_0 h_{\alpha\beta} \]

\[ = \frac{1}{4} \delta_0 (\partial \cdot D) h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{2} \partial \cdot D h^{\alpha\beta} \delta_0 h_{\alpha\beta} + \frac{1}{2} h^{\alpha\beta} D^\mu \partial_\mu \delta_0 h_{\alpha\beta} \]

\[ = \delta_0 h^{\alpha\beta} (\partial \cdot D h^{\alpha\beta} h_{\alpha\beta}) + h^{\alpha\beta} D^\mu \partial_\mu \partial_\alpha \xi_\beta. \]  

(A.3)

- **...**: 

\[ -\frac{1}{2} \partial^\gamma D^\beta \partial_\alpha h_{\beta\nu} \xi^\nu + \frac{1}{4} \partial^\mu D^\nu h_{\beta\nu} \partial_\mu \xi^\beta = -\frac{1}{4} \delta_0 D^\nu D^\beta h_{\beta\nu} + \frac{1}{4} \partial^\gamma \partial^\beta h_{\beta\nu} \xi^\nu \]

\[ + \frac{1}{4} \xi^\nu \partial_\beta \partial_\nu D^\beta \xi^\nu + \frac{1}{4} h_{\beta\nu} \xi^\nu D^\nu - \frac{1}{4} \partial^\nu (D^\nu h_{\beta\nu}) \xi^\nu - \frac{1}{4} \delta_0 D^\beta D^\nu h_{\beta\nu} \]

\[ = \frac{1}{4} F_{\beta\nu} D^\beta \xi^\nu + \frac{1}{2} \partial_\beta D^\nu D^\beta \xi^\nu + \frac{1}{2} \partial_\nu D^\beta D^\beta \xi^\nu - \frac{1}{2} \partial_\nu D^\nu D^\beta h_{\beta\nu}. \]  

(A.4)

- **____**: 

\[ -\frac{1}{2} \delta_0 D^\nu h^{\alpha\beta} \partial_\alpha h_{\beta\nu} + \frac{1}{4} \partial_\mu \delta_0 D^\beta h_{\mu\nu} = -\frac{1}{2} \partial_\alpha (\delta_0 \partial^\nu h_{\beta\nu}) h^{\alpha\beta} \]

\[ = \frac{1}{2} \delta_0 D^\nu h_{\beta\nu} D^\beta + \frac{1}{4} \delta_0 D^\nu h_{\beta\nu} \partial^\beta h. \]  

(A.5)

- **____**: 

\[ + \partial_\alpha D^\beta \partial_\beta h^{\alpha\beta} \xi^\nu + \frac{1}{2} \partial_\alpha \partial_\beta D^\beta h^{\alpha\beta} \xi^\nu = -\partial_\alpha D^\beta h^{\alpha\beta} \partial \cdot \xi - \xi^\nu \partial_\nu \partial_\alpha D^\beta h^{\alpha\beta} \]

\[ + \frac{1}{2} \xi^\nu \partial_\alpha \partial_\beta D^\beta h^{\alpha\beta} = -\frac{1}{2} h^{\alpha\beta} \partial_\alpha D^\beta \delta_0 h - \frac{1}{2} \xi^\nu \partial_\alpha \partial_\beta D^\beta h^{\alpha\beta} \]

\[ = \frac{1}{2} h^{\alpha\beta} D^\beta \delta_0 (\partial_\alpha h) + \frac{1}{2} D^\beta h \delta_0 h + \frac{1}{4} \partial^\beta h D^\beta h \delta_0 h - \frac{1}{2} \xi^\nu \partial_\alpha \partial_\beta D^\beta h^{\alpha\beta}. \]  

(A.6)

Next, we integrate by parts all the terms in \( \delta_0 \mathcal{L}_1^{TT} \) and in the expressions above which contain derivatives of \( \xi \), in order to isolate the gauge parameter. The list below contains all the necessary integration by parts, together with their results (terms proportional to the equations of motion are underlined as above):

- \[ h^{\mu\nu} \partial_\mu \partial_\nu \xi_\beta D^\beta = -\partial \cdot \partial \cdot h \xi_\beta D^\beta - h^{\mu\nu} \partial_\mu \partial_\nu D^\beta \xi^\beta - 2 \partial \cdot h^{\mu\nu} \partial_\mu D^\beta \xi^\beta \]

\[ = -\frac{1}{2} F_{\xi_\beta} D^\beta - 2 \partial \cdot D \xi_\beta D^\beta - h^{\mu\nu} \partial_\mu \partial_\nu D^\beta \xi^\beta - 2 D^{\mu\nu} \partial_\mu \partial_\nu D^\beta \xi^\beta - \partial^\mu h \partial_\mu D^\beta \xi^\beta, \]  

(A.7)
Finally, we substitute these results in the expression of $\delta_0 L^{TT}_1$:

$$\delta_0 L^{TT}_1 = \left\{ \frac{1}{2} F^{\alpha \beta} \partial_\alpha h_{\beta \gamma} \xi^\gamma - \frac{1}{2} \partial_\beta F_{\alpha \beta} h_{\alpha \gamma} \xi^\gamma + \frac{1}{2} \partial_\alpha F_{\beta \gamma} h_{\alpha \beta} \xi^\gamma - \frac{1}{2} F_{\alpha \beta} \partial_\beta h_{\alpha \gamma} \xi^\gamma \right\} + \frac{1}{2} F^{\mu \nu} h_{\beta \nu} \partial_\mu \xi^\beta - \frac{1}{2} F_{\beta \nu} h^{\mu \nu} \partial_\mu \xi^\beta - \frac{1}{2} F D^\beta \xi_\beta - \frac{1}{2} F \partial^\beta h \xi_\beta + \frac{1}{2} F_{\beta \nu} h^{\beta \nu} \xi_\nu.$$

Finally, we substitute these results in the expression of $\delta_0 L^{TT}_1$:
The terms in the curly brackets are proportional to the equations of motions, and they will be considered in the last step. The terms in the squared brackets can be summed and expressed as:

\[
\delta_0 \left( \frac{1}{4} \partial \cdot D h_{\alpha \beta} h^{\alpha \beta} + \frac{1}{2} h^{\alpha \beta} \partial_\alpha h D_\beta \right) - \frac{1}{2} \delta_0 h^{\alpha \beta} \partial_\alpha h D_\beta \\
= \delta_0 \left( \frac{1}{4} \partial \cdot D h_{\alpha \beta} h^{\alpha \beta} + \frac{1}{2} h^{\alpha \beta} \partial_\alpha h D_\beta \right) \\
+ \left\{ \frac{1}{2} F^\beta h^{\alpha \beta} + \partial \cdot D h_{\alpha \beta} \xi^\beta + \frac{1}{2} \partial_\alpha h \partial^\alpha D^\beta \xi^\beta + \frac{1}{2} \partial_\alpha h \partial D^\beta \xi^\alpha + \frac{1}{2} \partial_\alpha h \partial \cdot D^\xi_\alpha \right\}. \quad (A.19)
\]

We observe see the terms in the curly brackets of eq. (A.19) exactly cancel the ones in the round brackets of eq. (A.18). Therefore, the remaining part is:

\[
\delta_0 \mathcal{L}_{TT}^1 \approx \delta_0 \left( \frac{1}{4} \partial \cdot D h_{\alpha \beta} h^{\alpha \beta} + \frac{1}{2} h^{\alpha \beta} \partial_\alpha h D_\beta \right). \quad (A.20)
\]

Thus we have identified the terms which must be added to the TT Lagrangian in order to have on-shell gauge invariance for \( \mathcal{L}_1 \) at the linearised level. The corresponding Lagrangian is:

\[
\mathcal{L}_1 = \frac{1}{2} h^{\mu \nu} \partial_\mu \partial_\nu h_{\alpha \beta} h^{\alpha \beta} + h^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\alpha h_{\beta \nu} - \frac{1}{4} \partial \cdot D h_{\alpha \beta} h^{\alpha \beta} - \frac{1}{2} h^{\alpha \beta} \partial_\alpha h D_\beta. \quad (A.21)
\]
Appendix B

Noether procedure for the double-copy field

B.1 Completion of the double-copy cubic vertex

In this appendix, we write the computation which leads to the completion of the double-copy cubic vertex $L_1$, such that $\delta_0 L_1 \approx 0$, starting from the TT Lagrangian:

$$L_1^{TT} = H^{\mu\nu} \partial_\mu H_{\alpha\beta} H_{\alpha\beta}^{\alpha\beta} + H^{\mu\nu} \partial_\mu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} + H^{\mu\nu} \partial_\nu H^{\alpha\beta} \partial_\alpha H_{\mu\beta}. \quad (B.1)$$

Before starting the procedure, we collect some useful identities and gauge variations which will be employed extensively during the computation:

- $\partial \cdot \partial \cdot H = 2 \partial \cdot D = 2 \partial \cdot \bar{D}, \quad (B.2)$
- $\partial^\alpha E_{\alpha\beta} = \partial^\alpha E_{\beta\alpha} = 0, \quad (B.3)$
- $\square H = E + 2 \partial \cdot D, \quad (B.4)$
- $\partial^\gamma H_{\gamma\mu} = D_\mu + \partial_\mu \frac{\partial \cdot D}{\square} \quad (B.5)$
- $\partial^\gamma H_{\mu\gamma} = \bar{D}_\mu + \partial_\mu \frac{\partial \cdot D}{\square}, \quad (B.6)$
- $\square H_{\mu\nu} = E_{\mu\nu} + \partial_\mu D_\nu + \partial_\nu \bar{D}_\nu, \quad (B.7)$
- $\square \alpha_\mu = \delta D_\mu + \partial_\mu \partial \cdot \Lambda, \quad (B.8)$
- $\square \bar{\alpha}_\mu = \delta \bar{D}_\mu - \partial_\mu \partial \cdot \Lambda, \quad (B.9)$
- $\Lambda_\mu = \frac{1}{2} (\alpha_\mu - \bar{\alpha}_\mu), \quad (B.10)$
The variations of the terms in eq. (B.1) after a free gauge transformation are:

\[ \delta_0 H = \delta_0 (\frac{\partial \cdot D \cdot H}{\Box}) = 2\delta (\frac{\partial \cdot D}{\Box}) = \partial \cdot \alpha + \partial \cdot \tilde{\alpha}. \] (B.11)

The variations of the terms in eq. (B.1) after a free gauge transformation are:

\[ \delta_0 (H^{\mu \nu} \partial_{\mu} \partial_{\nu} H_{\alpha \beta} H^{\alpha \beta}) = (\partial^\nu \alpha^\nu + \partial^\nu \tilde{\alpha}^\nu) \partial_\mu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + H^{\mu \nu} \partial_\mu \partial_\nu (\partial_\alpha \alpha_\beta + \partial_\beta \tilde{\alpha}_\alpha) + \partial_\beta \tilde{\alpha}_\alpha) H^{\alpha \beta} + H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} \partial_\beta \tilde{\alpha}_\alpha + \frac{1}{2} \partial_\beta \tilde{\alpha}_\alpha = \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \alpha^\nu
\]

\[ + \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \tilde{\alpha}^\nu - \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} \partial_\beta \tilde{\alpha}_\alpha + H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu) \] (B.12)

\[ \delta_0 (H^{\mu \nu} \partial_{\mu} H^{\alpha \beta} \partial_{\nu} H_{\alpha \beta}) = (\partial^\nu \alpha^\nu + \partial^\nu \tilde{\alpha}^\nu) \partial_\mu H^{\alpha \beta} \partial_{\nu} H_{\alpha \beta} + H^{\mu \nu} \partial_\mu (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu) \partial_\nu H_{\alpha \beta} + \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \alpha^\nu
\]

\[ + \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \tilde{\alpha}^\nu - \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} \partial_\beta \tilde{\alpha}_\alpha + H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu) \] (B.13)

\[ \delta_0 (H^{\mu \nu} \partial_{\mu} H^{\alpha \beta} \partial_{\nu} H_{\alpha \beta}) = (\partial^\nu \alpha^\nu + \partial^\nu \tilde{\alpha}^\nu) \partial_\mu H_{\alpha \beta} \partial_{\nu} H_{\alpha \beta} + H^{\mu \nu} \partial_\mu H_{\alpha \beta} (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu)
\]

\[ + \partial_\nu \partial_\nu H_{\alpha \beta} (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu) \partial_\nu H_{\alpha \beta} + \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \alpha^\nu
\]

\[ + \frac{1}{2} \Box H^{\alpha \beta} \partial_\nu H_{\alpha \beta} \tilde{\alpha}^\nu - \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\nu \partial_\nu H_{\alpha \beta} \partial_\beta \tilde{\alpha}_\alpha + H^{\mu \nu} \partial_\mu H_{\alpha \beta} (\partial^\nu \alpha^\beta + \partial^\beta \tilde{\alpha}^\nu) \] (B.14)
Exploiting the substitutions listed in eqs. (B.2-B.11), we can derive:

\[
\delta_\ell \mathcal{L}_{TT}^1 = \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\mu H_{\alpha\beta} \alpha^\nu + \frac{1}{2} \partial^\nu D^\beta \partial_\nu H_{\alpha\beta} \alpha^\nu + \frac{1}{2} \partial^\beta \tilde{\mathcal{D}}^\alpha \partial_\alpha H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \delta_\nu \mathcal{D}^\nu \partial_\nu H_{\alpha\beta} \alpha^\nu \\
- \frac{1}{2} \partial^\nu \partial \cdot \Lambda \partial_\nu H_{\alpha\beta} H_{\alpha^\beta} - \frac{1}{2} \partial_\nu \varepsilon^{\alpha\beta} H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \partial_\nu \partial_\alpha \tilde{\mathcal{D}}_\alpha H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \partial_\nu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\beta} \alpha^\nu \\
+ \frac{1}{2} \partial^\mu \partial \cdot \Lambda \partial_\mu H_{\alpha\beta} H_{\alpha^\beta} \alpha^\nu + \frac{1}{2} \partial_\mu \varepsilon^{\alpha\beta} H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \partial_\mu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \partial_\mu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\beta} \alpha^\nu \\
- \frac{1}{2} \partial^\nu \partial \cdot \Lambda H_{\alpha\beta} \partial_\nu H_{\alpha\beta} - \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\beta H_{\alpha\nu} \partial_\alpha \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\nu} \partial_\beta \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\nu} \partial_\beta \alpha^\nu \\
+ \frac{1}{2} \partial^\nu \partial_\beta \cdot \Lambda H_{\alpha\beta} \partial_\nu H_{\alpha\beta} + \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\beta H_{\alpha\nu} \partial_\alpha \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\nu} \partial_\beta \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \tilde{\mathcal{D}}_\alpha H_{\alpha\nu} \partial_\beta \alpha^\nu \\
+ \frac{1}{2} \partial^\nu \partial \cdot \Lambda \partial_\nu \partial_\nu H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\alpha \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu \\
- \frac{1}{2} \partial^\nu \partial_\beta \cdot \Lambda \partial_\nu \partial_\nu H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\alpha \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu \\
- \frac{1}{2} \partial^\nu \partial_\beta \cdot \Lambda \partial_\nu \partial_\nu H_{\alpha\beta} \alpha^\nu - \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\alpha \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu - \frac{1}{2} \partial^\nu \partial_\beta \partial_\nu H_{\alpha\nu} \partial_\beta \alpha^\nu.
\]

(B.15)

In the previous expression, some terms were underlined in different ways. We collect terms with equal underlining and treat them separately:

- __ : terms proportional to the equations of motion. They will have the same underlin-
Appendix B. Noether procedure for the double-copy field

\[
\begin{align*}
\text{b) } &\quad - \frac{1}{2} \delta_0D^\nu D_\mu H_{\alpha \beta} H^{\alpha \beta} - \frac{1}{2} \delta_0D^\mu D_\mu H_{\alpha \beta} H^{\alpha \beta} - \tilde{D}^\nu D_\mu H^{\alpha \beta} H_{\alpha \beta} - D_\nu D_\mu H^{\alpha \beta} H_{\alpha \beta} \\
&\quad = - \frac{1}{2} (\delta_0D^\nu + \delta_0\tilde{D}^\nu) D_\mu (H^{\alpha \beta} H_{\alpha \beta}) + \partial \cdot DH^{\alpha \beta} (\partial_\alpha H_{\alpha \beta} + \partial_\beta H_{\alpha \beta}) + D^\mu H^{\alpha \beta} \partial_\mu H_{\alpha \beta} \\
&\quad + \tilde{D}^\mu H^{\alpha \beta} \partial_\mu \partial_\nu H_{\alpha \beta} = \delta_0 \left( \frac{1}{2} \partial \cdot \tilde{D} H^{\alpha \beta} H_{\alpha \beta} \right) + D^\mu H^{\alpha \beta} \partial_\mu H_{\alpha \beta} + \tilde{D}^\mu H^{\alpha \beta} \partial_\mu \partial_\nu H_{\alpha \beta}. \\
\text{B.16}
\end{align*}
\]

\[
\begin{align*}
\text{b) } &\quad - \frac{1}{2} \delta_0D^\nu \partial_\mu H_{\alpha \nu} H^{\alpha \nu} + \frac{1}{2} \partial_\mu D^\nu H_{\alpha \nu} \partial_\mu \tilde{D}^\alpha + \frac{1}{2} \partial_\mu D^\alpha \partial_\nu \tilde{D}^{\alpha \nu} + \frac{1}{2} \partial_\nu D^\alpha \partial_\mu \tilde{D}^{\mu \alpha} + \frac{1}{2} \partial_\nu D^\nu D_\mu H_{\alpha \beta} H^{\alpha \beta} - \frac{1}{2} \partial_\mu D^\nu \partial_\nu D_\mu H_{\alpha \beta} H^{\alpha \beta} \\
&\quad = - \frac{1}{4} \delta_0D^\nu D_\mu H_{\alpha \nu} H^{\alpha \nu} - \frac{1}{4} \delta_0D^\nu D_\mu H_{\alpha \nu} H^{\alpha \nu} + \frac{1}{4} \partial_\mu \partial \cdot \Lambda D^\alpha H_{\alpha \nu} + \frac{1}{4} \partial_\nu \partial \cdot \Lambda D^\nu H_{\alpha \nu} - \frac{1}{4} \partial_\nu \partial \cdot \Lambda D^\nu H_{\alpha \nu} \\
&\quad + \frac{1}{4} \partial_\mu \partial \cdot \Lambda H_{\alpha \beta} D^\beta, \\
\text{B.17}
\end{align*}
\]

\[
\begin{align*}
\text{b) } &\quad - \frac{1}{2} \delta_0D^\nu H^{\alpha \beta} \partial_\beta H_{\alpha \nu} + \frac{1}{2} H^{\nu \beta} H_{\alpha \nu} \partial_\mu \delta_0D^\alpha - \frac{1}{2} H^{\nu \beta} \partial_\nu \delta_0D^\alpha H_{\mu \beta} \\
&\quad = - \frac{1}{2} \delta_0D^\nu H^{\alpha \beta} \partial_\beta H_{\alpha \nu} + \frac{1}{2} H^{\nu \beta} H_{\alpha \nu} \partial_\mu \delta_0D^\alpha - \frac{1}{2} H^{\nu \beta} \partial_\nu \delta_0D^\alpha H_{\mu \beta} - \frac{1}{2} \delta_0D^\nu \partial_\nu \delta_0D^\alpha H_{\alpha \beta} - \frac{1}{2} \delta_0D^\nu \partial_\nu \delta_0D^\alpha H_{\alpha \beta} \\
&\quad + \frac{1}{2} \delta_0D^\nu H_{\alpha \beta} \partial_\beta \partial_\nu \Lambda, \\
\text{B.18}
\end{align*}
\]

\[
\begin{align*}
\text{b) } &\quad - \frac{1}{2} \delta_0D^\nu D^\alpha \partial_\beta H_{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} \\
&\quad = - \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} \\
&\quad - \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} = - \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} - \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} \\
&\quad + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \frac{1}{2} \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} = \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta} + \partial_\beta D^\alpha \partial_\beta H_{\alpha \beta} H^{\alpha \beta}.
\end{align*}
\]
\[ + \tilde{D}^\alpha H_{\alpha\beta}\delta_0(\partial^\beta \partial \cdot \mathcal{D}) + D^\beta D_\beta\delta_0(\partial \cdot \mathcal{D}) + \tilde{D}^\alpha \tilde{D}_\alpha\delta_0(\partial \cdot \mathcal{D}) \\
+ (D^\alpha + \tilde{D}^\alpha)\partial_\alpha \frac{\partial \cdot \mathcal{D}}{\delta_0} - \frac{1}{2} \partial^\mu \partial^\beta \tilde{D}^\alpha H_{\alpha\beta}\tilde{\alpha}_\mu - \frac{1}{2} \partial^\mu \partial^\alpha D^\beta H_{\alpha\beta}\alpha_\mu. \]

(B.19)

Terms with \( \partial \cdot \Lambda \) and terms proportional to the equations of motion will be considered separately. We integrate by parts all the other terms in \( \delta_0 L^T_{1 \text{TT}} \) and in the expressions above which contain derivatives of \( \alpha_\mu, \tilde{\alpha}_\mu \), in order to isolate the gauge parameters and sum the variations. The list below contains the results of such integrations:

- \[ - H^{\mu\nu} D^\beta \partial_\mu \partial_\nu \alpha_\beta = -2 \partial \cdot D^\beta \alpha_\beta - H^{\mu\nu} D_\mu \partial_\nu D^\beta \alpha_\beta - D^\nu \partial_\nu D^\beta \alpha_\beta, \]

\[ - \partial^\nu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\nu D^\beta \alpha_\beta - \tilde{\partial}^\nu \partial_\nu D^\beta \alpha_\beta - \partial^\mu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\mu D^\beta \alpha_\beta. \]

(B.20)

- \[ - H^{\mu\nu} D^\beta \partial_\mu \partial_\nu \tilde{\alpha}_\alpha = -2 \partial \cdot D^\beta \tilde{\alpha}_\alpha - H^{\mu\nu} D_\mu \partial_\nu D^\beta \tilde{\alpha}_\alpha - D^\nu \partial_\nu D^\beta \tilde{\alpha}_\alpha, \]

\[ - \partial^\nu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\nu \tilde{D}^\beta \tilde{\alpha}_\alpha - \tilde{\partial}^\nu \partial_\nu \tilde{D}^\beta \tilde{\alpha}_\alpha - \partial^\mu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\mu \tilde{D}^\beta \tilde{\alpha}_\alpha. \]

(B.21)

- \[ - H^{\mu\nu} D^\beta \partial_\mu \partial_\nu \tilde{\alpha}_\beta \tilde{\alpha}_\beta = -2 \partial \cdot D^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta - H^{\mu\nu} D_\mu \partial_\nu D^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta - D^\nu \partial_\nu D^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta, \]

\[ - \partial^\nu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\nu \tilde{D}^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta - \tilde{\partial}^\nu \partial_\nu \tilde{D}^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta - \partial^\mu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\mu \tilde{D}^\beta \tilde{\alpha}_\beta \tilde{\alpha}_\beta. \]

(B.22)

- \[ - H^{\mu\nu} D^\beta D_\nu \partial_\mu \alpha_\beta = D^\beta \partial_\beta D^\mu \alpha_\beta + \partial^\nu \frac{\partial \cdot \mathcal{D}}{\delta_0} \partial_\nu \beta D_\nu \alpha_\beta + H^{\mu\nu} \partial_\mu \partial_\nu \beta D_\nu \alpha_\beta; \]

(B.24)

- \[ - H^{\mu\nu} \partial_\beta D^\nu \partial_\nu \alpha_\beta = D^\nu \partial_\nu \alpha_\beta + \frac{\partial^\nu \partial \cdot \mathcal{D}}{\delta_0} \partial_\nu \beta \partial_\beta \partial_\nu \alpha_\beta + H^{\mu\nu} \partial_\mu \partial_\nu \beta \partial_\beta \partial_\nu \alpha_\beta, \]

(B.25)

- \[ \frac{1}{2} \partial^\nu D^\mu H_{\alpha\beta} \partial_\mu \tilde{\alpha}_\alpha = - \frac{1}{2} \partial^\nu \partial \cdot D^\mu H_{\alpha\beta} \tilde{\alpha}_\alpha - \frac{1}{2} \partial^\nu \tilde{D}^\mu \partial_\mu H_{\alpha\beta} \tilde{\alpha}_\alpha, \]

(B.26)

- \[ \frac{1}{2} \partial_\alpha D_\nu H^{\mu\nu} \partial_\mu \tilde{\alpha}_\alpha = \frac{1}{2} H^{\mu\nu} \partial_\nu \partial_\mu D_\nu \tilde{\alpha}_\alpha + \frac{1}{2} \partial_\nu D_\nu \partial^\nu \tilde{\alpha}_\alpha + \frac{1}{2} \partial_\alpha D_\nu \partial^\nu \partial \cdot \mathcal{D}_\nu \tilde{\alpha}_\alpha, \]

(B.27)
Appendix B. Noether procedure for the double-copy field

\[-\frac{1}{2} \partial_{\mu} \tilde{D}_{\alpha} H^{\mu \nu} \partial_{\nu} \tilde{\alpha}^{\alpha} = \frac{1}{2} H^{\mu \nu} \partial_{\mu} \tilde{D}_{\alpha} \tilde{\alpha}^{\alpha} + \frac{1}{2} \partial_{\nu} \tilde{D}_{\alpha} D^{\nu} \tilde{\alpha}^{\alpha} + \frac{1}{2} \partial_{\nu} \tilde{D}_{\alpha} \tilde{\alpha}^{\alpha} \partial D^{\alpha}, \tag{B.28}\]

\[-\partial^\mu \partial D_{\alpha} H_{\alpha \beta} \partial^\beta \tilde{\alpha}^{\alpha} = \tilde{\alpha}_{\alpha} \partial^\mu H^{\alpha \beta} \partial_{\mu} \partial_{\beta} \partial D^{\alpha} + \tilde{\alpha}_{\alpha} \partial^\mu \tilde{D}^\alpha \partial_{\mu} \partial D^{\alpha}, \tag{B.29}\]

\[+ \tilde{\alpha}_{\alpha} \partial_{\mu} \partial_{\nu} \partial^\mu \partial \partial D^{\alpha} \partial_{\beta} \partial D^{\alpha} = \frac{1}{2} \partial_{\nu} \tilde{D}^\alpha H_{\alpha \beta} \partial^\beta \partial D^{\alpha} - \frac{1}{2} \partial_{\nu} \partial \partial \partial D^{\alpha} \partial_{\beta} \partial D^{\alpha}, \tag{B.30}\]

\[-\frac{1}{2} \tilde{\alpha}_{\alpha} \partial^\mu \partial D^\beta \partial_{\beta} \partial D^{\alpha} - \frac{1}{2} \tilde{\alpha}_{\alpha} \partial^\mu \partial D^\beta \partial_{\beta} \partial D^{\alpha}, \tag{B.31}\]

\[-\frac{1}{2} \partial_{\mu} \tilde{D}_{\nu} H^{\mu \nu} \partial_{\nu} \partial \tilde{\alpha}^{\alpha} = \frac{1}{2} H^{\mu \nu} \partial_{\mu} \tilde{D}_{\nu} \tilde{\alpha}^{\alpha} + \frac{1}{2} \tilde{D}^\mu \partial_{\mu} \tilde{D}_{\nu} \tilde{\alpha}^{\alpha} + \frac{1}{2} \partial_{\nu} \partial D^\alpha \partial_{\mu} \tilde{D}_{\nu} \tilde{\alpha}^{\alpha}, \tag{B.32}\]

\[-H^{\mu \nu} \partial_{\mu} \tilde{D}_{\nu} \partial_{\alpha} \tilde{\alpha}^{\alpha} = \tilde{D}^\alpha \partial_{\mu} \tilde{D}_{\nu} \tilde{\alpha}^{\alpha} + \partial D^\alpha \partial_{\mu} \tilde{D}_{\nu} \tilde{\alpha}^{\alpha} + H^{\mu \nu} \partial_{\mu} \partial_{\nu} \partial \tilde{\alpha}^{\alpha}, \tag{B.33}\]

\[-H^{\mu \nu} \partial_{\mu} \partial D_{\nu} \partial \tilde{\alpha}^{\alpha} = \tilde{D}^\alpha \partial_{\mu} \partial D_{\nu} \tilde{\alpha}^{\alpha} + \partial D^\alpha \partial_{\mu} \partial D_{\nu} \tilde{\alpha}^{\alpha} + H^{\mu \nu} \partial_{\mu} \partial D_{\nu} \partial \tilde{\alpha}^{\alpha}, \tag{B.34}\]

\[-\partial^\nu \partial D_{\alpha} H_{\alpha \beta} \partial_{\beta} \partial \tilde{\alpha}^{\alpha} = \tilde{D}^\alpha \partial_{\nu} \partial D_{\beta} \tilde{\alpha}^{\alpha} + \partial D^\alpha \partial_{\nu} \partial D_{\beta} \tilde{\alpha}^{\alpha} + H^{\nu \beta} \partial_{\nu} \partial \partial \tilde{\alpha}^{\alpha}, \tag{B.35}\]

\[-D^{\nu} H_{\alpha \beta} \partial_{\nu} \partial \tilde{\alpha}^{\alpha} = \partial_{\alpha} \partial \partial D^{\alpha} \partial_{\beta} \partial \tilde{\alpha}^{\alpha} + \partial \partial D^{\alpha} \partial_{\beta} \partial \tilde{\alpha}^{\alpha}, \tag{B.36}\]

\[-D^{\alpha} \partial_{\beta} \partial \partial D^{\alpha} \partial_{\beta} \partial \tilde{\alpha}^{\alpha} = \tilde{D}^\alpha \partial_{\beta} \partial \partial D^{\beta} \tilde{\alpha}^{\alpha} + \partial \partial D^{\alpha} \partial_{\beta} \partial \tilde{\alpha}^{\alpha} + H^{\alpha \beta} \partial \partial \partial \tilde{\alpha}^{\alpha}, \tag{B.37}\]

\[-D^{\beta} D_{\beta} \partial_{\alpha} (\partial D^{\alpha}) = -\partial D^{\alpha} \partial_{\beta} \partial D^{\beta} - \tilde{\alpha}^{\alpha} \partial_{\alpha} \partial \partial D^{\beta}, \tag{B.38}\]
Appendix B. Noether procedure for the double-copy field

\[ \mathcal{D}_a \delta_0 (\mathcal{D}_{\beta} \mathcal{D}_{\beta} \mathcal{D}_{\beta} \mathcal{D}_{\beta}) = -\alpha^a \partial_a \mathcal{D}_{\beta} \mathcal{D}_{\beta} \mathcal{D}_{\beta} \mathcal{D}_{\beta}, \]  
(B.39)

\[ \mathcal{D}_a \partial_a \mathcal{D}_\delta \mathcal{D}_\delta = -\frac{1}{2}(\alpha_\alpha + \tilde{\alpha}_\alpha) \partial^\alpha \mathcal{D}_\beta \partial_\beta \mathcal{D}_\delta - \frac{1}{2}(\alpha^\alpha + \tilde{\alpha}^\alpha) \mathcal{D}_\beta \partial_\beta \mathcal{D}_\delta, \]  
(B.40)

\[ \mathcal{D}_a \partial_a \mathcal{D}_\delta \mathcal{D}_\delta = -\frac{1}{2}(\alpha_\alpha + \tilde{\alpha}_\alpha) \partial^\alpha \mathcal{D}_\beta \partial_\beta \mathcal{D}_\delta - \frac{1}{2}(\alpha^\alpha + \tilde{\alpha}^\alpha) \mathcal{D}_\beta \partial_\beta \mathcal{D}_\delta. \]  
(B.41)

Exploiting these in the expression of \( \delta_0 \mathcal{L}_{1}^{TT} \) yields the following result (apart from terms proportional to the equations of motion):

\[ \delta_0 \mathcal{L}_{1}^{TT} \approx \left\{ - \frac{1}{2} \partial^\alpha \partial \cdot \Lambda \mathcal{H}^\alpha{}^\beta \delta \mathcal{H}_{\alpha \nu} + \frac{1}{2} \partial^\mu \partial \cdot \Lambda \mathcal{H}^\alpha{}^\beta \partial_\alpha \mathcal{H}_{\mu \beta} + \frac{1}{2} \mathcal{H}^\mu \nu \mathcal{H}_{\nu \alpha} \partial_\mu \partial^\alpha \cdot \Lambda \right. \]

\[ - \frac{1}{2} \partial^\beta \partial \cdot \Lambda \mathcal{H}_{\alpha \beta} \delta \mathcal{D}_\alpha + \frac{1}{2} \partial^\alpha \partial \cdot \Lambda \mathcal{H}_\beta = \Lambda_\alpha \partial^\alpha \left[ \delta_0 \left( \frac{1}{2} \partial \cdot \mathcal{D} \delta \mathcal{H}_{\alpha \beta} \right) + \mathcal{D}^\beta \delta_0 \partial^\alpha \left( \partial \cdot \mathcal{D} \right) \mathcal{H}_{\alpha \beta} \right] \]

\[ + \delta_0 \mathcal{D}^\beta \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} + \tilde{\delta}_0 \mathcal{D}^\alpha \partial_\beta \delta \mathcal{D}_\beta \mathcal{H}_{\alpha \beta} + \delta_0 \mathcal{D}^\beta \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \]

\[ - \left( \alpha_\beta \partial^\alpha \mathcal{D}_{\beta} \partial_\alpha \mathcal{D}_\delta \mathcal{H}_{\alpha \beta} + \mathcal{D}^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \right) + \mathcal{D}^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \]

\[ + \partial \cdot \mathcal{D} \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} + \tilde{\mathcal{D}}^\alpha \partial_\beta \partial_\alpha \mathcal{D}_\beta \mathcal{H}_{\alpha \beta} + \lambda_\alpha \partial_\alpha \mathcal{D} \mathcal{H}_{\alpha \beta} \mathcal{H}_{\alpha \beta} + \partial \cdot \mathcal{D} \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \mathcal{H}_{\alpha \beta}. \]  
(B.42)

The curly brackets contain terms with \( \partial \cdot \Lambda \), and integrating by parts it can be shown that they cancel exactly. The terms in the square brackets can be summed and expressed as:

\[ \delta_0 \left( \frac{1}{2} \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \mathcal{H}_{\alpha \beta} + \mathcal{D}^\beta \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} + \tilde{\mathcal{D}}^\alpha \partial^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \right) \]

\[ - \mathcal{D}^\beta \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} - \tilde{\mathcal{D}}^\alpha \partial^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \]

\[ = \delta_0 \left( \frac{1}{2} \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \mathcal{H}_{\alpha \beta} + \mathcal{D}^\beta \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} + \tilde{\mathcal{D}}^\alpha \partial^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \right) \]

\[ + \left\{ \alpha_\beta \partial^\alpha \mathcal{D}_{\beta} \partial_\alpha \mathcal{D}_\delta \mathcal{H}_{\alpha \beta} + \mathcal{D}^\beta \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} + \partial \cdot \mathcal{D} \partial^\alpha \partial \cdot \mathcal{D} \mathcal{H}_{\alpha \beta} \right\}. \]  
(B.43)
There is an exact cancellation between the terms in the curly brackets of eq. (B.43) and the ones in the round brackets of eq. (B.42), hence the remaining part is:

$$\delta_0 \mathcal{L}_1^{TT} \approx \delta_0 \left( \frac{1}{2} \partial \cdot D H_{\alpha \beta} H^{\alpha \beta} + D^\alpha \partial^\alpha \frac{\partial}{\Box} H_{\alpha \beta} + D^\alpha \partial^\beta \frac{\partial}{\Box} H_{\alpha \beta} \right).$$  \hspace{1cm} (B.44)

Thus, we have identified the counterterms which must be added to $\mathcal{L}_1^{TT}$ and the resulting cubic vertex is:

$$\mathcal{L}_1 = H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} + H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\beta H_{\alpha \nu} + H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\alpha H_{\beta \mu}$$
$$- \frac{1}{2} \partial \cdot D H_{\alpha \beta} H^{\alpha \beta} - D^\alpha \partial^\alpha \frac{\partial}{\Box} H_{\alpha \beta} - D^\alpha \partial^\beta \frac{\partial}{\Box} H_{\alpha \beta}. \hspace{1cm} (B.45)$$

## B.2 Double-copy cubic vertex and $\mathcal{N} = 0$ Supergravity

We now show that the double-copy cubic vertex of eq. (B.45) is equivalent, in an appropriate basis, to the cubic vertex of the $\mathcal{N} = 0$ Supergravity, including the non-TT terms. As a first step, we decompose as usual $H_{\mu \nu} = h_{\mu \nu} + B_{\mu \nu} + \gamma \eta_{\mu \nu} \varphi$ and consequently expand $\mathcal{L}_1$ in terms of the gravity fields.

- $H^{\mu \nu} \partial_\mu \partial_\nu H_{\alpha \beta} H^{\alpha \beta} = h^{\mu \nu} \partial_\mu \partial_\nu h_{\alpha \beta} + h^{\mu \nu} \partial_\mu \partial_\nu B_{\alpha \beta} B^{\alpha \beta} + \gamma h^{\mu \nu} \partial_\mu \partial_\nu h \varphi$
  $$+ \gamma h^{\mu \nu} \partial_\mu \partial_\nu h \varphi + \gamma^2 D h^{\mu \nu} \partial_\mu \partial_\nu \varphi + \gamma \varphi \Box h^{\alpha \beta} h_{\alpha \beta} + \gamma \varphi \Box B^{\alpha \beta} B_{\alpha \beta}$$
  $$+ \gamma^2 \varphi \Box \varphi + \gamma^3 \varphi^2 \Box h + \gamma^3 D \varphi^2 \Box \varphi. \hspace{1cm} (B.46)$$

- $H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\beta h_{\alpha \nu} + H^{\mu \nu} \partial_\mu H^{\alpha \beta} \partial_\alpha h_{\beta \nu} + \gamma \eta_{\mu \nu} \varphi$ and consequently expand $\mathcal{L}_1$ in terms of the gravity fields.

- $- \frac{1}{2} \partial \cdot D H_{\alpha \beta} H^{\alpha \beta} = - \frac{1}{4} \partial \cdot \partial \cdot h \Box H_{\alpha \beta} H^{\alpha \beta} = - \frac{1}{4} \left( \partial \cdot \partial \cdot h + \gamma \Box \varphi \right) H_{\alpha \beta} H^{\alpha \beta}$
  $$= \left( - \frac{1}{2} \partial \cdot \partial \cdot h + \frac{1}{4} \Box h + \frac{1}{4} \left( \partial \cdot \partial \cdot h - \Box h \right) - \frac{1}{4} \gamma \Box \varphi \right) H_{\alpha \beta} H^{\alpha \beta}$$
  $$= - \frac{1}{2} \partial \cdot \partial \cdot h h^{\alpha \beta} h_{\alpha \beta} - \frac{1}{2} \partial \cdot \partial \cdot h B^{\alpha \beta} B_{\alpha \beta} - \frac{1}{2} D \gamma \partial \cdot \partial \cdot h \varphi^2 - \gamma \partial \cdot \partial \cdot h \varphi h$$
  $$+ \frac{1}{4} \Box h h^{\alpha \beta} h_{\alpha \beta} + \frac{1}{4} \Box h B^{\alpha \beta} B_{\alpha \beta} + \frac{1}{4} D \gamma^2 \varphi^2 + \frac{1}{2} \gamma \Box h \varphi h$$

All the terms in which the two-form appears an odd number of times ($BBB, Bhh, Bh\varphi, B\varphi\varphi$) cancel out in the sum for symmetry reasons.
Appendix B. Noether procedure for the double-copy field

\[ + \frac{1}{4} \gamma (D - 2) - 1 \square \varphi H^{\alpha \beta} H_{\alpha \beta}. \]  

(B.48)

We employed the identity \( \square h - \partial \cdot \partial \cdot h = [1 - \gamma (D - 1)] \square \varphi. \)

- \[ - D^\beta \partial^\alpha \partial^\alpha h_{\alpha \beta} - \bar{D}^{\alpha} \partial^\beta \partial^\beta H_{\alpha \beta} \]

\[ = - \frac{1}{2} h_{\alpha \beta} \{ \partial_\mu H^{\mu \alpha \beta} \partial^\alpha \partial^\beta H - \partial_\mu H^{\alpha \beta} \partial^\alpha \partial^\beta H - \partial^\alpha \partial^\beta H \partial^\alpha \partial^\beta \} \]

\[ = - h^{\alpha \beta} \partial^\alpha \partial^\beta h - (\gamma D - 1) h_{\alpha \beta} \partial^\alpha \partial^\beta \varphi - B_{\alpha \beta} \partial_\mu B^{\mu \alpha \beta} \partial^\alpha h \]

\[ - (\gamma D - 1) B_{\alpha \beta} \partial_\mu B^{\mu \alpha \beta} \partial^\alpha \varphi - \gamma \varphi \partial^\alpha \partial^\beta h - (\gamma D - 1) \varphi \partial^\alpha \partial^\beta \varphi - \gamma^2 \varphi \partial^\alpha \partial^\beta \varphi \]

\[ - (\gamma D - 1) \varphi \partial^\alpha \partial^\beta \varphi + \frac{1}{2} h_{\alpha \beta} \partial^\alpha h \partial^\beta h + (\gamma D - 1) h_{\alpha \beta} \partial^\alpha \partial^\beta \varphi \]

\[ + \frac{1}{2} (\gamma D - 1)^2 h_{\alpha \beta} \partial^\alpha \varphi \partial^\beta \varphi + \frac{1}{2} \gamma \varphi \partial^\alpha \partial^\beta h + \gamma (\gamma D - 1) \varphi \partial^\alpha \varphi \partial^\beta \varphi \]

\[ + \frac{1}{2} \gamma (\gamma D - 1)^2 \varphi \partial^\alpha \varphi \partial^\beta \varphi. \]  

(B.49)

Again, all the terms in which the two-form appears an odd number of times cancel out in the sum for symmetry reasons. Moreover, we employed the identity (already introduced in Section 4.1):

\[ \frac{\partial \cdot \partial \cdot H}{\square} = h + (\gamma D - 1) \varphi. \]  

(B.50)

We recall, once again, that interaction vertices proportional to the equations of motion are actually fake interactions, which can be eliminated by means of appropriate field redefinitions. We will exploit this fact several times in what follows, therefore we remind that the linearized equations of motion for the gravity fields are:

\[
\begin{align*}
\square h_{\mu \nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\mu + \partial_\mu \partial_\nu h &= 0, \\
\square B_{\mu \nu} - \partial_\mu \partial^\rho B_{\rho \nu} + \partial_\nu \partial^\rho B_{\rho \mu} &= 0, \\
\square \varphi &= 0.
\end{align*}
\]  

(B.51)

Thus, for instance, all the terms proportional to \( \square \varphi \) are fake interactions, and we will neglect them from now on. In order to compare the vertices we obtained with the ones of \( \mathcal{N} = 0 \) Supergravity, we collect separately interactions of different type and exploiting the freedom to integrate by parts we rearrange their expressions in order to make them as similar as possible to the vertices derived in Section 2.4.3.
Appendix B. Noether procedure for the double-copy field

• $\varphi^3$: all the terms of this kind are either $\sim \varphi^2 \Box \varphi$, which is proportional to the equations of motion of $\varphi$, or $\sim \varphi \partial^\mu \varphi \partial_\mu \varphi$, which is again in the form $\sim \varphi^2 \Box \varphi$ if integrated by parts. Therefore, there are no interactions among three scalars $\varphi$.

• $hhh$: the Lagrangian vertices are:

$$L_1^{hhh} = h^\mu_\nu \partial_\mu h h^{\alpha \beta} h_{\alpha \beta} + 2 h^\mu_\nu \partial_\mu h^{\alpha \beta} \partial_\nu h_{\alpha \beta} - h_{\alpha \beta} \partial^\alpha \partial^\beta h h_{\alpha \beta}.$$  

The last term can be rewritten as:

$$-\frac{1}{4} \partial \cdot \partial \cdot h h_{\alpha \beta} = \left[-\partial \cdot \partial \cdot h + \frac{1}{2} \Box h - \frac{1}{2} (\Box h - \partial \cdot \partial \cdot h)\right] h_{\alpha \beta}.$$  

(B.52)

Since $\Box h - \partial \cdot \partial \cdot h \propto \Box \varphi$, we can neglect it, and what remains is exactly the cubic vertex of the EH Lagrangian derived in appendix A.

• $BB\varphi$: the Lagrangian vertices are:

$$L_1^{BB\varphi} = H_\mu^\alpha \partial_\mu B_{\alpha \beta} + 2 \gamma B^{\mu \alpha} \partial_\mu B_{\alpha \beta} \partial^\beta \varphi + 2 \gamma B^{\mu \nu} \partial_\mu \varphi \partial^\alpha B_{\alpha \nu} + 2 \gamma \varphi \partial^\mu B^{\alpha \beta} \partial_\mu B_{\alpha \beta} - (\gamma D - 1) B_{\alpha \beta} \partial_\mu \partial^\mu \partial_\beta B_{\alpha \mu} + [\gamma B_{\alpha \beta} \partial_\mu \partial^\mu B_{\alpha \beta} + [(\gamma D - 1) - 2 \gamma \varphi \partial^\mu B_{\alpha \beta} \partial_\mu B_{\alpha \beta}].$$  

(B.53)

By comparison with the expansion of $\mathcal{N} = 0$ Supergravity action in Section 2.4.3, we can observe that the last term should cancel, and this is possible if

$$\gamma D - 1 - 2 \gamma = 0 \Rightarrow \gamma = \frac{1}{D - 2}.$$  

(B.54)

Moreover, making use of the equations of motion for $B_{\mu \nu}$ (we neglect vertices proportional to the free equations of motion):

$$\varphi B^{\alpha \beta} \partial_\alpha \partial^\mu B_{\mu \beta} = B^{\alpha \beta} \Box B_{\alpha \beta},$$  

(B.56)

and

$$\varphi B^{\alpha \beta} \Box B_{\alpha \beta} = -\partial_\mu \varphi \partial^\mu B_{\alpha \beta} B^{\alpha \beta} - \varphi \partial^\mu B^{\alpha \beta} \partial_\mu B_{\alpha \beta},$$  

(B.57)
but since
\[ \partial_\mu \varphi \partial^\mu B_{\alpha\beta}B^{\alpha\beta} = -\frac{1}{2} \Box \varphi B_{\alpha\beta}B^{\alpha\beta}, \quad (B.58) \]
we can ignore this term (it contains \( \Box \varphi \)). Summarizing, with \( \gamma = \frac{1}{D-2} \), we end up with:
\[ L_1^{BB\varphi} = -\frac{2}{D-2} \{ \partial^\mu B_{\alpha\beta} \partial_\mu B_{\alpha\beta} + 2 \partial^\mu B_{\alpha\beta} \partial_\mu B_{\alpha\beta} \}. \quad (B.59) \]

From now on, we will always use \( \gamma = \frac{1}{D-2} \), since also in the next cases it is the value which reproduces the correct vertices.

- \( hh\varphi \): the Lagrangian vertices are:
\[ L_1^{hh\varphi} = \gamma h_{\mu\nu} \partial_\mu \partial_\nu \varphi + \gamma h_{\mu\nu} \partial_\mu \partial_\nu \varphi h + \gamma \varphi \Box h, \]
\[ + 2 \gamma h_{\mu\nu} \partial_\mu \varphi \partial_\nu h + 2 \gamma \varphi \partial^\mu h_{\alpha\beta} \partial_\beta h_{\alpha\mu} - \gamma \partial \cdot \partial \varphi h + \frac{1}{2} \Box h_{\varphi} \]
\[ - (\gamma D - 1) h_{\alpha\beta} \partial \cdot h^\beta \varphi - \gamma \varphi \partial \cdot h^\alpha \partial_\alpha h - \gamma h_{\alpha\beta} \partial^\alpha \varphi \partial^\beta h \]
\[ + (\gamma D - 1) h_{\alpha\beta} \partial^\alpha h \partial^\beta \varphi + \frac{1}{2} \gamma \varphi \partial^\alpha \partial_\alpha h = \frac{1}{D-2} \varphi \{ h_{\mu\nu} \Box h_{\mu\nu} \}
\[ - \partial_\mu \partial \cdot h_{\nu} + \partial_\mu \partial_\nu h ] + \frac{1}{2} \Box h + \frac{1}{2} \partial^\alpha h \partial_\alpha h \}. \quad (B.60) \]

The first term in the curly brackets contains the linearized equations of motion of \( h_{\mu\nu} \), therefore we neglect it. For the remaining part, we observe that
\[ \frac{1}{2} \varphi \partial^\alpha h \partial_\alpha h = \frac{1}{4} \Box \varphi h^2 - \frac{1}{2} \varphi \Box h. \quad (B.61) \]

Therefore, neglecting also the term proportional to \( \Box \varphi \), the vertex reduces to \( L_1(hh\varphi) = 0 \).

- \( \varphi\varphi h \): the Lagrangian vertices are:
\[ L_1^{\varphi\varphi h} = \gamma^2 D h_{\mu\nu} \partial_\mu \partial_\nu \varphi \varphi + \gamma^2 \varphi \Box \varphi + \gamma^2 \varphi^2 \Box h + 4 \gamma^2 \varphi \partial \cdot h_{\beta} \partial^\beta \varphi \]
\[ + 2 \gamma^2 h_{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \gamma^2 \partial \cdot \partial \varphi h^2 + \frac{1}{4} \gamma^2 \Box \varphi h^2 - \gamma (\gamma D - 1) \varphi \partial \cdot h^\alpha \partial_\alpha \varphi \]
\[ - \gamma (\gamma D - 1) h_{\alpha\beta} \partial^\alpha \varphi \partial^\beta \varphi - \gamma^2 \varphi \partial_\alpha \varphi \partial^\alpha h + \frac{1}{2} \gamma (\gamma D - 1) h_{\alpha\beta} \partial^\alpha \varphi \partial^\beta \varphi \]
\[ + \gamma (\gamma D - 1) \varphi \partial^\alpha h \partial_\alpha \varphi = \frac{1}{(D-2)^2} \{ -(D-2) h_{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \]
Appendix B. Noether procedure for the double-copy field

\[-\frac{1}{2}(D + 2)h \partial^\alpha \varphi \partial_\alpha \varphi - \varphi^2 \partial h = \frac{1}{D - 2} \left( -h^{\mu \nu} + \frac{1}{2} \gamma^{\mu \nu} h \right) \partial_\mu \varphi \partial_\nu \varphi \]

(B.62)

- **BBh**: the Lagrangian vertices are:

\[
\mathcal{L}^{BBh}_1 = h^{\mu \nu} \partial_\mu \partial_\nu B_{\alpha \beta} B^{\alpha \beta} + 2h^{\mu \nu} \partial_\mu B^{\alpha \beta} \partial_\nu B_{\alpha \nu} + 2B^{\mu \nu} \partial_\mu B^{\alpha \beta} \partial_\nu h_{\alpha \nu} \\
+ 2B^{\mu \nu} \partial_\mu h^{\alpha \beta} \partial_\nu B_{\alpha \nu} - \frac{1}{2} \partial \cdot \partial \cdot h B^{\alpha \beta} B_{\alpha \beta} + \frac{1}{4} \Box h B^{\alpha \beta} B_{\alpha \beta} \\
- B_{\alpha \beta} \partial_\mu B^{\alpha \beta} \partial_\nu h = -2h^{\mu \nu} \left[ B_{\alpha \nu} \partial_\alpha \partial_\beta B_{\mu \beta} + B^{\alpha \beta} \partial_\alpha \partial_\beta B_{\mu \beta} \right] \\
+ h^{\mu \nu} \left[ -2\partial_\mu B^{\alpha \beta} \partial_\nu B_{\alpha \beta} - 2\partial_\beta B_{\alpha \nu} \partial_\nu B_{\mu \beta} - 2\partial_\alpha B^{\alpha \beta} \partial_\nu B_{\mu \beta} - \partial_\mu B^{\alpha \beta} \partial_\nu B_{\alpha \beta} \right] \\
+ h \left[ \frac{1}{2} B_{\alpha \beta} (\Box B_{\alpha \beta} + 2\partial_\alpha \partial_\beta B_{\mu \beta}) + \frac{1}{2} \partial_\mu B_{\alpha \beta} \partial_\nu B^{\alpha \beta} + \partial_\alpha B^{\alpha \beta} \partial_\mu B_{\mu \beta} \right].
\]

(B.63)

In the second square bracket, we integrate one term by parts:

\[-2h^{\mu \nu} \partial_\alpha B^{\alpha \beta} \partial_\nu B_{\mu \beta} = 2\partial_\alpha h^{\mu \nu} B^{\alpha \beta} \partial_\nu B_{\mu \beta} + 2h^{\mu \nu} B^{\alpha \beta} \partial_\alpha \partial_\nu B_{\mu \beta}.\]  

(B.64)

With this, the vertex can be written as:

\[
\mathcal{L}^{BBh}_1 = h \left[ \frac{1}{2} \partial_\mu B_{\alpha \beta} \partial_\nu B^{\alpha \beta} + \partial_\alpha B^{\alpha \beta} \partial_\mu B^{\alpha \beta} \right] + h^{\mu \nu} \left[ -4\partial_\mu B^{\alpha \beta} \partial_\beta B_{\mu \alpha} \\
- 2\partial_\beta B_{\alpha \nu} \partial_\alpha B_{\mu \beta} - 2h^{\mu \nu} B^{\alpha \beta} \partial_\alpha B_{\mu \beta} - \partial_\mu B^{\alpha \beta} \partial_\nu B_{\alpha \beta} \right] \\
+ \left\{ 2\partial_\alpha h^{\mu \nu} B^{\alpha \beta} \partial_\nu B_{\mu \beta} - 2h^{\mu \nu} B_{\alpha \nu} \partial_\alpha B_{\mu \beta} + 2h^{\mu \nu} \partial_\mu B^{\alpha \beta} \partial_\beta B_{\mu \alpha} \\
+ 2h^{\mu \nu} \partial_\alpha B_{\alpha \beta} \partial_\nu B_{\mu \beta} - h\partial_\mu B_{\alpha \beta} \partial_\alpha B_{\mu \beta} + \frac{1}{2} h B^{\alpha \beta} \Box B_{\alpha \beta} \\
+ h B^{\alpha \beta} \partial_\alpha \partial_\mu B_{\mu \beta} + h \partial_\alpha B_{\alpha \beta} \partial_\mu B^{\alpha \beta} \right\}.
\]

(B.65)

The terms in the square brackets match with the expected vertex of $N = 0$ Supergravity, therefore we must show that all the terms in the curly brackets cancel. To this end, we perform the following integrations by parts:

\[
\begin{align*}
* - h \partial_\mu B_{\alpha \beta} \partial_\alpha B^{\beta \mu} &= \partial_\alpha h \partial_\mu B^{\alpha \beta} B_{\mu \beta} + h \partial_\mu \partial_\alpha B_{\alpha \beta} B^{\beta \mu} \\
&= -\partial_\alpha \partial_\mu h B_{\alpha \beta} B^{\beta \mu} - \partial_\alpha h B^{\alpha \beta} \partial_\mu B_{\beta \mu} + h \partial_\mu \partial_\alpha B_{\alpha \beta} B^{\beta \mu}.
\end{align*}
\]

\[
\begin{align*}
* h \partial_\alpha B_{\alpha \beta} \partial_\mu B^{\beta \mu} &= -\partial_\alpha h B^{\alpha \beta} \partial_\mu B_{\beta \mu} - h B^{\alpha \beta} \partial_\alpha \partial_\mu B_{\beta \mu} \\
&= \partial_\alpha \partial_\mu h B_{\alpha \beta} B^{\beta \mu} + \partial_\alpha h \partial_\mu B^{\alpha \beta} B_{\beta \mu} - h B^{\alpha \beta} \partial_\alpha \partial_\mu B_{\beta \mu}.
\end{align*}
\]

(B.66)
\[ 2 \partial_\alpha h^{\mu\nu} B^{\alpha\beta} \partial_\nu B_{\mu\beta} = -2 \partial_\alpha \partial \cdot h^{\mu\nu} B^{\alpha\beta} B_{\mu\beta} - 2 \partial_\alpha h^{\mu\nu} \partial_\nu B^{\alpha\beta} B_{\mu\beta}. \] (B.67)

\[ 2 h^{\mu\nu} \partial^\rho B_{\mu\beta} \partial_\alpha B_\nu^\beta = \Box h^{\mu\nu} B_{\mu\beta} B_\nu^\beta - 2 h^{\mu\nu} \Box B_{\mu\beta} B_\nu^\beta. \] (B.68)

With these replacements, the terms which should cancel in eq. (B.65) become:

\[
\begin{align*}
\{ \ldots \} &= B^{\mu\beta} B_\beta^\nu \left[ \Box h^{\mu\nu} - \partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu \right] + \frac{1}{2} h B^{\alpha\beta} \left[ \Box B_{\alpha\beta} - \partial_\alpha \partial^\mu B_{\mu\beta} \right] \\
&\quad + \left\{ \partial_\mu \partial_\nu h B^{\alpha\beta} B_\beta^\nu + \partial_\nu h \partial_\mu \left( B^{\mu\beta} B_\beta^\nu \right) \right\} - 2 h^{\mu\nu} B_\beta^\nu \left( \Box B_{\mu\beta} + \partial_\beta \partial^\alpha B_{\alpha\mu} \right) \\
&\quad - 2 \partial_\alpha h^{\mu\nu} \partial_\nu B^{\alpha\beta} B_{\mu\beta} + 2 h^{\mu\nu} \partial_\nu B_{\alpha\beta} \partial^\beta B_\nu^\alpha.
\end{align*}
\] (B.69)

The terms in the square brackets of the first line contain respectively the equations of motion of \( h^{\mu\nu} \) and \( B_{\mu\nu} \), therefore we neglect them, moreover the curly brackets is identically zero if the second term inside it is integrated by parts. With another integration by parts:

\[
-2 \partial_\alpha h^{\mu\nu} \partial_\nu B^{\alpha\beta} B_{\mu\beta} = -2 h^{\mu\nu} \partial_\mu B_{\alpha\beta} \partial^\beta B_\nu^\alpha + 2 h^{\mu\nu} B_\beta^\nu \partial_\mu \partial^\alpha B_{\alpha\beta}.
\] (B.70)

With this, the last term cancels and the only leftover is:

\[
-2 h^{\mu\nu} B_\beta^\nu \left( \Box B_{\mu\beta} + \partial_\beta \partial^\alpha B_{\alpha\mu} - \partial_\mu \partial^\alpha B_{\alpha\beta} \right),
\] (B.71)

which contains the equations of motion of \( B_{\mu\nu} \) and is therefore another fake interaction. This completes the proof that this vertex is equivalent, modulo field redefinitions, to:

\[
\mathcal{L}_{1}^{BBh} = h \left[ \frac{1}{2} \partial_\mu B_{\alpha\beta} \partial^\mu B^{\alpha\beta} + \partial^\alpha B^{\mu\beta} \partial_\mu B^{\alpha\beta} \right] \\
+ h^{\mu\nu} \left[ -4 \partial_\mu B^{\alpha\beta} \partial_\nu B_{\alpha\beta} - 2 \partial_\beta B_{\alpha\nu} \partial^\alpha B_\mu^\beta - 2 \partial^\alpha B^{\mu\beta} \partial_\alpha B_\nu^\beta - \partial_\mu B^{\alpha\beta} \partial_\nu B_{\alpha\beta} \right].
\] (B.72)

Summarizing, the double-copy cubic vertex (B.45) is equivalent to the cubic vertices of the \( \mathcal{N} = 0 \) Supergravity when \( H_{\mu\nu} \) is expanded as:

\[
H_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu} + \frac{1}{D-2} h_{\mu\nu} \varphi,
\] (B.73)

modulo field redefinitions and the inclusion of the gravitational coupling constant \( \kappa \) in front of the cubic interactions.
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