Master degree course in Theoretical Physics

Master Degree Thesis

Gravitational Waves in Non-Fierz-Pauli Massive Gravities

Supervisors
prof. Gianluca Gemme
prof. Nicola Maggiore

Co-Supervisor
prof. Silvano Tosi

Candidate
Giulio Gambuti

Academic Year 2019-2020
Abstract

Modifying gravity at large distances by means of a massive graviton is an alternative to the cosmological constant for explaining the observed accelerated expansion of the Universe. The standard paradigm for Massive Gravity is the Fierz-Pauli theory, which, nonetheless, displays a few flaws in its massless limit. The most serious one perhaps is represented by the so called vDVZ discontinuity, which consists in a disagreement between the massless limit of Fierz-Pauli theory and General Relativity. Our approach is based on a field-theoretical treatment of Massive Gravity: General Relativity, in the weak field approximation, is treated as a gauge theory of a symmetric rank-2 tensor field. This leads us to propose an alternative theory of Massive gravity, describing five degrees of freedom of the graviton, with a good massless limit and without vDVZ discontinuity.
# Contents

Introduction

## Linearized Gravity and the Fierz-Pauli Theory

1. Linearized Gravity
   1.1 Linearization of the Einstein-Hilbert Action
   1.2 The Fierz-Pauli Action

## Covariant Gauge Fixing

2. Introduction to Gauge Fixing
2. Massive Propagators
   2.2.1 Lagrange multiplier method
   2.2.2 Quadratic gauge fixing method
   2.2.3 Direct gauge Fixing method
2. Degrees of Freedom
   2.3.1 Lagrange multiplier method
   2.3.2 Quadratic gauge fixing method
   2.3.3 Direct gauge fixing method
2. Summary of solutions
2.5 Gauge Fixing dependence of the Propagators
2.6 A new theory of Massive Gravity
   2.6.1 Consistency with Linearized Massless Gravity: zero-mass limit
2.6.2 Massive Gravity Theory Proposal .................................. 81
2.7 Cosmological Constant .................................................. 85

3 Lorentz Violating Mass Term ............................................. 89
  3.1 Covariant Gauge Fixing ............................................... 92
  3.2 Lorentz violating Gauge Fixing ........................................ 94
     3.2.1 Propagator and DOFs ......................................... 102

Conclusions ........................................................................... 105

Bibliography ................................................................. 111
Introduction

Motivations

It is nowadays an experimentally confirmed fact that the Universe is expanding at an accelerated rate [1, 2]. Within the theory of General Relativity (GR), the Cosmological Constant (CC) $\Lambda$ can explain this intriguing phenomenon. In a picture of the Universe seen as a perfect fluid, the CC can cause an accelerated expansion by acting as a negative energy density $\rho \sim \Lambda M_P^2$ and therefore providing a negative pressure, which is usually identified with the term Dark Energy. A fine tuning of the constant $\Lambda$ enables us to match the expansion predicted by GR to that observed empirically. The most recent estimates [3] give $\frac{\Lambda}{M_P^2} \sim 10^{-65}$, where $M_P = 1.22091 \times 10^{19} \frac{GeV}{c^2} = 2.17643 \times 10^{-8} \text{ kg}$ is the Planck mass. By contrast, the Quantum Field Theory (QFT) prediction of the CC seen as the vacuum energy density is $\frac{\Lambda}{M_P^2} \sim 6 \times 10^{54}$. Unfortunately, these two estimates disagree by about 120 orders of magnitude. This huge tension is known as the cosmological constant problem [4].

Modifying gravity at large distances by means of a massive graviton is an alternative to the cosmological constant for explaining the observed accelerated expansion of the Universe. The Yukawa potential for a massive field at large distances goes like $\propto \frac{1}{r} e^{-\alpha m r}$, where $m$ is the mass of the field and $\alpha$ a dimensional constant. At scales comparable to $\frac{1}{\alpha m}$, the exponential factor suppresses the potential and,
with it, the strength of interactions. This is the reason why long-range forces are associated with massless bosons and short-range forces with massive bosons. By means of a Yukawa-like potential, the gravitational effect of the vacuum energy density could be exponentially suppressed at large scales, explaining the disagreement between the QFT calculation and the observed cosmological value. Another way of understanding this point is to consider that a mass acts as a cutoff on interactions, effectively damping the effect of low frequency sources. The vacuum energy density can be considered constant, therefore its contribution to the evolution of the Universe would result greatly reduced by the introduction of a mass to the graviton, again confirming that in a picture where the graviton is massive the cosmological constant problem could be fixed by fine tuning the graviton mass.

Theories in which the graviton is massive are collectively referred to as Massive Gravity (MG).

In some sense MG is a substantial modification of GR, even if the mass of the graviton is chosen extremely close to zero. When adding a mass to a force-carrier particle, the 2 degrees of freedom (DOFs) carried by the massless boson become $2S + 1$ massive DOFs. For the graviton, which is the spin $S = 2$ representation of the Poincaré group, the two massless DOFs become five massive ones. Other approaches to modifying GR, like the F(R) theories [5], add only one additional scalar DOF to the theory and in this sense they can be considered more minimal modifications.

Phenomenological Limits

As previously said, long range forces are usually carried by massless bosons because the Yukawa exponential factor limits the range of interactions mediated by massive particles. Nevertheless, choosing a sufficiently small value of the graviton mass, we
could make sure that interactions at scales of the observable Universe remain essentially identical to those of ordinary gravity, while only interactions at greater scales are affected. Indeed, the observed accelerated rate of the Universe poses an upper limit to the mass of the graviton, which in [6] is shown to be of order $10^{-34} \frac{eV}{c^2}$, roughly an order of magnitude smaller than the Hubble parameter at our time. With such a limit, for interactions at distances much smaller than the radius of the observable Universe (which is estimated to be about $46.6 \times 10^9$ ly [7]), the Yukawa exponential drops to one, and the classical Newton potential $\propto \frac{1}{r}$ is recovered, as desired. Still, this cosmological upper limit becomes irrelevant when the mass is acquired through the condensation of some additional scalar field (see [6]). Another constraint on the mass of the graviton was obtained by the LIGO-Virgo collaboration through the analysis of binary black holes merger signals, comparing the time of arrival of the gravitational waves (detected by the interferometers) to that of their electromagnetic counterparts (traveling at the speed of light) [8]. In this way, the upper limit for the mass of the graviton is estimated to be $4.7 \times 10^{-23} \frac{eV}{c^2}$, which is several orders of magnitude higher than the limit derived in [6] by cosmological considerations.

**State of the Art**

In 1939 Fierz and Pauli [9] proposed a relativistic theory for a massive particle with arbitrary spin $f$, described by a symmetric rank-$f$ tensor field. They showed that “in the particular case of spin two, rest-mass zero, the equations agree in the force-free case with Einstein’s equations for gravitational waves in GR in first approximation; the corresponding group of transformations arises from the infinitesimal coordinate transformations”. This is what is known as Fierz-Pauli (FP) theory. After that, a variety of developments and alternative theories emerged. It was realized by van Dam, Veltman and Zakharov [10, 11] that the weak field (linearized) MG
theories exhibit a discontinuity with GR in their massless limit, known as vDVZ discontinuity. The Vainshtein mechanism [12] consists in reintroducing in the weak field approximation the non-linearities of GR. In this way it is possible to cure this discontinuity, which therefore can be interpreted as a pathology of the weak field theory. It was also pointed out by Boulware and Deser [13] that the most general MG theory contains a ghost mode, called Boulware-Deser (BD) ghost. The BD ghost problem, which for some time was thought to be present in all MG theories, was solved in the particular case of linearized MG with a particular choice of parameters [14, 15], referred to as the FP tuning, which will be explained in more detail in Chapter 1. However, in order to achieve a non-linear ghost-free MG more work was required. The realization that a theory of massless gravity in five spacetime dimensions produces a graviton which propagates 5 DOFs (which is the number of DOFs needed for a four spacetime dimensional massive graviton) led to the Dvali-Gabadadze-Porrati (DGP) model [16, 17]. A solution in four spacetime dimensions of the BD ghost problem is represented by the de Rham-Gabadadze-Tolley (dRGT) model [14], obtained by fine tuning the parameters of the non-linear theory. A detailed review on MG is presented in [14].

**Strategy**

This Thesis focuses on linearized MG, for which the standard paradigm is the FP theory, which, however, displays a few flaws in its massless limit. As anticipated, the most serious one perhaps is represented by the vDVZ discontinuity [10, 11], which consists in a disagreement between the massless limit of FP theory and GR. In particular, in [18] it is shown that the interaction of light with a massive body, like a star for instance, is 25% smaller in the massless limit of the FP theory than in GR. Consequently, the angle of deviation of light is also different by 25% in the two theories, a difference which is measurable.
In principle, this measurable disagreement between GR and the FP theory of MG would allow us to distinguish a strictly vanishing mass and one which has a very small value. Such a tension is crucial since GR has been extensively tested and as a consequence we should immediately rule out the FP theory as a theory of MG. This problem is solved through the Vainshtein mechanism [12], which hides the problems of the FP theory at small scales. The drawback of the Vainshtein mechanism is that it requires a further modification of the FP theory.

The approach described in this Thesis is based on a field-theoretical treatment of MG. We consider GR, in the weak field approximation, as a gauge theory of a symmetric rank-2 tensor field. Indeed, the linearization of the Einstein-Hilbert action (which yields the Einstein equations through a variation principle) leads to an action describing the dynamics of a spin 2 field, which is embedded in a symmetric rank-2 tensor field.

As in ordinary gauge field theory, a mass for the graviton is incompatible with gauge symmetry (which, in our case coincides with the infinitesimal diffeomorphism invariance).

In this Thesis an alternative theory of MG is proposed describing the five DOFs of the graviton, with a good massless limit and without the vDVZ discontinuity.

Hence, the mass is introduced by means of an additional term added to the action. Before this operation, it is necessary that the theory be well defined, which means that it must have a well defined partition function. In gauge field theory this is equivalent to requiring that the action is gauge fixed. The structure which will be adopted in this Thesis is: a massless theory with well defined propagators, hence gauge fixed, to which a mass term is added. By contrast, in the FP case a mass
term is added to the invariant action and effectively plays the double and unnatural role of mass term and gauge fixing. The massless limit of the FP theory is ill-defined because the propagator diverges.

**Summary**

This Thesis is organized as follows: in Chapter 1 it is explained how the weak field limit of GR is described by a rank-2 symmetric tensor field action and summarizes the main features of the FP theory. Chapter 2 contains a short introduction to the Faddeev-Popov gauge fixing procedure followed by the core of our work: a covariant gauge fixing applied to the invariant linearized gravity (LG) action and the introduction of a mass term. Then, the derivation of the propagator of the massive theory is presented and an analysis of the propagating DOFs is carried out using the Equations Of Motion (EOMs) of the massive action. Finally, it is shown that the approach of this Thesis to MG is free of discontinuities in the massless limit and the gauge fixed massive action has all the fundamental properties to describe a physical massive spin-2 particle, i.e. the massive graviton. Chapter 3 explores the possibility of a Lorentz violating mass term in order to find dispersion relations for the graviton whose signature on gravitational waves could be in principle measurable by an interferometer. Chapter Conclusions contains a discussion of the results of this Thesis and comments on possible future developments.
Chapter 1

Linearized Gravity and the Fierz-Pauli Theory

1.1 Linearized Gravity

The Einstein theory of GR is by now recognized as the correct theory to describe long range gravitational interactions. The Einstein equations, which describe the relationship between matter/energy and the space-time curvature, can be derived from an action principle. This action is the Einstein-Hilbert one:

\[ S_{EH} = k \int d^4x \sqrt{-g} R, \]  

(1.1)

where \( R \) is the Ricci scalar, \( g \) the determinant of the metric and \( k \) is a coupling constant.

For the scope of this Thesis we will focus on the approximation of flat space-time \( (i.e. \) Minkowski background metric) and we will consider the expansion of GR around small perturbations of the background. This approximation of GR around Minkowski space-time is called Linearized Gravity.
Mathematically, the metric can be written as

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) , \]  

where \( \eta_{\mu\nu} \) is the Minkowskian metric \( \text{diag}(-1,1,1,1) \) and \( h_{\mu\nu}(x) \) is the perturbation. Expanding GR around Minkowski spacetime means expanding the Einstein-Hilbert action in \( h_{\mu\nu} \) and the result is the LG action

\[ S[h] = -k \int d^4x \frac{1}{4} \left( \partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - 2 \partial_\mu h^{\mu\lambda} \partial_\nu h_\lambda^\nu + 2 \partial_\mu h^{\mu\nu} \partial_\nu h - \partial^\lambda h \partial_\lambda h \right) . \]  

Section 1.1.1 will be dedicated to the derivation of this action.

A key observation that can be made here is that the LG action is invariant under the gauge transformation

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \]  

This symmetry of the LG action is not a coincidence: it is a direct effect of the general coordinate transformation invariance of GR.

Following [19] we will now give a brief explanation of the origin of the connection between the symmetry (1.4) and general coordinate transformation in the LG setting.

We start by formalizing the idea of background metric identifying a background space-time \( M_b \) with the Minkowski metric \( \eta_{\mu\nu} \) and a physical space-time \( M_p \) with the physical metric \( g_{\mu\nu}(x) \). There exists a diffeomorphism

\[ \phi : M_b \rightarrow M_p \]  

between the two manifolds which we can use to move tensors from one space-time to the other. In particular, we want that our description of the physics take place in the background space-time in which we can use the pullback of the physical metric.
The perturbation $h_{\mu\nu}(x)$ can now be identified with the difference between this pullback and the flat metric $\eta_{\mu\nu}$, i.e.

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}. \quad (1.6)$$

As pointed out in [19], up to this point we have no reason to assume that $h_{\mu\nu}(x)$ is small and no gauge invariance has appeared yet. Still, if we are working in the weak field approximation there will exist some $\phi$ such that $|h_{\mu\nu}| \ll 1$. Therefore, we have to restrict the set of possible diffeomorphisms to those which satisfy this condition on $h_{\mu\nu}(x)$ which can thus be considered a perturbation of the background.

The point is that such a diffeomorphism is not unique: given the existence of a particular $\phi$ so that $|h_{\mu\nu}| \ll \epsilon$, then there is a whole class of them which satisfies this condition. If there is a coordinate transformation which maps the physical metric into a Minkowski metric plus a small perturbation, then there will exist infinitely many other coordinate transformations which do the same as long as they are close to the original transformation $\phi$. The way to formalize this is by composing $\phi$ with other small diffeomorphisms. These are obtained by introducing a vector field $\xi^\mu(x)$ on $M_b$ which generates a 1-parameter family of diffeomorphisms of the background space-time into itself

$$\psi_\epsilon : M_b \rightarrow M_b. \quad (1.7)$$

These are related to a background coordinate transformation of the type $x \rightarrow x + \epsilon \xi(x)$. As explained above, as long as this change of coordinates is small, $h_{\mu\nu}(x)$ will remain small in the new background coordinates. That is, for $\epsilon$ sufficiently small

$$h_{\mu\nu}^{(\epsilon)} = [\psi_\epsilon^* (\phi^* g)]_{\mu\nu} - \eta_{\mu\nu} \quad (1.8)$$

will still be a perturbation of the background: $|h_{\mu\nu}^{(\epsilon)}| \ll 1$. 

14
Now plugging

$$(\phi^* g)_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

into (1.8) we get

$$h^{(\epsilon)}_{\mu\nu} = \psi^*(\eta_{\mu\nu}) + \psi^*(h_{\mu\nu}) - \eta_{\mu\nu} = h_{\mu\nu} + \epsilon \frac{\psi^*(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon},$$

where in the second equality we have used the fact that for small $\epsilon$ and small $h_{\mu\nu}$ to first order $\psi^*(h_{\mu\nu}) = h_{\mu\nu}$. Also, multiplying and dividing by $\epsilon$ allows to recognize (in the limit $\epsilon \to 0$ of the fraction in (1.10)) the Lie derivative of $\eta_{\mu\nu}$ with respect to $\xi$. Considering that $\mathcal{L}_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, Eq. (1.10) gives

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + \epsilon (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

and the description of LG is invariant under the exchange $h_{\mu\nu} \to h^{(\epsilon)}_{\mu\nu}$ for small $\epsilon$. This is where general coordinate transformations invariance becomes the gauge invariance of LG. In order to get exactly the same expression of Eq. (1.4) we just need to absorb $\epsilon$ into $\xi^\mu(x)$ which means that we are now restricted to small values of the single components of the vector $\xi^\mu(x)$.

Here an important remark is in order: while invariance under general coordinate transformations implies invariance under the transformation (1.4) for small $\xi^\mu$, the LG action turns out to be invariant under the full gauge transformation, i.e. for arbitrary values of $\xi^\mu(x)$. There is more: not only the action (1.3) is invariant under the gauge transformation (1.4), but it is the only second order action that has this property. This tells us something about the mathematical description of gravity: while it is known that the equivalence principle and general coordinate transformation do not lead uniquely to GR and the Einstein-Hilbert action, the fact that LG should be a gauge theory describing a spin-2 particle strictly implies
the action (1.3). Therefore, it would seem that spin-2 and gauge invariance are more fundamental than the equivalence principle and, actually, imply it (see Chapter 13 of [4] for an explanation of this fact).

From what we have said we can deduce that LG is a gauge theory, but its gauge transformation is not completely free like that of the Maxwell action. In fact, we have seen that the gauge parameters $\xi^\mu(x)$ have to be small in order to preserve the perturbation condition $|h_{\mu\nu}| \ll 1$. Still, the explicit value of these gauge parameters never comes into play and we can treat LG as if the gauge symmetry was Eq. (1.4) with no constraints on the gauge parameters.

In the following Section the explicit linearization of the Einstein-Hilbert action is presented step by step.

### 1.1.1 Linearization of the Einstein-Hilbert Action

In this Section we will show that substituting

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$  \hspace{1cm} (1.12)

in the Einstein-Hilbert action yields, to second order in the perturbation $h_{\mu\nu}$, the LG action

$$S[h] = -k \int d^4x \frac{1}{4} \left( \partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - 2\partial_\mu h^{\mu\lambda} \partial_\nu h_\lambda^\kappa + 2\partial_\mu h^{\nu\mu} \partial_\nu h - \partial^\lambda h \partial_\lambda h \right).$$ \hspace{1cm} (1.13)

It is necessary to expand the action up to second order in $h_{\mu\nu}$ because the first order expansion yields terms which can be written as total derivatives and therefore give zero contribution to the action.

To find Eq. (1.13), we first recall the expression of the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right)$$  \hspace{1cm} (1.14)
in terms of which the Riemann tensor is expressed as

\[ R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \]  (1.15)

The Ricci tensor reads

\[ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\lambda\alpha} \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} - \Gamma^\lambda_{\nu\alpha} \Gamma^\alpha_{\lambda\mu}. \]  (1.16)

and the Ricci scalar is

\[ R = g^{\mu\nu} R_{\mu\nu}. \]  (1.17)

We start by showing how the Christoffel symbols are written in terms of \( h_{\mu\nu}(x) \). To do this we first note that

\[ \partial_\mu g_{\alpha\beta} = \partial_\mu \eta_{\alpha\beta} + \partial_\mu h_{\alpha\beta} = \partial_\mu h_{\alpha\beta} \]  (1.18)

since the Minkwoski metric is a constant tensor. Next we use the fact that

\[ g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + O(h^2), \]  (1.19)

where \( h_{\mu\nu}(x) \) is defined by

\[ h_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} h_{\alpha\beta}. \]  (1.20)

We will only need the expansion of \( g_{\mu\nu} \) up to second order in \( h_{\mu\nu} \) because the terms inside the parenthesis of the right side of Eq. (1.14) are already of first order because of (1.18).

Using (1.12), (1.18) and (1.20) we can now write down the Christoffel symbols to second order in \( h_{\mu\nu} \):

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda}) \left[ \partial_\mu(\eta_{\lambda\nu} + h_{\lambda\nu}) + \partial_\nu(\eta_{\mu\lambda} + h_{\mu\lambda}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu}) \right] \]

\[ = \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda}) \left[ \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} \right]. \]  (1.21)
We now use this result to work out the Ricci tensor to second order by plugging (1.22) into (1.16):

\[
R_{\mu\nu} =
\]

\[
-\frac{1}{2} \partial_{\lambda} h^{\lambda\beta}(\partial_{\mu} h_{\beta\nu} + \partial_{\nu} h_{\beta\mu} - \partial_{\beta} h_{\mu\nu}) + \frac{1}{2} (\eta^{\lambda\beta} - h^{\lambda\beta})(\partial_{\lambda} \partial_{\mu} h_{\beta\nu} - \partial_{\lambda} \partial_{\beta} h_{\mu\nu})
\]

\[
+ \frac{1}{2} \partial_{\nu} h^{\lambda\beta}(\partial_{\mu} h_{\beta\lambda} + \partial_{\lambda} h_{\beta\mu} - \partial_{\beta} h_{\mu\lambda}) - \frac{1}{2} (\eta^{\lambda\beta} - h^{\lambda\beta})(\partial_{\nu} \partial_{\mu} h_{\beta\lambda} - \partial_{\nu} \partial_{\beta} h_{\mu\lambda})
\]

\[
+ \frac{1}{4} \eta^{\alpha\beta} \eta^{\lambda\beta'}(\partial_{\mu} h_{\beta\nu} + \partial_{\nu} h_{\beta\mu} - \partial_{\beta} h_{\mu\nu})(\partial_{\lambda} h_{\beta\alpha} + \partial_{\alpha} h_{\beta\lambda} - \partial_{\beta} h_{\lambda\alpha})
\]

\[
- \frac{1}{4} \eta^{\lambda\beta} \eta^{\alpha\beta'}(\partial_{\nu} h_{\beta\alpha} + \partial_{\alpha} h_{\beta\nu} - \partial_{\beta} h_{\nu\alpha})(\partial_{\mu} h_{\beta\lambda} + \partial_{\lambda} h_{\beta\mu} - \partial_{\beta} h_{\mu\lambda}).
\]

(1.23)

The next step is to compute the Ricci scalar up to second order in $h_{\mu\nu}$. This can be achieved by plugging Eq. (1.23) straight into Eq. (1.17) and discarding all terms of order greater than 2. This results is easier to follow if we write

\[
R = g^{\mu\nu} R_{\mu\nu} = (\eta^{\mu\nu} - h^{\mu\nu}) R_{\mu\nu} = \eta^{\mu\nu} R_{\mu\nu} - h^{\mu\nu} R_{\mu\nu}.
\]

(1.24)

The right side of (1.24) is composed by two terms: the first term is just the Ricci tensor of Eq. (1.23) saturated with $\eta^{\mu\nu}$. Here we have to be careful about how we treat the saturated indices: indices appearing on $h_{\mu\nu}(x)$ are lowered and raised by $\eta^{\mu\nu}$, while indices on the partial derivatives, for instance $\partial_{\nu}$, are lowered and raised by $g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)$. Therefore, when contracting a partial derivative with $\eta_{\mu\nu}$ we have to make use of (1.12) to write $\eta^{\mu\nu} \partial_{\mu} = (g^{\mu\nu} + h^{\mu\nu}) \partial_{\mu} = \partial^{\nu} + h^{\mu\nu} \partial_{\nu}$.

However, notice that if the term in which this contraction appears is already of order 2 in $h_{\mu\nu}(x)$, then it is possible to just make the substitution $\eta^{\mu\nu} \partial_{\nu} \rightarrow \partial^{\mu}$ since the extra $h_{\mu\nu}(x)$ term coming from $g_{\mu\nu}(x)$ would give a contribution of order 3, which we discard. The second term consists of only the first order contributions to $R_{\mu\nu}$, which are

\[
\frac{1}{2} \eta^{\lambda\beta}(\partial_{\lambda} \partial_{\mu} h_{\beta\nu} - \partial_{\lambda} \partial_{\beta} h_{\mu\nu}) - \frac{1}{2} \eta^{\lambda\beta}(\partial_{\nu} \partial_{\mu} h_{\beta\lambda} - \partial_{\nu} \partial_{\beta} h_{\mu\lambda}),
\]

(1.25)
saturated with $h^{\mu\nu}(x)$. After some cancellations, the final result is

$$R =$$

$$- \frac{1}{2} \partial \lambda h^{\lambda\beta}(\partial_\mu h_\beta^\nu + \partial_\mu h_\beta^\nu - \partial_\beta h) + \frac{1}{2}(\eta^{\lambda\beta} - h^{\lambda\beta})(\partial_\lambda \partial_\mu h_\beta^\mu - \partial_\lambda \partial_\beta h)$$

$$+ \frac{1}{2} \partial_\mu h^{\lambda\beta}(\partial_\nu h_\beta^\lambda + \partial_\nu h_\beta^\lambda - \partial_\beta h_\mu^\lambda) - \frac{1}{2}(\eta^{\lambda\beta} - h^{\lambda\beta})(\partial_\lambda h_\beta^\mu + h_\nu^\nu \partial_\nu \partial_\mu h_\beta^\lambda - \partial_\lambda \partial_\beta h_\mu^\lambda)$$

$$+ \frac{1}{4}(2\partial_\mu h_\beta^\nu - \partial_\beta h)\partial_\beta h - \frac{1}{4}(\partial_\nu h_\lambda^\alpha + \partial_\nu h_\lambda^\alpha - \partial_\lambda h_\mu^\alpha)\partial_\lambda \delta_{\alpha\mu}$$

$$- \frac{1}{2}\eta^{\lambda\beta}h^{\mu\nu}(\partial_\lambda \partial_\mu h_\beta^\nu - \partial_\lambda \partial_\beta h_\mu^\nu) + \frac{1}{2}\eta^{\lambda\beta}h^{\mu\nu}(\partial_\nu \partial_\mu h_\beta^\lambda - \partial_\nu \partial_\beta h_\mu^\lambda) . \quad (1.26)$$

In order to finish our calculation of the linearized action we need to include the multiplicative term $\sqrt{-g}$. Since the Ricci scalar does not contain terms of order zero in $h_{\mu\nu}(x)$, we will only need to expand $\sqrt{-g}$ to first order. Let us write the expansion for $g$. By virtue of the fact that for a generic matrix $M$ the following identity holds

$$\text{Tr} \log M = \text{log det } M \implies \text{Tr}(M^{-1}\delta M) = \frac{1}{\text{det } M} \delta \text{det } M , \quad (1.27)$$

we find that

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \quad (1.28)$$

and using this we find

$$g = \text{det } \eta + h^{\mu\nu} \left[ \frac{\partial g}{\partial g^{\mu\nu}} \right]_{g=\eta} + O(h^2) = -1 - h + O(h^2) , \quad (1.29)$$

hence

$$\sqrt{-g} = 1 + \frac{h}{2} + O(h^2) . \quad (1.30)$$

Using this result we can finally get to the end of our computation writing

$$\sqrt{-g} R = (1 + \frac{h}{2}) R = R + \frac{h}{2} R^{(1)} + O(h^2) , \quad (1.31)$$

19
where $R^{(1)}$ stands for the first order part in $h_{\mu\nu}(x)$ of $R$. The term proportional to $R^{(1)}$ in Eq. (1.31) gives

$$R^{(1)} = \frac{1}{2} h(\partial_{\mu} \partial_{\nu} h^{\mu\nu} - \partial^2 h)$$  \hspace{1cm} (1.32)

Combining everything together we obtain the linearized action

$$S_{\text{lin}}[h] = k \int d^4x \sqrt{-g} R =$$

$$= k \int d^4x (\partial_{\mu} \partial_{\nu} h^{\mu\nu} - \partial^\mu \partial^\nu h)$$

$$- k \int d^4x \frac{1}{4} (\partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - 2\partial_{\mu} h^{\mu\lambda} \partial_\nu h_\lambda + 2\partial_{\mu} h^{\mu\nu} \partial_\nu h - \partial^\lambda h \partial_\lambda h) \ . \hspace{1cm} (1.33)$$

From this result it is clear why an expansion to second order in $h_{\mu\nu}$ is needed in order to find the linearized action. In fact, as mentioned before, the first order terms in the action (in the second line of (1.33)) are just a total derivative, which can be integrated away with proper boundary conditions (namely that the field $h_{\mu\nu}$ and its derivatives go to zero at infinity). The final result is

$$S_{\text{lin}}[h] = -k \int d^4x \frac{1}{4} (\partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - 2\partial_{\mu} h^{\mu\lambda} \partial_\nu h_\lambda + 2\partial_{\mu} h^{\mu\nu} \partial_\nu h - \partial^\lambda h \partial_\lambda h) \ ,$$

which exactly matches the LG action (1.13).

### 1.2 The Fierz-Pauli Action

The FP theory of MG is based on the action one obtains adding to the LG action (1.3) the mass term

$$S_m = \int d^4x \left[ \frac{1}{2} m_1^2 h_{\mu\nu} h^{\mu\nu} + \frac{1}{2} m_2^2 h^2 \right] \ . \hspace{1cm} (1.35)$$

For simplicity we rescale of a factor -2 the LG action and we set the coefficient $k$
to 1. The resulting action is

$$S[h,m_1,m_2] = \int d^4x \left[ \frac{1}{2} h \partial^2 h - h_{\mu\nu} \partial^\mu \partial^\nu h - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} + h^{\mu\nu} \partial_\sigma \partial^\sigma h_{\mu\nu} + \frac{1}{2} m_1^2 h_{\mu\nu} h^{\mu\nu} + \frac{1}{2} m_2^2 h^2 \right]$$  \hspace{1cm} (1.36)

Closely following [15], the action (1.36) can be rewritten in Hamiltonian form:

$$S = \int d^4x \mathcal{L} = \int d^4x \pi_{ij} \dot{h}_{ij} - \mathcal{H} + 2h_{0i} \partial_j \pi_{ij} - m_1^2 h_{0i}^2 + \frac{1}{2} (m_1^2 + m_2^2) h_{0i}^2 + h_{00} \left( \vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} + m_2^2 h_{ii} \right),$$  \hspace{1cm} (1.38)

where

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}}$$  \hspace{1cm} (1.39)

and

$$\mathcal{H} = \frac{1}{2} \pi_{ij}^2 - \frac{1}{4} \pi_{ii}^2 - \frac{1}{2} \partial_k h_{ij} \partial_k h_{ij} + \partial_i h_{jk} \partial_j h_{ik} - \partial_i h_{ij} \partial_j h_{kk} + \frac{1}{2} \partial_i h_{jj} \partial_j h_{kk} - \frac{1}{2} \left( m_1^2 h_{ij} h_{ij} + m_2^2 h_{ii}^2 \right).$$  \hspace{1cm} (1.41)

From (1.37) we see that imposing

$$m_1^2 = -m_2^2 = m_G^2$$  \hspace{1cm} (1.42)

$h_{00}$ appears only linearly in the action and acts as a Lagrange multiplier enforcing the constraint

$$\mathcal{C} = \vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} - m_G^2 h_{ii} = 0.$$  \hspace{1cm} (1.43)

The Poisson bracket of this constraint with the Hamiltonian $H = \int d^3x \mathcal{H}$ gives the secondary constraint

$$\{H, \mathcal{C} \} = \frac{1}{2} m_G^2 \pi_{ii} + \partial_i \partial_j \pi_{ij}.$$  \hspace{1cm} (1.44)
The two constraints (1.43) and (1.44) reduce the 12 dimensional phase space spanned
by $h_{ij}$ and $\pi_{ij}$ (each one has 6 components being a 3 dimensional symmetric rank-2
tensor) to a 10 dimensional phase space, corresponding to the 5 DOFs of the mas-
sive graviton. By contrast, if the condition on the mass parameters (1.42) is not
satisfied, the component $h_{00}$ appears quadratically in (1.37). This implies that no
constraints are present and the full 12 dimensional phase space, corresponding to
6 physical DOFs, is active.

We deduce that in order for the action (1.36) to describe a massive graviton the
condition (1.42) must be satisfied by the masses. Condition (1.42) is referred to
as Fierz-Pauli tuning. It is possible to arrive at the same conclusion through a
Stückelberg analysis of the action (1.36) which is carried out in [14].

Therefore, in order to describe a physical massive graviton we actually have only
one mass parameter at our disposal, namely $m_G$. In fact, plugging the FP tuning
in the massive action (1.36) and renaming $m_1$ to $m_G$ we get the FP action

$$S_{FP}[h, m_G] = \int d^4x \left[ \frac{1}{2} h_\mu^\nu h_\nu^\mu - h_\mu^\nu \partial_\mu \partial_\nu h - \frac{1}{2} h_\mu^\nu \partial_\mu h_\nu + h_\mu^\nu \partial_\nu h_\nu + \frac{1}{2} m_G^2 \left(h_\mu^\nu h_\nu^\mu - h^2\right) \right]$$  \hspace{1cm} (1.45)

Writing this action in momentum space as

$$S_{FP} = \int d^4p \tilde{h}_\mu^\nu \Omega_{\mu\nu,\alpha\beta} \tilde{h}^{\alpha \beta}$$  \hspace{1cm} (1.46)

we can invert the kinetic tensor $\Omega_{\mu\nu,\alpha\beta}$ to obtain the propagator

$$G_{\mu\nu,\alpha\beta} = \frac{2}{p^2 + m_G^2} \left[ \frac{1}{2} (P_{\mu\alpha} P_{\nu\beta} + P_{\nu\alpha} P_{\mu\beta}) - \frac{1}{3} P_{\mu\nu} P_{\alpha\beta} \right],$$  \hspace{1cm} (1.47)

where $P_{\mu\nu}$ is a transverse massive projector defined as

$$P_{\mu\nu} = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m_G^2}.$$  \hspace{1cm} (1.48)
From Eq. (1.47) it is clear that the FP theory has a bad massless limit: taking \( m_G \to 0 \) its propagator diverges. Still, the \( \frac{1}{m_G^2} \) divergence of the propagator (1.47) can be ignored in the FP theory because it is only contained in terms involving at least two 4-momenta \( p \). This is due to the fact that the EOMs of the FP action (1.45) imply the transversality of the graviton field, \( i.e. \)

\[
p_\mu \tilde{h}^{\mu\nu} = 0 .
\]  
(1.49)

So, when we compute the amplitude for the propagation of a graviton,

\[
\tilde{h}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{h}^{\alpha\beta},
\]  
(1.50)

every term proportional to \( \frac{1}{m_G} \) (which also contains a 4-momentum) drops out because of Eq. (1.49) and therefore it gives no contribution to the propagation of the field itself. Moreover, also the interaction of a graviton with a conserved energy-momentum tensor (\( T_{\mu\nu} \)) is free of such terms. That is because conservation of \( T_{\mu\nu} \) implies \( p_\mu T^{\mu\nu} = 0 \) and the same reasoning as for the field \( h_{\mu\nu} \) applies.

Thanks to transversality we can rewrite the propagator (1.47) in the more convenient form

\[
G_{\mu\nu,\alpha\beta} = \frac{1}{p^2 + m_G^2} \left[ \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - 2 \frac{2}{3} \eta_{\mu\nu} \eta_{\alpha\beta} + (p\text{-dependent terms}) \right] ,
\]  
(1.51)

where the terms containing 4-momenta have been isolated and only the non vanishing contributions to the field propagation and interaction with energy-momentum tensors are left explicit. As we will see in detail in Section 2.6 the coefficient \( \frac{2}{3} \) is the cause of the vDVZ discontinuity.
Chapter 2

Covariant Gauge Fixing

2.1 Introduction to Gauge Fixing

The most general action describing a symmetric rank-2 tensor in 4 dimensions invariant under the transformation

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \] (2.1)

is given by the local integrated functional \( S[h] \) (1.3). This action describes a gauge theory, meaning that the generating functional \( Z \) is not well defined, unless the path integral is restricted through a gauge fixing procedure. We will from now on refer to this invariant action as \( S_{\text{inv}} \).

In general, a gauge theory is a field theory for which the action is invariant under a particular transformation \( U \) (the gauge transformation), of which (2.1) is an example. For a gauge theory\(^1\), the configurations of the field can be sorted in equivalence classes: if two configurations \( \Phi_a \) and \( \Phi_a' \) are related by a particular

\(^1\)which describes a generic field \( \Phi_a \), where \( a \) is an index describing its tensorial and/or spinorial nature.
gauge transformation $U(\lambda)$ (where $\lambda$ is the transformation parameter)

$$U(\lambda) \Phi_a = \Phi'_a$$

(2.2)

we say that they are in the same equivalence class.

All gauge theories have in common the fact that, when calculating the generating functional of the theory

$$Z = \int \mathcal{D}\Phi e^{iS_{inv}} ,$$

(2.3)

one finds that the integral is infinite. This is because the integration for $Z$ extends over all possible configurations of the field, even those which are in the same equivalence classes. The infinity in the integral comes from the fact that for every equivalence class one can choose a representative $\bar{\Phi}(x)$ and separate the integration for $Z$ in the following way

$$Z = \int \mathcal{D}\Phi e^{iS_{inv}} = \int \mathcal{D}\bar{\Phi} e^{iS_{inv}} \int \mathcal{D}\lambda ,$$

(2.4)

where the integral $\int \mathcal{D}\lambda$ is infinite and represents the sum over all possible gauge transformations.

In this Section, we want to briefly introduce the well known topic of gauge fixing, which solves this issue by making $Z$ finite\(^2\).

The way to do this is through the Faddeev-Popov ($\Phi\Pi$) procedure [21], which manages to limit the integration to only one element of each equivalence class. Using the $\Phi\Pi$-procedure, the price one has to pay in order to obtain a finite $Z$ is the introduction of additional fields, satisfying an anticommuting Grassmannian algebra, called ghost and antighost, and the addition of a gauge fixing term $S_{gf}$ and a ghost term $S_{\text{ghost}}$ to the invariant action of the theory $S \equiv S_{inv}$, which gives the action

\(^2\)apart from the Gribov ambiguity [20], which is still an open problem, concerning mostly the non-perturbative approaches to gauge field theory.
2.1 – Introduction to Gauge Fixing

\[ S = S_{\text{inv}} + S_{gf} + S_{\text{ghost}} , \quad (2.5) \]

where \( S_{gf} \) is

\[ S_{gf} = - \int d^4x \; \frac{1}{2k} F_a^2(\Phi)(x) , \quad (2.6) \]

with \( k \) a gauge parameter and \( F_a \) a function of \( \Phi \) implementing the choice of representative \( \Phi \) for each equivalence class in the path integral \( Z \) (i.e. gauge condition)

\[ F_a[\phi](x) = C_a(x) , \quad (2.7) \]

with \( C_a \) arbitrary function. The ghost term \( S_{\text{ghost}} \) is a functional of anticommuting ghost \( c_b \) and antighost \( \bar{c}_a \) fields:

\[ S_{\text{ghost}} = - \int d^4x \; \bar{c}_a \left[ \frac{\delta F}{\delta \lambda} \right]_{\lambda=0}^{ab} F^{ab} c_b . \quad (2.8) \]

Summarizing, the \( \Phi \Pi \)-procedure yields

\[ Z = \int D\Phi Dc D\bar{c} \; e^{iS} = \]

\[ = \int D\Phi D\eta D\bar{\eta} \; \exp i \left\{ S_{\text{inv}} - \int d^4x \; \frac{1}{2k} F_a^2(\Phi) + S_{\text{ghost}} \right\} , \quad (2.9) \]

where the total action \( S \) is clearly no longer invariant under the original gauge transformation \( U \).

For abelian gauge theories, the ghost fields decouple from the field \( \Phi \) and, the integral being gaussian, their contribution can be factorized out from \( Z \), and normalized away. For this reason, we shall not bother about the ghost term \( S_{\text{ghost}} \) anymore.

The resulting generating functional is therefore

\[ Z = \int D\Phi \; \exp i \left\{ S_{\text{inv}} - \int d^4x \; \frac{1}{2k} F_a^2(\Phi) \right\} . \quad (2.10) \]

In the rank-2 symmetric tensor case, we shall use the vectorial gauge fixing function

\[ F_{k_1}^{\mu} = \partial_\mu h^{k_1} + k_1 \partial^{\mu} h , \quad (2.11) \]
Covariant Gauge Fixing

which depends on an additional gauge parameter $k_1$. Doing so, the gauge fixing term $S_{gf}$ finally reads

$$S_{gf} = -\frac{1}{2k} \int d^4x \left[ \partial_\mu h^{\mu\nu} + k_1 \partial^\nu h \right]^2. \tag{2.12}$$

The gauge fixing term (2.12) can be linearized by means of a Lagrange multiplier field, also called Nakanishi-Lautrup (NL) field [22, 23]. The reason to introduce a fourth field (besides the gauge field $h_{\mu\nu}$ and the ghost-antighost fields $c$ and $\bar{c}$) is to have a symmetry for the gauge fixed action $S$ (2.5). This new symmetry is the well known BRS symmetry [24]. Another good reason to linearize the gauge fixing term $S_{gf}$ is to render the Landau gauge $k = 0$ non-singular. The procedure is quite simple: the gauge fixing term (2.12) can be written as follows

$$S_{gf} = \int \mathcal{D}b \ exp i \left\{ \int d^4x \left[ b_a F_a[\Phi] + \frac{k}{2} b_a^2 \right] \right\}. \tag{2.13}$$

The field $b^\mu$ is the NL introduced to implement the gauge condition (2.11). It is a Lagrange multiplier, since it can be integrated away from the generating functional $Z$ by means of its EOM. Its vectorial nature is due to the particular form of the gauge fixing condition, which is vectorial. We stress that, since the physical properties of the theory depend on $Z$, the actions with and without the Lagrange multiplier $b$ describe the same physics. In other words, the gauge fixing terms (2.12) and (2.13) are completely equivalent.

### 2.2 Massive Propagators

As we have shown previously, the most general action\(^3\) in 4 dimensions describing a symmetric tensor field $h_{\mu\nu}$ and invariant under the diffeomorphism transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{2.14}$$

\(^3\)which can be directly derived by expanding the Einstein-Hilbert action to second order in $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as shown in Section 1.1.1

28
is
\[ S_{\text{inv}} = \int d^4x \left[ \frac{1}{2} h \partial^2 h - h_{\mu\nu} \partial^\mu \partial^\nu h - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} + h^{\mu\nu} \partial_\nu \partial^\rho h_{\mu\rho} + \frac{\Lambda}{2} h \right] \] (2.15)

where \( h = \eta^{\mu\nu} h_{\mu\nu} \) and \( \Lambda \) should be interpreted as the Cosmological Constant.

We shall first study this theory with \( \Lambda = 0 \). The cosmological constant will be discussed in a dedicated section.

As we discussed, our approach will be to add a gauge fixing term to this action in order to have a well-defined massless theory and only then add a mass-term. We think of it as a gauge fixed theory (for which a propagator can be found) onto which we introduce a mass term; what is really the point of this approach is evident as we consider the massless limit of the massive theory, which leads us to a perfectly defined, gauge fixed, theory, thus ensuring continuity in the mass parameter(s).

So the first step is to gauge fix the action. We shall use the two equivalent gauge fixing methods described in Section (2.1) in order to find the propagators of the theory. In addition, we shall also find the propagator directly by substituting the gauge condition
\[ \partial_\nu h^{\mu\nu} + k_1 \partial^\mu h = 0 \] (2.16)

into the action. We shall refer to this as ”direct gauge fixing”. The gauge fixed theories with “good” (in a sense which will be explained in what follows) propagators will be the starting point to build a new theory of MG.

Summarizing, in the following sections, the three approaches below will be presented:

- Lagrange multiplier gauge fixing;
- quadratic gauge fixing (the one given by (2.6));
- direct gauge fixing.
We stress that the third way of obtaining a gauge fixed action does not follow from the ΦΠ-procedure or any other formal argument. By fixing the gauge directly, we lose all symmetries and we do not expect to obtain physically viable results. In fact, we shall show that this is the case. Nevertheless, we believe it is instructive to develop this case together with the ones supported by the ΦΠ-procedure in order to compare the results and understand where the direct gauge fixing fails.

We shall now start with the computation of the propagators with the Lagrange multiplier gauge fixing.

### 2.2.1 Lagrange multiplier method

In this case, we implement the gauge fixing by adding to the invariant action $S_{\text{inv}}$ (2.15) the term (see (2.13))

$$S_{gf} = \int d^4x \left[ b^\mu \left( \partial^\nu h_{\mu \nu} + k_1 \partial_\mu h \right) + \frac{k}{2} b^\nu b_\nu \right],$$

(2.17)

where $k$ is the gauge parameter originating from the gauge fixing procedure discussed in (2.1) and for instance $k = 0$ gives the Landau gauge, while $k_1$ is a 2\textsuperscript{o} level gauge parameter, which indicates which specific gauge constraint we are enforcing with the function (2.11). A well-known case is $k_1 = -\frac{1}{2}$, which is called harmonic gauge.

The total gauge fixed action becomes

$$S = S_{\text{inv}} + S_{gf},$$

(2.18)

where $b^\nu$ is a vectorial Lagrange multiplier (or Lautrup-Nakanishi field) and the gauge condition we are enforcing on the tensor field in this way is (see Section (2.1))

$$\partial_\mu h^{\mu \nu} + k_1 \partial^\nu h = 0,$$

(2.19)
which guarantees that the theory has a propagator, or at least that the quadratic part of the action (2.18), which is the whole action in our case, can be inverted. The two things (existence of an inverse of the kinetic tensor and existence of a propagator) in general do not coincide because of the possible presence of tachyonic poles which should not appear in the physical propagators. We shall come back to this important point later on.

The next step is to add the most general mass term to the action, which is

\[ S_m = \int d^4x \frac{1}{2} \left[ m_1^2 h_{\mu\nu} h^{\mu\nu} + m_2^2 h^2 \right], \tag{2.20} \]

so that the action becomes

\[ S = S_{\text{inv}} + S_{gf} + S_m. \tag{2.21} \]

We are now ready to compute the propagators of the massive theory described by the action (2.21).

**Computation of the propagators**

We start by writing the action \( S \) (2.21) in momentum space

\[
S = \int d^4p \left\{ -\frac{1}{2} \tilde{h} p^2 \tilde{h} + \tilde{h}_{\mu\nu} p^\mu p^\nu \tilde{h} + \frac{1}{2} \tilde{h}^{\mu\nu} p^2 \tilde{h}_{\mu\nu} - \tilde{h}^{\mu\nu} p_\nu p_\rho \tilde{h}_{\mu\rho} + ight. \\
- i\tilde{\eta}^{\mu\nu} \left[ p^\nu \tilde{h}_{\mu\rho} + k_1 p_\mu \tilde{h} \right] + \frac{k_2}{2} \tilde{\eta}^{\mu\nu} \tilde{b}_\mu + \frac{1}{2} \left[ m_1^2 \tilde{h}_{\mu\nu} \tilde{h}^{\mu\nu} + m_2^2 \tilde{h}^2 \right] \right\}, \tag{2.22}
\]

where we decided to keep the same notation for the Fourier transformed tensor and trace since form now on we shall exclusively work in momentum space.
We can write this action in a compact form as follows

\[
S = \int d^4 p \left[ \tilde{h}_{\mu \nu} \Omega_{\mu \nu, \alpha \beta} \tilde{h}^{\alpha \beta} + \tilde{h}_{\mu \nu} \Lambda_{\mu \nu, \alpha} \tilde{b}^{\alpha} + \tilde{b}_{\mu} H^{\mu \alpha} \tilde{b}^{\alpha} \right] \tag{2.23}
\]

\[
= \int d^4 p \begin{bmatrix} \tilde{h}_{\mu \nu} \\ \tilde{b}_{\mu} \end{bmatrix} \begin{bmatrix} \Omega_{\mu \nu, \alpha \beta} & \Lambda_{\mu \nu, \alpha} \\ \Lambda^*_{\mu, \alpha \beta} & H_{\mu \alpha} \end{bmatrix} \begin{bmatrix} \tilde{h}^{\alpha \beta} \\ \tilde{b}^{\alpha} \end{bmatrix}, \tag{2.24}
\]

where

\[
\Omega_{\mu \nu, \alpha \beta} = \frac{1}{2} \left( m_2^2 - p^2 \right) \eta_{\mu \nu} \eta_{\alpha \beta} + \frac{1}{4} (p^2 + m_1^2) (\eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\nu \alpha} \eta_{\mu \beta}) + \frac{1}{2} (\eta_{\mu \nu} p_{\alpha} p_{\beta} + \eta_{\alpha \beta} p_{\mu} p_{\nu}) - \frac{1}{4} (e_{\mu \alpha} \eta_{\nu \beta} + e_{\nu \alpha} \eta_{\mu \beta} + e_{\mu \beta} \eta_{\nu \alpha} + e_{\nu \beta} \eta_{\mu \alpha}) p^2 \tag{2.25}
\]

\[
\Lambda_{\mu \nu, \alpha} = -i \left[ \frac{1}{2} (\eta_{\mu \alpha} p_{\nu} + \eta_{\nu \alpha} p_{\mu}) + k_1 \eta_{\mu \nu} p_{\alpha} \right] \tag{2.26}
\]

\[
H_{\mu \nu} = \frac{k}{2} \eta_{\mu \nu} \tag{2.27}
\]

Now, our aim is to obtain the propagators of this gauge fixed theory by inverting the momentum space kinetic operator

\[
\begin{bmatrix} \Omega_{\mu \nu, \alpha \beta} & \Lambda_{\mu \nu, \alpha} \\ \Lambda^*_{\mu, \alpha \beta} & H_{\mu \alpha} \end{bmatrix}. \tag{2.28}
\]

It is convenient to introduce the rank-2 projectors

\[
e_{\mu \nu} = \frac{p_{\mu} p_{\nu}}{p^2} \quad \text{and} \quad d_{\mu \nu} = \eta_{\mu \nu} - e_{\mu \nu} \tag{2.29}
\]

which are idempotent and orthogonal

\[
e_{\mu \lambda} e_{\lambda \mu} = e_{\mu \nu} \quad d_{\mu \lambda} d_{\lambda \mu} = d_{\mu \nu} \quad e_{\mu \lambda} d_{\lambda \nu} = 0 \tag{2.30}
\]

With these rank-2 projectors we can construct a basis of rank 4 tensors with the symmetry properties

\[
X_{\mu \nu, \alpha \beta} = X_{\nu \mu, \alpha \beta} = X_{\mu \alpha \beta \nu} = X_{\alpha \beta \mu \nu} \tag{2.31}
\]
The choice of basis we adopt is

\[ A_{\mu\nu,\alpha\beta} = \frac{d_{\mu\nu}d_{\alpha\beta}}{3} \quad (2.32) \]

\[ B_{\mu\nu,\alpha\beta} = \epsilon_{\mu\nu}\epsilon_{\alpha\beta} \quad (2.33) \]

\[ C_{\mu\nu,\alpha\beta} = \frac{1}{2} \left( d_{\mu\alpha}d_{\nu\beta} + d_{\mu\beta}d_{\nu\alpha} - \frac{2}{3}d_{\mu\nu}d_{\alpha\beta} \right) \quad (2.34) \]

\[ D_{\mu\nu,\alpha\beta} = \frac{1}{2} \left( d_{\mu\alpha}\epsilon_{\nu\beta} + d_{\mu\beta}\epsilon_{\nu\alpha} + \epsilon_{\mu\alpha}d_{\nu\beta} + \epsilon_{\mu\beta}d_{\nu\alpha} \right) \quad (2.35) \]

\[ E_{\mu\nu,\alpha\beta} = \eta_{\mu\nu}\eta_{\alpha\beta} \quad (2.36) \]

This set of tensors has the following properties:

- They sum to the identity:

\[ A_{\mu\nu,\alpha\beta} + B_{\mu\nu,\alpha\beta} + C_{\mu\nu,\alpha\beta} + D_{\mu\nu,\alpha\beta} = \mathcal{I}_{\mu\nu,\alpha\beta} \quad (2.37) \]

where \( \mathcal{I} \) is the rank-4 tensor identity

\[ \mathcal{I}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \quad (2.38) \]

- they are idempotent

- contractions between \( A, B, C \) and \( D \) vanish (i.e. \( A_{\mu\nu,\alpha\beta}B_{\alpha\beta\rho\sigma} = 0 \), etc.)

- the following relations hold

\[ A_{\mu\nu,\alpha\beta}E_{\alpha\beta}^{\rho\sigma} = \frac{d^{\mu\nu}\eta_{\rho\sigma}}{4} \quad (2.39) \]

\[ B_{\mu\nu,\alpha\beta}E_{\alpha\beta}^{\rho\sigma} = \frac{\epsilon^{\mu\nu}\eta_{\rho\sigma}}{4} \quad (2.40) \]

\[ C_{\mu\nu,\alpha\beta}E_{\alpha\beta}^{\rho\sigma} = 0 \quad (2.41) \]

\[ D_{\mu\nu,\alpha\beta}E_{\alpha\beta}^{\rho\sigma} = 0 \quad (2.42) \]
The tensors $\Omega_{\mu\nu,\alpha\beta}$ (2.25), $\Lambda_{\mu\nu,\alpha}$ (2.26) and $H_{\mu\alpha}$ (2.27) can be written in terms of the rank-2 and rank-4 projectors [(2.29),(2.32)–(2.36)] as:

$$\Omega_{\mu\nu,\alpha\beta} = tA_{\mu\nu,\alpha\beta} + uB_{\mu\nu,\alpha\beta} + vC_{\mu\nu,\alpha\beta} + zD_{\mu\nu,\alpha\beta} + wE_{\mu\nu,\alpha\beta},$$

(2.43)

where

$$t = -p^2 + \frac{1}{2}m_1^2$$

(2.44)

$$u = \frac{1}{2}m_1^2$$

(2.45)

$$v = \frac{1}{2}(p^2 + m_1^2)$$

(2.46)

$$z = \frac{1}{2}m_1^2$$

(2.47)

$$w = 2m_2^2,$$

(2.48)

and

$$\Lambda_{\mu\nu,\alpha} = -i\left[\frac{1}{2}(d_{\mu\alpha}p_\nu + d_{\nu\alpha}p_\mu) + k_1d_{\mu\nu}p_\alpha + (1 + k_1)e_{\mu\nu}p_\alpha\right]$$

(2.49)

$$H_{\mu\nu} = \frac{k}{2}(d_{\mu\nu} + e_{\mu\nu}).$$

(2.50)

We now define the propagators matrix

$$\begin{bmatrix} G_{\alpha\beta,\rho\sigma} & G_{\alpha\beta,\rho} \\ G_{\rho\sigma,\alpha}^* & G_{\alpha\rho} \end{bmatrix},$$

(2.51)

where

$$G_{\mu\nu,\rho\sigma}(p) \equiv \langle h_{\mu\nu}h_{\rho\sigma}\rangle(p)$$

(2.52)

$$G_{\mu\nu,\rho}(p) \equiv \langle h_{\mu\nu}b_{\rho}\rangle(p)$$

(2.53)

$$G_{\mu\nu}(p) \equiv \langle b_{\mu}b_{\nu}\rangle(p).$$

(2.54)
We then parametrize these tensors in the same way as for the kinetic tensors:

\[
G_{\mu\nu,\alpha\beta} = \hat{t} A_{\mu\nu,\alpha\beta} + \hat{u} B_{\mu\nu,\alpha\beta} + \hat{v} C_{\mu\nu,\alpha\beta} + \hat{z} D_{\mu\nu,\alpha\beta} + \hat{w} E_{\mu\nu,\alpha\beta}
\]

\[
G_{\mu\nu,\alpha} = i \left[ f(d_{\mu\alpha}p_\nu + d_{\nu\alpha}p_\mu) + gd_{\mu\nu}p_\alpha + le_{\mu\nu}p_\alpha \right]
\]

\[
G_{\mu\nu} = rd_{\mu\nu} + se_{\mu\nu}
\]

where \( \hat{t}, \hat{u}, \hat{v}, \hat{z}, \hat{w}, f, g, l, r \) and \( s \) are real constant parameters to be determined by imposing the following defining condition for the propagators matrix

\[
\begin{bmatrix}
\Omega_{\mu\nu,\alpha\beta} & \Lambda_{\mu\nu,\alpha} \\
\Lambda^*_{\mu,\alpha\beta} & H_{\mu\alpha}
\end{bmatrix}
\begin{bmatrix}
G^{\alpha\beta,\rho\sigma} & G^{\alpha\beta,\rho} \\
G^{*\rho\sigma,\alpha} & G^{*\alpha\rho}
\end{bmatrix}
= \begin{bmatrix}
I_{\mu\nu} & 0 \\
0 & \eta_{\mu\rho}
\end{bmatrix}.
\]

We shall now develop these equations in detail to find solutions for the ten coefficients previously defined.

We can now write equation (2.58) in components

\[
\begin{align}
\Omega_{\mu\nu,\alpha\beta} G^{\alpha\beta,\rho\sigma} + \Lambda_{\mu\nu,\alpha} G^{*\alpha,\rho\sigma} &= I_{\mu\nu}^\rho\sigma \\
\Omega_{\mu\nu,\alpha\beta} G^{\alpha\beta,\rho} + \Lambda_{\mu\nu,\alpha} G^{\alpha,\rho} &= 0 \\
\Lambda^*_{\mu,\alpha\beta} G^{*\alpha,\rho\sigma} + H_{\mu\alpha} G^{*\alpha\beta,\rho} &= 0 \\
\Lambda^*_{\mu,\alpha\beta} G^{*\alpha,\rho} + H_{\mu\alpha} G^{\alpha,\rho} &= \eta_{\mu}^\rho.
\end{align}
\]

Using the orthogonality relations between \( A, B, C, D \) and \( E \), we rewrite (2.59)-(2.62) in terms of the coefficients defined in (2.55), (2.56) and (2.57).
\( (2.59) \implies t \hat{A}_{\mu \nu}^{\rho \sigma} + u \hat{B}_{\mu \nu}^{\rho \sigma} + v \hat{C}_{\mu \nu}^{\rho \sigma} + z \hat{D}_{\mu \nu}^{\rho \sigma} + \)

\[
\frac{t}{4} \hat{d}_{\mu \nu}^{\eta \rho} + u \frac{e_{\mu \nu}^{\eta \rho}}{4} + w \eta_{\mu \nu}^{\eta \rho} + w \hat{u}_{\mu \nu}^{\eta \rho} + w \hat{v}_{\mu \nu}^{\eta \rho} + w \hat{z}_{\mu \nu}^{\eta \rho} - \]

\[
\left[ \frac{1}{2} (d_{\mu}^{\alpha} p_{\nu} + d_{\nu}^{\alpha} p_{\mu}) + k_1 p_{\alpha} d_{\mu \nu} + (1 + k_1) p_{\alpha} e_{\mu \nu} \right] \left[ f (d_{\alpha}^{\beta} p_{\rho} + d_{\rho}^{\beta} p_{\alpha}) + g p_{\alpha} d_{\mu \nu} + l p_{\alpha} e_{\mu \nu} \right] =
\]

\[
\frac{1}{2} (\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho}) . \quad (2.63)
\]

\( (2.60) \implies \left[ t \frac{d_{\mu \nu} d_{\alpha \beta}}{4} + u e_{\mu \nu} e_{\alpha \beta} + \frac{v}{2} \left( d_{\mu \alpha} d_{\nu \beta} + d_{\nu \alpha} d_{\mu \beta} - \frac{2}{3} d_{\mu \nu} d_{\alpha \beta} \right) + \)

\[
\frac{z}{2} \left( d_{\mu \alpha} e_{\nu \beta} + d_{\nu \alpha} e_{\mu \beta} + e_{\mu \alpha} d_{\nu \beta} + e_{\nu \alpha} d_{\mu \beta} + w \eta_{\mu \nu} \eta_{\alpha \beta} \right) \right] \left[ f (d_{\alpha}^{\beta} p_{\sigma} + d_{\beta}^{\rho} p_{\alpha}) + g p_{\alpha} d_{\mu \nu} + l p_{\alpha} e_{\mu \nu} \right] - \]

\[
\left[ \frac{1}{2} (d_{\mu \alpha} p_{\nu} + d_{\nu \alpha} p_{\mu}) + k_1 p_{\alpha} d_{\mu \nu} + (1 + k_1) p_{\alpha} e_{\mu \nu} \right] (r d_{\nu}^{\rho} + s e_{\nu}^{\alpha}) = 0 \implies \]

\[
\implies f z (d_{\mu}^{\rho} p_{\nu} + d_{\nu}^{\rho} p_{\mu}) + g t p_{\rho} d_{\mu \nu} + l u p_{\rho} e_{\mu \nu} - \frac{r}{2} (d_{\mu}^{\rho} p_{\nu} + d_{\nu}^{\rho} p_{\mu}) + \]

\[
\frac{w}{4} \left[ 3g + l \right] p_{\rho} \eta_{\mu \nu} - s \left[ k_1 p_{\alpha} d_{\mu \nu} + (1 + k_1) p_{\alpha} e_{\mu \nu} \right] = 0 . \quad (2.64)
\]

\( (2.61) \implies \left[ \frac{1}{2} (d_{\mu \alpha} p_{\beta} + d_{\nu \beta} p_{\alpha}) + k_1 d_{\alpha \beta} p_{\mu} + (1 + k_1) e_{\alpha \beta} p_{\mu} \right] \times \]

\[
\left[ \frac{k}{2} (d_{\mu}^{\alpha} + e_{\mu}^{\alpha}) \right] = 0 \implies \]

36
⇒ \(i k_1 p_\mu d_{\rho\sigma} + \bar{u}(1 + k_1)p_\mu e_{\rho\sigma} + \frac{\hat{z}}{2} (d_{\mu\rho} p_\sigma + d_{\mu\sigma} p_\rho) + \frac{\hat{w}}{2} (1 + 4k_1) \eta_{\rho\sigma} p_\mu - \)

\[
\frac{k}{2} \left[ f(d_{\mu\rho} p_\sigma + d_{\mu\sigma} p_\rho) + gp_\mu d_{\rho\sigma} + lp_\mu e_{\rho\sigma} \right] = 0 \quad (2.65)
\]

\[
(2.62) \quad \Rightarrow \left[ \frac{1}{2} (d_{\mu\alpha} p_\beta + d_{\nu\beta} p_\alpha) + k_1 d_{\alpha\beta} p_\mu + (1 + k_1) e_{\alpha\beta} p_\mu \right] \left[ f(d^{\rho\alpha} p^\beta d^{\beta\alpha} p^\rho) + gp^\rho d^{\alpha\beta} + lp^\rho e^{\alpha\beta} \right] + \\
\frac{k}{2} (d_{\mu\alpha} + e_{\mu\alpha})(rd^{\alpha\rho} + se^{\alpha\rho}) = \delta_\mu^\rho \quad \Rightarrow \\
\Rightarrow \quad -fp^2 d_{\mu\rho} - 3gk_1 p^2 e_{\mu\rho} - l(1 + k_1)p^2 e_{\mu\rho} + \frac{k}{2} (rd_{\mu\rho} + se_{\mu\rho}) = \eta_{\mu\rho} \quad . \quad (2.66)
\]

Thus, imposing equation (2.58) results into a set of 14 conditions (obtained by solving (2.63)-(2.66) for every linearly independent term individually) on the propagator parameters \((\hat{t}, \hat{u}, \hat{v}, \hat{z}, \hat{w}, f, g, l, r, s)\). Those equations can be divided as follows
Covariant Gauge Fixing

• nine equations for \((\hat{t}, \hat{u}, \hat{v}, \hat{w}, g, l, s)\)

\[
\frac{\dot{t}}{4}w + \frac{\dot{t}}{4}w + \frac{\dot{t}}{3} - gk_1p^2 = 0 \tag{2.67}
\]

\[
\frac{\dot{t}}{3} - \frac{4}{3} + gp^2 - lp^2 + uu = 0 \tag{2.68}
\]

\[
\frac{\dot{t}}{4}w + \frac{\dot{t}}{4}w - k_1lp^2 = 0 \tag{2.69}
\]

\[
\frac{\dot{t}}{4}w + \frac{\dot{u}}{4} - g(k_1 + 1)p^2 = 0 \tag{2.70}
\]

\[
lu - gt - s = 0 \tag{2.71}
\]

\[
\frac{w(3g + l)}{4} + gt - k_1s = 0 \tag{2.72}
\]

\[
\frac{4k_1 + 1}{4} + \frac{gk}{2} + k_1\dot{t} = 0 \tag{2.73}
\]

\[
\frac{4k_1 + 1}{4} + (k_1 + 1)u - \frac{kl}{2} = 0 \tag{2.74}
\]

\[
k_1s - 6gk_1p^2 - 2(k_1 + 1)lp^2 = 2. \tag{2.75}
\]

• four equations for \((\hat{z}, f, r)\)

\[
z\ddot{z} - fp^2 = 1 \tag{2.76}
\]

\[
fz - \frac{r}{2} = 0 \tag{2.77}
\]

\[
\dot{z} - kf = 0 \tag{2.78}
\]

\[
kr - 2fp^2 = 2. \tag{2.79}
\]

• one equation for \(\hat{v}\)

\[
v\dot{v} = 1. \tag{2.80}
\]
This last equation is trivially solved by
\[ \hat{v} = \frac{1}{v} = \frac{2}{p^2 + m_1^2}. \] (2.81)

Solving the four equations for \((\hat{z}, \hat{f}, r)\) one obtains
\[ \hat{z} = \frac{2k}{km_1^2 - 2p^2}, \quad f = \frac{2}{km_1^2 - 2p^2}, \quad r = \frac{2m_1^2}{km_1^2 - 2p^2}, \] (2.82)

which are solutions for all values of \(k, k_1, m_1^2\) and \(m_2^2\). We can here notice that these coefficients do not depend on the mass parameter \(m_2\). This is an important remark we shall comment on later.

Turning to the set of equations for \((\hat{t}, \hat{u}, \hat{v}, \hat{w}, g, l, s)\), we first start by looking at the solutions of two subsets of the nine equations.

The first subset is (2.71), (2.72), (2.75)
The solutions that this system gives for \((g, l, s)\) are:
\[ g = \frac{4 (m_2^2 - k_1 m_1^2)}{\text{DN}(m_1, m_2, k, k_1, p^2)} \] (2.83)
\[ l = -\frac{4 ((k_1 + 1) (m_1^2 - 2p^2) + 3m_2^2)}{\text{DN}(m_1, m_2, k, k_1, p^2)} \] (2.84)
\[ s = -\frac{2(m_1^4 - 2m_1^2p^2 + m_2^2(4m_1^2 - 2p^2))}{\text{DN}(m_1, m_2, k, k_1, p^2)} \] (2.85)

where
\[ \text{DN}(m_1, m_2, k, k_1, p^2) = -8(1 + k_1)^2p^4 + 2 \left[ 2 \left( 4k_1^2 + 2k_1 + 1 \right) m_1^2 + (km_1^2 + (6 + k)m_2^2) \right] p^2 - k m_1^2 (m_1^2 + 4m_2^2). \] (2.86)

The second subset we solve is that given by Eqs. (2.67), (2.70), (2.73), from which we obtain solutions for \((\hat{t}, g, \hat{w})\), namely:
\[ t = \frac{2km_1^2 - 32k_1^2 p^2 - 40k_1p^2 - 8p^2}{\text{dn}(m_1, m_2, k, k_1, p^2)} \] (2.87)

\[ g = -\frac{4 (k_1 (4m_2^2 - m_1^2) + m_2^2)}{\text{dn}(m_1, m_2, k, k_1, p^2)} \] (2.88)

\[ \dot{w} = -\frac{8km_2^2 + 32k_1^2 p^2 + 32k_1p^2}{\text{dn}(m_1, m_2, k, k_1, p^2)} \] (2.89)

where

\[ \text{dn}(m_1, m_2, k, k_1, p^2) = 8(1 + k_1)^2 p^4 + \left[-4 (4k_1^2 + 2k_1 + 1) m_1^2 - 12(4k_1 + 1)m_2^2 + 6km_2^2 - 2km_1^2 \right] p^2 + km_1^4 \] (2.90)

From (2.83) and (2.87) we obtain \( g \), but these two solutions are not trivially the same. To find a proper propagator solution we need to impose the equality of the two solutions for \( g \), so

\[ \frac{4 (m_2^2 - k_1m_1^2)}{\text{DN}(m_1, m_2, k, k_1, p^2)} = -\frac{4 (k_1 (4m_2^2 - m_1^2) + m_2^2)}{\text{dn}(m_1, m_2, k, k_1, p^2)} \] (2.91)

After some algebraic manipulations of this last equation we obtain an equation for a 4th degree polynomial in \( p \) of the kind

\[ ap^4 + bp^2 + c = 0 \] (2.92)

where

\[ a = -8k_1(1 + k_1)^2 m_2^2 \] (2.93)

\[ b = -(4k_1 + 1) k m_2^4 m_1^2 \] (2.94)

\[ c = 2(1 + k_1) (2k_1(4k_1 + 1)m_2^2 + km_2^2) m_2^2 \] (2.95)
and if the polynomial has to be zero for every value of $p^2$, then the three coefficients $a$, $b$ and $c$ all need to be zero individually. We remark that while looking for the possible solutions of Eq. (2.92) we are only interested in those which have at least one non-vanishing mass parameter since we are studying a massive theory. With this requirement the relevant solutions are

<table>
<thead>
<tr>
<th></th>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
<th>Solution 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>any</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$k$</td>
<td>any</td>
<td>0</td>
<td>0</td>
<td>any</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$\neq 0$</td>
<td>any</td>
<td>any</td>
<td>0</td>
</tr>
<tr>
<td>$m_2$</td>
<td>0</td>
<td>any</td>
<td>any</td>
<td>$\neq 0$</td>
</tr>
</tbody>
</table>

Let us consider first solution 1: $m_2 = 0$ and $m_1 \neq 0$ with arbitrary $k$ and $k_1$.

By substituting (2.83) and (2.87) (with $m_2 = 0$) in the remaining three equations, (2.68), (2.69) and (2.74), we obtain

$$
\hat{u} = \frac{2 k (m_1^2 - 2p^2) - 8k_1(4k_1 + 1)p^2}{km_1^3(m_1^2 - 2p^2) - 4 (4k_1^2 + 2k_1 + 1) m_1^2 p^2 + 8(k_1 + 1)^2 p^4}
$$

We now summarize the coefficients obtained for solution 1 in order of appearance in (2.55), (2.56) and (2.57)
\( m_2 = 0, \ m_1 \neq 0, \ k \) and \( k_1 \) arbitrary:

\[
\hat{t} = \frac{2km_1^2 - 8(1 + k_1)(1 + 4k_1)p^2}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.97}
\]

\[
\hat{u} = -\frac{-2km_1^2 + 8k_1(1 + 4k_1)p^2 + 4kp^2}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.98}
\]

\[
\hat{v} = \frac{2}{p^2 + m_1^2} \tag{2.99}
\]

\[
\hat{z} = \frac{2k}{km_1^2 - 2p^2} \tag{2.100}
\]

\[
\hat{w} = \frac{32k_1(1 + k_1)p^2}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.101}
\]

\[
f = \frac{2}{km_1^2 - 2p^2} \tag{2.102}
\]

\[
g = \frac{4k_1m_1^2}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.103}
\]

\[
l = \frac{4(1 + k_1)(m_1^2 - 2p^2)}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.104}
\]

\[
r = \frac{2m_1^2}{km_1^2 - 2p^2} \tag{2.105}
\]

\[
s = \frac{2(m_1^4 - 2m_1^2p^2)}{km_1^2(m_1^2 - 2p^2) - 4(4k_1^2 + 2k_1 + 1)m_1^2p^2 + 8(1 + k_1)^2p^4} \tag{2.106}
\]

We now turn to solution 2. This time we obtain \( \hat{u} = 0 \), so the coefficients of the propagators are
2.2 – Massive Propagators

$k = 0, k_1 \neq 0, m_1$ and $m_2$ arbitrary:

\[ \hat{i} = \frac{2}{-2p^2 + m_1^2 - 3m_2^2} \]  \hspace{1cm} (2.107)

\[ \hat{u} = 0 \]  \hspace{1cm} (2.108)

\[ \hat{v} = \frac{2}{p^2 + m_1^2} \]  \hspace{1cm} (2.109)

\[ \hat{z} = \hat{w} = 0 \]  \hspace{1cm} (2.110)

\[ f = -\frac{1}{p^2} \]  \hspace{1cm} (2.111)

\[ g = \frac{m_2^2}{p^2 [-2p^2 + m_1^2 - 3m_2^2]} \]  \hspace{1cm} (2.112)

\[ l = -\frac{1}{p^2} \]  \hspace{1cm} (2.113)

\[ r = -\frac{m_1^2}{p^2} \]  \hspace{1cm} (2.114)

\[ s = \frac{-m_1^2(m_1^2 + 4m_2^2) + 2(m_1^2 + m_2^2)p^2}{2p^2 [-2p^2 + m_1^2 - 3m_2^2]} \]  \hspace{1cm} (2.115)

Moving to solution 3 and solution 4, we find that when solving (2.68), (2.69) and (2.74) with the parameters of solution 4 there is a solution for $\hat{u}$ only if we set $k = 0$. Therefore, we can regard solution 4 as a special case of solution 3 (i.e. $k_1 = -1, k = 0$ and $m_1 = 0$). We thus only need to study solution 3, the one with $m_1 = 0$. This gives the following coefficients for the propagator.
Covariant Gauge Fixing

\( k = 0, \ k_1 = -1, \ m_1 \) and \( m_2 \) arbitrary:

\[
\hat{t} = 0 \tag{2.116}
\]

\[
\hat{u} = \frac{2}{m_1^2 + m_2^2} \tag{2.117}
\]

\[
\hat{v} = \frac{2}{p^2 + m_1^2} \tag{2.118}
\]

\[\hat{z} = \hat{w} = 0 \tag{2.119}\]

\[f = -\frac{1}{p^2} \tag{2.120}\]

\[g = \frac{1}{3p^2} \tag{2.121}\]

\[
l = -\frac{m_2^2}{(m_1^2 + m_2^2)p^2} \tag{2.122}\]

\[r = -\frac{m_1^2}{p^2} \tag{2.123}\]

\[s = -\frac{m_1^2(m_1^2 + 4m_2^2) + 2(m_1^2 + m_2^2)p^2}{6p^2(m_1^2 + m_2^2)} \tag{2.124}\]

From the denominators of \( \hat{u}, \ l \) and \( s \) we can see that this solution diverges when \( m_1 \to 0 \) and \( m_2 \to 0 \), so solution 3 yields a propagator without a good massless limit, which therefore we must discard.

Summarizing, we found two possible solutions for the propagator: one with \( m_2 = 0, \ m_1 \neq 0 \) and arbitrary \( k \) and \( k_1 \) (solution 1), and another with \( k_1 = k = 0 \) and arbitrary \( m_1 \) and \( m_2 \) (solution 2). This is a remarkable result: we obtained a solution for the propagator (solution 1) which does not include the FP tuning of
the mass parameters \( (m_1^2 = -m_2^2) \) and is valid for any value of the gauge fixing parameters \( k \) and \( k_1 \). A second solution (solution 2) contains the FP case, but is obtained for particular values of the gauge fixing parameters. We shall further comment this when we discuss the DOFs propagated by this theory.

We also point out that many of the coefficients obtained have poles dependent on the gauge fixing parameters and are, moreover, potentially tachyonic. We shall address these issues in a dedicated section.

### 2.2.2 Quadratic gauge fixing method

In the previous section we implemented the gauge fixing on the action through the Lagrange multiplier method. The aim of this section is to compute the propagators, but with a quadratic gauge fixing term, which is the one obtained directly from the \( \Phi\Pi \)-procedure described in Section (2.1).

In order to implement the gauge constraint

\[
\partial_{\mu}h^{\mu\nu} + k_1 \partial^n h = 0 \tag{2.125}
\]

on the invariant action we add, as anticipated before, the term

\[
S'_{gf} = -\frac{1}{2k} \int d^4x \left[ \partial_{\mu}h^{\mu\nu} + k_1 \partial^n h \right]^2, \tag{2.126}
\]

so that the whole action, with the mass term (2.20), becomes

\[
S = S_{inv} + S'_{gf} + S_m. \tag{2.127}
\]

The gauge fixing term \( S'_{gf} \) (2.126) can be obtained from \( S_{gf} \) (2.17) by eliminating the Lagrange multiplier \( b^\mu \) through its EOM

\[
k\tilde{b}^\mu = ip_\nu \tilde{h}^{\nu\mu} + ik_1 p^\mu \tilde{h}. \tag{2.128}
\]
which corresponds to integrating $b^\mu$ away from the generating functional $Z[J]$ (2.3).

We stress that the linear action term $S_{gf}$ (2.17) follows from the need of a symmetry (the BRS symmetry) which replaces the diffeomorphisms-gauge symmetry (2.14). In addition, the NL formulation of the gauge fixing term renders non-singular the Landau gauge choice $k = 0$.

**Computation of the propagator**

The action (2.127) can be written in momentum space as

$$S = \int d^4p \ \tilde{h}_{\mu \nu} \Omega^{\mu \nu, \alpha \beta} \hat{h}_{\alpha \beta},$$

(2.129)

where the kinetic operator $\Omega$ is

$$\Omega_{\mu \nu, \alpha \beta} = \frac{1}{2} \left[ m^2 - \left( 1 + \frac{k^2}{k} \right) p^2 \right] \eta_{\mu \nu} \eta_{\alpha \beta} + \frac{1}{2} \left( 1 - \frac{k_1^2}{k} \right) (\eta_{\mu \nu} e_{\alpha \beta} + \eta_{\alpha \beta} e_{\mu \nu}) p^2 +$$

$$\frac{1}{2}(p^2 + m_1^2) \Omega_{\mu \nu, \alpha \beta} - \left( 1 + \frac{k}{2k_1} \right) \frac{1}{4} (e_{\mu \alpha} \eta_{\nu \beta} + e_{\nu \alpha} \eta_{\mu \beta} + e_{\mu \beta} \eta_{\nu \alpha} + e_{\nu \beta} \eta_{\mu \alpha}) p^2,$$

(2.130)

and $\Omega$ is the rank-4 tensor identity (2.38).

Now, the propagator $G_{\alpha \beta, \rho \sigma}$ must satisfy the equation

$$\Omega_{\mu \nu}^{\alpha \beta} G_{\alpha \beta, \rho \sigma} = \Omega_{\mu \nu, \rho \sigma}.$$  

(2.131)

Without showing the explicit calculations, we obtain the propagator

$$G_{\mu \nu, \alpha \beta} = \hat{A}_{\mu \nu, \alpha \beta} + \hat{B}_{\mu \nu, \alpha \beta} + \hat{C}_{\mu \nu, \alpha \beta} + \hat{D}_{\mu \nu, \alpha \beta} + \hat{E}_{\mu \nu, \alpha \beta},$$

(2.132)

where
\[
\hat{t} = \frac{2(1 + k_1)(1 + 4k_1)p^2 - 2k(m_1^2 + 4m_2^2)}{\text{DN}(m_1, m_2, k, k_1, p^2)} \\
\hat{u} = \frac{2[k_1(1 + 4k_1) + 2k]p^2 - 2k(m_1^2 + 4m_2^2)}{\text{DN}(m_1, m_2, k, k_1, p^2)} \\
\hat{v} = \frac{2}{p^2 + m_1^2} \\
\hat{z} = \frac{-4k}{p^2 - 2km_1^2} \\
\hat{w} = \frac{8km_2^2 - 8k(1 + k_1)p^2}{\text{DN}(m_1, m_2, k, k_1, p^2)},
\]

with

\[
\text{DN}(m_1, m_2, k, k_1, p^2) = -2(1 + k_1)^2p^4 + (1 + 2k_1 + 4k_1^2 + 2k)m_1^2 + (3 + 2k)m_2^2 p^2 - 4km_1^2m_2^2 - km_1^4.
\]

Again, the poles of this propagator are gauge dependent and potentially tachyonic, but, as previously mentioned, we shall discuss and solve this issue later.

### 2.2.3 Direct gauge Fixing method

We now come to the direct gauge fixing. We remind the reader that this approach is not equivalent to the other two discussed in Sections (2.2.1) and (2.2.2). In fact, it does not originate from a formal field theoretical argument, like the \(\Phi\Pi\)-procedure. Instead, it should be regarded as a way for us to check that this way of fixing the
Covariant Gauge Fixing

gauge leads to unwanted results. Nevertheless, we proceed by plugging the gauge fixing condition (2.16) directly into the action $S_{inv} (2.15)$. No additional action term is added this time. Doing so, we obtain the action

$$S_{direct} = \int d^4x \left[ \left( \frac{1}{2} + k_1(1 + k_1) \right) \tilde{h} \partial^2 \tilde{h} - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} \right].$$  \hspace{1cm} (2.138)

We now define the parameter

$$\beta = 2k_1(1 + k_1) + 1 \in \left[ \frac{1}{2}, +\infty \right),$$  \hspace{1cm} (2.139)

The harmonic gauge, for instance, corresponds to $\beta = \frac{1}{2}$.

We can write the action (2.138) more cleanly as

$$S_{direct} = \int d^4x \left[ \frac{\beta}{2} \tilde{h} \partial^2 \tilde{h} - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} \right],$$  \hspace{1cm} (2.140)

and after adding the mass term (2.20)

$$S = S_{direct} + S_m = \int d^Dx \left[ \frac{1}{2} \tilde{h} \left( m_2^2 + \beta \partial^2 \right) \tilde{h} - \frac{1}{2} h^{\mu\nu} (\partial^2 - m_1^2) h_{\mu\nu} \right].$$  \hspace{1cm} (2.141)

Computation of the propagator

The action (2.141) in momentum space is

$$S = \int d^4p \left[ \frac{1}{2} \tilde{h} \left( m_2^2 - \beta p^2 \right) \tilde{h} + \frac{1}{2} \tilde{h}^{\mu\nu} \left( p^2 + m_1^2 \right) \tilde{h}_{\mu\nu} \right],$$  \hspace{1cm} (2.142)

and can be written as

$$S = \int d^4p \tilde{h}_{\mu\nu} \Omega^{\mu\nu,\alpha\beta} \tilde{h}_{\alpha\beta},$$  \hspace{1cm} (2.143)

where the kinetic operator $\Omega$ is

$$\Omega_{\mu\nu,\alpha\beta} = \frac{1}{2} (m_2^2 - \beta p^2) \eta_{\mu\nu} \eta_{\alpha\beta} + \frac{1}{2} (p^2 + m_1^2) \mathcal{I}_{\mu\nu,\alpha\beta} \hspace{1cm} (2.144)$$
The kinetic tensor is easily invertible and the propagator is

\[ \langle h_{\mu\nu} h_{\alpha\beta} \rangle (p) = G_{\mu\nu,\alpha\beta} = \]

\[ = \frac{2 \beta p^2 - m_2^2}{p^2 + m_1^2 (1 - 4\beta)p^2 + m_1^4 + 4m_2^2 \eta_{\mu\nu} \eta_{\alpha\beta} + \frac{2}{p^2 + m_1^2} I_{\mu\nu,\alpha\beta}} \quad (2.145) \]

The \( \beta \)-dependence in the pole of the first term and its potential tachyonic nature will be addressed further on.
2.3 Degrees of Freedom

As explained in [14,15], a realistic theory of MG needs 5 propagating massive DOFs. Since our end-goal is to build a MG theory, we shall now consider the EOMs originating from the three actions used in Sections (2.2.1), (2.2.2) and (2.2.3). This is because EOMs can give us insight on the DOFs propagated by these theories and on how these DOFs manifest themselves.

In order for our theory to be consistent, we must require that the propagating DOFs should be independent from the gauge fixing parameters, while they can of course depend on the mass parameters.

We will see that the solutions meeting this requirement also satisfy the necessary (although not sufficient) condition to have 5 DOFs.

We shall start again from the actions (2.22), (2.129) and (2.142) and derive the EOMs for each method.

2.3.1 Lagrange multiplier method

Let us write the EOMs obtained from the action (2.22)

\[ \frac{\delta S}{\delta \tilde{h}_{\mu\nu}} = -\eta^{\mu\nu} p^2 \tilde{h} + p^\mu p^\nu \tilde{h} + \eta^{\mu\nu} p^\rho p^\beta \tilde{h}_{\alpha\beta} + p^2 \tilde{h}_{\mu\nu} - p^\mu p^\alpha \tilde{h}_\alpha^\nu - p^\nu p^\alpha \tilde{h}_\alpha^\mu + \frac{i}{2} [p^\rho \tilde{b}^\nu + p^\rho \tilde{b}^\mu] + ik_1 \eta^{\mu\nu} p^\alpha \tilde{b}_\alpha + m_1^2 \tilde{h} + m_2^2 \eta^{\mu\nu} \tilde{h} = 0 \]  

(2.146)

\[ \frac{\delta S}{\delta \tilde{b}_\mu} = -ip_\nu \tilde{h}^{\nu\mu} - ik_1 p^\rho \tilde{h}_\rho + \bar{\tilde{b}}^{\mu} = 0. \]  

(2.147)

Saturating (2.146) with \( \eta_{\mu\nu} \) and \( \epsilon_{\mu\nu} \), we obtain
\[ \eta_{\mu\nu} : \left[ 2p^2 - (m_1^2 + 4m_2^2) \right] \tilde{h} - 2p^2 e_{\mu\nu} \tilde{h}^{\mu\nu} = i(1 + 4k_1) p^\alpha \tilde{b}_\alpha \] (2.148)

\[ e_{\mu\nu} : m_1^2 e_{\mu\nu} \tilde{h}^{\mu\nu} + m_2^2 \tilde{h} = -i(1 + k_1) p^\alpha \tilde{b}_\alpha . \] (2.149)

We shall now cross (2.148) and (2.149) with the two solutions we obtained for the propagator which we recall here:

**Solution 1** : \( m_2 = 0, \ m_1 \neq 0, \) arbitrary \( k \) and \( k_1 \) (2.150)

**Solution 2** : \( k = 0, \ k_1 = 0, \) arbitrary \( m_1 \) and \( m_2 \) (2.151)

**Degrees of freedom for Solution 1**: \( m_2 = 0, \ m_1 \neq 0, \) arbitrary \( k \) and \( k_1 \)

In this case, we need an additional equation, which we obtain by contracting (2.146) with \( p_\mu \) and plugging (2.147) into it. Doing so, we get

\[ -\frac{1}{2k} \left[ p^2 p_\alpha \tilde{h}^{\mu\alpha} + p^\mu p_\alpha p_\beta \tilde{h}^{\alpha\beta} + 2k_1 p^\mu p^2 \tilde{h} \right] - \frac{k_1}{k} \left[ p_\alpha p_\beta \tilde{h}^{\alpha\beta} + k_1 p^2 \tilde{h} \right] p^\mu + m_1^2 p_\mu \tilde{h}^{\mu\nu} + m_2^2 p^\mu \tilde{h} = 0 , \] (2.152)

which, after rearranging the terms, becomes

\[ \left( m_1^2 - \frac{p^2}{2k} \right) p_\alpha \tilde{h}^{\mu\alpha} - \frac{1}{k} \left( \frac{1}{2} + k_1 \right) p_\alpha p_\beta \tilde{h}^{\alpha\beta} p^\mu - \left( \frac{k_1}{k} (1 + k_1) p^2 - m_2^2 \right) \tilde{h} p^\mu = 0 . \] (2.153)

If we set \( m_2 = 0 \), equations (2.148), (2.149) and (2.153) reduce to
\[ (2p^2 - m_1^2)\hat{h} - 2p^2 e_{\mu\nu} \hat{h}^{\mu\nu} = i(1 + 4k_1) p^\alpha \tilde{b}_\alpha \quad (2.154) \]

\[ m_1^2 e_{\mu\nu} \hat{h}^{\mu\nu} = -i(1 + k_1) p^\alpha \tilde{b}_\alpha \quad (2.155) \]

\[ \left( m_1^2 - \frac{p^2}{2k} \right) p_\alpha \hat{h}^{\mu\alpha} - \frac{1}{k} \left( \frac{1}{2} + k_1 \right) p^2 e_{\alpha\beta} \hat{h}^{\alpha\beta} p^\mu - \left( \frac{k_1}{k} (1 + k_1) p^2 \right) \hat{h} p^\mu = 0 . \quad (2.156) \]

Solving (2.155) for \( p^\alpha \tilde{b}_\alpha \) and substituting it back into (2.154) we have

\[ \left( 2p^2 - \frac{1 + 4k_1}{1 + k_1} m_1^2 \right) e_{\mu\nu} \hat{h}^{\mu\nu} = \left( 2p^2 - m_1^2 \right) \hat{h} , \quad (2.157) \]

while by plugging (2.147) into (2.155) we find

\[ \left( p^2 - \frac{k}{1 + k_1} m_1^2 \right) e_{\mu\nu} \hat{h}^{\mu\nu} = -k_1 p^2 \hat{h} . \quad (2.158) \]

From these last 2 equations, since \( m_1 \neq 0^4 \), it is clear that the following conditions must hold

\[ e_{\mu\nu} \hat{h}^{\mu\nu} = \hat{h} = 0 . \quad (2.159) \]

We can now plug \( e_{\mu\nu} \hat{h}^{\mu\nu} = \hat{h} = 0 \) into (2.156) to obtain

\[ \left( m_1^2 - \frac{p^2}{2k} \right) p_\mu \hat{h}^{\mu\nu} = 0 . \quad (2.160) \]

The general solution to this equation, assuming \( k \neq 0 \) is

\[ p_\mu h^{\mu\nu} = \xi^\nu \delta \left( p^2 - 2km_1^2 \right) , \quad (2.161) \]

\(^4\text{This is because } m_2 \text{ is already zero and we want at least one of the two mass parameters to be non-vanishing.}\)
where $\xi^\nu$ is an arbitrary 4-vector. Since we cannot accept a gauge dependent mass, we are forced to set $\xi^\nu = 0$. Therefore, Eq. (2.160) implies
\[ p_\mu \tilde{h}^{\mu\nu} = 0 . \] (2.162)
To make this more precise, notice that if the field $h_{\mu\nu}(x)$ has a mass then it can be written as
\[ h_{\mu\nu}(x) = \int d^4p \ e^{-ipx} \tilde{h}_{\mu\nu}(p) , \] (2.163)
where $h_{\mu\nu}(p)$ contains a delta factor $\delta(p^2 + M^2)$, with $M$ the mass of the field, which cannot depend on the gauge fixing. We can rewrite the Fourier transformed field as
\[ \tilde{h}_{\mu\nu}(p) = \delta(p^2 + M^2) \bar{h}_{\mu\nu} , \] (2.164)
where we have extracted the delta dependence which fixes the mass of the field.
Using this we obtain
\[ \partial_\mu h^{\mu\nu}(x) = -i \int d^4p \ e^{-ipx} \ p_\mu \tilde{h}^{\mu\nu}(p) = -i \int d^4p \ e^{-ipx} \delta(p^2 - M^2) p_\mu \tilde{h}^{\mu\nu}(p) . \] (2.165)
Form here, if we consider Eq. (2.161) with $\xi_\nu \neq 0$ we obtain that
\[ \partial_\mu h^{\mu\nu}(x) = -i \int d^4p \ e^{-ipx} \delta(p^2 + M^2) \delta(p^2 - 2km_1^2) \xi^\nu . \] (2.166)
Whenever $M^2 \neq -2km_1^2$ the above equation yields $\partial_\mu h^{\mu\nu} = 0$. Still, given a fixed value for $M^2$ and for $m_1^2$ we can always find a $k$ such that $M^2 = -2km_1^2$. In that case, the integral in Eq. (2.166) would have two identical delta factors and would be equivalent to
\[ \partial_\mu h^{\mu\nu}(x) = -i \delta(0) \int d^4p \ e^{-ipx} \delta(p^2 + M^2) \xi^\nu , \] (2.167)
which, unless we set $\xi^\nu = 0$, is a nonsensical expression because of the $\delta(0)$ factor. We conclude that, for any possible value of the real mass of the graviton $M^2$, Eq. (2.160) implies $\partial_\mu \hat{h}^{\mu \nu} = 0$ which in turn means $p_\mu \hat{h}^{\mu \nu} = 0$.

We have thus proven that for the solution with $m_2 = 0$ we have the following constraints\(^5\)

\[
\begin{aligned}
\tilde{h} &= 0 \\
p_\mu \tilde{h}^{\mu \nu} &= 0 .
\end{aligned}
\tag{2.168}
\tag{2.169}
\]

The two conditions above mean that solution 1 gives the propagation of 5 DOFs. Plugging (2.168) and (2.169) into the EOMs (2.146) and (2.147) we obtain

\[
b^{\mu} = 0 
\tag{2.170}
\]

and

\[
(p^2 + m_1^2)\hat{h}^{\mu \nu} = 0 .
\tag{2.171}
\]

**Degrees of freedom for Solution 2: $k = 0$, $k_1 = 0$, arbitrary $m_1$ and $m_2$**

This time, imposing $k = 0$, the EOM for $b_\mu$ (2.147) gives directly the gauge fixing condition

\[
p_\nu \hat{h}^{\mu \nu} + k_1 p^{\mu} \hat{h} = 0 ,
\tag{2.172}
\]

and by setting also $k_1 = 0$ we obtain

\[
p_\nu \hat{h}^{\mu \nu} = 0 .
\tag{2.173}
\]

\(^5\)note that in order to get to this point we ignored the case $k_1 = -1$. With this value of the gauge fixing parameter, Eq. (2.155) gives $\epsilon_{\mu \nu} \hat{h}^{\mu \nu} = 0$ which plugged (with $k_1 = -1$) in (2.156) directly yields Eq. (2.159) which in turn gives $p_\mu \hat{h}^{\mu \nu} = 0$. Now substituting this result into Eq. (2.147) and using Eq. (2.154) we get $b = 0$, again using that the mass cannot depend on the gauge fixing parameters (in this case $k$). So, also in the $k_1 = -1$ we recover the same result, given by Eqs. (2.168) and (2.169).
If we set \( k = 0 \) and \( k_1 = 0 \), equations (2.148), (2.149) reduce to

\[
\begin{align*}
2p^2 - (m_1^2 + 4m_2^2)\hat{h} - 2p^2 e_{\mu\nu}\hat{h}^{\mu\nu} &= ip^\alpha b_\alpha \\
(2.174)
\end{align*}
\]

\[
\begin{align*}
m_1^2 e_{\mu\nu}\hat{h}^{\mu\nu} + m_2^2 \hat{h} &= -ip^\alpha b_\alpha . \\
(2.175)
\end{align*}
\]

Solving (2.175) for \( p^\alpha b_\alpha \) and substituting it back into (2.174) together with condition (2.173) we get

\[
\begin{align*}
\left[2p^2 - (m_1^2 + 3m_2^2)\right] \hat{h} &= 0 . \\
(2.176)
\end{align*}
\]

Now, since in solution 2 the gauge parameters are fixed \textit{a priori}, we cannot infer the constraint \( h = 0 \) from Eq. (2.176) using the fact that we cannot accept a gauge dependent mass.

This means that solution 2 does not guarantee the propagation of 5 DOFs and we must discard it. Therefore we are left with only solution 1 which is summarized by

\[
\begin{array}{c|c}
\text{Condition} & \text{DOFs Propagated} \\
\hline
m_2 = 0, m_1 \neq 0, \text{arbitrary } k \text{ and } k_1 & 5
\end{array}
\]

We see that the condition above is characterized by a fixed mass parameter \( m_2 = 0 \) and it is gauge independent, a feature that we demand for a good MG theory. In fact, independently of the gauge choice, we always obtain 5 DOFs, a necessary property of a MG theory.
2.3.2 Quadratic gauge fixing method

The EOM we get from the action $S$ (2.129) is

$$\frac{\delta S}{\delta \tilde{h}_{\mu\nu}} = - \left( 1 + \frac{k_1^2}{k} \right) \eta_{\mu\nu} \tilde{p}^2 \tilde{h} + \left( 1 - \frac{k_1}{k} \right) P^\mu p^\nu \tilde{h} + \left( 1 - \frac{k_1}{k} \right) \eta_{\mu\nu} p^\alpha p^\beta \tilde{h}_{\alpha\beta} +$$

$$p^2 \tilde{h}^\mu_{\nu} - \left( 1 + \frac{1}{2k} \right) (p^\mu p^\alpha \tilde{h}_{\alpha\nu} + p^\nu p^\alpha \tilde{h}_{\mu\alpha}) + m_1^2 \tilde{h}^\mu_{\nu} + m_2^2 \eta_{\mu\nu} \tilde{h} = 0. \quad (2.177)$$

In order to study the propagating DOFs, we saturate the EOM (2.177) with $\eta_{\mu\nu}$ and $e_{\mu\nu}$, respectively, to get

$$\eta_{\mu\nu} : \quad \left[ (m_1^2 + 4m_2^2) - \left( 2 + \frac{k_1}{k} (1 + 4k_1) \right) p^2 \right] \tilde{h} + \left( 2 - \frac{1}{k} (1 + 4k_1) \right) p^2 e_{\mu\nu} \tilde{h}^\mu_{\nu} = 0$$

$$e_{\mu\nu} : \quad \left[ m_1^2 - \frac{k_1}{k} (1 + k_1) p^2 \right] \tilde{h} + \left[ m_1^2 - \frac{1}{k} (1 + k_1) p^2 \right] e_{\mu\nu} \tilde{h}^\mu_{\nu} = 0. \quad (2.178)$$

From these two equations we deduce that, if $m_1 \neq 0$, the only solution is

$$\begin{cases} 
\tilde{h} = 0 \\
e^\mu_{\nu} \tilde{h}^\mu_{\nu} = 0 ,
\end{cases} \quad (2.180)$$

$$e^\mu_{\nu} \tilde{h}^\mu_{\nu} = 0 , \quad (2.181)$$

Then, substituting these two constraints into the EOM (2.177) we get

$$(p^2 + m_1^2) \tilde{h}^\mu_{\nu} - \left( 1 + \frac{1}{2k} \right) (p^\mu p^\alpha \tilde{h}_{\alpha\nu} + p^\nu p^\alpha \tilde{h}_{\mu\alpha}) = 0 , \quad (2.182)$$

which, saturated with $p^\nu$, yields

$$\left( m_1^2 - \frac{1}{2k} p^2 \right) p^\nu \tilde{h}^\mu_{\nu} = 0 . \quad (2.183)$$

We stop a moment to notice that this last equation exactly matches Eq. (2.160), which we obtained using the Lagrange multiplier method EOMs. This reassures us
that, the Lagrange multiplier and the quadratic gauge fixing approaches are in fact physically equivalent.

Now, like in the Lagrange multiplier case (Section 2.3.1), Eq. (2.183) implies

\[ p^\nu \tilde{h}_{\mu\nu} = 0 . \quad (2.184) \]

because we cannot accept a gauge dependent mass.

This, together with (2.180), gives again the two constraints

\[
\begin{align*}
\tilde{h} &= 0 \\
p^\mu \tilde{h}_{\mu\nu} &= 0 ,
\end{align*}
\]

which ensure the propagation of 5 DOFs. These two constraints inserted into (2.182) give the massive propagation of \( h_{\mu\nu} \)

\[ (p^2 + m_1^2)\tilde{h}_{\mu\nu} = 0 \quad (2.187) \]

Conversely, if \( m_1 = 0 \), the two equations (2.178) and (2.179) have a non-trivial solution only if they reduce to the same equation. To achieve this, we want the coefficient of \( h \) in (2.178) to be proportional to that in (2.179), and the same for the coefficients of \( e^{\mu\nu} \tilde{h}_{\mu\nu} \). This leads to

\[ k_1 = -1 \quad (2.188) \]

\[ k = -\frac{3}{2} , \quad (2.189) \]

which, plugged back into (2.178) and (2.179) with \( m_1 = 0 \), yields

\[ \tilde{h} = 0 . \quad (2.190) \]
Substituting $m_1 = 0$ and $\tilde{h} = 0$ in the EOMs (2.177) we notice that all the dependence on the mass parameters vanishes. This leaves EOMs from which we can only derive the propagation of a massless field or, at most, the propagation of a field with a mass dependent on the gauge parameters $k$ and $k_1$. For this reason we decide to exclude the case $m_1 = 0$ from the possible values of $m_1$, because it does not represent a good candidate for a MG theory.

To summarize, what we obtained in the quadratic gauge fixing case is

<table>
<thead>
<tr>
<th>Condition</th>
<th>DOFs Propagated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1 \neq 0$</td>
<td>arbitrary $m_2$</td>
</tr>
<tr>
<td></td>
<td>arbitrary $k$</td>
</tr>
<tr>
<td></td>
<td>arbitrary $k_1$</td>
</tr>
</tbody>
</table>

It is straightforward to see that this condition shows no dependence on the gauge parameters and thus satisfies the request of independence of the DOFs on the gauge parameters $k$ and $k_1$.

### 2.3.3 Direct gauge fixing method

The EOM we get from the variation of the action (2.142) are

$$\frac{\delta S}{\delta \tilde{h}_{\mu\nu}} = (m_2^2 - \beta p^2) \tilde{h} \eta^{\mu\nu} + (p^2 + m_1^2) \tilde{h}^{\mu\nu} = 0$$

(2.191)

and the gauge fixing condition is

$$p_{\mu} \tilde{h}_{\mu\nu} + k_1 p^{\nu} \tilde{h} = 0.$$  

(2.192)
To study the DOFs of this theory, we start by taking the trace of (2.191) from which we get

\[ 4(m_2^2 - \beta p^2) + p^2 + m_1^2 \tilde{h} = 0, \]  

(2.193)

where we recall that \( \beta \) is given by (2.139).

On the other hand we can also contract (2.191) with \( p_\nu \) and using \( p_\mu h^{\mu \nu} + k_1 p^\nu h = 0 \) we find

\[ [m_2^2 - \beta p^2 - k_1(p^2 + m_1^2)] \tilde{h} = 0. \]  

(2.194)

Let us now rewrite Eq. (2.193) in the following way

\[ (m_2^2 - \beta p^2) h = -\frac{1}{4}(p^2 + m_1^2) \tilde{h} \]  

(2.195)

and substitute this in the EOM (2.191) to obtain

\[ (p^2 + m_1^2)(\tilde{h}^{\mu \nu} - \tilde{h} 4 \eta^{\mu \nu}) = 0. \]  

(2.196)

Plugging (2.195) in the left-hand side of (2.194) we have

\[ \left( \frac{1}{4} + k_1 \right)(p^2 + m_1^2) \tilde{h} = 0. \]  

(2.197)

We are now able to find how many DOFs are propagating. If \( k_1 = -\frac{1}{4} \), eq. (2.197) trivializes. Thanks to (2.196), we know that the traceless tensor

\[ \tilde{h}^{\mu \nu} \equiv \tilde{h}^{\mu \nu} - \tilde{h} 4 \eta^{\mu \nu} \]  

(2.198)

satisfies

\[ (p^2 + m_1^2) \tilde{h}^{\mu \nu} = 0. \]  

(2.199)

Then we can notice that

\[ p_\nu \tilde{h}^{\mu \nu} = p_\nu (\tilde{h}^{\mu \nu} - \tilde{h} 4 \eta^{\mu \nu}) = -k_1 p^\nu \tilde{h} - \tilde{h} 4 p^\nu = -\left( \frac{1}{4} + k_1 \right) p^\nu \tilde{h} = 0, \]  

(2.200)

59
where we have used the gauge fixing condition (2.192) and $k_1 = -\frac{1}{4}$. We can summarize what we have just shown for $k_1 = -\frac{1}{4}$ with the equations

$$\begin{cases} 
(p^2 + m_1^2) \tilde{h}^{\mu\nu} = 0 \\
\tilde{h} = 0 \\
p_\nu \tilde{h}^{\mu\nu} = 0.
\end{cases} \quad (2.201)$$

In other words $\tilde{h}^{\mu\nu}$ is a transverse, traceless and symmetric tensor that propagates 5 DOFs.

Let us now turn to the $k_1 \neq -\frac{1}{4}$ case. Then from (2.197) we have

$$(p^2 + m_1^2) \tilde{h} = 0. \quad (2.204)$$

By substituting this in (2.194), we get

$$\left(\beta m_1^2 + m_2^2\right) \tilde{h} = 0 \quad (2.205)$$

Now, if $\beta m_1^2 + m_2^2 \neq 0$, we have again that the tensor $h^{\mu\nu}$ is traceless and transverse, i.e.

$$\begin{cases} 
\tilde{h} = 0 \\
p_\nu \tilde{h}^{\mu\nu} = 0
\end{cases} \quad (2.206)$$

and thus propagates 5 massive DOFs.

If instead $\beta m_1^2 + m_2^2 = 0$, from (2.204) and (2.205) we get

$$\left(\beta p^2 - m_1^2\right) \tilde{h} = 0, \quad (2.208)$$

which plugged into the EOM gives

$$(p^2 + m_1^2) \tilde{h}^{\mu\nu} = 0, \quad (2.209)$$
which implies (2.204). Now, to explicitly find the DOFs propagated, we define

\[ \hat{h}_{\mu\nu} \equiv \tilde{h}_{\mu\nu} + k_1 \tilde{h}\eta_{\mu\nu} \]  (2.210)

which is transverse thanks to the gauge condition (2.192)

\[ p_\nu \hat{h}^{\mu\nu} = p_\nu \tilde{h}^{\mu\nu} + k_1 p^\mu \tilde{h} = 0, \]  (2.211)

but not traceless

\[ \eta_{\mu\nu} \hat{h}^{\mu\nu} = (1 + 4k_1) \tilde{h}, \]  (2.212)

and therefore represents 6 massive DOFs.

Here we list the possible parameter conditions we found for the direct gauge fixing method and the DOFs propagated in every case

<table>
<thead>
<tr>
<th>Condition</th>
<th>DOFs Propagated</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_1 = -\frac{1}{4} )</td>
<td>5</td>
</tr>
<tr>
<td>( k_1 \neq -\frac{1}{4}, \beta m_1^2 + m_2^2 \neq 0 )</td>
<td>5</td>
</tr>
<tr>
<td>( k_1 \neq -\frac{1}{4}, \beta m_1^2 + m_2^2 = 0 )</td>
<td>6</td>
</tr>
</tbody>
</table>

Looking at the table above, it might seem that the propagated DOFs depend on the gauge fixing parameter \( k_1 \). This it true unless we set the mass parameters to specific values in order to eliminate this dependence. In particular, being \( \beta \) a strictly positive real number, if we set either \( m_1 \) or \( m_2 \) to zero (while keeping the other different from zero) the condition \( \beta m_1^2 + m_2^2 \neq 0 \) is always satisfied and the first and second conditions reported in the table above merge into a single case with 5 DOFs. In Section (2.4) we shall explain this in more detail.
2.4 Summary of solutions

In the attempt to find a propagator for the gauge fixed massive theory (i.e. with at least one between \(m_1\) and \(m_2\) non vanishing) we have found different conditions on the mass and gauge parameters for the Lagrange multiplier gauge fixing, the quadratic gauge fixing (originating from the \(\Phi \Pi\)-procedure) and what we called the direct gauge fixing (not motivated by any formal argument). In the following pages we have collected all these results to make it easier for future reference.

Lagrange multiplier gauge fixing

The propagators found using the Lagrange multiplier method are of the form

\[
\begin{bmatrix}
G_{\mu \nu, \alpha \beta} & G_{\mu \nu, \alpha} \\
G^{*}_{\mu, \alpha \beta} & G_{\mu \alpha}
\end{bmatrix},
\tag{2.213}
\]

where

\[
\langle h_{\mu \nu} h_{\alpha \beta} \rangle (p) = G_{\mu \nu, \alpha \beta} \tag{2.214}
\]

\[
\langle h_{\mu \nu} b_{\alpha} \rangle (p) = G_{\mu \nu, \alpha} \tag{2.215}
\]

\[
\langle b_{\mu} b_{\alpha} \rangle (p) = G_{\mu \alpha} \tag{2.216}
\]
2.4 - Summary of solutions

and

\[ G_{\mu\nu,\alpha\beta} = iA_{\mu\nu,\alpha\beta} + \hat{u}B_{\mu\nu,\alpha\beta} + \hat{v}C_{\mu\nu,\alpha\beta} + \hat{z}D_{\mu\nu,\alpha\beta} + \hat{w}E_{\mu\nu,\alpha\beta} \]  

(2.217)

\[ G_{\mu\nu,\alpha} = i[f(d_{\mu\alpha}p_{\nu} + d_{\nu\alpha}p_{\mu}) + gd_{\mu\nu}p_{\alpha} + le_{\mu\nu}p_{\alpha}] \]  

(2.218)

\[ G_{\mu\nu} = rd_{\mu\nu} + se_{\mu\nu} . \]  

(2.219)

Valid solutions for the coefficients of (2.217), (2.218) and (2.219) exist only in the following two cases:

- \( m_2 = 0, \ m_1 \neq 0, \) arbitrary \( k \) and \( k_1 \) (solution 1)

- \( k = 0, \ k_1 = 0, \) arbitrary \( m_1 \) and \( m_2 \) (solution 2)

The coefficients obtained for solution 1 are listed in (2.97)–(2.106), while those of solution 2 are listed in (2.107)–(2.115).

A case against solution 2 is the fact that we see the massive theory as a perturbation of the massless one and we ask that it should be valid for all gauge choices, i.e. for every value of \( k \) and \( k_1 \). It therefore comes natural to exclude solution 2 as it fixes both gauge fixing parameters, not meeting our request.

Also, as seen in 2.3, the DOFs propagated by solution 1 are 5, while those propagated by solution 2 are only guaranteed to be at most 6. Therefore we cannot be sure that the theory has the right DOFs to describe the propagation of a massive graviton.

In conclusion, gauge fixing the action using a Lagrange multiplier field we obtained the condition
\[ m_2 = 0, \ m_1 \neq 0, \text{ arbitrary } k \text{ and } k_1 \]

**Quadratic gauge fixing**

In the quadratic gauge fixing gauge we found a propagator of the form

\[
\langle h_{\mu\nu}h_{\alpha\beta}\rangle(p) = G_{\mu\nu,\alpha\beta} = \hat{t}A_{\mu\nu,\alpha\beta} + \hat{u}B_{\mu\nu,\alpha\beta} + \hat{v}C_{\mu\nu,\alpha\beta} + \hat{z}D_{\mu\nu,\alpha\beta} + \hat{w}E_{\mu\nu,\alpha\beta} \quad (2.220)
\]

where the coefficients \( \hat{t}, \hat{u}, \hat{v}, \hat{z} \) and \( \hat{w} \) are given by (2.133)–(2.137).

The study of the DOFs propagated by this theory yielded only one case:

\[ m_1 \neq 0, \ m_2 \text{ any, arbitrary gauge parameters } k \text{ and } k_1 \implies 5 \text{ DOFs} \]

As already mentioned in Section (2.3.2), we recall that the mass parameter choice \( m_1 = 0 \) was excluded because it implied either massless propagation (which is not in our interest) or, possibly, massive propagation with a mass dependent on the gauge choice. Therefore, we are only left with \( m_1 \neq 0 \) for which the corresponding solution has no dependence on the gauge parameters and 5 massive DOFs are propagated. It should be noted that the FP tuning between the mass parameters \( (m_1^2 + m_2^2 = 0) \) is included in this case.

**Direct gauge fixing**

Using a direct gauge fixing on the action we found the following propagator

\[
\langle h_{\mu\nu}h_{\alpha\beta}\rangle(p) =
\frac{2}{p^2 + m_1^2 (1 - 4\beta)}p^2 p^2 p^2 + m_2^2 + 4m_2^2 \eta_{\mu\nu}\eta_{\alpha\beta} + \frac{2}{p^2 + m_1^2} \mathcal{I}_{\mu\nu,\alpha\beta}, \quad (2.221)
\]

where \( \beta \) is give by (2.139) and depends on the gauge fixing parameter \( k_1 \).

For the DOFs we found again three 3 cases:

\[ \text{Case 1: } k_1 = -\frac{1}{4} \implies 5 \text{ DOFs} \]
• Case 2: $k_1 \neq -\frac{1}{4}$, $\beta m_1^2 + m_2^2 \neq 0 \implies 5$ DOFs

• Case 3: $k_1 \neq -\frac{1}{4}$, $\beta m_1^2 + m_2^2 = 0 \implies 6$ DOFs

As in the quadratic gauge fixing case, it might seem that the DOFs always depend on the gauge parameters choice, but, as we mentioned in Section (2.3.3), for specific values of the mass parameters the DOFs become gauge independent, i.e. do not depend on the gauge parameter $k_1$. In particular, there are two such choices for the mass parameters:

• $m_1 = 0$, $m_2 \neq 0$
• $m_2 = 0$, $m_1 \neq 0$

For both choices, the condition

$$\beta m_1^2 + m_2^2 \neq 0$$

is always satisfied and, unifying case 1 and case 2 we get the propagation of 5 massive DOFs, with arbitrary $k_1$.

Referring to Section (2.3.3) and in particular to (2.199) and (2.209) It should be noted that with, the $m_1 = 0$ mass choice, the propagation of $h^{\mu\nu}$ is massless (i.e. $p^2 h^{\mu\nu} = 0$), while with $m_2 = 0$ the propagation is massive and of the usual form

$$(p^2 + m_1^2) h^{\mu\nu} = 0.$$  (2.223)

So, for the quadratic gauge fixing method we restricted ourselves to the solutions of the propagator with either of the two conditions

• $m_1 = 0$, $m_2 \neq 0$ giving massless propagation
• $m_2 = 0$, $m_1 \neq 0$ giving massive propagation with mass $m_1$
Comment on $m_2$

It is instructive to point out the fact that whenever we obtain a solution which represents a proper theory for a rank-2 massive tensor and ask that the massive propagating DOFs does not dependent on the gauge choice, we always get the equation

\[(p^2 + m_1^2)\tilde{h}^\mu^\nu = 0 \quad (2.224)\]

and 5 massive DOFs. This last point is important: we find 5 DOFs without imposing it as a preliminary request. We shall comment more on this in another section.

We also see that $m_2$, the mass parameter associated in the action with the trace $h$, is irrelevant: it never appears to contribute to the mass of the propagating DOFs even if (like in the quadratic gauge fixing case) it has a value different from zero.

Comment on the Fierz-Pauli tuning

A remarkable result of this section is that, following the criteria we adopted, the FP relation between the mass parameters

\[m_1^2 + m_2^2 = 0 \quad (2.225)\]

is excluded both in the Lagrange multiplier approach and with the direct gauge fixing. In fact, using the Lagrange multiplier we found $m_2 = 0$ (see the beginning of Section (2.4)) which excludes the FP tuning between the mass parameters, while with the direct gauge fixing we found (above) two acceptable conditions on the mass parameters which both exclude the FP tuning. The FP tuning is instead included in the quadratic gauge fixing method as pointed out before.

Though, it should be noted that since the actions we are working with have been gauge fixed before the addition of a mass, they are dramatically different from the ones used for instance in [15] and [18], which have no gauge fixing at all, or rather
use the mass parameters as gauge fixing parameters. This implies that the relation between the mass parameters cannot be assumed to have the same meaning as in the FP MG. Indeed, while the FP MG turns out to have a discontinuity with linearized GR (the vDVZ discontinuity), our properly gauge fixed theories (with the Lagrange multiplier or quadratic terms) do not show any such issue. So, the fact that the quadratic approach includes the FP tuning between (2.225) the mass parameters does not mean at all that the FP MG is a sub-case of our theory. In fact, we shall give a proof of this fact in Section (2.6).
2.5 Gauge Fixing dependence of the Propagators

It is straightforward to see that in the propagators we found there is a heavy dependence on the gauge fixing parameters. We have previously noted that this gauge dependence also extends to the poles of the propagators and this represents a concern.

The problem lies in the fact that the mass of a free field is defined as the pole of its propagator which therefore cannot depend on the gauge choice. Gauge fixing before adding a mass term is motivated by the fact that in the Fierz-Pauli approach the mass parameters themselves are used as gauge fixing parameters and hence cannot be regarded as physical parameters anymore. With this in mind, when we obtain poles that depend on the gauge fixing parameters, which are not physical by definition, we might be alarmed that the theory is therefore not valid.

The aim of this section is to show that this issue is solved by noticing that the gauge dependent poles are associated with projections of $h^\mu\nu$ which are identically zero and thus do not propagate.

We shall address one method at a time.

Lagrange multiplier gauge fixing

When we studied the equations of motion for the cases described by solution 1 we obtained

\[
\begin{align*}
\tilde{h} &= 0 \\
 p_\nu \tilde{h}^{\mu\nu} &= 0.
\end{align*}
\] (2.226) (2.227)

We recall that solution 1 corresponds to $m_2 = 0$ and arbitrary $k$ and $k_1$. 68
The last two equations are crucial to understand which are the propagating DOFs and to explain why the gauge dependence in the poles of the propagator does not represent a problem. We now show how this can be done.

Computing the amplitude for the $h^{\mu\nu}$ propagation (the 2-point Green function with external legs) we have something proportional to

$$\tilde{h}^{\mu\nu}G_{\mu\nu,\alpha\beta}\tilde{h}^{\alpha\beta}, \quad (2.228)$$

and, using (2.55), this is

$$\tilde{h}^{\mu\nu}\left(\hat{v}A_{\mu\nu,\alpha\beta} + \hat{u}B_{\mu\nu,\alpha\beta} + \hat{v}C_{\mu\nu,\alpha\beta} + \hat{z}D_{\mu\nu,\alpha\beta} + \hat{w}E_{\mu\nu,\alpha\beta}\right)\tilde{h}^{\alpha\beta}. \quad (2.229)$$

Now, using (2.227), this expression further simplifies to

$$\tilde{h}^{\mu\nu}\left(\hat{v}I_{\mu\nu,\alpha\beta}\right)\tilde{h}^{\alpha\beta} \quad (2.230)$$

revealing that the only relevant pole for the propagation of $h^{\mu\nu}$ is the one contained in $\hat{v}$. Recalling that for solution 1, i.e. the one we selected in Section (2.4) as the only valid solution for the Lagrange multiplier gauge fixing, we have (see (2.99))

$$\hat{v} = \frac{2}{p_1^2 + m_1^2}. \quad (2.231)$$

We are left with only one physical pole

$$p^2 = -m_1^2 \quad (2.232)$$

with no dependence on the gauge fixing parameters. So this means that, at least at the propagator level, we can be assured that poles with a dependence on the gauge fixing parameters do not represent a contradiction, solving this pressing issue.
Not only: the relevant mass parameter is $m_1$. It appears that a MG theory is unsensitive to the mass related to the trace $h$ of the field.

**Quadratic gauge fixing**

In Section (2.4), we isolated only one acceptable condition on the mass parameters which yields a gauge independent determination of the propagated DOFs:

$$m_1 \neq 0.$$

With this choice we have again

$$\left\{ \begin{array}{l}
\tilde{h} = 0 \\
p_\mu \tilde{h}^{\mu \nu} = 0,
\end{array} \right. \quad (2.233)$$

$$\left\{ \begin{array}{l}
\tilde{h} = 0 \\
p_\mu \tilde{h}^{\mu \nu} = 0,
\end{array} \right. \quad (2.234)$$

which as before implies that the amplitude of the $h^{\mu \nu}$ propagation is proportional to (2.230), because we parametrized the propagator in exactly the same way as in the Lagrange multiplier approach. Also, the solution for $\hat{v}$ in this case is given by (2.135) and it is the same as that obtained in the Lagrange multiplier case

$$\hat{v} = \frac{2}{p^2 + m_1^2}.$$ \quad (2.235)

So, also in the quadratic gauge fixing approach, we found just one physical massive pole, which is

$$p^2 = -m_1^2.$$ \quad (2.236)
independently of the value taken by \( m_2 \) and we discovered that we can safely accept the dependence of some of the poles of the propagator on the gauge choice since they do not play any role on the propagation of the field itself.

### Direct gauge fixing

In Section (2.4), for this gauge fixing method we obtained that the only two choices we have to obtain a theory propagating DOFs which do not depend on the gauge choice are

- \( m_1 = 0, \ m_2 \neq 0 \)
- \( m_2 = 0, \ m_1 \neq 0 \),

which yield

\[
\begin{aligned}
\tilde{h} &= 0 \\
p_\nu \tilde{h}^{\mu\nu} &= 0 .
\end{aligned}
\]  

When calculating the amplitude of propagation of \( h^{\mu\nu} \) we find that it is proportional to (2.228). Looking at the propagator (2.221) we found using this method, which we recall here

\[
G_{\mu\nu,\alpha\beta} = \frac{2}{p^2 + m_1^2} \frac{\beta p^2 - m_2^2}{(1 - \beta D)p^2 + m_1^2 + Dm_2^2} \eta_{\mu\nu} \eta_{\alpha\beta} + \frac{2}{p^2 + m_1^2} \mathcal{T}_{\mu\nu,\alpha\beta} 
\]  

thanks to (2.237) and (2.238), (2.228) reduces to

\[
\frac{2}{p^2 + m_1^2} \tilde{h}^{\mu\nu} \tilde{h}_{\mu\nu} .
\]  

As in the two previous methods, the only physical pole is in

\[
\frac{2}{p^2 + m_1^2} .
\]
which in the $m_1 = 0$ case gives massless propagation, while in the $m_2 = 0$ case gives 5 massive propagating DOFs, as it should.
2.6 A new theory of Massive Gravity

In this section our goal is to use the results obtained so far to build a proper MG theory. In order to promote a theory describing a rank-2 symmetric tensor to a theory for a massive graviton, a list of requirements should be satisfied:

1. finite massless limit,

   which is true for all the propagators listed in Section (2.4) since we discarded solutions for the propagator which did not have a finite massless limit;

2. 5 massive DOFs,

   as one can check again in Section (2.4), this point is also met asking that the number of DOFs should not depend on a particular gauge choice. The fact that for every solution we obtained the propagation \((p^2 + m_1^2) h_{\mu\nu} = 0\) implies that if we want massive DOFs we must reject all solutions with \(m_1 = 0\);

3. independence from the gauge fixing parameters,

   this property is also satisfied: we recall the results of Section (2.5), where we showed that using the constraints obtained from the EOMs, all terms in the propagator whose poles are gauge dependent vanish as one calculates the propagation amplitude;

4. absence of tachyonic poles in the propagator,

   this requirement is met by setting \(m_1^2 > 0\) since, in each of the three propagators we found, the only relevant pole is \(p^2 = -m_1^2\);
5. provide physical predictions which are continuous with linearized GR,

this property will be the focus of next pages.

We shall stress that the Fierz-Pauli theory of MG does not match the first and, as we shall see shortly, the fifth requirement, as it is well known (we are referring to the vDVZ discontinuity thoroughly discussed in [15,18] and firstly presented in [10,11].

2.6.1 Consistency with Linearized Massless Gravity: zero-mass limit

We now turn to the zero mass limit of the theories obtained in the previous sections: Lagrange multiplier, quadratic gauge fixing and direct gauge fixing. In particular, we are interested in the solutions we found for the propagators of these theories, in order to study whether or not they are consistent with GR. We start by first introducing a major problem of the FP approach to MG.

Non-relativistic interactions: vDVZ discontinuity

A well known flaw of the FP theory is the Van Dam-Veltman-Zakharov discontinuity [10,11,15,18]. This discontinuity appears when one studies the zero-mass limit of the FP theory. In doing so, one finds a coupling of $h_{\mu\nu}$ with a non relativistic energy-momentum tensor $T_{\mu\nu}$ which is different from that of massless linearized gravity. A striking example of this fact is the prediction of light bending, which differs from that found in GR by 25%. As it is discussed in [18], the FP propagator takes the form
2.6 – A new theory of Massive Gravity

\[ FP : \quad G_{\mu\nu,\alpha\beta} = \frac{1}{p^2 + m_G^2} \left[ \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} - \frac{2}{3} \eta_{\mu\nu} \eta_{\alpha\beta} + (p\text{-dependent terms}) \right] \]

(2.242)

where \( m_G^2 = m_1^2 = -m_2^2 \), while in GR (i.e. linearized massless GR) it is

\[ GR : \quad G_{\mu\nu,\alpha\beta} = \frac{1}{p^2} \left[ \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} - \eta_{\mu\nu} \eta_{\alpha\beta} + (p\text{-dependent terms}) \right] \] . \quad (2.243)

The interaction between two non relativistic bodies in the two cases (FP and GR) is computed by contracting the propagator with the energy-momentum tensors (which are conserved, i.e. \( p_\nu \tilde{T}^{\mu\nu} = 0 \)) of the two bodies, \( T_{\mu\nu} \) and \( T'_\alpha\beta \), of which only the 00-components are non-negligible. So, in momentum space

\[ \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}'^{\alpha\beta} \approx \tilde{T}_{00} G_{00,00} \tilde{T}'_{00} \] . \quad (2.244)

Taking into account the coupling constants of the two theories, which can \emph{a priori} be different, we get the couplings

\[ GR : \quad G_{GR} \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}'^{\alpha\beta} = G_{GR} \tilde{T}_{00} \tilde{T}'_{00} \frac{1}{p^2} \] \quad (2.245)

\[ FP : \quad G_{FP} \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}'^{\alpha\beta} = \frac{4}{3} G_{FP} \tilde{T}_{00} \tilde{T}'_{00} \frac{1}{p^2 + m_G^2} \] . \quad (2.246)

The constant \( G_{GR} \) is the one that has been experimentally measured to a certain value \( G_{GR} \equiv G_{\text{Newton}} \). Taking the zero-mass limit of (2.246) and imposing that the couplings become the same as we take \( m_G \to 0 \), we get

\[ G_{FP} = \frac{3}{4} G_{GR} = \frac{3}{4} G_{\text{Newton}} \] \quad (2.247)

If we now take in consideration the energy-momentum tensor of an electromagnetic wave (say \( T' \)), which is traceless, and consider its interaction with that of a massive
body (T) we obtain

\[ GR : \quad G_{GR} \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} = G_{GR} \tilde{T}^{\mu}_{00} \tilde{T}^{\mu}_{00} \frac{2}{p^2} \quad (2.248) \]

\[ FP : \quad G_{FP} \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} = G_{FP} \tilde{T}^{\mu}_{00} \tilde{T}^{\mu}_{00} \frac{2}{p^2 + m^2_G} \quad (2.249) \]

This result clearly shows how the strength of interaction with light in the FP theory, taken in the zero-mass limit, is \( \frac{3}{4} \) of that in GR, which is therefore not acceptable. One way of fixing this fact is through the Vainshtein mechanism [12, 14, 15], which implies the re-introduction of the non-linearities coming from GR in order to provide a prediction for the coupling in the massless limit that is coherent with that of LG.

This fact is referred to as vDVZ discontinuity and, after having explained it in the FP framework, we investigate whether it is present in our gauge fixed massive theories or not.

Lagrange multiplier method: absence of the vDVZ discontinuity

We now want to show that the vDVZ discontinuity is not present in the theory obtained with the Lagrange multiplier approach.

What we need to compute is

\[ \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} T^{\alpha\beta} , \quad (2.250) \]

where \( T \) and \( T' \) are non-relativistic energy-momentum tensors. We use the
from the remaining terms (and considering only the 00-components of the energy momentum tensors) we obtain

\[ \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx \left[ \frac{1}{3} \hat{t} + \frac{2}{3} \hat{v} + \frac{1}{4} \hat{w} \right] \hat{T}_{00} \hat{\tilde{T}}_{00} . \] (2.252)

If we now plug the two solutions \( m_2 = 0 \) or \( k = 0, k_1 = 0 \) into (2.252) and compute the zero-mass limit what we obtain in both cases is

\[ G' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx G' \tilde{T}_{00} \tilde{T}_{00} \frac{1}{p^2} . \] (2.253)

This implies that, in order to have the same coupling of GR in the massless limit, we just need to set \( G = G_{\text{Newton}} \). From this we can immediately see that the coupling strength is the same as GR and when we consider the interaction between an electromagnetic wave (traceless energy-momentum tensor) and a non-relativistic source (for which only 00-components are non-negligible) the only contributing tensor is \( C_{\mu\nu,\alpha\beta} \) and in the zero-mass limit we immediately get

\[ G' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx G' \tilde{T}_{00} \tilde{T}_{00} \hat{\tilde{v}} \rightarrow G' \tilde{T}_{00} \tilde{T}_{00} \frac{2}{p^2} , \] (2.254)

which, considering \( G = G_{\text{Newton}} \), exactly matches the GR prediction (2.248).

Remark  The fact that with this gauge fixed massive theory we did not find a discontinuity with GR is extremely important. It encourages to believe that this way of approaching MG is the right one. There is more: although the absence of the vDVZ discontinuity is a specific test and it does not prove in...
general that the theory is continuous with GR for every measurable quantity, the way the theory itself was constructed does prove it. When trying to calculate any observable quantity in linearized GR, we are forced to gauge fix the action in order to obtain a propagator. A possible way to do this is through a Lagrange multiplier, as we did here. Now, the massive action we used, in the zero-mass limit becomes exactly the same action we would have used to calculate the propagator and consequently every other quantity in linearized GR. So it does not come as a surprise that the coupling between non-relativistic matter and light is continuous with GR, but we can also safely say that every other observable is, indeed, continuous in the massless limit.

**Quadratic gauge fixing method: absence of the vDVZ discontinuity**

To check if the vDVZ discontinuity is present within this approach, we can use directly (2.252), but this time with the coefficients (2.133)-(2.137). The interaction between two non-relativistic massive objects in the zero-mass limit is, similarly to the Lagrange multiplier method,

\[
G'' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}''^{\alpha\beta} \approx G'' T_{00}^{00} \frac{1}{p^2}.
\]

In order for the coupling strength to be the same of \( G \), we need to set

\[
G'' = G_{\text{Newton}}.
\]

As for the Lagrange multiplier case, when one calculates the interaction between a massive object and an electromagnetic wave in the massless limit of the theory, the only contributing tensor is \( C_{\mu\nu,\alpha\beta} \) and we get
\[ G'' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx G'' \tilde{T}_{00} \tilde{T}'_{00} \hat{v} \rightarrow G'' \tilde{T}_{00} \tilde{T}'_{00} \frac{2}{p^2}, \]

(2.256)

which, with \( G'' = G_{\text{Newton}} \), gives exactly the same result as the one obtained in GR.

**Remark** Again, as in the Lagrange multiplier method, we found that no discontinuity is present. The same reasoning used in the Lagrange multiplier case applies here too: every observable of this theory is continuous with GR in the zero-mass limit since the the two ways of realizing the gauge fixing restriction of the path integral (Lagrange multiplier approach and quadratic approach) are equivalent. For this reason, it should not come as a surprise that the physical results do not differ. The absence of the vDVZ discontinuity in these two equivalent methods confirms that applying a gauge fixing before adding a mass term to the action is a good way of constructing a MG theory.

**Direct gauge fixing method: presence of the vDVZ discontinuity**

We have just shown that in the Lagrange multiplier and quadratic gauge fixing approaches we have a good massless limit and the vDVZ discontinuity is not present, which makes our theories legitimate candidates for MG. we shall see that in this case the vDVZ discontinuity is present, confirming that this way of gauge fixing the action is not valid and provides incorrect physical predictions. The coupling between two non-relativistic bodies in the massless limit in this case is

\[ G''' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx G''' \tilde{T}_{00} \tilde{T}'_{00} \left( 2 + \frac{2\beta}{1 - 4\beta} \right) \frac{1}{p^2}, \]

(2.257)
Covariant Gauge Fixing

In order to have consistency with the linearized GR coupling, we must set

$$G''(2 + \frac{2\beta}{1 - 4\beta}) = G_{\text{Newton}}.$$  \hspace{1cm} (2.258)

On the other hand, the coupling between light and a massive object (computed in the zero-mass limit) gives

$$G'' \tilde{T}^{\mu\nu} G_{\mu\nu,\alpha\beta} \tilde{T}^{\alpha\beta} \approx G'' \tilde{T}_{00} \tilde{T}_{00} \frac{2}{p^2}$$  \hspace{1cm} (2.259)

which is consistent with GR only if $G'' = G_{\text{Newton}}$. So, the discontinuity is absent if we impose

$$\left(2 + \frac{2\beta}{1 - 4\beta}\right) = 1$$  \hspace{1cm} (2.260)

or

$$\beta = \frac{1}{2}$$  \hspace{1cm} (2.261)

which, according to (2.139), corresponds to $k_1 = -\frac{1}{2}$. So, only for a particular value of the gauge fixing parameter the vDVZ discontinuity for this theory disappears. Notice that $k_1 = -\frac{1}{2}$ gives the harmonic gauge condition

$$\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h = 0.$$  \hspace{1cm} (2.262)

and plugging $k_1 = -\frac{1}{2}$ in the direct gauge fixed action (2.141) we obtain exactly the same action we would get from setting $k_1 = k = -\frac{1}{2}$ in (2.127), the quadratic gauge fixing action.

So, we found that the only case in which the vDVZ discontinuity is absent in the direct gauge fixing method (the one with $k_1 = -\frac{1}{2}$) is a special case of the quadratic gauge fixing method.
A gauge theory must provide the same physical results independently of the gauge choice. Therefore, the absence of the vDVZ discontinuity for the particular value of the gauge fixing parameter $k_1 = -\frac{1}{2}$ violates this requirement. Note that even if we were to ignore the case $k_1 = -\frac{1}{2}$, we would still find that the direct gauge fixing does not eliminate the vDVZ discontinuity, hence providing a theory which is not continuous with GR.

In other words, we have obtained a proof of the fact that substituting a gauge constraint into the action is not a proper way of gauge fixing the theory.

We point out that the only reason why the discontinuity disappears when $k_1 = -\frac{1}{2}$ is that, for this value of the gauge fixing parameter, the action becomes exactly the same as the one we would obtain by setting $k_1 = k = -\frac{1}{2}$ in the quadratic gauge fixing case (2.127), which we have proven to be continuous with GR.

In conclusion, with the direct gauge fixing we obtain a theory providing physical predictions which are not continuous with linearized GR (not fulfilling the fifth requirement at the beginning of this Section). Because of this, we have to exclude this approach as a candidate for a MG theory, as expected. Also, every result that we have obtained so far through direct gauge fixing should be discarded from now on.

### 2.6.2 Massive Gravity Theory Proposal

Armed of our results, we are now able to make a proposal for a theory of MG, alternative to the standard FP theory.

We started by finding the propagators with three ways of realizing the gauge fixing condition (2.16), with all the caveats we discussed. Studying the DOFs in these three cases, we excluded all solutions which depend on the gauge choice, since we
want our theory (in the massless limit) to be continuous with linearized GR, whose
measurable quantities do not depend on the gauge parameters. In Section (2.6), we
investigated the vDVZ discontinuity in each case and we concluded that the direct
gauge fixing has to be discarded.

We are now left with two options: the Lagrange multiplier approach and the
quadratic approach.

In Section (2.1), we explained how these two methods of implementing the gauge
fixing restriction on the path integral are equivalent, although the former has the
advantage of eliminating the singularity at $k = 0$ and, more importantly, of substi-
tuting the diffeomorphism symmetry (2.14), which is not a symmetry of the gauge
fixed action, with the BRS symmetry. In order to leave open the possibility of
quantizing the theory by means of the algebraic BRS method [25], we choose the
Lagrange multiplier approach for our MG theory. Nevertheless, as long as we are
interested in the classical results of this theory which do not need the BRS symme-
try, the quadratic approach is totally equivalent and represents a good alternative
to the Lagrange multiplier approach, with the advantage of implying less cumber-
some calculations.

For the Lagrange multiplier case, following the five criteria listed at the beginning
of this Section, we found that the mass parameter $m_2$ has to vanish. Therefore, the
action describing our theory is

$$S_{MG} = \int d^4x \left[ \frac{1}{2} \partial^2 h - h_{\mu\nu} \partial^\mu \partial^\nu h - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} + h^{\mu\nu} \partial_\nu \partial^\rho h_{\mu\rho} \right] +$$

$$\int d^4x \left[ b^\mu (\partial^\nu h_{\mu\nu} + k_1 \partial_\mu h) + \frac{k}{2} b^\mu b_\mu \right] + \int d^4x \frac{1}{2} m_1^2 h_{\mu\nu} h^{\mu\nu} . \quad (2.263)$$
This action can be written in momentum space as

\[ S_{MG} = \int d^4p \begin{bmatrix} \tilde{h}^{\mu\nu} & \tilde{b}^\mu \\ \Lambda_{\mu,\alpha} & H_{\mu\alpha} \end{bmatrix} \begin{bmatrix} \Omega_{\mu\nu,\alpha\beta} & \Lambda_{\mu\nu,\alpha} \\ \Lambda^*_{\mu,\alpha\beta} & H^*_{\mu\alpha} \end{bmatrix} \begin{bmatrix} \tilde{h}^{\alpha\beta} \\ \tilde{b}^\alpha \end{bmatrix} \] (2.264)

and inverting the matrix of kinetic tensors we obtained the matrix of propagators

\[
\begin{bmatrix}
\langle h_{\mu\nu} h_{\alpha\beta} \rangle (p) & \langle h_{\mu\nu} b_{\alpha} \rangle (p) \\
\langle h_{\alpha\beta} b_{\mu} \rangle (p) & \langle b_{\mu} b_{\alpha} \rangle (p)
\end{bmatrix} = \begin{bmatrix} G_{\mu\nu,\alpha\beta} & G_{\mu\nu,\alpha} \\
G^*_{\mu,\alpha\beta} & G_{\mu\alpha}
\end{bmatrix},
\] (2.265)

where

\[
G_{\mu\nu,\alpha\beta} = \hat{t} A_{\mu\nu,\alpha\beta} + \hat{u} B_{\mu\nu,\alpha\beta} + \hat{v} C_{\mu\nu,\alpha\beta} + \hat{z} D_{\mu\nu,\alpha\beta} + \hat{w} E_{\mu\nu,\alpha\beta}
\] (2.266)

\[
G_{\mu\nu,\alpha} = i [f (d_{\mu\alpha} p_\nu + d_{\nu\alpha} p_\mu) + g d_{\mu\nu} p_\alpha + l e_{\mu\nu} p_\alpha]
\] (2.267)

\[
G_{\mu\nu} = r d_{\mu\nu} + s e_{\mu\nu}.
\] (2.268)

The coefficients \( \hat{t}, \hat{u}, \hat{v}, \hat{z}, \hat{w}, f, g, l, r \) and \( s \) can be found at (2.97)—(2.106).

Studying the EOMs (Section (2.3)) given by the action (2.263), we found that this theory propagates five DOFs, as a good theory for a massive graviton should do. In particular, we derived from the EOMs the five constraints

\[
\left\{ \begin{array}{l}
\tilde{h} = 0 \\
p_\nu \tilde{H}^{\mu\nu} = 0
\end{array} \right. \] (2.269, 2.270)

which, subtracted from the original 10 DOFs of the symmetric tensor \( h^{\mu\nu} \), leave the 5 DOFs of a massive spin-2 field. Moreover, thanks to the constraints (2.269) and (2.270) the gauge dependent poles of the propagators (2.266), (2.267) and (2.268)
Covariant Gauge Fixing

do not represent an issue. In fact, when computing the propagation amplitude

\[ \tilde{h}_{\mu \nu} G^{\nu \alpha \beta} \tilde{h}_{\alpha \beta}, \]  

we found, quite remarkably, that only the term with a gauge independent pole does not vanish.

A crucial result is that, as discussed in Section (2.6), this theory has no vDVZ discontinuity, i.e. the massless limit coupling strength between two non-relativistic energy-momentum tensors and between a non-relativistic energy-momentum tensor and an electromagnetic wave is consistent with the predictions of GR. And this comes naturally, without utilising a ghost DOF and non linearities of GR, which through the Vainshtein screening mechanism are invoked to get rid of the vDVZ discontinuity.

In addition, we provided an argument to prove that every physical result deriving form the action (2.263) is consistent with GR in the zero-mass limit. For clarity we now briefly recall that argument.

If our aim is to evaluate measurable quantities in linearized GR, what we would first have to do is to gauge fix the action \( S_{inv} \) (2.15) in order to compute the propagators. Doing so, we would obtain the action \( S \) (2.18).

The central point is to notice that taking the action (2.263) in the limit \( m_1 \to 0 \) yields a well defined theory (i.e. with well defined propagators) which is exactly the gauge fixed action (2.18), the one used to find physical results in linearized GR. So it does not come as a surprise that, in the zero-mass limit, our theory gives the same results as linearized GR, in contrast with the FP action, for which this reasoning does not hold.

Finally, concerning the relation between the mass parameter \( m_1 \) and the mass with which the tensor field \( h^{\mu \nu} \), i.e. a gravitational wave (GW), propagates, with Eq.
we found that
\[ p^2 = -m_1^2, \quad (2.272) \]
which, since \( p^\mu = (E, \vec{P}) \), means
\[ E^2 = m_1^2 + p^2. \quad (2.273) \]

So the mass carried by the graviton (or GW) is, unsurprisingly, \( m_1 \).

## 2.7 Cosmological Constant

In the previous Sections we developed our calculations with the cosmological constant \( \Lambda \) introduced in the action (2.15) set to zero.

The aim of this section is to understand the effect of using a non-vanishing cosmological constant.

First of all we would like to explain why we call it cosmological constant. In the Introduction we show how the action (except for the \( \Lambda \) term) (2.15) can be derived from the linearization of the Einstein-Hilbert action

\[
S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \quad (2.274)
\]

In GR a Cosmological Constant term can be added to this action which becomes

\[
S_{GR} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right), \quad (2.275)
\]

where the coefficient in front of the constant \( \Lambda \) is conventional. The term proportional to \( \Lambda \) turns out to be a way of tuning the rate of expansion of the Universe [19].
Now, expanding the term involving $\Lambda$ in $S_{GR}$ to the second order in $h_{\mu\nu}$ and multiplying by $-8\pi G$, like we did to obtain the action (2.15), we get

$$S_\Lambda = \int d^4x \left( \Lambda + \frac{\Lambda}{2} h + \frac{\Lambda}{8} h^2 + \frac{\Lambda}{4} h_{\mu\nu} h_{\mu\nu} \right).$$  \hspace{1cm} (2.276)

The constant term can be ignored as it does not contribute to EOMs and the quadratic terms in $h$ can be later reabsorbed in the mass terms with parameters $m_1$ and $m_2$. This way we are left only with the linear term $\frac{\Lambda}{2} h$, which is the one reported in (2.15).

Now that we have established the identity of $\Lambda$, we turn to its effects on the theory. In the previous section we used three ways of gauge fixing the action, the Lagrange multiplier method, the quadratic gauge fixing and the direct gauge fixing. We also showed that this last approach gives a theory which is unacceptable according to our request of continuity with GR. The other two methods are instead valid and, indeed, equivalent. In order to show the effect of $\Lambda$ on the theory we will only use the quadratic gauge fixing (which we briefly recall below), as it is the most natural, directly deriving from the $\Phi \Pi$ gauge fixing procedure.

Therefore, to the action $S_{inv}$ (2.15) we add a quadratic gauge fixing and a mass term. Doing so, we get

$$S = \int d^4x \left[ h_{\mu\nu} \Omega^{\mu\nu,\alpha\beta} h_{\alpha\beta} + \frac{\Lambda}{2} h \right]$$ \hspace{1cm} (2.277)

where the kinetic tensor $\Omega^{\mu\nu,\alpha\beta}$ is given by Eq. (2.130).

The propagator, being defined as the inverse of $\Omega^{\mu\nu,\alpha\beta}$, is not affected by the introduction of the Cosmological Constant, and is the one of Eq. (2.132).

What is instead changed by the $\Lambda$ term are the EOMs, which now read
\[
\frac{\delta S}{\delta h_{\mu\nu}} = - \left(1 + \frac{k_1^2}{k}\right) \eta^{\mu\nu} p^2 \tilde{h} + \left(1 - \frac{k_1}{k}\right) p^\mu p^\nu \tilde{h} + \left(1 - \frac{k_1}{k}\right) \eta^{\mu\nu} p^\alpha p^\beta \tilde{h}_{\alpha\beta} + \\
p^2 \tilde{h}_{\mu\nu} - \left(1 + \frac{1}{2k}\right) \left(p^\mu p^\nu \tilde{h}_\alpha + p^\nu p^\alpha \tilde{h}_\mu\right) + m_1^2 \tilde{h}_{\mu\nu} + m_2^2 \eta^{\mu\nu} \tilde{h} = - \frac{\Lambda}{2} \eta^{\mu\nu}.
\]

(2.278)

Proceeding exactly in the same way as for the \( \Lambda = 0 \) case of Section (2.3) we saturate the EOMs first with \( \eta_{\mu\nu} \) and then with \( e_{\mu\nu} \):

\[
\eta_{\mu\nu} : \left[ (m_1^2 + 4m_2^2) - \left(2 + \frac{k_1}{k} (1 + 4k_1)\right) p^2 \right] \tilde{h} + \left(2 - \frac{1}{k} (1 + 4k_1)\right) p^2 e_{\mu\nu} \tilde{h}_{\mu\nu} = -2\Lambda
\]

(2.279)

\[
e_{\mu\nu} : \left[ m_2^2 - \frac{k_1}{k} (1 + k_1) p^2 \right] \tilde{h} + \left[ m_1^2 - \frac{1}{k} (1 + k_1) p^2 \right] e_{\mu\nu} \tilde{h}_{\mu\nu} = - \frac{\Lambda}{2}.
\]

(2.280)

We now subtract 4 times Eq. (2.280) to Eq. (2.279) to obtain

\[
\eta_{\mu\nu} : \left[ m_1^2 + \left(\frac{3k_1}{k} - 2\right) p^2 \right] \tilde{h} + \left[ \left(2 + \frac{3}{k}\right) p^2 - 4m_1^2 \right] e_{\mu\nu} \tilde{h}_{\mu\nu} = 0
\]

(2.281)

\[
e_{\mu\nu} : \left[ m_2^2 - \frac{k_1}{k} (1 + k_1) p^2 \right] \tilde{h} + \left[ m_1^2 - \frac{1}{k} (1 + k_1) p^2 \right] e_{\mu\nu} \tilde{h}_{\mu\nu} = - \frac{\Lambda}{2}.
\]

(2.282)

which are clearly not solved anymore by

\[
\tilde{h} = e^{\mu\nu} \tilde{h}_{\mu\nu} = 0.
\]

(2.283)

Since the above equations are not trivial as the ones we found with \( \Lambda = 0 \), we cannot conclude that the graviton described by the action \( S \) (2.277) propagates 5 DOFs.
Chapter 3

Lorentz Violating Mass Term

The aim of this Chapter is to study the effects of a Lorentz violating mass term in MG. We will also explore the possibility of a Lorentz violating gauge fixing condition, such as the transverse gauge or the synchronous gauge, which we will recall in due course.

The motivation behind this Chapter is the idea that a Lorentz violating mass term could potentially yield a Lorentz violating dispersion relation of the type

$$E^2 = m^2 + \sum_{n \geq 0} \alpha_n |p|^n$$  \hspace{1cm} (3.1)

with $\alpha_n \neq 0$ for some $n$. Some examples theories in which this kind of dispersion relation is found are:

- Double Special Relativity [26–29]: $E^2 = m_g^2 c^4 + p^2 c^2 + \eta_{drst} E^3 + ...$, where $\eta_{drst}$ is a parameter of the order of the Planck length;

- Extra-Dimensional Theory [30]: $E^2 = m_g^2 c^4 + p^2 c^2 - \alpha_{edt}$, where $\eta_{drst}$ is a constant of the order of the Planck length;

- Horava-Lifshitz Theory [31–34]: $E^2 = p^2 c^2 + \frac{k_h \mu_{hl}}{16} p^4 + ...$, where $k_{hl}$ and $\mu_{hl}$ are constants of the theory;
• Non-Commutative Geometries Theories [35–37]: \( E^2 g_1(E) = m^2 c^4 + p^2 c^2 \), with 
\[ g_1 = \left( 1 - \frac{\sqrt{\alpha_{\text{neg}}}}{2} \right) \exp \left( -\alpha \frac{E^2}{E_0^2} \right), \] 
with \( \alpha_{\text{neg}} \) a constant.

This type of relation between energy and momentum of a wave leaves a signature on the phase evolution of GWs, which can be experimentally detected [38].

The reason why we expect a relation of the type (3.1) after breaking the Lorentz invariance of the action is quite simple. We define the mass of a particle as the simple pole of the propagator of the corresponding field, which depends on the 4-momentum \( p^\mu \). If Lorentz invariance is respected, the dependence on \( p^\mu \) of the denominator of the propagator can only be through \( p^2 \), since it is the only scalar we can build from \( p^\mu \). Therefore, finding the poles of such a propagator can only lead to something of the type

\[ p^2 = -M^2. \] (3.2)

If instead we relax the Lorentz invariance condition on the mass term, seen as a small perturbation of the theory, we can hope to find poles with more complicated expressions, which, in turn, could lead to a dispersion relation like that of Eq. (3.1).

We will limit ourselves to a time-space Lorentz breaking, isolating the time components from the spatial components of the field \( h_{\mu\nu} \).

Before writing the most generic mass term of this type we need to rewrite the field \( h_{\mu\nu} \) by defining new useful fields. Conventionally [19] these are:

\[
\begin{align*}
h_{00} &= -2\Phi \quad \text{(3.3)} \\
h_{0i} &= w_i \quad \text{(3.4)} \\
h_{ij} &= 2(s_{ij} - \Psi\delta_{ij}) \quad \text{(3.5)}
\end{align*}
\]

Inverting these relations one obtains
Lorentz Violating Mass Term

\[
\begin{align*}
\Phi &= -\frac{1}{2} h_{00} \tag{3.6} \\
\omega_i &= h_{0i} \tag{3.7} \\
\sigma_{ij} &= \frac{1}{2} (h_{ij} + 2\delta_{ij}\Psi) \tag{3.8} \\
\Psi &= -\frac{1}{6} \delta_{ij} h_{ij} \tag{3.9}
\end{align*}
\]

So, in this reparametrization, \( \Phi \) represents the 00-component of the GW, \( \omega_i \) represents the vectorial components of \( h_{\mu\nu} \), \( \Psi \) is proportional to the spatial trace of \( h_{\mu\nu} \) and \( \sigma_{ij} \) is proportional to the traceless part of \( h_{ij} \). This decomposition is the standard choice in the massless case [19].

In terms of the new fields, the invariant action (2.15) reads, in momentum space,

\[
S_{\text{inv}} = \int d^4p \quad 6\tilde{\Psi}(2\tilde{p}^2 - 3p^2)\tilde{\Psi} - \tilde{p}^2\tilde{\omega}_i\tilde{\omega}_i + p_ip_j\tilde{\omega}_i\tilde{\omega}_j + 8p_0p_i\tilde{\omega}_i\tilde{\Psi} + 4p_ip_j\tilde{\sigma}_{ij}(\tilde{\Phi} - \tilde{\Psi}) + 8\tilde{p}^2\tilde{\Phi}\tilde{\Psi} - 4p_ip_k\tilde{\sigma}_{ij}\tilde{\sigma}_{ik} + 2p^2\tilde{\sigma}_{ij}\tilde{\sigma}_{ij} + 4p_0p_i\tilde{\sigma}_{ij}\tilde{\omega}_j \tag{3.10}
\]

and the most general Lorentz violating mass term (which preserves rotational invariance of the action) is of the form

\[
S_m = \int d^4x \quad (m_0^2\Phi^2 + m_1^2\Psi^2 + 2m_2^2\Phi\Psi + m_3^2\omega_i\omega_i + m_4^2\sigma_{ij}\sigma_{ij}) \quad . \tag{3.11}
\]

Following the ideas presented in the Introduction and Chapter 1, in addition to a mass term, we need a gauge fixing term so that our theory is well defined and continuous with LGR in the massless limit, \( \text{i.e.} \) the limit in which all masses go to zero.

For the choice of the gauge fixing we have two options:

- implementing a covariant gauge fixing, identical to those of Chapter 2;
- implementing a Lorentz violating gauge fixing, such as the transverse gauge.
In the following Sections we will develop both these options, even though we will particularly focus on the second. In both cases we will fix the gauge by introducing a gauge fixing term according to the $\Phi\Pi$ exponentiation procedure.

3.1 Covariant Gauge Fixing

The most generic covariantly gauge fixed massless action is the one of Eq. (2.127),

\[
S = S_{inv} + S_{gf} = \int d^4x \left[ \frac{1}{2} h \partial^2 h - h_{\mu\nu} \partial^\mu \partial^\nu h - \frac{1}{2} h^{\mu\nu} \partial^2 h_{\mu\nu} + h^{\mu\nu} \partial_\rho \partial^\rho h_{\mu\nu} - \frac{1}{2k} (\partial_\nu h^{\mu\nu} + k_1 \partial^\nu h)^2 \right] \tag{3.12}
\]

For sake of simplicity we decide to fix the gauge parameters to $k = k_1 = -\frac{1}{2}$. We will develop the whole calculation only for this choice of parameters. The harmonic gauge choice yields the gauge fixed action

\[
S_H = \int d^4x \left[ \frac{1}{4} h \partial^2 h - \frac{1}{2} h_{\mu\nu} \partial^2 h^{\mu\nu} \right], \tag{3.13}
\]

which, in terms of the fields (3.6)—(3.9), is

\[
S = \int d^4x \left[ -\Phi \partial^2 \Phi + 3\Psi \partial^2 \Psi - 6\Phi \partial^2 \Psi + w_i \partial^2 w_i - 2s_{ij} \partial^2 s_{ij} \right]. \tag{3.14}
\]

Adding the mass term $S_m$ of Eq. (3.11) we obtain the MG action

\[
S_{MG} = S_H + S_m = \int d^4x \left[ -\Phi(\partial^2 - m_0^2)\Phi + 3\Psi \left( \partial^2 - \frac{m_1^2}{3} \right) \Psi - 6\Phi \left( \partial^2 - \frac{m_2^2}{3} \right) \Psi + w_i(\partial^2 - m_3^2)w_i - 2s_{ij} \left( \partial^2 - \frac{m_1^2}{2} \right) s_{ij} \right]. \tag{3.15}
\]
Notice that with the harmonic gauge choice the fields $w_i$ and $s_{ij}$ decouple, while the fields $\Phi$ and $\Psi$ are coupled.

The massive action (3.15) can be rewritten in momentum space as

$$S_{MG} = \int \mathrm{d}^4 p \ \begin{bmatrix} \tilde{s}_{ij} & \tilde{w}_i & \tilde{\Psi} & \tilde{\Phi} \end{bmatrix} \begin{bmatrix} \Omega_{ij,lm} & 0 & 0 & 0 \\ 0 & \Omega_{il} & 0 & 0 \\ 0 & 0 & \Omega & \Lambda \\ 0 & 0 & \Lambda & \Gamma \end{bmatrix} \begin{bmatrix} \tilde{s}_{lm} \\ \tilde{w}_l \\ \tilde{\Psi} \\ \tilde{\Phi} \end{bmatrix},$$

where

$$\Omega_{ij,lm} = 2(p^2 + m_4^2)\mathcal{I}_{ij,lm}$$

$$\Omega_{il} = -(p^2 + m_3^2)\delta_{il}$$

$$\Omega = 6(p^2 + m_2^2)$$

$$\Lambda = -\frac{3}{2}(p^2 + m_1^2)$$

$$\Gamma = (p^2 + m_0^2).$$

From Eq. (3.17) it is easy to find the propagator:

$$G = \begin{bmatrix} G_{ij,lm} & 0 & 0 & 0 \\ 0 & G_{il} & 0 & 0 \\ 0 & 0 & G & \Theta \\ 0 & 0 & \Theta & \Delta \end{bmatrix}$$

imposing
Solving (3.23) we find

\[ G_{ij,lm} = \langle s_{ij}s_{lm} \rangle (p) = \frac{1}{2} \frac{1}{p^2 + m_0^2} \mathcal{I}_{ij,lm} \]  \hspace{1cm} (3.24) 

\[ G_{il} = \langle w_iw_l \rangle (p) = -\frac{1}{3} \frac{1}{p^2 + m_0^2} \delta_{il} \]  \hspace{1cm} (3.25) 

\[ G = \langle \Psi^*\Psi \rangle (p) = \frac{4}{3} \frac{(p^2 + m_0^2)}{(p^2 + m_0^2)(p^2 + m_0^2) - 3(p^2 + m_1^2)^2} \]  \hspace{1cm} (3.26) 

\[ \Theta = \langle \Psi^*\Phi \rangle (p) = \frac{2}{8} \frac{(p^2 + m_1^2)}{(p^2 + m_0^2)(p^2 + m_0^2) - 3(p^2 + m_1^2)^2} \]  \hspace{1cm} (3.27) 

\[ \Delta = \langle \Phi^*\Phi \rangle (p) = \frac{8}{8} \frac{(p^2 + m_2^2)}{(p^2 + m_0^2)(p^2 + m_0^2) - 3(p^2 + m_1^2)^2} \]  \hspace{1cm} (3.28) 

We see that despite the Lorentz breaking mass term (3.11) we obtained a propagators which do not lead to an explicit Lorentz violating dispersion relation of the kind (3.1). We now move to the case of a non-covariant gauge fixing condition.

### 3.2 Lorentz violating Gauge Fixing

We now go back to the action \( S_{inv} \) of Eq.(3.10). A commonly adopted choice of Lorentz violating gauge is the transverse gauge (see [19]),

\[ \partial_i s_{ij} = 0; \quad \partial_i w_i = 0. \]  \hspace{1cm} (3.29)
The corresponding $\Phi\Pi$ gauge fixing term is

$$S_{gf} = -\int d^4x \left[ \frac{1}{2\xi_1} (\partial_i s_{ij})^2 + \frac{1}{2\xi_2} (\partial_i w_i)^2 \right], \quad (3.30)$$

where two different gauge fixing parameters $\xi_1$ and $\xi_2$ have been used.

The complete massive action is

$$S_{MG} = S_{inv} + S_{gf} + S_m, \quad (3.31)$$

where $S_m$ is given by (3.11) We are interested in inverting the kinetic tensor of this action in order to obtain the propagators. In momentum space we have

$$S_{MG} = \int d^4p \begin{bmatrix} \tilde{s}_{ij} & \tilde{w}_i & \tilde{\Psi} & \tilde{\Phi} \end{bmatrix} \begin{bmatrix} \Omega_{ij,lm} & \Omega_{lm,i} & \Omega_{lm} & \Lambda_{lm} \\ \Omega_{ij,l} & \Theta_{il} & \Gamma_l & 0 \\ \Omega_{ij} & \Gamma_l & \Omega & \Lambda \\ \Lambda_{ij} & 0 & \Lambda & \Gamma \end{bmatrix} \begin{bmatrix} \tilde{s}_{lm} \\ \tilde{w}_l \\ \tilde{\Psi} \\ \tilde{\Phi} \end{bmatrix}, \quad (3.32)$$

where
Lorentz Violating Mass Term

\[ \Omega_{ij,lm} = \left( \frac{1}{2\xi_1} - 4 \right) \frac{1}{4} \left( \delta_{il} p_j p_m + \delta_{jl} p_i p_m + \delta_{lm} p_j p_i + \delta_{jm} p_i p_l \right) + \]

\[ (2p^2 + m_4^2) \frac{1}{2} \left( \delta_{il} \delta_{jm} + \delta_{jl} \delta_{im} \right) \quad (3.33) \]

\[ \Omega_{ij,l} = p_0 (p_i \delta_{jl} + p_j \delta_{il}) \quad (3.34) \]

\[ \Omega_{ij} = -2p_i p_j \quad (3.35) \]

\[ \Lambda_{ij} = 2p_i p_j \quad (3.36) \]

\[ \Theta_{il} = (m_3^2 - p^2) \delta_{il} + \left( 1 + \frac{1}{2\xi_2} \right) p_i p_l \quad (3.37) \]

\[ \Gamma_i = 4p_0 p_i \quad (3.38) \]

\[ \Omega = 4(2p^2 - 3p^2) \quad (3.39) \]

\[ \Lambda = 4p^2 + m_2^2 \quad (3.40) \]

\[ \Gamma = m_0^2 . \quad (3.41) \]

As before, we define the propagators matrix

\[ G = \begin{bmatrix}
G_{lm, kh} & L_{kh, l} & F_{kh} & G_{kh} \\
L_{lm, k} & E_{lk} & D_k & Z_k \\
F_{lm} & D_l & C & B \\
G_{lm} & Z_l & B & A
\end{bmatrix} \quad (3.42) \]

by imposing
Looking at the expansion of Eqs. (2.55), (2.56) and (2.57) and recalling that we are now working with spatial indices \( D = 3 \), we can decompose the propagator tensors (3.42) by defining (just like in Section 2) the basis of symmetric rank-2 tensors

\[
e_{ij} = \frac{p_i p_j}{\vec{p}^2} \quad d_{ij} = \delta_{ij} - e_{ij} ,
\]

and that of symmetric rank-4 tensors

\[
A_{lm,kh} = \frac{d_{lm} d_{kh}}{2} \quad (3.45)
B_{lm,kh} = e_{lm} e_{kh} \quad (3.46)
C_{lm,kh} = \frac{1}{2} (d_{lk} d_{mh} + d_{lh} d_{mk} - d_{ln} d_{kh}) \quad (3.47)
D_{lm,kh} = \frac{1}{2} (d_{lk} e_{mh} + d_{lh} e_{mk} + e_{lk} d_{mh} + e_{lh} d_{mk}) \quad (3.48)
E_{lm,kh} = \frac{\delta_{lm} \delta_{kh}}{3} \quad (3.49)
\]

These tensors satisfy identical properties to those of the covariant case (Eq. (2.37), etc.).

Using this basis, the decomposition of the propagators reads:
Lorentz Violating Mass Term

\[ G_{lm,kh} = \alpha A_{lm,kh} + \beta B_{lm,kh} + \gamma C_{lm,kh} + \delta D_{lm,kh} + \eta E_{lm,kh} \quad (3.50) \]

\[ L_{kh,l} = f (p_k d_{hl} + p_l d_{kh}) + g p_l d_{kh} + l p_l e_{kh} \quad (3.51) \]

\[ E_{lk} = r d_{lk} + s e_{lk} \quad (3.52) \]

\[ F_{kh} = a d_{kh} + b e_{kh} \quad (3.53) \]

\[ G_{kh} = c d_{kh} + d e_{kh} \quad (3.54) \]

\[ D_k = m p_k \quad (3.55) \]

\[ Z_k = n p_k \quad (3.56) \]

Our task is now to work out the values of the coefficients in Eqs. (3.50)—(3.56) using the defining equation for the propagator matrix, Eq. (3.43).

The first step is to write out explicitly Eq. (3.43) into 16 tensor equations. A clever way of grouping these equations is the following. We consider the 4 sets of 4 equations corresponding to the individual rows of the propagator matrix in Eq. (3.43) multiplying one by one all 4 columns of the kinetic tensor. For instance the first set is composed of the equations resulting from the first row of the propagator matrix (the one containing \( G_{lm,kh} \)) multiplying the 4 columns of the kinetic tensor. In this way every set of 4 equations allows us to compute all the coefficients contained in it.

After some simplification these become:

Set 1
\[2e_{ij}((a - c)\delta_{lm} + (b - d)e_{lm})p^2 +
\]
\[p_0^2\left[f(d_{ij}e_{im} + d_{mj}e_{il} + d_{li}e_{jm} + d_{mi}e_{jl}) + g_{e_{ij}d_{lm}} + le_{ij}e_{lm}\right] +
\]
\[\left(\frac{1}{2\xi_1} - 4\right)\left[\beta e_{ij}e_{im} + \frac{\delta}{4}(d_{im}e_{jl} + d_{il}e_{jm} + d_{jm}e_{il} + d_{jl}e_{im}) + \frac{\eta}{3}e_{ij}\delta_{lm}\right]p^2 +
\]
\[2p^2 + m_1^2\left[\frac{\alpha}{2}d_{ij}d_{lm} + \beta e_{ij}e_{im} + \frac{\gamma}{2}(d_{il}d_{jm} + d_{jl}d_{im} - d_{ij}d_{lm}) +
\]
\[\frac{\delta}{2}(d_{im}e_{jl} + d_{il}e_{jm} + d_{jm}e_{il} + d_{jl}e_{im}) + \frac{\eta}{3}d_{ij}\delta_{lm}\right] = 0 \quad (3.57)
\]
\[2e_{ij}p_i(n - m) + p_0(r p_i d_{jl} + s p_i e_{jl} + r p_j d_{il} + s p_j e_{il}) +
\]
\[\left(\frac{1}{2\xi_1} - 4\right)\left[\frac{f}{2}(d_{il}p_j + d_{ij}p_i) + p_i e_{ij}\right]p^2 +
\]
\[2p^2 + m_1^2\left[f(p_i d_{jl} + p_j d_{il}) + g p_id_{ij} + p_i e_{ij}\right] = 0 \quad (3.58)
\]
\[2e_{ij}(B - C + p_0 m) + e_{ij}p^2\left(\frac{1}{2\xi_1} - 4\right)d + (2p^2 + m_1^2)(c d_{ij} + d e_{ij}) = 0 \quad (3.59)
\]
\[2e_{ij}(A - B + p_0 n) + e_{ij}p^2\left(\frac{1}{2\xi_1} - 4\right)b + (2p^2 + m_1^2)(a d_{ij} + b e_{ij}) = 0 ; \quad (3.60)
\]

Set 2

99
4p₀pᵢ(credᵢm + dᵢeᵢm) + (mₓ² - p²)[f(pᵢdᵢm + pₘdᵢ) + gₚᵢdᵢm + lₚᵢeᵢm] + 

\[ \left( 1 + \frac{1}{2\xi_2} \right) pᵢ\bar{p}²(gdᵢm + leᵢm) + 2p₀\left[ βpᵢeᵢm + \frac{δ}{2}(pₘdᵢ + pᵢdₘ + \frac{η}{3}pᵢδᵢm) \right] \right] = 0 \quad (3.61)
Set 4

\[ m_0^2(a d_{lm} + b e_{lm}) + (4p^2 + m_2^2)(c d_{lm} + d e_{lm}) + 2p^2 \left( \beta e_{lm} + \frac{\eta}{3} \delta_{lm} \right) \]  

(3.69)

\[ m_0^2 n + (4p^2 + m_2^2)m + 2t p^2 = 0 \]  

(3.70)

\[ m_0^2 B + (4p^2 + m_2^2)C + 2p^2 d = 0 \]  

(3.71)

\[ m_0^2 A + (4p^2 + m_2^2)B + 2p^2 b = 1 . \]  

(3.72)

The coefficients in the above equations are in bold type.

Taking Eq. (3.57) as an example, one can find relations between the coefficients by bringing the identity to the left-hand side of the equation and then setting to zero the coefficients of every linearly independent component in the expression: \( \delta_{ij} \delta_{lm} \), \( \delta_{ij} e_{lm} \), \( \delta_{il} \delta_{jm} \), \( \delta_{im} e_{jm} \), etc. Of course not all of the resulting equations are independent because of the symmetries between indices. One must then do the same for every equation of the 4 sets above. Set 1 yields 11 independent equations for the coefficients \( \alpha, \beta, \gamma, \delta, \eta, a, b, c, d, f \), and \( l \); Set 2 gives 7 equations for the coefficients \( s, r, f, g, l, m, n \); Set 3 gives 4 equations for \( B, C, d, m \); Set 4 yields another 4 equations for \( A, B, b, n \). Of course the solutions for the coefficients which appear in more than one set must coincide.

With the aid of Mathematica we are able to find solutions for all 16 coefficients and therefore obtain the whole propagator, without conditions on the mass or gauge.
Lorentz Violating Mass Term

parameters required.

Unfortunately the explicit expressions for the coefficients are too long to report here. We will limit ourselves to write the propagator for specific values of the gauge parameters. In the "Landau-Landau" gauge $\xi_1 = \xi_2 = 0$, we obtain the following propagator:

$$G = \begin{pmatrix}
\frac{1}{2}(d_{lh}d_{m,h} + d_{lh}d_{m,k}) & 0 & 0 & 0 \\
0 & \frac{d_{lk}}{m_3^2 - \vec{p}^2} & 0 & 0 \\
0 & 0 & \frac{-m_0^2}{\Delta} & \frac{m_2^2 + 4\vec{p}^2}{\Delta} \\
0 & 0 & \frac{m_2^2 + 4\vec{p}^2}{\Delta} & \frac{4\vec{p}^2 - 12p_0^2}{\Delta}
\end{pmatrix} \tag{3.73}$$

with

$$\Delta = -12p_0^2 + 16\vec{p}^2 + 4(2m_2^2 + m_0^2)\vec{p}^2 + m_4^2. \tag{3.74}$$

Notice that the Landau-Landau propagator is particularly simple compared to the general result.

In the next Section we will comment this propagator and analyze the propagating DOFs.

### 3.2.1 Propagator and DOFs

In Chapter 2 we used the EOMs to deduce the DOFs propagated by the theory. An alternative way to count how many DOFs are propagated by a field is to look at the propagator and try to understand how many constraints it satisfies.

Initially we start from 10 components of the original $h_{\mu\nu}(x)$ symmetric tensor. We expect to have no more than 6 DOFs left after imposing the 4 gauge conditions

$$\partial_i s_{ij} = 0 , \tag{3.75}$$
3.2 – Lorentz violating Gauge Fixing

\[ \partial_i w_i = 0. \quad (3.76) \]

through the "Landau-Landau" gauge fixing term.

Straight away we notice that the pole of the \( w_i \) sector of the propagator \( E_{ik} \) is non dynamic: the fields \( w_i \) do not propagate and, in absence of sources, are identically zero.

On the contrary, the propagator \( G_{ij,kh} \) is dynamic. Its pole is

\[ p^2 = -\frac{m_i^2}{2}. \quad (3.77) \]

This is not surprising because \( m_i^2 \) is the mass parameter of the quadratic term \( s_{ij} s_{ij} \) in the action (3.31). On its own, \( s_{ij} \), being a symmetric traceless spatial tensor, carries 5 DOFs. The 3 conditions (3.75) reduce the 5 DOFs to only 2 DOFs.

The fields \( \Phi \) and \( \Psi \) have the same dynamic pole (for \( m_0 \neq 0 \))

\[ \Delta = 0 \implies p_0^2 = \frac{m_i^4}{12m_0^2} + \frac{2m_i^2 + m_0^2}{3m_0^2} p^2 + \frac{4}{3m_0^2} p^4 \quad (3.78) \]

which is a dispersion relation of the type we were looking for (see Eq. (3.1)).

Moreover, the fields \( \Phi \) and \( \Psi \) have mixed propagator terms and no constraints: they propagate 2 DOFs. Summarizing:
<table>
<thead>
<tr>
<th>Field</th>
<th>DOFs Propagated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{ij}$</td>
<td>2</td>
</tr>
<tr>
<td>$w_i$</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>1</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>1</td>
</tr>
<tr>
<td>Total DOFs</td>
<td>4</td>
</tr>
</tbody>
</table>

From the table above we can see that the total number of DOFs propagated by this theory is 4, while the required DOFs for a spin 2 massive particle is 5.

Furthermore we see that, depending on the gauge fixing we used, harmonic or transverse, we obtained different poles in the propagator. In the first case, the harmonic gauge fixing, we did not find poles with a dispersion relation of the type (3.1), while in the second case, the Landau-Landau transverse gauge, we obtained such poles. We point out that the coefficient of the $\vec{p}^4$ term in (3.78) cannot be set to zero by a choice mass parameters, therefore it is not possible to impose a constraint on the mass term in order to eliminate this feature of the propagator.

We deduce that, although the gauge choice is unphysical in a massless theory, in our massive theory it seems to acquire a physical role.
Conclusions

This Thesis presented an alternative approach to the standard treatment of linearized MG, based on a field theoretical treatment. The FP theory was shown to be based on an action characterized by a divergent massless limit, which, in addition, is discontinuous with GR. The main reason of this major flaw is that the FP action (1.45) is the result of adding a mass term directly to the invariant action. The drawback is that in the massless limit the propagator diverges and the predictions of the theory are discontinuous with those of GR as $m_G$ goes to 0. This is called the vDVZ discontinuity and we explained how it manifests itself in the interactions between energy-momentum tensors in Section 2.6.1. The way to solve the vDVZ discontinuity is by reintroducing the non-linearities of GR (the Vainshtein mechanism [12,15]).

The claim discussed in this Thesis is that the discontinuity is due to the fact that the mass parameter $m_G$ in the FP action (1.45) plays the double role of mass parameter and gauge fixing parameter. In fact, the FP propagator exists (i.e. the theory has a dynamics) only due to the presence of the mass term (1.35), which, for this reason, serves as gauge fixing as well. The contradiction is in the fact that a gauge fixing parameter should not be physical, whereas a mass parameter is a measurable quantity: the two things evidently cannot coincide.

This issue was solved in this Thesis by writing an action which includes a gauge fixing term in addition to the mass term which breaks the gauge symmetry (1.4),
as customary in any gauge field theory. The mechanism through which the gravi-
ton should acquire mass goes beyond the scope of this Thesis and requires further investigation.

Following the ΦΠ procedure, we adopted a covariant gauge fixing, see Eqs. (2.126) and (2.127), and we obtained a theory with a massive propagator and with a well defined massless limit, in a way independent of the gauge parameters. In the mass-
less limit, the propagator (2.132) becomes exactly the one we would have obtained in classic LG, provided that we used the same gauge fixing term (2.126). This is an encouraging result, suggesting the absence of the vDVZ discontinuity. In fact, this is exactly the case: we proved that the massive action (2.127) describes a mas-
sive theory which is continuous with LG in the massless limit. In Section 2.6.1 we explicitly checked that the interactions between two massive bodies and between a massive body and electromagnetic radiation are continuous with LG as the masses $m_1$ and $m_2$ vanish, but a direct consequence of the structure of the massive action proposed in this Thesis is that every physical observable is continuous with LG in the massless limit. An interpretation of this result might be that the addition of the gauge fixing term in the action protects the theory and its physical predictions in the massless limit, avoiding the reintroduction of invariance in the action.

In Chapter 2 we studied the EOMs deriving from the massive gauge fixed action. Requiring that the mass of the graviton field $h_{\mu\nu}$ does not depend on the gauge fixing parameters, we obtained that the massive action (2.127) describes a transverse traceless symmetric tensor field which therefore propagates 5 massive DOFs. In particular, thanks to transversality and tracelessness of the field, in Chapter 2 we could observe that the propagation amplitude of the field $h_{\mu\nu}$ loses any dependence on the gauge fixing parameters.

Another interesting feature of the massive action (2.127) (and of its counterpart Eq.
Conclusions

(2.21) where the gauge fixing is obtained through a vectorial Lagrange multiplier \( b_\mu \) is the fact that the mass parameter \( m_2 \) plays a marginal role in the propagation of the graviton field itself. In other words, we found that while the mass \( m_1 \) must not vanish in order to have massive propagation of \( h_{\mu\nu} \), the mass \( m_2 \) is completely free to vary and, in particular, it might vanish entirely. Looking at the propagator (2.132) and considering the fact that the field \( h_{\mu\nu} \) was shown to be transverse and traceless, we find that the mass \( m_2 \) only appears in sectors of the propagator which drop out once they are contracted with the graviton field, exactly like for the gauge fixing parameters. This was shown in Section 2.3.2. Tracelessness and transversality are implied by the EOMs, i.e. when the field is on-shell. In the eventuality of an attempt to quantize MG, one should consider that the dependence on the gauge fixing and on the mass \( m_2 \) would reappear and would have to be dealt with. Notice that the freedom on \( m_2 \) means that the FP tuning is included in our theory but represents only a very special case, while in the FP theory it was the only available option. This remarkable fact, which reproduces the result of the FP theory in a completely different context, might suggest that the dependence of the theory on the mass parameter \( m_2 \) is confined in an unphysical sector, like a kind of gauge parameter. One of the possible extensions and applications of this Thesis is the full BRS formulation of MG, which should be possible since MG is the gauge field theory of a symmetric tensor field. The anomalous role of \( m_2 \) would be explained by a relation of this type

\[
\frac{\partial}{\partial m_2} S_{MG} = s \hat{S},
\]

where \( S_{MG} \) is the action of MG, \( s \) is the BRS operator and \( \hat{S} \) is a functional with \( \Phi \Pi \)-charge -1. But this is left for the future.

It is a remarkable fact that, even in a theory which is fundamentally different from the FP theory of MG, there is still only one relevant mass parameter. Moreover, taking into account Eqs. (2.231) and (2.235), if the mass of a field is to be identified
with the pole(s) of its propagator, only $m_1$ should be regarded as a true mass while $m_2$ should only be considered a free parameter of the theory whose effect is yet to be identified.

In Section 2.6.1 we showed that, in the massless limit of our gauge fixed MG theory, interactions between energy-momentum tensors are gauge independent. However, when the mass parameter $m_1$ and $m_2$ do not vanish, the interaction between two energy-momentum tensors does depend on the gauge fixing parameters as well as on the mass $m_2$. Still, if the two energy momentum tensors are conserved and traceless (like those of electromagnetic waves) the independence from the gauge parameters and $m_2$ is preserved. More exploration in this direction is needed to understand what is the role played by the gauge fixing parameters and $m_2$ in the massive theory.

Chapter 3 we evaluated the idea of adding a Lorentz-violating mass term in order to obtain a dispersion relation of the type (3.1) for the graviton, which would leave a measurable signature on the gravitational waveforms detected at interferometers such as Virgo and LIGO. Indeed, we managed to obtain such a dispersion relation, see (3.78). This result was obtained with arbitrary values of the mass parameters, while we fixed a particular gauge for simplicity.

Unfortunately, in Section 3.2.1 we showed that the action producing this dispersion relation also propagates 4 DOFs, while a massive graviton should propagate 5 DOFs. Additionally, we found that the Lorentz-violating theory predictions depend on the gauge fixing choice. Choosing the harmonic gauge fixing (3.13) the theory propagates at most 6 DOFs, while with the transverse gauge (3.30) only 4 are propagated. A way of fixing this could be studying the relation between gauge fixing and mass parameters. Like in the covariant case, one could hope to find constraints on the mass parameters such that the gauge dependence of the physical predictions vanishes. The non-covariant formulation of MG and the search for a
Lorentz-violating dispersion relation remains an open question, which we leave for the future.

Summarizing, the main result of this Thesis is the proposal of a new Massive Gravity theory which solves the vDVZ discontinuity problem and has a well-defined massless limit because of the addition of a gauge fixing term into the action.
Bibliography


