Doctorate of Philosophy in
PHYSICS

Classical Moduli Spaces of
Non-Abelian Vortices

Ph.D Thesis
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Introduction

At the begin of the mid seventies, mainly thanks to the discovery of asymptotic freedom, people started to believe that QCD could be a good model to describe strong interactions. After that, a striking collection of experimental data was collected, which confirmed the predictions of QCD. To recognize QCD as a correct model was a breakthrough in the understanding of strong interactions, but this discovery forced physicists to face an difficult problem: the study of strongly coupled gauge theories. In fact, no one knows how to use QCD to explain the most elementary experimental observations for strong interactions: color confinement.

While we can always rely on perturbation theory to extract predictions from a weakly coupled gauge theory, there are no general techniques to extract exact results for the physics of strongly coupled theories. Nevertheless, physicists identified many non-perturbative mechanisms, and their effects, which generically occur in gauge theories. Maybe one of the first and most known example is the resolution of the so-called $U(1)$ problem. ’t Hooft solved the problem by taking into account the instantonic contributions to the QCD functional integral \[1\]. Instantons are a particular type of solitons, extended classical configurations which can give relevant contributions to the path integral. It is thus widely believed that solitons play crucial role in the dynamics of non-Abelian gauge theories. This is true for the microscopic dynamics as well as for the evolution of the universe (cosmic defect).

Another, striking, example of the importance of solitons in the non-perturbative dynamics of gauge theories is the ’t Hooft and Mandelstam mechanism of confinement called dual superconductivity \[2\]: the QCD vacuum is a kind of dual superconductor, in which the role of electric and magnetic charges is exchanged. If magnetic monopoles condense in the vacuum, the electric sources are confined by tubes of electric flux, just like in an ordinary superconductor magnetic charges would be confined by Abrikosov-Nielsen-Olesen vortices (Meissner effect). This idea is strongly supported by experimental observations like Regge trajectories and lattice simulations which show a linear quark-antiquark potential.
This mechanism is very simple, but it is prohibitive to demonstrate its effective occurrence: it involves solitonic objects, like magnetic monopoles and vortices, which appear in a phase of the gauge theory which is usually strongly coupled. For these reasons this picture of confinement in non-Abelian gauge theories remained just a qualitative sketch for many years.

In the nineties, Seiberg and Witten reached a striking goal: they could find the exact low energy effective action of supersymmetric gauge theories with extended supersymmetry ($\mathcal{N} = 2$) \cite{3,4}. To reach their result they collected together properties of supersymmetric gauge theories like non-renormalization theorems, holomorphicity of superpotentials, appearance of moduli space of vacua, with general properties of gauge theories like perturbative and instantonic effects. In their study the electro-magnetic duality plays a crucial role. Using their result it is possible to rigourously demonstrate the actual occurrence of dual superconductivity in a large class of supersymmetric gauge theories. The appearance of this mechanism in close supersymmetric relatives of QCD strengthened the idea that dual superconductivity could be the actual mechanism of color confinement chosen by Nature. Even though the first models studied with these techniques were very interesting, they had some crucial qualitative differences with respect to QCD. One of the most important is the excessive richness of the hadronic spectrum. In fact, the generic quantum vacuum of an $\mathcal{N} = 2$ $SU(N)$ gauge, always undergoes dynamical abelianization. This means that the gauge group is dynamically broken to $U(1)^{N-1}$. Thus, an infinite tower of mesons, for each of the $N-1$ abelian strings, exists. This is clearly not the case of QCD.

We can significantly improve this picture considering $\mathcal{N} = 2$ theories with fundamental matter fields. In these theories, for some values of the bare masses of the matter fields, there exist the so called $r$-vacua, where the low energy physics is described by a dual non-Abelian $SU(r)$ gauge theory. In this vacua there appear massless degrees of freedom which can be identified as quantum monopoles. When we add a mass perturbation, these monopole condense, providing a dual mechanism of non-Abelian Meissner effect. It is possible to study this mechanism in a semi-classical regime. If we have a sufficient number of flavor, and we take the bare mass of the squarks to be high enough (much bigger than the strong coupling scale of the theory), we can get a system with the following hierarchical symmetry breaking pattern:

$$SU(N + 1) \xrightarrow{v_1} SU(N) \times U(1) \xrightarrow{v_2} \emptyset.$$  

The theory is weakly coupled at all scales, and the hierarchy is huge: $v_1 \gg v_2$. Topological arguments show that at the scale $v_1$ the theory supports monopoles while at lower scales, $v_2$, it supports vortices. At the same time,
if one consider the theory at every scale, one can see that there are no truly stable monopoles, neither vortices. This argument show that a vortex can exist only if it ends on a monopole-antimonopole pair, thus monopoles are confined. It is important to stress that this is a non-Abelian generalization of the Meissner effect. Both monopoles and vortices are truly non-Abelian objects. It is also important to observe that, thank to supersymmetry, this model is fully reliable also at the quantum level. Notice that we now have only one $U(1)$ factor that provides the existence of vortices at low energies. This leads to a significant reduction of the meson spectrum with respect to models in which dynamical abelianization take place.

Eventually, we want to study the model when strong coupling effects take place, taking the bare masses very small: monopoles becomes massless and condense. The Meissner effect now should take place in the dual theory and the original electric degrees of freedom are confined. Along the path to the strong coupling regime, we can recall the famous holomorphic dependence of physical quantities with respect to the couplings that is a fundamental property of supersymmetric gauge theories. We can thus learn a lot about the strong coupling regime, and about a mechanism of confinement which can be relevant for QCD, studying the semiclassical one. Furthermore, one finds that non-Abelian monopoles and vortices are deeply connected, so that we can learn much of one object studying the other, even at the semiclassical level. This observation is important if we consider that non-Abelian monopoles are still mysterious object. They usually possess non-normalizable modes which make the study of their quantum properties subtle. According to the well known GNOW conjecture, non-Abelian monopoles should form, at the quantum level, multiplets of a dual gauge group. Because of the aforementioned difficulties, it is difficult to check explicitly this conjecture.

Non Abelian vortices attracted much attention also for another interesting property: they provide a map between theories in two and four dimensions. It has been well known for many years that two dimensional sigma models and four dimensional gauge theories share many features, included the non-perturbative ones. The progresses in the study of supersymmetric theories made possible to prove indeed strong quantitative relations: the BPS spectrum of the mass deformed two-dimensional $\mathcal{N} = (2, 2)$ $\text{CP}^{N-1}$ sigma-model coincides with the BPS spectrum of four-dimensional $\mathcal{N} = 2$ $SU(N)$ supersymmetric QCD. This relation is not a coincidence: the two-dimensional sigma model is exactly the effective theory which lives on the vortex worldsheet which appears in the higgs phase of the four-dimensional gauge theory. In some sense, the vortex theory must “reproduce” the physics of the bulk theory. Many close relations have been found also for theories with less supersymmetry.
Is thus hard to underestimate the importance of non-Abelian vortices in the strongly coupled dynamics of supersymmetric gauge theories. We believe that some of the non-perturbative mechanism in which non-Abelian vortices play crucial role can be of relevance also for non-supersymmetric theories like QCD.

This Ph.D. Thesis is devoted to describe our efforts to improve the present knowledge about non-Abelian vortices. We will be mainly focused on the study of supersymmetric solitons. Supersymmetry is a sufficient property to make a soliton “BPS” (Bogomol’yi-Prasad-Sommerfeld). This property drastically simplify the study of these objects. First of all, the equations of motion reduce to first order partial differential equations, instead of being of the second order. Secondly, the tension of a BPS soliton is completely given by boundary terms which depends only by the topology of the configurations, becoming thus proportional to an integer which identify a topological winding. This property has an important consequence: the energy of composite configurations of solitons does not depend on their relative separations, thus there are no net static forces between them. This also implies the existence of a continuous set of configurations, degenerate in energy, which all together form the so-called “moduli space“ of solutions. The knowledge of the moduli space and its properties is crucial to understand the low-energy physics of the vortex. Generically, in fact, this physics will be described by a two dimensional non linear sigma model, whose target space is given by the moduli space. This sigma model will describe the relevant classical and quantum physics of the vortex. Usually one determines only the bosonic moduli and degrees of freedom. Supersymmetry is in fact sufficient to recover the fermionic sector, if needed.

We will use the so called “moduli-matrix formalism”, and the equivalent “Kahler-quotient construction” to systematically explore the bosonic zero modes and the corresponding bosonic sector of a wide range of vortex configurations. We will consider, as laboratory model, a $U(N_c)$ gauge theory with $N_f \geq N_c$ flavour in the fundamental representation. We will choose its parameters so that it could be embedded into an $\mathcal{N} = 2$ QCD, and it will admit a degenerate set of vortex solutions.

In Part I we will introduce the model and the basic construction of non-Abelian vortices. We will review some of the most important efforts to use non-Abelian vortices for a better understanding of the 4-d dynamics of non-Abelian gauge theories.

In Part II we will describe our results about the properties of the moduli space of non-Abelian vortices in gauge theories with unitary gauge group, $U(N_c)$. In Chapter 4 we will study configurations of composite vortices, important in the problem of interactions. In Chapter 5 we will study the effect
on moduli space structure when an arbitrary number of flavor is added. In these case there appear the so-called “semi-local” vortices. The connections with lump solutions are also discussed. We will use these knowledge in Chapter 6 to study the interactions of vortices, when some non-BPS correction are added to our model. The same non-BPS correction are also introduced in models with several flavors in Chapter 7, to discuss the stability of semi-local vortices. The knowledge about the moduli space of composite vortices is used to discuss a dual model of confinement in $SO(N)$ gauge theories in Chapter 8, and to solve the problem of the reconnection of cosmic strings in Chapter 9.

In Part III, we will introduce a more general derivation of the moduli space, which will enable us to construct non-Abelian vortices in a set of non-Abelian gauge theories with a generic gauge group. In Chapter 10 we will derive this generalized construction, using the deep connection between vortices and lumps in related sigma models. In Chapter 11 we will use the construction for a detailed study of vortices in non-Abelian gauge theories with orthogonal and symplectic gauge groups. These new vortex solutions provide semiclassical hints for the emergence of a GNOW duality, which will be discussed in Chapter 12.

In the last chapter we will conclude and discuss, future, possible development of this PhD thesis work.
Part I

A Brief Introduction
Chapter 1

Construction and Properties

1.1 Basic set-up

The basic model for the construction of non-Abelian vortices is an $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theory with $N_f = N_c = N$ hypermultiplets (quarks) $Q_f, \tilde{Q}_f$ ($f = 1, \ldots, N$). The field content is as follows. There are a $U(1)$ gauge field $A_\mu$ and $SU(N)$ gauge fields $A_{i\mu}$ (where $i = 1, \ldots, N^2 - 1$), together with the Weyl fermion superpartners: $(\lambda^1, \lambda^2)$ and $(\tilde{\lambda}^1, \tilde{\lambda}^2)$, where $\alpha$ is the spinorial index. The $\mathcal{N} = 2$ vector multiplet includes also a complex singlet $a$ and a complex field in the adjoint representation $a_i$. The matter hypermultiplets contain $N$ pair of complex fields, $(Q_f, \tilde{Q}_f)$, together with the fermionic superpartners, $(\psi_f, \tilde{\psi}_f)$, all in the fundamental representation. The superpotential is completely determined by supersymmetry:

$$W = \frac{1}{\sqrt{2}} \left[ \tilde{Q}_f (a_0 + a_i \tau^i + \sqrt{2} m_f) Q_f + W_{FI} (a_0) \right], \quad (1.1.1)$$

where

$$W_{FI} = -\frac{N}{2} \xi a_0. \quad (1.1.2)$$

$m_f$ are the (bare) quark masses. $\xi$ is the $F$-term Fayet-Iliopoulos (FI) parameter and can take an arbitrary value. This term is equivalent to the standard $D$-term FI term $[5]^1$. As well-known, it does not break any supersymmetry $[11, 15]$. This kind of system naturally arises in the $\mathcal{N} = 2$, $1$In the terminology used in Davis et al. $[6]$ in the discussion of the Abelian vortices in supersymmetric models, this model corresponds to an F model while the models of $[7, 8, 9, 10]$, which we will also consider in the next Sections, correspond to a D model. The two models are equivalent: they are related by an $SU_R(2)$ transformation $[11, 12, 13, 14]$. 

8
SU(N + 1) SQCD (see Sec. 2 for more discussions on this), with SUSY softly broken down to \( \mathcal{N} = 1 \) with a mass term for the adjoint fields of the form 

\[ W = \kappa \text{Tr}\Phi^2 \]

The bosonic part of the Lagrangian is [10] (we use the same symbols for the scalars as for the corresponding superfields):

\[
\mathcal{L} = -\frac{1}{4 g^2} (F_{\mu \nu})^2 - \frac{1}{4 e^2} (F^0_{\mu \nu})^2 + \frac{1}{g^2} |D_\mu a_i|^2 + \frac{1}{e^2} |\partial_\mu a_0|^2 + (D_\mu Q_f)^1 D^\mu Q_f + D_\mu \bar{Q}_f (D^\mu \bar{Q}_f)^1 - V(Q, \bar{Q}, a_i, a_0),
\]

where \( e \) is the \( U(1) \) gauge coupling and \( g \) is the \( SU(N) \) gauge coupling. The covariant derivatives and field strengths, respectively, are defined by

\[
D_\mu \left( Q_f, \bar{Q}_f^* \right) = \left( \partial_\mu - i A^i_\mu \frac{\tau^i}{2} - \frac{i}{2} A^0_\mu \right) \left( Q_f, \bar{Q}_f^* \right), \quad D_\mu a_i = \partial_\mu a_i + e^{ijk} A^j_\mu a_k,
\]

\[
F_{\mu \nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e^{ijk} A^j_\mu A^k_\nu, \quad F^0_{\mu \nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu,
\]

where the generators are normalized as

\[
\text{Tr} (\tau^a \tau^b) = 2 \delta^{ab}.
\]

The potential \( V \) is the sum of the following \( D \) and \( F \) terms

\[
V = V_1 + V_2 + V_3 + V_4;
\]

\[
V_1 = \frac{g^2}{8} \left( \frac{2}{g^2} e^{ijk} a_j a_k + \text{Tr}_f [Q^i f \tau^i Q] - \text{Tr}_f [\bar{Q} f \tau^i \bar{Q}^i] \right)^2;
\]

\[
V_2 = \frac{e^2}{8} \left( \text{Tr}_f [Q^i f Q] - \text{Tr}_f [\bar{Q} f \bar{Q}] \right)^2;
\]

\[
V_3 = \frac{g^2}{2} \left| \text{Tr}_f [\bar{Q} f \tau^i Q] \right|^2 + \frac{e^2}{2} \left| \text{Tr}_f [\bar{Q} f Q] - \frac{N}{2} \xi \right|^2;
\]

\[
V_4 = \frac{1}{2} \sum_{f=1}^{N_f} \left| a_0 + \tau^i a_i + \sqrt{2} m_f Q_f \right|^2 + \frac{1}{2} \sum_{f=1}^{N_f} \left| a_0 + \tau^i a_i + \sqrt{2} m_f \bar{Q}_f^i \right|^2.
\]

(1.1.6)

\( \text{Tr}_f \) denotes a trace over the flavor indices. For generic, unequal quark masses,

\[
M = \text{diag}(m_1, m_2, \ldots, m_N),
\]

the vacuum expectation value of the adjoint field:

\[
\langle a_0 + a^i \tau^i \rangle = -\sqrt{2} \begin{pmatrix}
m_1 & 0 & \cdots & 0 \\
0 & m_2 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & m_N
\end{pmatrix},
\]

(1.1.8)
breaks the gauge group down to $U(1)^N$.

Non-Abelian vortices are constructed in the most symmetric situation in which the squark multiplets have the same bare masses. The value of the common mass can be chosen to be equal to zero, after a proper shift of the adjoint scalars:

$$m_f = 0 .$$  \hspace{1cm} (1.1.9)

At the classical level, the theory as a unique vacuum. Up to gauge and flavor rotations, we can choose the following VEV for the scalar fields

$$Q = \tilde{Q}^\dagger = \sqrt{\frac{\xi}{2}} N, \quad a_0 = 0 , \quad a = 0 ,$$  \hspace{1cm} (1.1.10)

where $a \equiv a_i \tau_i / 2$. The gauge symmetry is completely broken by the squarks expectation value. The vacuum is invariant under the following global color-flavor locked rotations ($U_c \in SU(N)$, $U_f \in SU(N)$):

$$Q \to U_c Q U_f^\dagger , \quad \tilde{Q} \to U_f^\dagger \tilde{Q} U_c , \quad a \to U_c a U_c^\dagger , \quad F_{\mu\nu} \to U_c F_{\mu\nu} U_c^\dagger .$$  \hspace{1cm} (1.1.11)

Let us briefly consider the main quantum effects. If all bare masses are taken to be zero, theory (1.1.3) is asymptotically free, and the running of its gauge coupling in terms of the energy scale $\mu$ is given by:

$$\frac{8\pi}{g^2(\mu)} = N \log \frac{\mu}{\Lambda_{SU(N)}} .$$  \hspace{1cm} (1.1.12)

The theory thus necessarily runs in a strong coupling regime at energies below than the dynamically generated scale $\Lambda_{SU(N)}$. Nevertheless, it is possible to extract exact informations about this regime. Using the well-known Seiberg-Witten solution of $N = 2$ gauge theories [3, 4] it is possible to show that strong coupling effects would dynamically break gauge symmetry down to $U(1)^N$. There are two possible ways to prevent the flowing of the theory to this strong coupling regime. One is to take different masses, so that $\Delta m \gg \Lambda_{SU(N)}$. In this case, the gauge symmetry is semiclassically broken down to $U(1)^N$. The various $U(1)$ sector are then infrared free. Another technique is to keep equal masses but taking a large FI term $\xi \gg \Lambda_{SU(N)}$. In fact, the theory will enter the Higgs phase at the energy scales equal to the masses of the gauge and scalar fields ($\sim g\sqrt{\xi}$). Because of the mass gap, all the degrees of freedom are frozen below this scale, and the gauge coupling will stop its running at lower energies. We will consider this second way to study non-Abelian vortices, from now on, because it preserves the full
SU($N_f$) flavor symmetry (on the existence of such a symmetry relies the very existence of non-Abelian vortices\footnote{To be more precise, in the next Section we will construct reliable semiclassical vortex solutions in a weakly coupled theory, provided we choose: }}. To be more precise, in the next Section we will construct reliable semiclassical vortex solutions in a weakly coupled theory, provided we choose:

$$\xi^{N/2} \gg \Lambda_{SU(N)}^N = \xi^{N/2} \exp\left(-\frac{8\pi^2}{g^2(\xi)}\right). \quad (1.1.13)$$

### 1.2 The vortex solution

The model that we considered in the previous Section admits BPS vortex solutions, e.g., supersymmetric, string-like, classical configurations. This can be easily understood if we notice that the theory (1.1.3) in the chosen vacuum (6.1.4) undergoes a gauge symmetry breaking which has the correct topology that supports vortices:

$$U(N) \xrightarrow{\xi} \mathbb{1} \quad \pi_1(U(N)) = \mathbb{Z}. \quad (1.2.1)$$

These kind of solutions have been thoroughly studied in literature \cite{15, 11, 17, 16, 7, 18, 19}.

It turns out that for a semiclassical vortex solution, the adjoint fields are identically zero, and $Q = \tilde{Q}^1$. Using this in the Lagrangian (1.1.3) we can rewrite the full action after a Bogomol’nyi completion \cite{20}. For static solutions, like those we are looking for, it reduce to the expression for the energy (tension) of the string:

$$T = \int d^2x \frac{1}{2} \left\{ \frac{1}{g} F^i_{ij} + g \text{Tr}_f [Q^i \tau^i Q] \epsilon_{ij} \right\}^2 +$$

$$+ \left[ \frac{1}{e} F^0_{ij} + e \left( \text{Tr}_f [Q^i Q] - \frac{N}{2} \epsilon_{ij} \right) \epsilon_{ij} \right]^2 + |D_i Q^A + i\epsilon_{ij} D_j Q^A|^2 + N\xi F^0_{ij} \right\}$$

$$\geq N\xi \int d^2x F^0_{ij} = 2\pi \xi \kappa \quad (1.2.2)$$

where $\kappa$ is the quantized Abelian magnetic flux, which gives the lower bound for the vortex tension. This bound (Bogomol’ny bound) is saturated by

\footnote{Studying the case with large mass differences gives precious informations too, in particular when one consider quantum aspects. We will discuss this case in Section (1.4)}
solutions of the first order BPS equations for the vortex:

\[ F^i_{12} + g^2 \text{Tr}_f [Q^i Q^i] = 0, \quad i = 1, \ldots, N^2 - 1; \]

\[ F^0_{12} + e^2 \left( \text{Tr}_f [Q^0 Q^0] - \frac{N}{2} \xi \right) = 0; \]

\[ \mathcal{D}_1 Q_f + i \mathcal{D}_2 Q_f = 0, \quad f = 1, \ldots, N. \quad (1.2.3) \]

The fundamental \((\kappa = 1)\) vortex is explicitly constructed by the use of the following standard Ansatz, which gives a static, cylindrically invariant solution \[16\]:

\[
Q = \begin{pmatrix}
\phi_0(r)e^{i\theta} & 0 & \ldots & 0 \\
0 & \phi_1(r) & \ldots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \ldots & \phi_1(r)
\end{pmatrix},
\]

\[
A_i = A_i^\alpha x^\alpha = -\frac{1}{N} \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \ldots & -(N-1)
\end{pmatrix} \frac{\epsilon_{ij} x_j}{r^2} [1 - f_i(r)],
\]

\[
A_0^i = -\frac{1}{N} \frac{\epsilon_{ij} x_j}{r^2} [1 - f_0(r)]. \quad (1.2.4)
\]

By plugging the ansatz \[6.1.10\] into Eq. \[1.2.3\], we get the following set of first order differential equations for the four unknown functions \(\{\phi_0, \phi_1, f_0, f_1\}\).

\[
\frac{r}{d} \frac{d}{dr} \phi_0(r) - \frac{1}{N} (f_0(r) + (N - 1)f_1(r)) \phi_0(r) = 0,
\]

\[
\frac{r}{d} \frac{d}{dr} \phi_1(r) - \frac{1}{N} (f_0(r) - f_1(r)) \phi_1(r) = 0,
\]

\[
-\frac{1}{r} \frac{d}{dr} f_1(r) + \frac{e^2 N}{4} (\phi_0(r)^2 + (N - 1)\phi_1(r)^2 - N\xi) = 0,
\]

\[
-\frac{1}{r} \frac{d}{dr} f_3(r) + \frac{g^2}{2} (\phi_0(r)^2 - \phi_1(r)^2) = 0,
\]

which must be solved with the boundary conditions:

\[
f_1(0) = 1, \quad f_0(0) = 1, \quad f_1(\infty) = 0, \quad f_0(\infty) = 0, \quad \phi_0(\infty) = 1, \quad \phi_1(\infty) = 1
\]

\[1.2.6\]
and the following behavior for small \( r \)

\[
\phi_0 \propto \mathcal{O}(r), \quad \phi_1 \propto \mathcal{O}(1).
\]  

(1.2.7)

A numerical solution is shown in Fig. 1.1. The tension saturate the BPS bound:

\[
T_{\text{BPS}} = 2\pi \xi.
\]  

(1.2.8)

Figure 1.1: Profile functions for a vortex in a \( U(2) \) gauge theory in terms of the radial direction \( r \): \( f_0 \) (red), \( f_1 \) (dashed red), \( \phi_0 \) (blue), \( \phi_1 \) (dashed blue). The numerical values are taken as \( \xi = 2, \epsilon = 2, g = 1 \).

A fundamental property of non-Abelian vortices is the possession of a continuum set of degenerate solutions which represent the arbitrary orientation of the flux of the vortex inside the non-Abelian gauge group \[10, 7\]. This set of solution is called “moduli space”. The study of the moduli space is the main tool to understand the physics of non-Abelian vortices, and it will be the main subject of all our forthcoming discussions. To determine the moduli space in the case considered here, let us start from the solution of Eq. 6.1.10 and then apply to it the \( SU(N)_{c+f} \) symmetry \( (7.2.3) \) of the vacuum. This symmetry is partially broken by the vortex solution, with an \( SU(N - 1) \times U(1) \) subgroup which leave the solution unchanged. Thus, we get for the moduli space of solutions

\[
\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N - 1) \times U(1)}.
\]  

(1.2.9)
Notice that the existence of such a degeneracy is not related to the BPS saturation of the solitons. On the contrary, it represents zero modes interpreted as massless goldstone bosons which come from the breaking of the global color-flavor symmetry of the vacuum.

1.3 Effective world-sheet theory

In the previous Section we have derived the moduli space for non-Abelian vortices, which is given by the projective complex space \( \mathbb{C}P^{N-1} \). The effective theory is thus a non-linear sigma model with that target space. Let us thus study this effective theory in more detail [16, 7, 9, 8, 21].

Let us introduce the following parametrization for \( \mathbb{C}P^{N-1} \):

\[
\frac{1}{N} \begin{pmatrix} U_{c+f} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & -(N-1) \end{pmatrix} U_{c+f}^\dagger \bigg|_p = -n^l n^*_p + \frac{1}{N} \delta^l_p \quad (1.3.1)
\]

where \( n^l \) is a complex vector in the fundamental representation of \( SU(N) \), with \( k = 1, \cdots, N \) and \( n^*_l n^l = 1 \). Also, the phase of \( n \) is redundant and must be gauged away. We are ready to write the full set of solutions for a generic orientation \( n^k \):

\[
A^0_i = \frac{1}{N} \frac{\epsilon_{ij} x^j}{r^2} [1 - f_0], \\
A = \left( n \cdot n^* - \frac{1}{N} \right) \frac{\epsilon_{ij} x^j}{r^2} [1 - f_1], \\
Q = \tilde{Q}^l = \frac{1}{N} ((N-1)\phi_1 + \phi_0) + (\phi_0 - \phi_1) \left( n \cdot n^* - \frac{1}{N} \right)
\]

(1.3.2)

The presence of orientational zero-modes on the non-Abelian string leads to a non-trivial interacting physics, when we consider the low-energy effective theory on the string background. The basic procedure to derive the effective action is to promote the vector \( n^l \) to be a slowly varying function of the 1+1 dimensional world-sheet coordinates: \( n^l(x_i), i = 0, 3 \). As soon as we consider non-static solutions, we have to generalize the Ansatz of Eq. 1.3.2 for the gauge fields, in order to include the now non-zero fields \( A_i, i = 0, 3 \) [16, 9]:

\[
A_i = -i [\partial_i n \cdot n^* - n \cdot \partial_i n^* - 2n \cdot n^*(n^* \partial_i n)] \rho(r)
\]

(1.3.3)
where $\rho(r)$ is a function which must be determined with a variational procedure. After some long calculations, one gets the following action for the orientational modes:

$$S_{eff}^{1+1} = 2\beta \int dt \, dz \left\{ (\partial_i n^* \partial_i n) + (n^* \partial_i n)^2 \right\}$$  \hfill (1.3.4)

which is the well-known action for a bosonic $\mathbb{C}P^{N-1}$ non-linear sigma model. The two-dimensional coupling $\beta$ is related to the four-dimensional non-Abelian gauge coupling $g$ by the relation:

$$\beta = \frac{2\pi}{g^2}. \hfill (1.3.5)$$

The action above includes only the bosonic zero modes, parameterized by the complex vector $n^l$. The true effective action includes also fermionic zero-modes, which can be completely determined by the use of supersymmetry. The complete effective theory is the $\mathcal{N} = (2,2)$ supersymmetric extension of the $\mathbb{C}P^{N-1}$ sigma model.

Let us briefly discuss some of the quantum aspects of the supersymmetric $\mathbb{C}P^{N-1}$ sigma model obtained \cite{22}. Again, the most important quantum effect is the running of the coupling constant $\beta$, which is asymptotically free. It is given by

$$4\pi \beta(E) = N \log \frac{E}{\Lambda_\sigma}, \hfill (1.3.6)$$

where $\Lambda_\sigma$ is a strong coupling scale dynamically generated. The running of the couplings implies that we have to specify a scale at which the relation \hbox{(1.3.5)} is satisfied. It turns out that the correct scale is given by $g\sqrt{\xi}$. This is the scale defined by the mass of the $SU(N)$ gauge bosons in the Higgs phase, where we constructed the vortex. It also roughly gives the inverse of the width of the vortex itself. This scale length has to interpreted as an UV cut-off which limit the validity of the effective action \hbox{(1.3.4)}. In fact, the gauge coupling runs down to this scale and it freeze. A t lower scales the sigma model coupling take its place, starting the running. The matching of the running couplings at this scales

$$4\pi \beta(g\sqrt{\xi}) = N \log \frac{g\sqrt{\xi}}{\Lambda_\sigma} = \frac{8\pi}{g^2(g\sqrt{\xi})} = N \log \frac{g\sqrt{\xi}}{\Lambda_{SU(N)}} \hfill (1.3.7)$$

gives the following important result:

$$\Lambda_{SU(N)}^N = \Lambda_\sigma^N \hfill (1.3.8)$$
There are several ways to study the strong coupling regime of the $\mathbb{C}P^{N-1}$ non-linear sigma model. It can be solved exactly in the limit of large $N$, for example [23]. It also possible to work out a “mirror” description in terms of an equivalent, weakly coupled theory [24, 25, 26, 27]. The main result, in all the approaches, thanks to the supersymmetric nature, is the computation of the exact superpotential, which is non-perturbatively generated by instantonic effects. The minimization of this potential gives the exact quantum vacua of the model. In general, it turns out that the supersymmetric $\mathbb{C}P^{N-1}$ non-linear sigma model has $N$ isolated vacua, which all possess a mass gap. It is important to say that in these quantum vacua, the expectation value of the orientational field is zero: $\langle n^l \rangle = 0$. This can be considered as a consequence of a well-known theorem which forbids the existence of Goldstone bosons in two-dimensional theories [28].

We have already encountered many “correspondences” between the 4-dimensional gauge theory and the 2-dimensional vortex theory. In the next section we will briefly review the physical picture which explain why this is not a “coincidence”.

1.4 Non-Abelian vortices and 4-D gauge dynamics

Let us explore in more detail the physical intuition which lies behind the correspondences found in the previous Section between the bulk theory and the effective vortex theory, following Ref. [29]. To this end, we start adding big mass differences to the bulk theory, e.g. we consider $\Delta m_{ij} \gg \Lambda_{SU(N)}$. We keep the theory on its Coulomb phase, by setting the FI term to zero: $\xi = 0$. As already said, the bulk theory is now weakly coupled, and undergoes the following symmetry breaking pattern:

$$U(N) \xrightarrow{\Delta m_{ij}} U(1)^N.$$  \hspace{1cm} (1.4.1)

This breaking supports ’t Hooft-Polyakov monopoles which are characterized by the second homotopy group

$$\pi_2 \left( U(N)/U(1)^N \right) = \mathbb{Z}^{N-1}. $$ \hspace{1cm} (1.4.2)

There are thus $N-1$ distinct monopoles, and each of them can be constructed in the semiclassical limit. Their masses are exactly determined (as always for BPS solitons) by boundary terms proportional to the quantized magnetic flux:

$$M_{mon} = \frac{4\pi}{g^2} \Delta m_{ij}. $$ \hspace{1cm} (1.4.3)
Let us now turn on the FI term. This will put the theory on the Higgs branch. Vortices now will be present in the Higgs vacuum. In the presence of non-zero mass differences, vortices loose their orientational moduli because the \( SU(N)_{c+f} \) symmetry is broken down to \( U(1)^{N-1} \). In this case we obtain \( N \) distinct vortices which are to be considered as different embedding of an abelian (ANO-like) vortex. These vortices must be present, and must confine the magnetic flux of the monopoles. This picture is confirmed by the fact that the fluxes of monopoles and vortices coincide in various different models \cite{30, 31, 32, 33, 34, 35, 36, 37, 12, 38}. As soon as \( \Delta m_{ij} \gg \sqrt{\xi} \), the situation is nicely showed in the upper picture of Figure (1.2): the size of the monopoles (\( \sim \Delta m_{ij}^{-1} \)) is much smaller than the transverse size of the vortex (\( \sqrt{\xi}^{-1} \)). Monopoles are almost unconfined. If we increase the value of the FI term, the size of vortices will shrink, and eventually will be so small that the attached monopoles will be squeezed inside the vortex. Monopoles are now strongly confined (left-bottom picture of Figure 1.2). However, the theory is still in a weakly coupled regime. We can study this configuration from the point of view of the \( CP^{N-1} \) sigma model: if the mass differences are much smaller than the gauge bosons masses, the effective theory is reliable (\( \Delta m_{ij} \ll \sqrt{\xi} \)). The effective lagrangian \( (1.3.4) \) will be deformed by the introduction of non-zero mass differences. This deformation was found in
Ref. [9] and correspond to the inclusion of the following superpotential:
\[
V_{\mathbb{CP}^N} = 2\beta \left\{ \sum_l |\Delta m_l|^2 |n_l|^2 - \left| \sum_l \Delta m_l |n_l|^2 \right|^2 \right\},
\]  
(1.4.4)
where \( \Delta m_l = m_l - \frac{1}{N} \sum_l m_l \). This deformation corresponds to the inclusion of the so-called twisted masses in the two-dimensional sigma model [39-42].

The potential admits \( N \) distinct minima, which correspond to \( N \) isolated semiclassical vacua. This implies the existence of \( N-1 \) kinks (domain walls in one dimension) which interpolate between two of them. Again, it is possible to exactly calculate the mass of these kinks, which turns out to be:

\[
M_{\text{kink}} = 2\beta \Delta m_{ij}.
\]  
(1.4.5)

The dependence of the mass of the kinks in terms of the twisted masses is the same as that of the unconfined monopoles in Eq. (1.4.3). This has a natural explanation if we interpret the kinks as the sigma model description of the monopoles confined on the string. This interpretation is strengthened by the observation that both expression (1.4.3) and (1.4.5) do not depend by the FI term \( \xi \) (there are standard argument in supersymmetric theories which forbids the dependence of BPS quantities, like the masses of BPS solitons, by the FI term [11, 15]). The identification between free monopoles in the Coulomb phase and kinks on the string in the Higgs phase leads to the relation:

\[
M_{\text{mon}}^{\text{Coulomb}} = M_{\text{mon}}^{\text{Higgs}} = M_{\text{kink}}.
\]  
(1.4.6)
As it is easy to check, this relation is equivalent to the previously mentioned relation between the two and four dimensional couplings Eq. (1.3.5).

One may now go to the strong coupling regime, turning all the masses off:
\( \Delta m_{ij} \ll \Lambda \) (right-bottom picture of Figure 1.2). In this regime the semiclassical description of both monopoles and kinks is not reliable. Nonetheless, it is possible to determine their exact masses:

\[
M_{\text{mon}} = \frac{2}{\pi} \Lambda_{\text{SU}(N)}, \quad M_{\text{kink}} = \frac{2}{\pi} \Lambda_{\sigma}.
\]  
(1.4.7)

Equating the two values above, we again obtain equivalence between the strong coupling scales: \( \Lambda_{\text{SU}(N)} = \Lambda_{\sigma} \).

The physical rationale which lies behind the correspondence of monopoles and kinks can be generalized for all the BPS states of the two theories. It
was shown that the full BPS spectrum of the four dimensional gauge theory, at the root of the baryonic branch, coincides with that of the supersymmetric $\mathbb{C}P^{N-1}$ sigma model [39, 43]. We interpret the states of the effective theory as states of the bulk theory which are confined on the string. This interpretation work for the solitonic sector for the perturbative one too. The gauge bosons are massless in the Coulomb phase. When we enter the Higgs phase, all of them acquire non zero masses. But in the presence of a vortex the gauge symmetry is (partially) restored at the core vortex. Thus, gauge bosons may show themselves as massless excitation confined inside the vortex. They are in fact interpreted as the orientational modes of the $\mathbb{C}P^{N-1}$ sigma model.

These statements can be formalized as follows. First of all we consider an alternative description of the vortex theory. The $\mathbb{C}P^{N-1}$ of (1.3.4) is obtained by taking the strong coupling limit of the following $U(1), N = 2$, gauge theory with $N$ chiral hypermultiplets ($\ell = 1, \ldots, N$) [22]:

$$S_{\text{eff}}^{1+1} = \int dt \, dz \left\{ 2\beta |D_\mu n^l|^2 + \frac{1}{4e_{1+1}^2} F_{\mu\nu}^2 + \frac{1}{e_{1+1}^2} |\partial_\mu \sigma|^2 + 4\beta |\sigma - \frac{\Delta m}{\sqrt{2}}|^2 |n^l|^2 + 2e_{1+1}^2 \beta^2 (|n^l|^2 - 1)^2 \right\}, \quad (1.4.8)$$

where we have included non-zero masses. As we take the strong coupling limit, the photon and the $\sigma$ field become massive and can be integrated out. At the same time the condition $|n^l|^2 - 1 = 0$ is implemented, and gives us a $\mathbb{C}P^{N-1}$ as moduli space of vacua of the theory. The quantities we are interested in, like masses of BPS states, do not depend by the two dimensional gauge coupling $e_{1+1}$. Thus we are allowed to study the linear gauged sigma model (1.4.8) at finite coupling, which is much easier. The important result found in Ref. [39, 43] is that the exact BPS spectrum of the theory (1.4.8) (vortex theory) coincides with that of the theory (1.1.3) (bulk theory), once the bare masses of the hypermultiplets and the strong coupling scales of the two models are identified. The correspondence holds both at the semiclassical and quantum level. At the time of its discovery, this result seemed really puzzling. As we briefly review discussed, non-Abelian vortices provides the fundamental connection ring between the two theories.
Chapter 2

Confinement and Dualities in 4-d Gauge Theories

A system in which the gauge symmetry is spontaneously broken

\[ G \xrightarrow{\langle v_1 \rangle \neq 0} H \]  

(2.0.1)

where \( H \) is some non-Abelian subgroup of \( G \), possesses a set of regular magnetic monopole solutions in the semi-classical approximation, which are natural generalizations of the 't Hooft-Polyakov monopoles [44, 45] found in the system \( G = SO(3), H = U(1) \). A straightforward generalization of the Dirac's quantization condition leads to the GNOW (Goddard-Nuyts-Olive-E.Weinberg) conjecture, i.e., that they form a multiplet of the group \( \tilde{H} \), dual of \( H \). The group \( \tilde{H} \) is generated by the dual root vectors

\[ \alpha^* = \frac{\alpha}{\alpha \cdot \alpha}, \]  

(2.0.2)

where \( \alpha \) are the non-vanishing roots of \( H \) [46]. There are however well-known difficulties in such an interpretation. The first concerns the topological obstruction discussed in [47, 48, 49, 50]: in the presence of the classical monopole background, it is not possible to define a globally well-defined set of generators isomorphic to \( H \). As a consequence, no “colored dyons” exist. In the simplest example of a system with the symmetry breaking,

\[ SU(3) \xrightarrow{\langle v_1 \rangle \neq 0} SU(2) \times U(1), \]  

(2.0.3)

this means that no monopoles exist which carry the quantum number, e.g.,

\[ (2,1^*) \]  

(2.0.4)
where the asterisk indicates the dual, magnetic $U(1)$ charge.

The second can be regarded as the infinitesimal version of the same difficulty: certain bosonic zero-modes around the monopole solution, corresponding to the $H$ gauge transformations, are non-normalizable (behaving as $r^{-1/2}$ asymptotically). Thus the standard procedure of semiclassical quantization leading to the $H$ multiplet of the monopoles does not work. Some progress on the check of GNO duality along this orthodox approach has been reported nevertheless in [51] for $\mathcal{N} = 4$ supersymmetric gauge theories, which however requires the consideration of particular multi-monopole systems neutral with respect to the non-Abelian group (more precisely, non-Abelian part of) $H$.

Both of these difficulties concern the transformation properties of the monopoles under the subgroup $H$, while the truly relevant question is how they transform under the dual group, $\tilde{H}$. As field transformation groups, $H$ and $\tilde{H}$ are relatively non-local; the latter should look like a non-local transformation group in the original, electric description.

Another related question concerns the multiplicity of the monopoles; take again the case of the system with breaking pattern Eq. (2.0.3). One might argue that there is only one monopole, as all the degenerate solutions are related by the unbroken gauge group $H = SU(2)$. Or one might say that there are two monopoles as, according to the semiclassical GNO classification, they are supposed to belong to a doublet of the dual $SU(2)$ group. In short, what is the multiplicity of the monopoles? The answer to this question needs a deeper understanding of the quantum properties of non-Abelian monopoles. Clearly the very concept of the dual gauge group or dual gauge transformation must also be better understood. In attempting to gain such an improved insight on the nature of these objects, we are naturally led to several general considerations.

The first is the fact when $H$ and $\tilde{H}$ groups are non-Abelian the dynamics of the system should enter the problem in an essential way. It should not be surprising if the understanding of the concept of non-Abelian duality required a full quantum mechanical treatment of the system.

For instance, the non-Abelian $H$ interactions can become strongly-coupled at low energies and can break itself dynamically. This indeed occurs in pure $\mathcal{N} = 2$ super Yang-Mills theories (i.e., theories without quark hypermultiplets), where the exact quantum mechanical result is known in terms of the Seiberg-Witten curves [52, 52, 53, 54, 55, 56, 57]. Consider for instance, a pure $\mathcal{N} = 2$, $SU(N + 1)$ gauge theory. Even though partial breaking, e.g.,

\[1\text{This interpretation however encounters the difficulties mentioned above. Also there are cases in which degenerate monopoles occur, which are not simply related by the group } H, \text{ see below.}\]
$SU(N + 1) \rightarrow SU(N) \times U(1)$ looks perfectly possible semi-classically, in an appropriate region of classical degenerate vacua, no such vacua exist quantum mechanically. In all vacua the light monopoles are Abelian, the effective, magnetic gauge group being $U(1)^N$.

Generally speaking, the concept of a dual group multiplet is well-defined only when $\tilde{H}$ interactions are weak (or, at worst, conformal). This however means that one must study the original, electric theory in the regime of strong coupling, which would usually make the task of finding out what happens in the system at low energies exceedingly difficult. Fortunately, in $\mathcal{N} = 2$ supersymmetric gauge theories, the exact Seiberg-Witten curves describe the fully quantum mechanical consequences of the strong-interaction dynamics in terms of weakly-coupled dual magnetic variables. And this is how we know that the non-Abelian monopoles do exist in fully quantum theories [58]: in the so-called $r$-vacua of softly broken $\mathcal{N} = 2$ SQCD, the light monopoles interact as a point-like particle in a fundamental multiplet $r$ of the effective, dual $SU(r)$ gauge group. In the system of the type Eq. (2.0.3) with appropriate number of quark multiplets ($N_f \geq 4$), we know that light magnetic monopoles carrying the non-Abelian quantum number

$$(2^*, 1^*)$$

under the dual $SU(2) \times U(1)$ appear in the low-energy effective action (cfr. Eq. (2.0.4)). The distinction between $H$ and $\tilde{H}$ is crucial here.

In general $\mathcal{N} = 2$ SQCD with $N_f$ flavors, light non-Abelian monopoles with $SU(r)$ dual gauge group appear for $r \leq \frac{N_f}{2}$ only. Such a limit clearly reflects the dynamical properties of the soliton monopoles under renormalization group: the effective low-energy gauge group must be either infrared free or conformal invariant, in order for the monopoles to emerge as recognizable low-energy degrees of freedom [59, 60].

A closely related point concerns the phase of the system. Even if there is ample evidence for the non-Abelian monopoles, as explained above, we might still wish to understand them in terms of something more familiar, such as semiclassical ’t Hooft-Polyakov solitons. An analogous question can be (and should be) asked about the Seiberg’s “dual quarks” in $\mathcal{N} = 1$ SQCD [61]. Actually, the latter can be interpreted as the GNOW monopoles becoming light due to the dynamics, at least in $SU(N)$ theories [62, 63].

Dynamics of the system is thus a crucial ingredient: if the dual group were in Higgs phase, the multiplet structure among the monopoles would get lost, generally. Therefore one must study the dual ($\tilde{H}$) system in confinement phase. But then, according to the standard electromagnetic duality argument, one must analyze the electric system in Higgs phase. The monopoles will...
appear confined by the confining strings which are nothing but the vortices in the $H$ system in Higgs phase.

We are thus led to study the system with a hierarchical symmetry breaking,

$$ G \xrightarrow{(v_1)\neq 0} H \xrightarrow{(v_2)\neq 0} 1, $$

(2.0.6)

where

$$ |\langle v_1 \rangle| \gg |\langle v_2 \rangle|, $$

(2.0.7)

instead of the original system Eq. (2.0.1). The smaller VEV breaks $H$ completely. Also, in order for the degeneracy among the monopoles not to be broken by the breaking at the scale $|\langle v_2 \rangle|$, we assume that some global color-flavor diagonal group

$$ H_{c+f} \subset H_{\text{color}} \otimes G_f $$

(2.0.8)

remains unbroken.

It is hardly possible to emphasize the importance of the role of the massless flavors too much. This manifests in several different aspects.

(i) In order that $H$ must be non-asymptotically free, there must be sufficient number of massless flavors: otherwise, $H$ interactions would become strong at low energies and $H$ group can break itself dynamically;

(ii) The physics of the $r$ vacua [59, 60, 64] indeed shows that the non-Abelian dual group $SU(r)$ appear only for $r \leq N_f^2$. This limit can be understood from the renormalization group: in order for a non-trivial $r$ vacuum to exist, there must be at least $2r$ massless flavors in the fundamental theory;

(iii) Non-Abelian vortices [7, 8, 16], which as we shall see are closely related to the concept of non-Abelian monopoles, require a flavor group. The non-Abelian flux moduli arise as a result of an exact, unbroken color-flavor diagonal symmetry of the system, broken by individual soliton vortex.

In this Chapter we will review the idea that the properties of non-Abelian monopoles related to the dual group (such as multiplicities and transformation properties) can be inferred by the study of the vortices which form our monopole-vortex complex [36].
2.1 $SU(N + 1)$ model with hierarchical symmetry breaking

We consider now a concrete model in which the set-up discussed in the previous section is explicitly realized. It is similar to that we studied in the previous Chapter for the construction of vortices (1.1.3). Actually, it contains the vortex theory as a low energy approximation. It is the standard $\mathcal{N} = 2$ SQCD with $N_f$ quark hypermultiplets, with gauge symmetry $SU(N + 1)$, which is broken at a much larger mass scale as

$$SU(N + 1) \xrightarrow{v_1 \neq 0} SU(N) \times U(1) \bigotimes_{\mathbb{Z}_N}. \quad (2.1.1)$$

The remaining gauge symmetry is then completely broken at a lower mass scale.

Clearly one can attempt a similar embedding of the vortex theory in a larger gauge group broken at some higher mass scale, in the context of a non-supersymmetric model, even though in such a case the potential must be judiciously chosen and the dynamical stability of the scenario would have to be carefully monitored. Here we have chosen to study the softly broken $\mathcal{N} = 2$ SQCD for concreteness, and above all because the dynamical properties of this model are well understood: this will provide us with a non-trivial check of our results. Another motivation is purely of convenience: it gives a definite potential with desired properties.

The bosonic sector of our model is given by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2}(F_{\mu\nu})^2 + \frac{1}{g^2}|D_\mu a_i|^2 + (D_\mu Q_f)^\dagger D^\mu Q_f + D_\mu \tilde{Q}_f (D^\mu \tilde{Q}_f)^\dagger - V(Q, \tilde{Q}, a_i, a_0), \quad (2.1.2)$$

where the potential is basically given by the non-Abelian part of the expressions in Eq. 1.1.6. The only difference comes from the superpotential. In the present case, in fact, we add a mass term for the adjoint chiral multiplet, which softly breaks the supersymmetry to $\mathcal{N} = 1$.

$$W_\mu = \frac{1}{2} \mu a^i a^i \quad (2.1.3)$$

Recent developments [65, 66, 67, 68, 69, 70, 71] allow us actually to consider systems of this sort within a much wider class of $\mathcal{N} = 1$ supersymmetric models, whose infrared properties are very much under control. We stick ourselves to the standard $\mathcal{N} = 2$ SQCD, however, for concreteness.
which changes $V_3$ in Eq. (1.1.6) as

$$V_3 = \frac{g^2}{2} \left| \text{Tr}_f (\hat{Q} \tau^i Q) + \mu a^i \right|^2,$$

(2.1.4)

In order to keep the hierarchy of the gauge symmetry breaking scales, Eq. (2.0.7), we choose the masses such that

$$m_1 = \ldots = m_{N_f} = m,$$

(2.1.5)

$$m \gg \mu \gg \Lambda.$$

(2.1.6)

Although the theory described by the above Lagrangian has many degenerate vacua, we are interested in the vacuum [64]

$$\langle a^i \tau^i \rangle = -\sqrt{2} \begin{pmatrix} m & 0 & 0 & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \ldots & m & 0 \\
0 & \ldots & 0 & -N m \end{pmatrix},$$

(2.1.7)

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} d & 0 & 0 & 0 & \ldots \\
0 & \ddots & 0 & \vdots & \vdots \\
0 & 0 & d & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \end{pmatrix}, \quad d = \sqrt{(N+1)\mu m \over \sqrt{2}}.$$

(2.1.8)

This is a particular case of the so-called $r$ vacuum, with $r = N$. Although such a vacuum certainly exists classically, the existence of the quantum $r = N$ vacuum in this theory requires $N_f \geq 2N$, which we shall assume [1].

To start with, ignore the smaller squark VEV, Eq. (2.1.8). As $\pi_2(G/H) \sim \pi_1(H) = \pi_1(SU(N) \times U(1)) = \mathbb{Z}$, the symmetry breaking Eq. (2.1.7) gives rise to regular magnetic monopoles with mass of order of $O(v_1 g) (v_1 \sim m)$,

$$3$$

This might appear to be a rather tight condition as the original theory loses asymptotic freedom for $N_f \geq 2N + 2$. This is not so. An analogous discussion can be made by considering the breaking $SU(N) \rightarrow SU(r) \times U(1)^{N-r}$. In this case the condition for the quantum non-Abelian vacuum is $2N > N_f \geq 2r$, which is a much looser condition. Also, although the corresponding $U(N)$ theory Eq. (1.1.3) with such a number of flavor has semi-local strings [72, 73, 74, 75, 6], these moduli are not directly related to the discussion about monopoles and dual gauge symmetry, which is our interest in this Chapter. We will study the fate of semi-local moduli in Section [7].
whose continuous transformation property is our main concern here. The semiclassical formulas for their mass and fluxes are well known \[46, 37\] and will not be repeated here.

At scales much lower than \( v_1 = m \) but still neglecting the smaller squark VEV \( v_2 = d \sim \sqrt{(N + 1) \mu m} \ll v_1 \), the theory reduces to an \( SU(N) \times U(1) \) gauge theory with \( N_f \) light quarks \( q_i, \tilde{q}_i \) (the first \( N \) components of the original quark multiplets \( Q_i, \tilde{Q}_i \)). By integrating out the massive fields, after a suitable redefinition of fields, the effective Lagrangian valid between the two mass scales is precisely the vortex model \[1.1.3\] that we introduced in Chapter \[1\], where the FI term is given by

\[
\xi = \sqrt{2} \mu m (N + 1) \tag{2.1.9}
\]

The splitting of the coupling constant into two different couplings is accomplished by the renormalization group flow, under which the Abelian and the non-Abelian couplings run differently towards the infrared. To exactly get the model \[1.1.3\], we have to ignore contributions to the action which are quadratic in the mass parameter \( \mu \). Within this approximation, \( N = 2 \) supersymmetry is unbroken. The presence of higher order corrections, however, makes the low-energy vortices not strictly BPS (and this will be important in the discussion of Section \[7\] about their stability).

\section{2.2 Vortex-monopole complex}

The fact that there must be a continuous set of monopoles, which transform under the color-flavor \( G_{c+f} \) group, follows from the following exact homotopy sequence

\[
\cdots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \xrightarrow{f} \pi_1(G) \to \cdots, \tag{2.2.1}
\]

applied to our systems with a hierarchical symmetry breaking, Eq. \[2.0.6\], with an exact unbroken symmetry, Eq. \[2.0.8\]. \( \pi_2(G) = 1 \) for any Lie group, and \( \pi_1(G) \) depends on the group considered. Eq. \[2.2.1\] was earlier used to obtain the relation between the regular, soliton monopoles (represented by \( \pi_2(G/H) \)) and the singular Dirac monopoles, present if \( \pi_1(G) \) is non-trivial. The isomorphism

\[
\pi_1(G) \sim \pi_1(H)/\pi_2(G/H) \tag{2.2.2}
\]

implied by Eq. \[2.2.1\] shows that among the magnetic monopole configurations \( A_\alpha^a(x) \) classified according to \( \pi_1(H) \) \[76\], the regular monopoles correspond to the kernel of the map \( f : \pi_1(H) \to \pi_1(G) \) \[77\].
When the homotopy sequence Eq. (2.2.1) is applied to a system with hierarchical breaking, in which $H$ is completely broken at low energies, $G \xrightarrow{v_1} H \xrightarrow{v_2} 1$, it allows an interesting re-interpretation. $\pi_1(H)$ classifies the quantized flux of the vortices in the low-energy $H$ theory in Higgs phase. Vice versa, the high-energy theory (in which the small VEV is negligible) has 't Hooft-Polyakov monopoles quantized according to $\pi_2(G/H)$. However, there is something of a puzzle: when the small VEV’s are taken into account, which break the “unbroken” gauge group completely, these monopoles must disappear somehow. A related puzzle is that the low-energy vortices with $\pi_1(H)$ flux, would have to disappear in a theory where $\pi_1(G)$ is trivial.

What happens is that the massive monopoles are confined by the vortices and disappear from the spectrum; on the other hand, the vortices of the low-energy theory end at the heavy monopoles once the latter are taken into account, having mass large but not infinite Fig. (2.2). The low-energy vortices become unstable also through heavy monopole pair productions which break the vortices in the middle (albeit with small, tunneling rates [78]), which is really the same thing. Note that, even if the effect of such string breaking is neglected, a monopole-vortex-antimonopole configuration is not topologically stable anyway: its energy would become smaller if the string becomes shorter (so such a composite, generally, will get shorter and shorter and eventually disappear).

In the case $G = SU(N + 1)$, $H = \frac{SU(N) \times U(1)}{\mathbb{Z}_N}$ we have a trivial $\pi_1(G)$, so $\pi_2\left(\frac{SU(N + 1)}{U(N)}\right) = \pi_2(\mathbb{C}P^N) \sim \pi_1(U(N)) = \mathbb{Z}$ : (2.2.3) each non-trivial element of $\pi_1(U(N))$ is associated with a non-trivial element of $\pi_2(\frac{SU(N+1)}{U(N)})$. Each vortex confines a regular monopole. The monopole transformation properties follow from those of the vortices, as will be more concretely studied in the next section.

In theories with a non-trivial $\pi_1(G)$ such as $SO(N)$, the application of these ideas is slightly subtle: these points will be discussed in Chapter [8].

In all cases, as long as the group $H$ is completely broken at low energies and because $\pi_2(G) = 1$ always, none of the vortices (if $\pi_1(G) = 1$) and monopoles are truly stable, as static configurations. They can be only approximately so, in an effective theory valid in respective regions ($v_1 \simeq \infty$ or $v_2 \simeq 0$).

However, this does not mean that a monopole-vortex-antimonopole composite configuration cannot be dynamically stabilized, or that they are not
relevant as a physical configuration. A rotation can stabilize easily such a configuration dynamically, except that it will have a small non-vanishing probability for decay through a monopole-pair production, if such a decay is allowed kinematically.

After all, we believe that the real-world mesons are quark-string-antiquark bound states of this sort, the endpoints rotating almost with a speed of light! An excited meson can and indeed do decay through quark pair productions into two lighter mesons (or sometimes to a baryon-antibaryon pair, if allowed kinematically and by quantum numbers). Only the lightest mesons are truly stable. The same occurs with our monopole-vortex-antimonopole configurations. The lightest such systems, after the rotation modes are appropriately quantized, are truly stable bound states of solitons, even though they might not be stable as static, semiclassical configurations.

Our model is thus a reasonably faithful (dual) model of the quark confinement in QCD.

A related point, more specific to the supersymmetric models we consider here as a concrete testing ground, is the fact that monopoles in the high-energy theory and vortices in the low-energy theory, are both BPS saturated. It is crucial in our argument that they are both BPS only approximately; they are almost BPS but not exactly\textsuperscript{4} They are unstable in the full theory. But the fact that there exists a limit (of a large ratio of the mass scales, $v_1/v_2 \to \infty$) in which these solitons become exactly BPS and stable, means that the magnetic flux through the surface of a small sphere surrounding the monopole and the vortex magnetic flux through a plane perpendicular to the

\textsuperscript{4} The importance of almost BPS soliton configurations have also been emphasized by Strassler [63].
vortex axis, must match exactly. These questions (the flux matching) have been discussed extensively already in [36].

Our argument, applied to the simplest case, $G = SO(3)$, and $H = U(1)$, is precisely the one adopted by ’t Hooft in his pioneering paper [44, 45] to argue that there must be a regular monopole of charge two (with respect to the Dirac’s minimum unit): as the vortex of winding number $k = 2$ must be trivial in the full theory ($\pi_1(SO(3)) = \mathbb{Z}_2$), such a vortex must end at a regular monopole. What is new here, as compared to the case discussed by ’t Hooft [44, 45] is that now the unbroken group $H$ is non-Abelian and that the low-energy vortices carry continuous, non-Abelian flux moduli. The monopoles appearing as the endpoints of such vortices must carry the same continuous moduli Fig. (2.2).

The fact that the vortices of the low-energy theory are BPS saturated (which allows us to analyze their moduli and transformation properties elegantly, as discussed in the next section), while in the full theory there are corrections which make them non BPS (and unstable), could cause some con-
cern. Actually, the rigor of our argument is not affected by those terms which can be treated as perturbation. The attributes characterized by integers such as the transformation property of certain configurations as a multiplet of a non-Abelian group which is an exact symmetry group of the full theory, cannot receive renormalization. This is similar to the current algebra relations of Gell-Mann which are not renormalized. Conserved vector current (CVC) of Feynman and Gell-Mann \[79\] also hinges upon an analogous situation.\(^{5}\) The results obtained in the BPS limit (in the limit \(v_2/v_1 \to 0\)) are thus valid at any finite values of \(v_2/v_1\).

### 2.3 Transformation properties of monopoles

The concepts such as the low-energy BPS vortices or the high-energy BPS monopole solutions are thus only approximate: their explicit forms are valid only in the lowest-order approximation, in the respective kinematical regions. Nevertheless, there is a property of the system which is exact and does not depend on any approximation: the full system has an exact, global \(SU(N)_{c+f}\) symmetry, which is neither broken by the interactions nor by both sets of VEVs, \(v_1\) and \(v_2\). This symmetry is broken by individual soliton vortex, endowing the latter with non-Abelian orientational moduli, analogous to the translational zero-modes of a kink. As we have already emphasized, the vortex breaks the color-flavor symmetry as

\[
SU(N)_{c+f} \to SU(N - 1) \times U(1),
\]

leading to the moduli space of the minimum vortices which is

\[
\mathcal{M} \simeq \mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N - 1) \times U(1)}.
\]

The fact that this moduli coincides with the moduli of the quantum states of an \(N\)-state quantum mechanical system, is a first hint that the monopoles appearing at the endpoint of a vortex, transform as a fundamental multiplet \(N\) of a group \(SU(N)\).

The moduli space of the vortices is described by the complex vector \(n^I\) which we introduced in (6.1.11). It transforms like a fundamental vector of the \(SU(N)_{c+f}\) symmetry

\[
n \to U_{c+f} n
\]

\(^{5}\) The absence of “colored dyons” \[47, 48, 49, 50\] mentioned earlier can also be interpreted in this manner.
If we remember the constraint $|n|^2 = 1$ and the phase identification $n \sim e^{i\phi} n$, we get the parametrization of the moduli space $\mathbb{C}P^{N-1}$. The points on this space represent all possible $k = 1$ vortices. Note that points on the space of a quantum mechanical $N$-state system,

$$|\Psi\rangle = \sum_{i=1}^{N} a_i |\psi_i\rangle, \quad a_i \sim \lambda a_i, \quad \lambda \in \mathbb{C}, \quad (2.3.4)$$

can be put in one-to-one correspondence with the points of a $\mathbb{C}P^{N-1}$.

The fact that the vortices (seen as solitons of the low-energy approximation) transform as in the $N$ representation of $SU(N)_{c+f}$, implies that there exist a set of monopoles which transform accordingly, as $\overline{N}$. The existence of such a set follows from the exact $SU(N)_{c+f}$ symmetry of the theory, broken by the individual monopole-vortex configuration. Such a kind of considerations can answer questions like the one we pose at the beginning of this Chapter, about the quantum multiplicity of non-Abelian monopoles. Note that in our derivation of continuous transformations of the monopoles, the explicit, semiclassical form of the latter is not utilized.

A subtle point is that in the high-energy approximation, and to lowest order of such an approximation, the semiclassical monopoles are just certain non-trivial field configurations involving $a(x)$ and $A_i(x)$ fields, and therefore apparently transform under the color part of $SU(N)_{C+F}$ only. When the full monopole-vortex configuration $a(x), A_i(x), q(x)$ Fig. (2.2) are considered, however, only the combined color-flavor diagonal transformations keep the energy of the configuration invariant. In other words, the monopole transformations must be regarded as part of more complicated transformations involving flavor, when higher order effects in $O(v_1 v_2)$ are taken into account.

And this means that the transformations are among physically distinct states, as the vortex moduli describe obviously physically distinct vortices.

6 Another independent effect due to the massless flavors is that of Jackiw-Rebbi [80]: due to the normalizable zero-modes of the fermions, the semi-classical monopole is converted to some irreducible multiplet of monopoles in the flavor group $SU(N_f)$. The “clouds” of the fermion fields surrounding the monopole have an extension of $O(\frac{1}{v_1})$, which is much smaller than the distance scales associated with the infrared effects discussed here and should be regarded as distinct effects.
2.4 Non-Abelian duality requires an exact flavor symmetry

In the $\mathcal{N} = 2$ supersymmetric QCD, the presence of massless flavor and the exact color-flavor diagonal symmetry is fundamental for the emergence of the dual (non-Abelian) gauge transformations. It is well known in fact that the continuous non-Abelian vortex flux moduli - hence the non-Abelian vortex - disappear as soon as non-zero mass differences $m_i - m_j$ are introduced. \[7\] Also in order for the $SU(N)_{C+F}$ color-flavor symmetry not to be destroyed by the gauge dynamics itself, it is necessary to have the number of flavors such that $N_f \geq 2N$. These points have been emphasized already in the first paper on the subject [16].

It is illuminating that the same phenomenon can be seen in the fully quantum behavior of the theory of Section (2.1), in another regime,

$$\mu, m_i \sim \Lambda$$ \hspace{1cm} (2.4.1)

(cfr. Eq. (2.1.6)). Indeed, this model was analyzed thoroughly in this regime in [64]. The so-called $r$ vacua with the low-energy effective $SU(r) \times U(1)^{N+1-r}$ gauge symmetry emerges in the equal mass limit $m_i \to m$ in which the global symmetry group $SU(N_f) \times U(1)$ of the underlying theory become exact. When the bare quark masses are almost equal but distinct, the theory possesses a group of \(\binom{N_f}{r}\) nearby vacua, each of which is an Abelian $U(1)^N$ theory, with $N$ massless Abelian magnetic monopole pairs. The jump from the $U(1)^N$ to $SU(r) \times U(1)^{N+1-r}$ theory in the exact $SU(N_f)$ limit might appear a discontinuous change of physics, but is not so. What happens is that the range of validity of Abelian description in each Abelian vacuum, neglecting the light monopoles and gauge bosons (including massless particles of the neighboring vacua, and other light particles which fill up a larger gauge multiplet in the limit the vacua coalesce), gradually tends to zero as the vacua collide. The non-Abelian, enhanced gauge symmetry of course only emerges in the strictly degenerate limit, in which the underlying theory has an exact $SU(N_f)$ global symmetry.

\[7\] Such an alignment of the vacuum with the bare mass parameters is characteristic of supersymmetric theories, familiar also in the $\mathcal{N} = 1$ SQCD [83]. In real QCD we do not expect such a strict alignment.
Chapter 3

Moduli Space of Non-Abelian Vortices

In this Chapter we will review the construction of the most general non-Abelian vortex solution. A composite state of several BPS vortices develops a complicate moduli space of degenerate solutions. This property is a consequence of the BPS saturation, which implies that the static forces between BPS solitons are exactly zero. The full moduli space, for a generic number of vortices, was first obtained in certain D-brane configuration in string theory [7, 8], and later in a field theory framework [10] as well.

3.1 The moduli matrix formalism

To study the most general vortex solution, it is convenient to slightly change notations. First of all we will make an $SU(2)_R$ rotation on the vortex theory (1.1.3) to align the FI term along the D-term direction. After the rotation, the VEVs of the squark fields become:

$$\langle H \rangle = \xi, \quad \langle \tilde{H}^\dagger \rangle = 0,$$

with

$$\begin{pmatrix} H \\ \tilde{H}^\dagger \end{pmatrix} = U_R \begin{pmatrix} Q \\ Q_i \end{pmatrix},$$

(3.1.1)

while the VEV of the rotated adjoint fields still remain zero. It is easy to show that the fields with zero VEV, like the adjoint fields and the anti-chiral squark multiplet $\tilde{H}$ are trivial (equal to their VEVs) for every BPS vortex configuration. To study the classical properties of vortices, it thus suffices to
consider a truncated bosonic sector of theory (1.1.3):

\[ \mathcal{L} = -\frac{1}{4e^2} F^\mu_\nu F^{\mu\nu} - \frac{1}{4g^2} F^a_\mu F^{a\mu} + (D_\mu H_f)^\dagger D^\mu H_f \\
- \frac{e^2}{2} \left| H_f^\dagger H_f - \frac{v^2}{\sqrt{2N_c}} \right|^2 - \frac{g^2}{2} |H_f^\dagger H_f|^2. \]  

(3.1.2)

We can consider, for a general discussion, a generic number of flavors: \( f = 1, \ldots, N_f \) with \( N_f \geq N_c \equiv N \) To obtain the expression above from (1.1.3) one has to conveniently redefine the gauge coupling \( e \), the abelian fields \( A_0^a \) and \( a \) and the FI term. The abelian generator is now normalized as

\[ t_0 = \frac{1}{\sqrt{2N_c}} \mathbb{1}_{N_c}, \]  

(3.1.3)

while field strength and covariant derivative are given by

\[ F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + i [W_\mu, W_\nu], \quad D_\mu H = (\partial_\mu + i W_\mu) H. \]  

(3.1.4)

in terms of the gauge fields \( W_\mu \). The vacuum equation \( H H^\dagger = \frac{v^2}{1} \mathbb{1}_{N_c} \) implies that \( H \) has maximal rank and so, after quotienting out the gauge equivalent configurations, \( H \) defines a Grassmannian manifold. Therefore the model has a continuous Higgs branch

\[ \mathcal{V}_{\text{Higgs}} = Gr_{N_f, N_c} \simeq \frac{SU(N_f)}{SU(N_c) \times SU(N_f - N_c) \times U(1)} \]  

(3.1.5)

and no Coulomb vacuum\footnote{The gauge symmetry is completely broken while an exact global \( SU(N_c)_{\text{C+F}} \) color-flavor symmetry remains unbroken. We define a dual theory by the same Lagrangian \( (3.1.2) \) with different gauge group \( U(N_c) \) \( (\tilde{N}_c \equiv N_f - N_c) \) and the same number of flavors. From the last expression of \( (3.1.5) \) the Higgs branch of the dual theory is obviously identical to that of the original theory. We refer to this duality as “Seiberg-like dual”, or simply “Seiberg dual”. In the Hanany-Witten type D-brane realization \[87\] this duality can be understood as exchange of two NS5-branes, while the original Seiberg duality in \( \mathcal{N} = 1 \) theory can be understood by this procedure with one NS5-brane rotated.}

In the context of \( \mathcal{N} = 2 \) supersymmetry the Higgs branch is the cotangent bundle over the Grassmannian manifold, \( T^* Gr_{N_f, N_c} \) \[84\]\[85\] obtained as a hyper-Kähler quotient \[86\].
The Bogomol’nyi completion for independent configurations reads in this notation:

\[
T = \int d^2 x \left[ \frac{1}{2e^2} \left| F_{12}^0 - e^2 \left( H_A^\dagger t^0 H_A - \frac{v^2}{\sqrt{2N}} \right) \right|^2 + 4 |D_z H|^2 + \frac{1}{2g^2} \left| F_{12}^a - g^2 H_A^\dagger t^a H_A \right|^2 - \frac{v^2}{\sqrt{2N}} F_{12}^0 \right] \\
\geq - \frac{v^2}{\sqrt{2N}} \int d^2 x F_{12}^0 ,
\]

(3.1.6)

where a complex coordinate \( z \equiv x^1 + ix^2 \) has been introduced. We obtain the BPS vortex equations

\[
D_\bar{z} H = 0 , \quad F_{12}^0 - \frac{e^2}{\sqrt{2N}} \left( \text{tr} (H H^\dagger) - v^2 \right) = 0 , \quad F_{12}^a t^a - \frac{g^2}{2} \left[ H H^\dagger - \frac{1}{N} \text{tr} (H H^\dagger) \right] = 0 .
\]

(3.1.7, 3.1.8, 3.1.9)

To simplify the algebra, we take the gauge couplings to be equal \( e = g^2 \).

The BPS equations can be casted in a simple matrix form:

\[
D_\bar{z} H = 0 , \quad F_{12} - \frac{g^2}{2} \left( v^2 I_{N_c} - H H^\dagger \right) = 0 .
\]

(3.1.10)

Actually these equations possess a continuous moduli space of solutions. Since every solution is characterized by a topological (winding) number valued in \( \pi_1(U(N_c)) = \mathbb{Z} \), the moduli space is divided into topological sectors. Upon having introduced the complex parametrization of the plane, \( z = x_1 + ix_2 \), coordinates on the moduli space are conveniently collected in a unique mathematical object \( H_0(z) \), a holomorphic \( N_c \times N_f \) matrix called the moduli matrix which is defined by [81]

\[
H = S^{-1}(z, \bar{z}) H_0(z) , \quad W_1 + i W_2 = -2 i S^{-1}(z, \bar{z}) \partial_\bar{z} S(z, \bar{z})
\]

(3.1.11)

where \( S(z, \bar{z}) \) is an \( N_c \times N_c \) invertible matrix. The elements of \( H_0(z) \) are polynomials in \( z \) whose coefficients are good coordinates (in the sense of [SS]) on the moduli space. A configuration has winding number \( k \) if

\[
\det H_0 H_0^\dagger = O(|z|^{2k})
\]

(3.1.12)

\(^2\)In the next Chapters we will generalize some discussions to the case of different couplings.

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for large $z$. The tension of the corresponding vortex configuration is given by Eq. (3.1.6)

$$T = 2\pi v^2 \frac{k}{N}$$  \hspace{1cm} (3.1.13)

From the definition (3.1.11) one sees that $H_0(z)$ and $S(z, \bar{z})$ are only determined up to the so-called $V$-equivalence given by

$$H_0(z) \to V(z) H_0(z), \quad S^{-1}(z, \bar{z}) \to S^{-1}(z, \bar{z}) V(z)^{-1}$$  \hspace{1cm} (3.1.14)

where $V(z)$ is a $GL(N, C)$ matrix whose elements are polynomials in $z$. Eq. (3.1.12) implies that if one fixes the winding number $k$, then $\det V$ must be constant.

The first BPS equation is automatically solved by the ansatz (3.1.11), while the second can be rewritten as

$$\partial_z (\Omega^{-1} \bar{\partial}_z \Omega) = \frac{g^2}{4} \left( v^2 1_{N_C} - \Omega^{-1} H_0 H^\dagger_0 \right),$$  \hspace{1cm} (3.1.15)

where $\Omega \equiv S(z, \bar{z}) S^\dagger(z, \bar{z})$. We will refer to Eq. (3.1.15) as the master equation, and assume that it has a unique solution with a boundary condition $\Omega \to v^{-2} H_0 H^\dagger_0$. This assumption has only been proven in the abelian case and for vortices on compact Riemann surfaces; however, there are arguments that it extends to general vortices on $C$ (see [73] for a more detailed discussion).

The moduli space of the vortex equations Eq. (3.1.10) is obtained as the quotient space $\mathcal{M}_{\text{total}} = \{ H_0(z) \}/GL(N, C)$. This space is infinite dimensional and can be decomposed into topological sectors according to the vortex number $k$. The $k$-th topological sector $\mathcal{M}_{N,k}$, the moduli space of $k$ vortices, is determined by the condition that $\det H_0(z)$ is of order $z^k$:

$$\mathcal{M}_{N,k} \simeq \left\{ H_0(z) \Big| \det H_0(z) = \mathcal{O}(z^k) \right\} /\{V(z)\}. $$  \hspace{1cm} (3.1.16)

Notice that the rank of $H_0$ gets reduced by one at vortex positions $z_i$ when all the vortices are separated, $z_i \neq z_j$ for $i \neq j$. A constant vector defined by

$$H_0(z = z_i) \tilde{\phi}_i = 0$$  \hspace{1cm} (3.1.17)

The tension of the ANO vortex is, in this notation $2\pi k$, which differs from (1.2.8) by a factor of $1/N$. The difference is simply due to the different normalization a the $U(1)$ generator.

3The tension of the ANO vortex is, in this notation $2\pi k$, which differs from (1.2.8) by a factor of $1/N$. The difference is simply due to the different normalization a the $U(1)$ generator.
is associated with each component vortex at \( z = z_i \). An overall constant of \( \vec{\phi}_i \) cannot be determined from Eq. (3.1.17) so we should introduce an equivalence relation “\( \sim \)”, given by

\[
\vec{\phi}_i \sim \lambda \vec{\phi}_i, \quad \text{with } \lambda \in \mathbb{C}^*.
\] (3.1.18)

Thus, each vector \( \vec{\phi}_i \) takes a value in the projective space

\[
CP^{N-1} = SU(N)/[SU(N - 1) \times U(1)].
\] (3.1.19)

This space can be understood as a space parameterized by Nambu-Goldstone modes associated with the symmetry breaking,

\[
SU(N)_{G+F} \rightarrow U(1) \times SU(N - 1),
\] (3.1.20)

caused by the presence of a vortex [7, 8, 16, 89, 10, 75]. We call \( \phi_i \) the orientational vector.

### 3.2 Kahler quotient construction

Following [75], it is possible to organize the moduli into a Kähler quotient [59]. First, let us write the moduli matrix as

\[
H_0(z) = (D(z), Q(z)) \quad (3.2.1)
\]

where \( D(z) \) is an \( N_c \times N_c \) matrix and \( Q(z) \) is a \( N_c \times \tilde{N}_c \) matrix (\( \tilde{N}_c \equiv N_f - N_c \)). Defining

\[
P(z) \equiv \det D(z),
\] (3.2.2)

we set \( \deg P(z) = k \), whereas all other minor determinants of \( H_0(z) \) have degree at most \( k - 1 \): this guarantees that Eq. (3.1.12) is satisfied and the winding number is equal to \( k \). One can show that only a subset of the minor determinants are independent, namely \( P(z) \) and the determinants of the minor matrices obtained by substituting the \( r \)-th column of \( D(z) \), \( r = 1, \ldots, N_c \), with the \( A \)-th column of \( Q(z) \), \( A = 1, \ldots, \tilde{N}_c \). The matrix having these minor determinants as its \((rA)\) elements is denoted by \( F(z) \),

\[
F_{rA} = \sum_{k=1}^{N_c} Q_{kA} (\text{Cof } D)_{rk} = P(z) \sum_{k=1}^{N_c} Q_{kA} (D^{-1})_{rk},
\]

\[
F(z) = P(z) D^{-1} Q(z).
\] (3.2.3)
Consider the equation
\[ D(z) \vec{\phi}(z) = J(z)P(z) = 0 \mod P(z) \] (3.2.4)
for \( \vec{\phi} \) modulo \( P(z) \), where the components of \( \vec{\phi} \) are polynomials at most of degree \( z^k - 1 \). We find \( k \) linearly independent such vectors \( \vec{\phi}_i(z) \), \( i = 1, \ldots, k \), each of which is a solution with a suitable \( J_i(z) \). In matrix form,
\[ D(z) \Phi(z) = J(z)P(z) = 0 \mod P(z) \] (3.2.5)
where \( \Phi(z) \) and \( J(z) \) are \( N_c \times k \) matrices made of the \( \{ \vec{\phi}_i \} \) and the \( \{ J_i \} \), respectively. We are naturally free to choose a basis in the linear space of solutions to Eq. (3.2.4) using the equivalence relation
\[ \Phi(z) \sim \Phi(z) V^{-1}, \ V \in GL(k, \mathbb{C}). \] (3.2.6)
Since \( \Phi(z) \) has the maximal rank, the \( \{ \vec{\phi}_i \} \) being linearly independent, the \( GL(k, \mathbb{C}) \) action is free\(^4\)
\[ \Phi(z) = \Phi(z) V^{-1} \Rightarrow V^{-1} = 1_k. \] (3.2.7)

Now consider the product \( z \vec{\phi}_i(z) \), whose degree is in general less than or equal to \( k \). The polynomial division by \( P(z) \) leads to a constant quotient and a remainder that must be a linear combination of \( \{ \vec{\phi}_i \} \), as it must satisfy Eq. (3.2.4). In matrix form, these are summarized as
\[ z \Phi(z) = \Phi(z)Z + \Psi P(z). \] (3.2.8)
By multiplying \( D(z)/P(z) \) from the left and using Eq. (3.2.5), we also find,
\[ zJ(z) = J(z)Z + D(z)\Psi. \] (3.2.9)
This defines uniquely the constant matrices \( Z \) and \( \Psi \), of sizes \( k \times k \) and \( N_c \times k \) respectively. They enjoy an equivalence relation due to (3.2.6)
\[ (Z, \Psi) \sim (VZV^{-1}, \Psi V^{-1}), \ V \in GL(k, \mathbb{C}), \] (3.2.10)
where the \( GL(k, \mathbb{C}) \) action is free (Eq. (3.2.7)). Eigenvalues of \( Z \) describe \( k \) positions of vortices, and thus there is an equality \( P(z) = \det D(z) = \det(z - Z) \). Roughly speaking, each column of \( \Psi \) parametrizes an orientation in \( \mathbb{C}P^{N_c - 1} \) of each corresponding vortex. This would be the whole story in

\(^4\)A group \( G \) is said to act freely on a space \( M \), if for any point \( x \in M \), \( gx = x \) (\( g \in G \)) implies \( g = 1 \).
the local case, but in the semi-local case there are extra moduli coming from $Q(z)$.

Using the relation

$$D(z)F(z) = Q(z)P(z)$$  \hspace{1cm} (3.2.11)

which follows from Eq. (3.2.3) and the condition $\text{deg } F(z)_r \leq k − 1$ one finds that the columns of $F(z)$ are linear combinations of those of $\Phi(z)$,

$$F(z) = \Phi(z)\tilde{\Psi},$$  \hspace{1cm} (3.2.12)

where $\tilde{\Psi}$ is a $k \times \tilde{N}_c$ constant matrix. Comparing Eq. (3.2.5) and Eq. (3.2.11) we find

$$Q(z) = J(z)\tilde{\Psi}.$$  \hspace{1cm} (3.2.13)

As $\Phi(z)$ is defined modulo equivalence relation Eq. (3.2.6), it follows that $\tilde{\Psi}$ is also defined up to

$$\tilde{\Psi} \sim \mathcal{V}\tilde{\Psi}, \quad \mathcal{V} \in \text{GL}(k, \mathbb{C}).$$  \hspace{1cm} (3.2.14)

Having identified the moduli space parameters with the space defined by Eqs. (3.2.10) and (3.2.14), it is straightforward to identify the same space as the Higgs branch of a $(1+1)$-dimensional, $\mathcal{N} = 1$ supersymmetric, non-Abelian $U(k)$ gauge theory, with $N_c$ fundamental hypermultiplets $\Psi$, $\tilde{N}_c$ anti-fundamental multiplets $\tilde{\Psi}$, and an adjoint multiplet $Z$. The theory has also a positive FI terms, which put the theory on the Higgs Branch. The D-term constraint is given by:

$$D = [Z^\dagger, Z] + \Psi^\dagger\Psi - \tilde{\Psi}\tilde{\Psi}^\dagger - r = 0.$$  \hspace{1cm} (3.2.15)

The constraint, as usual for a supersymmetric theory, can be considered as a fixing condition for the complexified part of the gauge symmetry. The 2-dimensional FI term is related to the 4-dimensional gauge coupling by the relation:

$$r = \frac{2\pi}{g^2}$$  \hspace{1cm} (3.2.16)

### 3.3 Fundamental vortex revised

Let us first discuss again single non-Abelian vortex in $U(2)$ gauge theory, within the moduli matrix formalism. We will obtain again the results of Section (1.2).
The condition on the moduli matrix $H_0$ is $\det H_0 = \mathcal{O}(z)$. Modulo $V$-equivalence relation Eq. [3.1.14], the moduli matrix can be brought to one of the following two forms [89]:

$$H_0^{(1,0)}(z) = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \quad H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix} \quad (3.3.1)$$

with $b, b'$ and $z_0$ complex parameters. Here $z_0$ gives the position moduli whereas $b$ and $b'$ give the orientational moduli as we see below. The two matrices in Eq. (3.3.1) describe the same single vortex configuration but in two different patches of the moduli space. Let us denote them $U^{(1,0)} = \{z_0, b'\}$ and $U^{(0,1)} = \{z_0, b\}$. The transition function between these patches is given, except for the point $b' = 0$ in the patch $U^{(1,0)}$ and $b = 0$ in $U^{(0,1)}$, by a $V$-transformation of the form [89]

$$V = \begin{pmatrix} 0 & -1/b' \\ b' & z - z_0 \end{pmatrix} \in GL(2, \mathbb{C}). \quad (3.3.2)$$

This yields the transition function

$$b = \frac{1}{b'}, \quad (b, b' \neq 0). \quad (3.3.3)$$

$b'$ and $b$ are seen to be the two patches of a $\mathbb{C}P^1$, leading to the conclusion that the moduli space of the single non-Abelian vortex is

$$\mathcal{M}_{N=2, k=1} \simeq \mathbb{C} \times \mathbb{C}P^1, \quad (3.3.4)$$

where the first factor $\mathbb{C}$ corresponds to the position $z_0$ of the vortex.

The same conclusion can be reached from the orientation vector, defined by $H_0(z = z_0) \vec{\phi} = 0$: $\vec{\phi}$ is given by

$$\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix}. \quad (3.3.5)$$

We see that the components of $\vec{\phi}$ are the homogeneous coordinates of $\mathbb{C}P^1$; $b, b'$ are the inhomogeneous coordinates.\footnote{Similarly, in the case of $U(N)$ gauge theory, the components of $\vec{\phi}$ correspond to the homogeneous coordinates of $\mathbb{C}P^{N-1}$.}

The individual vortex breaks the color-flavor diagonal symmetry $SU(2)_{c+f}$, so that it transforms nontrivially under it. The transformation property of the vortex moduli parameters can be conveniently studied by the $SU(2)_{f}$ flavor transformations on the moduli matrix, as the color transformations.
acting from the left can be regarded as a \( V \) transformation. The flavor symmetry acts on \( H_0 \) as \( H_0 \to H_0 U \) with \( U \in SU(2)_f \). A general \( SU(2) \) matrix

\[
U = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix},
\]

(3.3.6)

with \( u, v \in \mathbb{C} \) satisfying \( |u|^2 + |v|^2 = 1 \), acts for instance on \( H_0^{(0,1)} \) in Eq. (3.3.1) as

\[
H_0^{(0,1)}(z) \to H_0^{(0,1)}(z) U = \begin{pmatrix} u + v^* b & -u^* b + v \\ -v^* (z - z_0) & u^* (z - z_0) \end{pmatrix}.
\]

(3.3.7)

The right hand side should be pulled back to the form \( H_0^{(0,1)} \) in Eq. (3.3.1) by using an appropriate \( V \)-transformation. This can be achieved by

\[
V_U H_0^{(0,1)}(z) U = \begin{pmatrix} 1 - \frac{u^* b - v}{v^* b + u} \\ 0 \\ z - z_0 \end{pmatrix},
\]

\[
V_U = \begin{pmatrix} (u + v^* b)^{-1} & 0 \\ v^* (z - z_0) & u + v^* b \end{pmatrix} \in GL(2, \mathbb{C}).
\]

(3.3.8)

The \( SU(2)_{c+f} \) transformation law of \( b \) is then

\[
b \to \frac{u^* b - v}{v^* b + u},
\]

(3.3.9)

which is the standard \( SU(2) \) transformation law of the inhomogeneous coordinate of \( CP^1 \).

In terms of the orientational vector \( \vec{\phi} \) in Eq. (3.3.5), the transformation law Eq. (3.3.9) can be derived more straightforwardly. According to the definition Eq. (3.1.17), \( \vec{\phi} \) is transformed in the fundamental representation of \( SU(2)_f \):

\[
\vec{\phi} \to U^\dagger \vec{\phi}, \quad U \in SU(2)_F.
\]

(3.3.10)

---

\( ^6 \) The coordinate \( b \) is invariant (more precisely the orientational vector receives a global phase) under the \( U(1) \) transformations generated by \( n \cdot \vec{\sigma}/2 \),

\[ n = \frac{1}{(1+|b|^2)} \begin{pmatrix} 2 \Re b & 2 \Im b & 0 \\ 2 \Im b & 2 \Re b & 0 \end{pmatrix}, \]

\[ |b|^2 - 1. \]

This implies the coset structure \( CP^1 \simeq SU(2)_{c+f}/U(1) \).
Part II

Vortices in $U(N)$ Gauge Theories
Chapter 4

Non-Abelian vortices of higher winding numbers

The moduli subspace of $k = 2$ axially symmetric vortices was studied by two groups: Hashimoto and Tong [90] and Auzzi, Shifman and Yung [91]. The former concluded that it is $\mathbb{C}P^2$ by using the brane construction [7, 8] whereas the latter found $\mathbb{C}P^2/\mathbb{Z}_2$ by using a field theoretical construction. This discrepancy is crucial when one discusses the interactions of vortices because the latter contains an orbifold singularity (see Chapter (9) for a discussion about the reconnection of cosmic strings). In this Section we study the moduli space of $k = 2$ axially symmetric non-Abelian vortices of $U(N)$ gauge theories, by using the method of moduli matrix. For simplicity we will take here $N_f = N_c = N$. We definitely solve the discrepancy between the two results.

4.1 $k = 2$ Vortices in $U(2)$ Gauge Theory

Configurations of $k = 2$ vortices at arbitrary positions are given by the moduli matrix whose determinant has degree two, $\det H_0 = \mathcal{O}(z^2)$. By using $V$-transformations Eq. (3.1.14) the moduli matrix satisfying this condition can be brought into one of the following three forms [10, 75]

$$H_0^{(2,0)} = \begin{pmatrix} z^2 - \alpha' & z - \beta' & 0 \\ -a' & -b' & 1 \end{pmatrix}, \quad H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \tilde{\phi} \end{pmatrix},$$

$$H_0^{(0,2)} = \begin{pmatrix} 1 & -a z - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}. \quad (4.1.1)$$

These define the three patches $U^{(2,0)} = \{a', b', \alpha', \beta'\}$, $U^{(1,1)} = \{\phi, \tilde{\phi}, \eta, \tilde{\eta}\}$, $U^{(0,2)} = \{a, b, \alpha, \beta\}$ of the moduli space $\mathcal{M}_{N=2,k=2}$. The transition from
$U^{(1,1)}$ to $U^{(0,2)}$ is given via the $V$-transformation $V = \left( \begin{array}{cc} 0 & -1/\tilde{\eta} \\ \tilde{\eta} & z - \phi \end{array} \right)$:

$$a = \frac{1}{\tilde{\eta}}, \quad b = -\frac{\tilde{\phi}}{\tilde{\eta}}, \quad \alpha = \phi + \tilde{\phi}, \quad \beta = \eta \tilde{\eta} - \phi \tilde{\phi}. \quad (4.1.2)$$

Similarly the transition from $U^{(2,0)}$ to $U^{(0,2)}$ is given by

$$V = \left( \begin{array}{cc} a' z^2 & a' \alpha \beta - a' b' \alpha - b' \beta \\ a' \beta - a' \alpha \beta - b' \beta - \alpha \beta \end{array} \right), \quad (4.1.3)$$

which yields

$$a = \frac{a'}{a'^2 \beta' - a' \beta' \alpha' - b'^2}, \quad b = -\frac{b' + a' \alpha'}{a'^2 \beta' - a' \beta' \alpha' - b'^2}, \quad \alpha = \alpha', \quad \beta = \beta'. \quad (4.1.4)$$

Finally those between $U^{(1,1)}$ and $U^{(2,0)}$ are given by the composition of the transformations Eq. (4.1.2) and Eq. (4.1.4). Let us now discuss the moduli space of $k = 2$ vortices separately for the cases where the two vortex centers are 1) distinct ($z_1 \neq z_2$), and 2) coincident ($z_1 = z_2$).

1) At the vortex positions $z_i$ the orientational vectors are determined by Eq. (3.1.17). The orientational vectors $\vec{\phi}_i$ ($i = 1, 2$) are then obtained by

$$\vec{\phi}_i \sim \left( \begin{array}{c} az_i + b \\ 1 \end{array} \right) \sim \left( \begin{array}{c} z_i - \phi \\ \eta \end{array} \right) \sim \left( \begin{array}{c} \eta \\ z_i - \phi \end{array} \right) \sim \left( \begin{array}{c} 1 \\ a' z_i + b' \end{array} \right). \quad (4.1.5)$$

The two ($i = 1, 2$) parameters defined by $b_i \equiv a z_i + b$ (or $b'_i \equiv a' z_i + b'$) parameterize the two different $\mathbb{C}P^1$’s separately. Conversely if the two vortices are separated $z_1 \neq z_2$, then moduli parameters $a, b$ are described by $b_1, b_2$ with positions of vortices $z_1, z_2$ as

$$a = \frac{b_1 - b_2}{z_1 - z_2}, \quad b = \frac{b_2 z_1 - b_1 z_2}{z_1 - z_2}, \quad \alpha = z_1 + z_2, \quad \beta = -z_1 z_2, \quad (4.1.6)$$

(and similar relations for the primed variables). Thus in the case of separated vortices $\{b_1, b_2, z_1, z_2\}$ ($\{b'_1, b'_2, z_1, z_2\}$) can be taken as appropriate coordinates of the moduli space, instead of $\{a, b, \alpha, \beta\}$ ($\{a', b', \alpha', \beta'\}$). The transition functions are also obtained by applying the equivalence relation Eq. (3.1.18) to Eq. (4.1.5), for instance,

$$b_i = \frac{1}{b'_i} \quad (b_i, b'_i \neq 0). \quad (4.1.7)$$

It can be shown that these are equivalent to Eqs. (4.1.4) by use of Eq. (4.1.6) and analogous relation for the primed parameters. The coordinates in $U^{(1,1)}$
are also the orientational moduli. If we take \( \{b_1, b'_2, z_1, z_2\} \) as a set of independent moduli and substitute Eq. (4.1.6) and \( b_2 = 1/b'_2 \) to Eq. (4.1.2), then we obtain, for \( b_1 b'_2 \neq 1 \)

\[
\phi = \frac{z_2 - b_1 b'_2 z_1}{1 - b_1 b'_2}, \quad \eta = \frac{z_1 - z_2}{1 - b_1 b'_2} b_1, \quad \tilde{\phi} = \frac{z_1 - b_1 b'_2 z_2}{1 - b_1 b'_2}, \quad \tilde{\eta} = -\frac{z_1 - z_2}{1 - b_1 b'_2} b'_2.
\]

(4.1.8)

It can be seen that the representation Eq. (4.1.6) implies that \( U(0,2) \) and \( U(2,0) \) are suitable for describing the situation when two orientational moduli are parallel or nearby. On the other hand, Eq. (4.1.8) implies that \( U(1,1) \) is suitable to describe the situation when orientational moduli are orthogonal or close to such a situation, while not adequate for describing a parallel set. Therefore, the moduli space for two separated vortices are completely described by the positions and the two orientational moduli \( b_1, b_2, (b'_1, b'_2) \); the moduli space for the composite vortices in this case is given by \( M_{N=2, k=2} \approx \mathbb{C} \times \mathbb{C} P^1 / \mathcal{G}_2 \),

(4.1.9)

where \( \mathcal{G}_2 \) permutes the centers and orientations of the two vortices.

2) We now focus on coincident (co-axial) vortices \((z_1 = z_2)\), with the moduli space denoted by

\[
\tilde{M}_{N=2, k=2} \equiv M_{N=2, k=2} \big|_{z_1 = z_2}.
\]

(4.1.10)

As an overall translational moduli is trivial, we set \( z_1 = z_2 = 0 \) without loss of generality. According to Eqs. (4.1.6) and Eq. (4.1.8), all points in the moduli space tend to the origin of \( U^{(1,1)} \) in the limit of \( z_2 \to z_1 \), as long as \( b_1 \) and \( b_2 \) take different values. A more careful treatment is needed in this case. In terms of the moduli matrix, the condition of coincidence is given by \( \det H_0(z) = z^2 \). We have

\[
\{\alpha = 0, \ \beta = 0\}, \ \{\tilde{\phi} = -\phi, \ \phi \tilde{\phi} - \eta \tilde{\eta} = 0\} \quad \text{and} \quad \{\alpha' = 0, \ \beta' = 0\},
\]

(4.1.11)
in \( U^{(2,0)}, U^{(1,1)} \) and \( U^{(0,2)} \), respectively. \( \tilde{M}_{N=2, k=2} \) is covered by the reduced patches \( \tilde{U}^{(2,0)}, \tilde{U}^{(1,1)} \) and \( \tilde{U}^{(0,2)} \), defined by the moduli matrices

\[
H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a' z - b' & 1 \end{pmatrix}, \quad H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \quad H_0^{(0,2)} = \begin{pmatrix} 1 & -a z - b \\ 0 & z^2 \end{pmatrix}.
\]

(4.1.12)
The following constraint exists among the coordinates in $\tilde{U}^{(1,1)}$:

$$\phi^2 + \eta \tilde{\eta} = 0. \quad (4.1.13)$$

The transition functions between $\tilde{U}^{(0,2)}$ and $\tilde{U}^{(1,1)}$ are given by

$$a = \frac{1}{\tilde{\eta}}, \quad b = \frac{\phi}{\tilde{\eta}} = -\frac{\eta}{\phi}, \quad (4.1.14)$$

and those between $\tilde{U}^{(0,2)}$ and $\tilde{U}^{(2,0)}$ by

$$a = -\frac{a'}{b'^2}, \quad b = \frac{1}{b'}. \quad (4.1.15)$$

All the patches defined by Eq. (4.1.12) are parameterized by two independent complex parameters. The reduced patches $\tilde{U}^{(2,0)}$ and $\tilde{U}^{(0,2)}$ are locally isomorphic to $C^2$ with $a, b$ or $a', b'$ being good coordinates. However, $\tilde{U}^{(1,1)}$ suffers from the constraint Eq. (4.1.13) which gives the $A_1$-type ($Z_2$) orbifold singularity at the origin and therefore $\tilde{U}^{(1,1)} \simeq C^2/Z_2$ locally (See Eq. (4.1.18), below). Note that the moduli matrix $H_0(z)$ is proportional to the unit matrix at the singularity: $H_0(z) = z 1_2$. This implies that configurations of the physical fields ($H$ and $F_{12}$) are also proportional to the unit matrix where the global symmetry $SU(2)_{G+F}$ is fully recovered at that singularity. The full gauge symmetry is also recovered at the core of coincident vortices.

**Remark:** A brief comment on the orientational vectors. We could extract a part of moduli in the moduli matrix as the orientational vector at $z = 0$, as in the case of separated vortices discussed above:

$$\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} \eta \\ -\phi \end{pmatrix} \sim \begin{pmatrix} \phi \\ \tilde{\eta} \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix}. \quad (4.1.16)$$

From the identification $\vec{\phi} \sim \lambda \vec{\phi}$ ($\lambda \in C^*$) with the transition functions given in Eqs. (4.1.14) and Eq. (4.1.15), we find that the orientational moduli again parameterizes $CP^1$. However, the orientational vectors in Eq. (4.1.16) are not sufficient to pick up all the moduli parameters in the moduli matrix $H_0$. For instance $a$ is lost in the $\tilde{U}^{(0,2)}$ patch. It is even ill-defined at the singular point, as $H_0^{(1,1)}(z = 0) = 0$.

To clarify the whole structure of the space $\tilde{M}_{N=2,k=2}$, let us define new coordinates, solving the constraint Eq. (4.1.13)

$$XY \equiv -\phi, \quad X^2 \equiv \eta, \quad Y^2 \equiv -\tilde{\eta}. \quad (4.1.17)$$

This clarifies the structure of the singularity at the origin. The coordinates $(X,Y)$ describe the patch $\tilde{U}^{(1,1)}$ correctly modulo $Z_2$ identification

$$(X,Y) \sim (-X, -Y). \quad (4.1.18)$$
Using the transition functions Eq. (4.1.14) and Eq. (4.1.15), the three local domains are patched together as in Table (4.1). In terms of the new coordinates \((X, Y)\), the orientational vector defined at \(z = 0\) is given by

\[
\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix}
\]  

(4.1.19)

with \(\vec{\phi} \sim \lambda \vec{\phi}'\) \((\lambda \in \mathbb{C}^*)\). This equivalence relation recovers the transition functions between \(b, b'\) and \((X, Y)\) in the Table (4.1). These are coordinates on the \(\mathbb{C}P^1\) as was mentioned above. But this \(\mathbb{C}P^1\) is only a subspace of the moduli space \(\tilde{\mathcal{M}}_{N=2,k=2}\).

The full space \(\tilde{\mathcal{M}}_{N=2,k=2}\) can be made visible by attaching the remaining parameters \(a, a'\) to \(\mathbb{C}P^1\). We arrange the moduli parameters in the three patches \(\tilde{\mathcal{U}}^{(2,0)}, \tilde{\mathcal{U}}^{(1,1)}\) and \(\tilde{\mathcal{U}}^{(0,2)}\) as

\[
\begin{pmatrix} a' \\ 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} \sim \begin{pmatrix} -a \\ b \\ 1 \end{pmatrix},
\]  

(4.1.20)

respectively, with the equivalence relation “\(\sim\)”, defined by

\[
\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \sim \begin{pmatrix} \lambda^2 \phi_0 \\ \lambda \phi_1 \end{pmatrix} \text{ with } \lambda \in \mathbb{C}^*.
\]  

(4.1.21)

All the transition functions in Table (4.1) are then nicely reproduced. The equivalence relation Eq. (4.1.21) defines a weighted complex projective space with the weights \((2, 1, 1)\). We thus conclude that the moduli space for the coincident (coaxial) \(k = 2\) non-Abelian vortices is a weighted projective space,

\[
\tilde{\mathcal{M}}_{N=2,k=2} \simeq \mathbb{W}CP^2_{(2,1,1)}.
\]  

(4.1.22)

While the complex projective spaces with common weights, \(\mathbb{C}P^n\), are smooth, weighted projective spaces have singularities. In fact, we have
shown that $\tilde{U}^{(1,1)} \simeq \mathbb{C}^2/\mathbb{Z}_2$, and it has a conical singularity at the origin by $(1, X, Y) \sim (1, -X, -Y)$, whose existence was first pointed out by ASY [91]. The origin of the conical singularity can be seen clearly from the equivalence relation Eq. (4.1.21). As mentioned above the transition functions in Table (4.1) are reproduced via the equivalence relation Eq. (4.1.21). In fact, one finds that $\lambda = \frac{1}{X}$ gives $(\lambda^2, \lambda X, \lambda Y) = (a', 1, b')$ and $\lambda = \frac{1}{Y}$ gives $(\lambda^2, \lambda X, \lambda Y) = (-a, b, 1)$. Note that $\lambda$ in the equivalence relation Eq. (4.1.21) is completely fixed in the patches $\tilde{U}(2,0)$ and $\tilde{U}(0,2)$ given in Eq. (4.1.20). However, in the middle patch $(1, X, Y)$ we still have a freedom $\lambda = -1$ which leaves the first component 1 untouched, but changes $(1, X, Y) \rightarrow (1, -X, -Y)$.

The relation between our result and that in [91] becomes clear by defining $\xi^2 \equiv \phi_0$ ($\xi = \pm \sqrt{\phi_0}$). Now the parameters $(\xi, \phi_1, \phi_1)$ have a common weight $\lambda$, so they can be regarded as the homogeneous coordinates of $\mathbb{C}P^2$. But one must identify $\xi \sim -\xi$ clearly, and this leads to the $\mathbb{Z}_2$ quotient $(\xi, \phi_1, \phi_1) \sim (\xi, -\phi_1, -\phi_1)$. Therefore our moduli space can also be rewritten as

$$\tilde{\mathcal{M}}_{N=2,k=2} \simeq \mathbb{C}P^2/\mathbb{Z}_2$$

reproducing the result of [91]. Such a $\mathbb{Z}_2$ equivalence, however, does not change the homotopy of $\mathcal{M}_{N=2,k=2}$: it remains $\mathbb{C}P^2$ [90]. This is analogous to an $(x, y) \sim (-x, -y)$ equivalence relation (with real $x, y$) introduced in one local coordinate system of $\mathbb{C}P^1$ (a sphere), which leads to a sphere with two conic singularities (a rugby ball, or a lemon) instead of the original smooth sphere.

4.1.1 $SU(2)$ transformation law of co-axial $k = 2$ vortices

The complex projective space $\mathbb{C}P^2 \simeq \frac{SU(3)}{SU(2) \times U(1)}$ with the Fubini-Study metric has an $SU(3)$ isometry. On the other hand, the weighted projective space $W\mathbb{C}P^2_{(2,1,1)}$ can have an $SU(2)$ isometry at most due to the difference of the weights Eq. (4.1.21). This matches with the fact that we have only $SU(2)_{G+F}$ symmetry acting on the moduli space. In this Section we investigate the $SU(2)_{G+F}$ transformation laws of the moduli for the co-axial two vortices, as was done for the fundamental vortex in Section (3.3).

Let us start with the patch $\tilde{U}^{(0,2)}$, with the moduli matrix

$$H_0^{(0,2)} = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}$$

\(^1\)For instance, it is easily seen that $\mathcal{M}_{N=2,k=2} \simeq \mathbb{C}P^2/\mathbb{Z}_2$ remains simply connected. The higher homotopy groups cannot change by a discrete fibration [92].
An $SU(2)$ matrix $U$ like Eq. [3.3.6] acts on the above $H_0$ from the right,

$$H_0^{(0,2)} \to H_0^{(0,2)} U = \begin{pmatrix} v^* a z + u + v^* b & -u^* a z + v - u^* b \\ -v^* z^2 & u^* z^2 \end{pmatrix}. \tag{4.1.25}$$

A $V$-transformation $V = V_1 V_2$, where

$$V_1 = \begin{pmatrix} -\frac{v^* a}{(u+v^* b)^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} z - \frac{u v^* b}{v^* z^2 a} & a \\ v^* a z + u + v^* b & v^* a z + u + v^* b \end{pmatrix} \tag{4.1.26}$$

brings the result back to the upper right triangle form,

$$H_0^{(0,2)} \to H_0^{(0,2)} U \sim V H_0^{(0,2)} U = \begin{pmatrix} 1 & -\frac{a}{(u+v^* b)^2} z + \frac{v-u^* b}{u+v^* b} \\ 0 & \frac{a}{(u+v^* b)^2} z + \frac{v-u^* b}{u+v^* b} \end{pmatrix}. \tag{4.1.27}$$

The $SU(2)$ transformation laws of the parameters $a, b$ are then

$$a \to \frac{a}{(v^* b + u)^2}, \quad b \to \frac{u^* b - v}{v^* a z + u + v^* b}. \tag{4.1.28}$$

As in the previous Section, $b$ can be regarded as an inhomogeneous coordinate of $CP^1$; in fact, $b$ is invariant under a $U(1)$ subgroup and this means that $b$ parameterizes $CP^1 \simeq SU(2)/U(1)$. On the other hand, the transformation law of the parameter $a$ can be re-written as

$$a \to \left[ \frac{d}{db} \left( \frac{u^* b - v}{v^* b + u} \right) \right] a, \tag{4.1.29}$$

showing that the parameter $a$ is a tangent vector on the base space $CP^1$ parameterized by $b$. This is very natural. First recall the situation for separated vortices. The moduli parameters are extracted from their positions and their orientations, defined at the vortex centers. However, once the vortices overlap exactly ($z_i \to z_j$), the positions and the orientations only do not have enough information. When $l (\leq k)$ vortices are coincident, we need $1, 2, \ldots, l - 1$ derivatives at the coincident point in order to extract all the information. They define how vortices approach each other ($b_i \to b_j$) \[10\].

Some $SU(2)$ action sends the points in the patch $\tilde{U}^{(0,2)}$ to where a better description is in the patch $\tilde{U}^{(2,0)}$, and vice versa. Compare Eq. (4.1.28) with $u = 0, v = i$, with Eq. (4.1.15). This shows indeed that

$$\tilde{U}^{(0,2)} \cup \tilde{U}^{(2,0)} \simeq TCP^1. \tag{4.1.30}$$

Next consider the patch $\tilde{U}^{(1,1)}$ with

$$H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \quad \phi^2 + \eta \tilde{\eta} = 0. \tag{4.1.31}$$
Figure 4.1: Toric diagram of $WCP^2_{(2,1,1)}$ and their three patches $\tilde{U}^{(2,0)}, \tilde{U}^{(1,1)}$ and $\tilde{U}^{(0,2)}$. The diagram is drawn under a gauge fixing condition (called the $D$ term constraint) $\sum a=0 q^a |\phi_a|^2 = 1$ where $U(1)^C$ charges are $q_a = (2, 1, 1)$ for $WCP^2_{(2,1,1)}$ while $q_a = (1, 1, 1)$ for the ordinary $CP^2$. The triangle with the broken line and $O$ (without singularity) denotes the ordinary $CP^2$.

It is convenient to rewrite this as
\[ H_{0}^{(1,1)} = z 1_2 - \vec{X} \cdot \vec{\sigma} \] (4.1.32)
where $\vec{\sigma}$ are the Pauli matrices and
\[ \phi \equiv X_3, \quad \eta \equiv X_1 - iX_2, \quad \tilde{\eta} \equiv X_1 + iX_2. \] (4.1.33)

$X_1, X_2, X_3$ are then complex coordinates with a constraint $X_1^2 + X_2^2 + X_3^2 = 0$.

To keep the form Eq. (4.1.32) under $SU(2)_H$ transformation, we perform a $V$-transformation with $V = U^\dagger$: $H_0^{(1,1)} \rightarrow U^\dagger H_0^{(1,1)} U$. Equivalently, we study the transformation property of the vortex under $SU(2)_{G+F}$. We find
\[ \vec{X} \cdot \vec{\sigma} \rightarrow U^\dagger \left( \vec{X} \cdot \vec{\sigma} \right) U, \] (4.1.34)
that is, the vector $\vec{X}$ transforms as an adjoint (triplet) representation, except at $\vec{X} = 0$. This last point - singular point of $WCP^2_{(2,1,1)}$ - or the origin of the patch $\tilde{U}^{(1,1)}$, is a fixed point of $SU(2)$ (a singlet). Note also that the
transition functions between the patches $\tilde{U}^{(0,2)}$ and $\tilde{U}^{(1,1)}$ are given by

$$X_3 = \frac{b}{a}, \quad X_1 - iX_2 = -\frac{b^2}{a}, \quad X_1 + iX_2 = \frac{1}{a}. \quad (4.1.35)$$

The patch $U^{(1,1)}$ does not cover points at “infinity”, namely the subspace defined by $a = 0$ in the patch $U^{(0,2)}$. That submanifold is nothing but $\mathbb{CP}^1$ parameterized by $b$ which is an edge of $\mathbb{WC}_{(2,1,1)}$. See Fig. (4.1). One can verify that the transformation law for $a, b$ in Eq. (4.1.28) and that for $\phi, \eta, \tilde{\eta}$ in Eq. (4.1.34) are consistent through the transition function Eq. (4.1.35). These results confirm those in [91].

4.2 $k = 2$ Vortices in $U(N)$ Gauge Theory

In this Section the composition of two non-Abelian vortices in a $U(N)$ gauge theory is systematically investigated. Up to now we made use of the direct form of the moduli matrix $H_0(z)$ for studying the moduli space structure. Another method for studying the latter will be developed and used to determine the moduli space below.

Let $\mathbf{Z}$ and $\mathbf{\Psi}$ be $k$ by $k$ and $N$ by $k$ constant complex matrices, respectively. We consider the $GL(k, \mathbb{C})$ action defined by

$$\mathbf{Z} \to V \mathbf{Z} V^{-1}, \quad \mathbf{\Psi} \to \mathbf{\Psi} V^{-1}, \quad V \in GL(k, \mathbb{C}). \quad (4.2.1)$$

It was shown in [75] that the moduli space $\mathcal{M}_{N,k}$ of $k$ vortices can be written as the Kähler quotient [86] defined by

$$\mathcal{M}_{N,k} \simeq \{ \mathbf{Z}, \mathbf{\Psi} \} \sslash GL(k, \mathbb{C}), \quad (4.2.2)$$

where $GL(k, \mathbb{C})$ action is free on these matrices.\footnote{The relation between the moduli matrix $H_0(z)$ and the two matrices $(\mathbf{Z}, \mathbf{\Psi})$ is given by the following ADHM-like equation

$$\nabla^\dagger L = 0, \quad \det(z - \mathbf{Z}) = \det H_0(z), \quad (4.2.3)$$

where $L^\dagger \equiv (H_0(z), \mathbf{J}(z))$ and $\nabla^\dagger \equiv (-\mathbf{\Psi}^\dagger, \bar{z} - \mathbf{Z}^\dagger)$. Here $\mathbf{J}(z)$ is $N$ by $k$ matrix whose elements are holomorphic function of $z$. $(\mathbf{Z}, \mathbf{\Psi})$ and $\mathbf{J}(z)$ can be uniquely determined from a given $H_0(z)$ [75].} The moduli space $\mathcal{M}_{N,k}$ given by the moduli matrix $H_0(z)$ in Eq. (3.1.16) and hence by the complex Kähler quotient in Eq. (4.2.2) is identical to that obtained by use of the D-brane construction by Hanany-Tong [7, 8]. The concrete correspondence between them is obtained by fixing the imaginary part of $GL(k, \mathbb{C})$ in...
Eq. (4.2.2) by the moment map $[Z^\dagger, Z] + \Psi^\dagger \Psi$:

$$\mathcal{M}_{N,k} \simeq \{ (Z, \Psi) | [Z^\dagger, Z] + \Psi^\dagger \Psi \propto 1_k \} / U(k),$$  \hspace{1cm} (4.2.4)

where $Z$ and $\Psi$ are again in the adjoint and fundamental representations of $U(k)$ group, respectively.

We now use the Kähler quotient construction to generalize the discussion to general $N$. The authors in [90] and [91] used the expression Eq. (4.2.4) but Eq. (4.2.2) is easier to deal with. Let us discuss the moduli space of co-axial vortices in terms of Eq. (4.2.2). A subspace of the moduli space $\mathcal{M}_{N,k}$ for coincident vortices at the origin of the $x^1$-$x^2$ plane is given by putting the constraint

$$\det(z-Z) = z^k,$$

that is,

$$\text{Tr}(Z^n) = 0, \quad \text{for } n = 1, 2, \ldots k. \hspace{1cm} (4.2.5)$$

To understand the subspace clearly we need to solve the above constraint by taking appropriate coordinates with $k^2 - k$ complex parameters.

In the case of $k = 2$, $U(N)$ vortices, the constraints Eq. (4.2.5) are equivalent to the constraint Eq. (4.1.13) for the $N = 2$ case. $Z$ can be solved in this case as

$$Z = \epsilon v v^T, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \hspace{1cm} (4.2.6)$$

with $v$ a column two-vector with complex components. The fact that the above form of $Z$ transforms as in the adjoint representation under $SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$ means that $v$ is a fundamental representation of $SL(2, \mathbb{C})$ since $2$ and $2^*$ is equivalent in the $k = 2$ case. Let us define a complex $2 \times (N+1)$ matrix by

$$M = \begin{pmatrix} \Psi^T & v \end{pmatrix}. \hspace{1cm} (4.2.7)$$

The $GL(2, \mathbb{C}) = SL(2, \mathbb{C}) \times \mathbb{C}^*$ action with elements $S \in SL(2, \mathbb{C})$ and $\lambda \in \mathbb{C}^*$ read

$$M \rightarrow SM, \quad \begin{pmatrix} \Psi^T & v \end{pmatrix} \rightarrow \begin{pmatrix} \lambda \Psi^T & v \end{pmatrix}. \hspace{1cm} (4.2.8)$$

The quotient by $GL(2, \mathbb{C})$ results in a kind of complex Grassmannian manifold whose $\mathbb{C}^*$ action has weights $(1, \ldots, 1, 0)$. The moduli subspace of coincident two vortices in $U(N)$ gauge theory is therefore found to be a weighted Grassmannian manifold,

$$\mathcal{M}_{N,k=2}^{\text{coincident}} \simeq \text{WGr}_{N+1,2}^{(1, \ldots, 1, 0)}. \hspace{1cm} (4.2.9)$$
We note again that the elements $S = -1_2$ and $\lambda = -1$ acting as

$$
( \Psi^T \quad v ) \rightarrow ( \Psi^T \quad -v )
$$

is precisely the $\mathbb{Z}_2$ action which gives orbifold singularities. Note that, although the ordinary complex Grassmannian manifold $Gr_{N+1,2} \simeq SU(N+1)_{SU(N-1) \times SU(2) \times U(1)}$ naturally enjoys an $SU(N+1)$ symmetry, the weighted Grassmannian manifold $WG_{N+1,2}^{(1,\cdots,1,0)}$ can have an $SU(N)$ symmetry at most, due to the difference of $U(1)$ charges. This is consistent with the existence of the $SU(N)_{G+F}$ symmetry acting on the moduli space in the $U(N)$ case. In cases of $N > 2$ the orbifold singularities are not isolated points but form a submanifold given by $v = 0$, which is the ordinary complex Grassmannian manifold $Gr_{N,2} \subset WGr_{N,2}^{(1,\cdots,1,0)}$ reflecting the $SU(N)_{G+F}$ symmetry.

As an illustration consider again the case of $U(2)$ theory. 2 by 2 matrices $Z$ and $\Psi$ correspond to the moduli space of $k = 2$ non-Abelian vortices in the $U(2)$ gauge theory with the equivalence relation Eq. (4.2.10). The double co-axial vortices are described by $\det (z_1 Z - z_2) = z^2$. This can be rewritten as $\text{Tr} Z = \text{Tr} Z^2 = 0$. These conditions are easily solved and we find that these vortices are described by the following two 2 by 2 matrices $Z$ and $\Psi$:

$$
Z = \epsilon^T v^T = \begin{pmatrix} v_1 v_2 & v_2^2 \\ -v_1^2 & -v_1 v_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}
$$

where $v^T = (v_1, v_2)$ and $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These obey the equivalence relation $GL(2, \mathbb{C})$ given in Eq. (3.2.10). At this stage the matter would become simple if we consider $v$ rather than $Z$. Since $Z$ is in the adjoint representation of $GL(2, \mathbb{C})$, $v$ is in the fundamental representation of $SL(2, \mathbb{C})$. Notice that $v$ is not charged under the overall $U(1)^{C} \subset GL(2, \mathbb{C})$.

It is natural to define $k (= 2)$ by $N + 1 (= 2 + 1)$ matrix

$$
M = (\Psi^T, v) = \begin{pmatrix} \psi_{11} & \psi_{21} & v_1 \\ \psi_{12} & \psi_{22} & v_2 \end{pmatrix}.
$$

This matrix $M$ transforms under $GL(2, \mathbb{C}) = U(1)^C \times SL(2, \mathbb{C})$ as follows

$$
M = (\Psi^T, v) \sim (S \lambda \Psi^T, S v)
$$

where $S \in SL(2, \mathbb{C})$ and $\lambda \in U(1)^C$. If the vector $v$ had a charge 1 under $U(1)^C$ (that is, $v \rightarrow \lambda v$), the above identification would correspond to the complex Grassmannian $Gr_{3,2} \simeq M/GL(2, \mathbb{C})$ which is same as $\mathbb{C}P^2$. But since $v$ is not charged under $U(1)^C$, the manifold is not a Grassmannian. Eq. (4.2.13) is an example of a weighted Grassmannian manifold and we
denote it by $WGr_{3/2}^{(1,1,1,0)}$. Here the numbers $(1, 1, 0)$ denote the $U(1)^c$-charges (the weights) of columns of $M$.

We choose appropriate $GL(2, \mathbb{C})$ matrices to obtain various patches on the moduli space for the composite vortices. Let us define the 2 by 2 minors $M_{ij}$ and their determinants as

$$M_{ij} = \begin{pmatrix} M_{i1} & M_{j1} \\ M_{i2} & M_{j2} \end{pmatrix}, \quad \tau_{ij} = \det M_{ij}. \quad (4.2.14)$$

There are 3 minors $M_{[12]}$, $M_{[23]}$ and $M_{[13]}$. Using the $GL(2, \mathbb{C})$, one of them can be brought to identity. So one has 3 patches as follows.

- $M_{[23]} = 1_2$ patch.

  First act $S = (\tau_{23}^0 0) M_{[23]}^{-1}$ to the matrix $M$ in Eq. (4.2.12), and after that by $\lambda = \tau_{23}^{-1}$:

$$M \rightarrow \begin{pmatrix} \tau_{13} & \tau_{23} & 0 \\ -\tau_{12} & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \tau_{13} & 1 & 0 \\ -\tau_{12} & 0 & 1 \end{pmatrix}. \quad (4.2.15)$$

- $M_{[12]} = 1_2$ patch.

  First one acts $S = (\tau_{12}^{1/2} M_{[12]}^{-1}$ to the matrix $M$ in Eq. (4.2.12), and after that then by $\lambda = (\tau_{12})^{-2}$:

$$M \rightarrow \begin{pmatrix} \sqrt{\tau_{12}} & 0 & -\tau_{23} \\ 0 & \sqrt{\tau_{12}} & \tau_{13} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\tau_{23} \\ 0 & 1 & \tau_{13} \end{pmatrix}. \quad (4.2.16)$$

  Notice that the element $S = -1_2$ with $\lambda = -1$, has not been fixed, thus one has a $\mathbb{Z}_2$ symmetry in this patch

$$\begin{pmatrix} -\tau_{23} \\ \sqrt{\tau_{12}} \\ \tau_{13} \end{pmatrix} \sim \begin{pmatrix} \tau_{23} \\ \sqrt{\tau_{12}} \\ -\tau_{13} \end{pmatrix} \quad (4.2.17)$$

- $M_{[13]} = 1_2$ patch.

  Act first by $S = (\tau_{13}^0 0) M_{[13]}^{-1}$ to the matrix $M$ in Eq. (4.2.12), and then by $\lambda = \tau_{13}^{-1}$:

$$M \rightarrow \begin{pmatrix} \tau_{13} & \tau_{23} & 0 \\ 0 & \tau_{12} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \tau_{23} & 0 & 0 \\ \tau_{13} & \tau_{12} & 1 \end{pmatrix}. \quad (4.2.18)$$
The corresponding matrices \((\Psi^\top Z^\top )\) for the above three patches are summarized as follows:

\[
\begin{pmatrix}
\tau_{13} & \tau_{12} \\
\tau_{23} & \tau_{22}
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
-\tau_{23}\tau_{13} & \tau_{13}^2 \\
\tau_{12} & \tau_{12}
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
\tau_{12} & \tau_{12}
\end{pmatrix}.
\] (4.2.19)

The leftmost one corresponds to the \(M_{[23]} = 1\) patch, the middle to the \(M_{[12]} = 1\) and the rightmost one to the \(M_{[13]} = 1\) patch. These must be identified respectively with the patches: \(\tilde{U}^{(0,2)}, \tilde{U}^{(1,1)}\) and \(\tilde{U}^{(2,0)}\). The concrete identification is

\[
\begin{pmatrix}
\tau_{12} \\
\tau_{23}
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix},
\begin{pmatrix}
\tau_{13} \\
\sqrt{\tau_{12}}
\end{pmatrix} = \begin{pmatrix}X \\
Y
\end{pmatrix},
\begin{pmatrix}
\tau_{12} \\
\tau_{23}
\end{pmatrix} = \begin{pmatrix}a' \\
b'
\end{pmatrix}.
\] (4.2.20)

Now we are ready to understand a little mysterious relation Eq. (4.1.21) which gave us the weighted complex projective space \(WCP^2_{(2,1,1)}\). Ordinary complex Grassmannian \(Gr_{3,2}\) is known to be equivalent to \(CP^2\) and the weighted cases are quite analogous. Because all the parameters in Eq. (4.2.19) are functions of the determinants of the minors \(M_{[12]}, M_{[23]}\) and \(M_{[13]}\), it would be natural to consider that the manifold is naturally parameterized by them. In particular the origin of the weighted equivalence relation Eq. (4.1.21) becomes clear since \(\tau_{ij}\) are invariant under \(SL(2, C)\) while transforming under \(U(1)^C\) as

\[
\begin{pmatrix}
\phi_0 \\
\phi_1
\end{pmatrix} = \begin{pmatrix}
\tau_{12} \\
\tau_{13}
\end{pmatrix} \sim \begin{pmatrix}
\lambda^2 \tau_{12} \\
\lambda \tau_{13}
\end{pmatrix}.
\] (4.2.21)

This is nothing but Eq. (4.1.21). The \(\tilde{U}^{(0,2)}\) patch is obtained by fixing with \(\lambda = \tau_{12}^{-1/2}\), the \(\tilde{U}^{(1,1)}\) patch by \(\lambda = \tau_{23}^{-1/2}\) and the \(\tilde{U}^{(2,0)}\) patch by \(\lambda = \tau_{13}^{-1/2}\). Thus

\[
\tilde{M}_{N=2,k=2} \simeq WCP^2_{(2,1,1)} \simeq WGr^{(1,1,0)}_{3,2}.
\] (4.2.22)

### 4.3 Summary

The moduli subspace of two co-axial vortices in the \(U(N)\) gauge theories with \(N\) flavors, is found to be a weighted Grassmannian manifold, Eq. (4.2.9). In the case of \(U(2)\) gauge theory, it reduces to a weighted projective space
$WC_{P^2_{(2,1,1)}} \simeq CP^2/\mathbb{Z}_2$ ($CP^2$ homotopically), in agreement with the known results [90, 91]. This space contains a $\mathbb{Z}_2$ orbifold (conic) singularity at the origin of the (1,1) patch. In the case of $U(N)$ gauge theory, it contains singularities along a subspace which is $Gr_{N,2}$.

The presence of this kind of orbifold singularities is a general feature of weighted Grassmannian manifolds. This fact implies the necessity to reconsider the reconnection of non-Abelian vortices. This issue was studied considering the moduli space of $k = 2$ co-axial vortices smooth everywhere [90]. We saw that this is not the case and we need to reconsider the problem. We will solve it in Chapter [9].
Chapter 5

Semi-local strings and lumps

Semi-local vortices interpolate between ANO vortices and sigma model lumps \cite{74,93}, to which they reduce in two different limits. As usual their total magnetic flux is quantized in terms of their topological charge, however the magnetic field of a semi-local vortex does not decay exponentially in the radial direction, instead it falls off according to a power law. Moreover the transverse size of the flux tube is not fixed but becomes a modulus. This feature gives rise to questions about the stability of these objects; in \cite{74} (see also \cite{94}) it is argued that they are stable if the quartic coupling in the potential is less than or equal to the critical (BPS) value, \textit{i.e.}, if the mass of the scalar is less than or equal to the mass of the photon. It is possible to study their dynamics \cite{95} in the moduli space approximation \cite{96} and also in the lump limit \cite{97}. It turns out that, in general, the fluctuations of some zero-modes corresponding to the global size of the configuration actually cost infinite energy. These zero-modes have to be fixed to make moduli space dynamics meaningful.

It is natural to ask what emerges for semi-local non-Abelian vortices. The number of zero modes, namely the dimension of the moduli space, of winding number $k$ was calculated to be $2kN_F$ in \cite{7,8}. The problem of (non-)normalizability of these zero modes, or the construction of the effective theory, has been considered in detail for $\mathcal{N} = 2$ supersymmetric $U(2)$ gauge theories with $N_f = 3, 4$ in \cite{6} by Shifman and Yung. They found BPS solutions for single non-Abelian semi-local vortices and then they used symmetry arguments to develop an effective theory on the vortex worldsheet. They noted that single semi-local vortices have only non-normalizable zero-modes except for the position modulus: not only the size modulus, but also the orientational moduli undergo this pathology, which is somewhat more surprising. This behavior is manifest because the effective worldsheet theory is a two dimensional sigma model, whose target space has a divergent metric.
unless an infrared regulator is provided. In this respect the geodesic motion on the moduli space seems essentially frozen, in contrast with the local case.

The aim of this Chapter is to use the moduli matrix approach to study in more detail semi-local vortices (a first application of this method to semi-local strings is found in [81]). Our considerations here apply to vortex configurations of generic winding number $k$ in any $U(N_c)$ gauge theory with $N_f$ flavors and the critical quartic coupling, in the case of $N_f > N_c$.

## 5.1 Moduli spaces and quotients

To discuss the moduli space of semi-local vortices, it is much more convenient to consider the Kähler quotient construction. According to this construction reviewed in Section (3.2), for configurations of winding number $k$ in a $U(N_c)$ gauge theory with $N_f$ fundamental flavors, the moduli are conveniently collected into the triplet $(Z, \Psi, \tilde{\Psi})$, where $Z$ is a $k \times k$, $\Psi$ an $N_c \times k$ and $\tilde{\Psi}$ a $k \times \tilde{N}_c$ complex matrix. They are defined modulo the $GL(k, \mathbb{C})$ equivalence relation

$$(Z, \Psi, \tilde{\Psi}) \sim (VZV^{-1}, \Psi V^{-1}, \tilde{V} \tilde{\Psi}), \quad V \in GL(k, \mathbb{C}). \quad (5.1.1)$$

The $GL(k, \mathbb{C})$ action is free on the set $\{Z, \Psi, \tilde{\Psi}\}$, in fact it is even free on the subset $\{Z, \Psi\}$. This is enough to define a good Kähler quotient and, indeed, the $k$-vortex moduli space turns out to be

$${\mathcal M}_{N_c,N_f;k} = \{(Z, \Psi, \tilde{\Psi}) : GL(k, \mathbb{C}) \text{ free on } (Z, \Psi)\}/GL(k, \mathbb{C}). \quad (5.1.2)$$

Let us, instead, consider the quotient

$${\tilde{\mathcal M}}_{N,c,N_f;k} \equiv \{Z, \Psi, \tilde{\Psi}\}/GL(k, \mathbb{C}) \quad (5.1.3)$$

where the $GL(k, \mathbb{C})$ acts freely. Now, while any free action of a compact group produces a reasonable quotient, this is not always the case for a non-compact group, like $GL(k, \mathbb{C})$. The corresponding quotient can indeed present some pathologies: in particular it becomes typically non-Hausdorff [22]. The absence of the Hausdorff property may appear to be just a mathematical detail but it is actually crucial to the physics. As is well known, in certain kinematical regimes, the dynamics of solitons (and vortices among them) can be described by geodesic motion on their moduli space. If this moduli space is non-Hausdorff, two distinct points may happen to lie at zero relative distance, in such a way that a geodesic can end at, or simply touch,
both of them at once. This is physically meaningless because two different points in the moduli space correspond to two distinguishable physical configurations.

In general, it is possible to “regularize” a non-Hausdorff quotient space (i.e., to make it Hausdorff) by removing some points \((GL(k, C)\) orbits for us). This can be done in more than one way; indeed, as intuition suggests, if two distinct points do not have disjoint neighborhoods, one could remove either one point or the other.

Let us describe this phenomenon from another point of view. It is possible to associate to the quotient Eq. (5.1.3) a moment map

\[
D = [Z^\dagger, Z] + \tilde{\Psi}^\dagger \tilde{\Psi} - \tilde{\Psi}^\dagger \tilde{\Psi} - r. 
\]

(5.1.4)

Setting \(D = 0\), which corresponds to fixing the imaginary part of the gauge group \(GL(k, C) = U(k)C\), and further dividing by the real part, which is \(U(k)\), leads to the symplectic quotient

\[
\{Z, \Psi, \tilde{\Psi} | D = 0\} / U(k).
\]

(5.1.5)

Now, the symplectic quotient depends on the value of \(r\) in Eq. (5.1.4). In particular, its topology is related to the sign of \(r\). There are three cases: \(r > 0\), \(r < 0\) and \(r = 0\), which represent three possible regularizations of the space (5.1.3), i.e., three possible ways to obtain Hausdorff spaces. In fact, if we choose \(r = 0\), the point \((Z, \Psi, \tilde{\Psi}) = (0, 0, 0)\), which would be a fixed point of \(U(k)\), will be an element of (5.1.5). This point would have to be excluded by hand. It corresponds to a small lump singularity as discussed below. The choice \(r \neq 0\) guarantees a non-singular space automatically.

A large class of examples of such quotients consists of weighted projective spaces. Consider for instance the simple example \(WCP^1_{(1,-1)}\). This is the space \(\{y_1, y_2\} / C^*\) defined by the equivalence relations \((y_1, y_2) \sim (\lambda y_1, \lambda^{-1} y_2), \lambda \in C^*\). After removing the origin \((0,0)\) the remaining sick points are \((0, y_2)\) and \((y_1, 0)\), which are each in every open neighborhood that contains the other. The two possible regularizations are:

1) \(WCP^1_{(1,-1)} = \{(y_1, y_2) | y_1 \neq 0\} / C^*\)

Introducing the moment map \(D = |y_1|^2 - |y_2|^2 - r\) with \(r > 0\), \(WCP^1_{(1,-1)}\) is seen to be equivalent to the symplectic \(U(1)\) quotient \(\{D = 0\} / U(1)\).

We have introduced a notation in which the underlined coordinates are the ones which cannot all vanish. This is because \(D\) restricted to a single \(C^*\) orbit,

\[
\tilde{D}(\lambda) = |\lambda|^2 |y_1|^2 - |\lambda|^{-2} |y_2|^2 - r,
\]

(5.1.6)
is a monotonic function of $|\lambda|$; for $|\lambda| \to +\infty$, $\tilde{D}$ goes to $+\infty$ and for $|\lambda| \to 0$ it goes to $-\infty$ or $-r$ if $y_2 \neq 0$ or $y_2 = 0$, respectively. This implies that there is a unique value of $|\lambda|$ which gives $\tilde{D} = 0$, unambiguously fixing the imaginary part of $U(1)^C = C^*$.

Note that the $C^*$ action is free on the set $\{y_1\}$, as the point $y_1 = 0$ is excluded.

ii) $WCP^1_{(1,-1)} = \{(y_1, y_2) | y_2 \neq 0\}/C^*$

This is, instead, equivalent to the symplectic $U(1)$ quotient obtained by setting $r < 0$ in the moment map of i).

Now the $C^*$ action is free on the set $\{y_2\}$.

Both i) and ii) turn out to be isomorphic to $C$, but, as mentioned above, two different regularizations of a complex quotient lead in general to different spaces. Note that in the case with $r = 0$, the fixed point $(0,0)$ will be a solution of (5.1.6) and the resulting space will be a singular conifold. This conifold is resolved into a regular space by setting $r \neq 0$. Similar phenomena occur in the general case of (5.1.5).

Based on considerations similar to those above, we are led to claim that, for any $k$, the $k$-vortex moduli spaces of two Seiberg-like dual theories (in the sense of Section 3.1) correspond to the two different regularization of the parent space in Eq. (5.1.3); these regularized spaces appear after a symplectic reduction as the quotients Eq. (5.1.5) with $r > 0$ and $r < 0$ respectively. Indeed, in the Seiberg dual theory the representations of the moduli $\{Z, \Psi, \tilde{\Psi}\}$ under $GL(k, C)$ are replaced by their complex conjugates, which is formally equivalent to flipping the sign of the Fayet-Iliopoulos parameter in Eq. (5.1.4).

In fact, adding the condition that $GL(k, C)$ is free on the subset $\{Z, \Psi\}$ (resp. $\{Z, \tilde{\Psi}\}$) to Eq. (5.1.3), as imposed by the moduli matrix construction of Section 3.1, turns out to be equivalent to selecting the specific regularization corresponding to the symplectic quotient (5.1.5) with $r > 0$ (resp. $r < 0$), that was first found in brane theory [7, 8]. The quotient

$$\mathcal{M}_{N_c,Nf,k} = \{(Z, \Psi, \tilde{\Psi}) : GL(k, C) \text{ free on } (Z, \Psi)\}/GL(k, C). \quad (5.1.7)$$

is isomorphic to the symplectic quotient

$$\{(Z, \Psi, \tilde{\Psi}) : D = [Z^\dagger, Z] + \Psi^\dagger \Psi - \tilde{\Psi} \tilde{\Psi}^\dagger - r = 0\}/U(k). \quad (5.1.8)$$

with $r > 0$.

Obviously, an analogous result holds with the following substitutions:

1. $N_c \to \tilde{N}_c \ (= N_f - N_c)$
2. \( GL(k, \mathbb{C}) \) free on \((Z, \Psi) \rightarrow GL(k, \mathbb{C}) \) free on \((Z, \tilde{\Psi}) \)

3. \( r > 0 \rightarrow r < 0 \).

5.2 The lump limit

Lumps are well-known objects, arising as static finite energy configurations of codimension two in non-linear sigma models (see, for example, [98]). Lumps typically are partially characterized by a size modulus, which in particular implies that the set of lump solutions is closed with respect to finite rescaling. Formally one would also include the solution with vanishing size modulus, but a physical configuration of zero width (and an infinitely spiked energy density) makes no sense and must be discarded. Such limiting situations are known as small lump singularities, and they actually represent singularities in the moduli space of lumps, which is then geodesically incomplete. In contrast, semi-local vortices do not present this kind of pathology because they have a minimum size equal to the ANO radius \( \frac{1}{g v} \).

The situation is particularly clear for \( \mathbb{C}P^1 \) lumps (related to our model with \( N_c = 1 \) and \( N_f = 2 \)). In fact, the set of lump configurations in the two dimensional \( \mathbb{C}P^1 \) sigma model was found to be in one-to-one correspondence with the set of holomorphic rational maps of the type [99]

\[
R(z) = \frac{p(z)}{q(z)}
\]

where \( p(z) \) and \( q(z) \) are two polynomials with no common factors and \( \deg p < \deg q \). The topological charge of the configuration \( \pi_2(\mathbb{C}P^1) = \mathbb{Z} \) is given by \( \deg q \). It is clear that, in order to define a fixed topological sector of the lump moduli space, one must consider the space of pairs of polynomials \( E = \{(p(z), q(z))\} \) of appropriate degree, subject to the constraint that the resultant is non-vanishing,

\[
\text{Res } [p(z), q(z)] \neq 0, \quad \text{or, } \quad |p(z)|^2 + |q(z)|^2 \neq 0 \quad \forall z.
\]

This condition excludes the singular points at which \( p(z) \) and \( q(z) \) share a common factor, implying that the corresponding state is not physical.

In the limit of infinite gauge coupling, \( g^2 \rightarrow \infty \), our model Eq. (3.1.2) reduces to a sigma model whose target space is the Higgs branch \( \mathcal{V}_{\text{Higgs}} = \)

\[1\] These considerations can be extended also to general \( \mathbb{C}P^n \) lumps using holomorphic rational maps.

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Grassmannian lumps. On the other hand, semi-local vortices therein reduce, upon compactification of the $z$ plane, to Grassmannian lumps \[7, 8, 100\], which are topologically supported by $\pi_2(Gr_{N_f, N_c}) = \mathbb{Z}$. A rational map in this case is extended to holomorphic $N_C \times \tilde{N}_C$ matrix given by

$$R(z) \equiv \frac{1}{D(z)} Q(z) = \frac{F(z)}{P(z)} = \Psi \frac{1_k}{z - Z} \tilde{\Psi}, \quad (5.2.3)$$

which is invariant under the $V$-transformation \[3.1.14\] and gives a holomorphic map from $S^2$ to $Gr_{N_f, N_c}$. Since $Gr_{N_f, N_c} = Gr_{N_f, N_f - N_c}$, the Seiberg dual theory of Eq. \[3.1.2\] is the same Grassmannian sigma model in the dual infinite gauge coupling limit, $\tilde{g}^2 \to \infty$; moreover, its semi-local vortex configuration tends to the same lump solutions. Actually, the extended rational map in the last form in Eq.\[5.2.3\], which is obtained by using Eq.\[3.2.13\] and Eq.\[3.2.9\], is manifestly invariant under the Seiberg-like duality. Of course the two dual limits cannot physically co-exist: here we are interested in the mathematical correspondences among moduli spaces of topological string-like objects of two dual theories in the various limits.

In the end we expect the two dual vortex moduli spaces to be deformed and/or modified in the (respective) infinite gauge coupling limits in such a way that they reduce to the same moduli space of Grassmannian lumps. Indeed, in the lump limit, the master equation \[3.1.15\] can be solved algebraically by

$$\Omega(z, \bar{z}) = v^{-2} H_0(z) H_0^\dagger(\bar{z}). \quad (5.2.4)$$

det $H_0 H_0^\dagger$ must be non-vanishing in order to have non-singular configurations. A set of parameters for which det $H_0 H_0^\dagger = 0$ at some point on the $z$ plane corresponds to a small lump singularity which must be discarded. In terms of $R(z)$, such unphysical singularities can be avoided by means of the constraint

$$\forall z : \ |P(z)|^2 \det \left(1_{N_C} + R(z) R^\dagger(z)\right) \neq 0, \quad (5.2.5)$$

that is nothing but the generalization of Eq. \[5.2.2\], to which it correctly reduces for $N_c = 1$ and $N_f = 2$. This nicely completes the extension of the holomorphic rational map approach to Grassmannian lumps.

It is possible to show that the “lump” condition det $H_0 H_0^\dagger \neq 0$ is equivalent to statement that $(Z, \tilde{\Psi})$ is $GL(k, \mathbb{C})$ free, so that the moduli space of

\[2\] A Seiberg-like dual pair of solitons was previously found for domain wall solutions \[101\] \[102\] and was then nicely understood in a D-brane configurations by exchanging of positions of two NS5-branes along one direction \[103\].
\( k \)-lumps is given by:

\[
\mathcal{M}_{N_c,N_f;k}^{\text{lump}} = \left\{ (Z, \Psi, \tilde{\Psi}) : GL(k, C) \text{ free on } (Z, \Psi) \text{ and } (Z, \tilde{\Psi}) \right\} / GL(k, C)
\]

\[
= \mathcal{M}_{N_c,N_f;k} \cap \mathcal{M}_{\tilde{N}_c,\tilde{N}_f;k}.
\]

(5.2.6)

Namely, the moduli space of \( k \)-lumps is the intersection of the moduli space of \( k \)-vortices in one theory with that of the Seiberg dual. The physical interpretation is the same as for singularities in the \( CP^1 \) lump moduli space. Increasing \( g^2 \) the semi-local vortex moduli space is deformed and approaches that of Grassmannian lumps and, in the infinite coupling limit, it only develops small lump singularities, as expected. These singularities correspond exactly to the presence of local vortices, whose sizes, \( 1/(gv) \), shrink to zero in the infinite gauge coupling limit. The same occurs in the Seiberg dual theory. What is left after the removal of the singular points is nothing but the intersection of the two vortex moduli spaces. In other words, the moduli space of semi-local vortices in each dual theory is given by the same moduli space of lumps in which we “blow-up” the small lump singularities with the insertion of the local vortex moduli subspace of the respective theory. From these considerations it is easy to convince ourselves that (5.2.6) is correct: taking the intersection in (5.2.6) eliminates the local vortices of both dual theories, leaving us with the moduli space of lumps.

In terms of the symplectic quotient (5.1.8), one must identify the parameter \( r \) with the gauge coupling of the four dimensional gauge theory as in Eq. 3.2.16 \( r = 2\pi/g^2 \). As one may expect, the lump limit is seen to be formally equivalent to taking the limit \( r \to 0 \). This limit is singular, in fact it develops singularities that correspond to the already mentioned small lump singularities. We can write for the moduli space of lumps:

\[
\mathcal{M}_{N_c,N_f;k}^{\text{lump}} = \left\{ (Z, \Psi, \tilde{\Psi}) : D = [Z^\dagger, Z] + \Psi^\dagger\Psi - \tilde{\Psi}\tilde{\Psi}^\dagger = 0, \ U(k) \text{ free } \right\} / U(k),
\]

(5.2.7)

were we have excluded “by hand” the small lump singularities by considering only points for which the \( U(k) \) action is free.

Coming back to our example \( WC\!P^1_{(1,-1)} \), from Section 5.1, we see that we must take away both of the pathological points that spoil the Hausdorff property, instead of only one. The net result is the intersection of the two regularized spaces \( WC\!P^1_{(1,-1)} \) and \( WC\!P^1_{(1,-1)} \), e.g. \( C^* \). This is also the space that we obtain if we eliminate the singularity of the conifold, in agreement with the general statement Eq. (5.2.7).

\(^3\)A free quotient of a compact group is always smooth.
The moduli space duality and the lump limit are then summarized by the following “diamond” diagram:

\[ \hat{M}_{N_c,N_f;k} \leftarrow \mathcal{M}_{N_c,N_f;k} \quad \text{Seiberg duality} \quad \mathcal{M}_{\tilde{N}_c,N_f;k} \rightarrow \mathcal{M}_{lump} \]

\[ g^2 \rightarrow \infty \]

\[ \tilde{g}^2 \rightarrow \infty \]

5.3 Some examples

In this Section we consider the topological sector \( k = 1 \), which consists of fundamental (single) semi-local vortices and lumps. The basic mathematical objects in this case are the weighted projective spaces with both positive and negative weights. For these we adopt the notation \( \mathbb{WCP}^{n-1}[Q_{w_1}, \ldots, Q_{w_l}] \), where the \( Q_i \), \( i = 1, \ldots, l(\leq n) \), represents the weight and \( w_i \) the number of homogeneous coordinates carrying that weight; clearly \( \sum_{i=1}^l w_i = n \).

This particular kind of toric variety plays a fundamental role in gauged sigma models in two dimensions \[22\] and their solitons \[93\]. As was noted in \[22\], when the set of weights includes both positive and negative integers (recall that multiplying all of the weights by a common integer number has no effect), the space is non-Hausdorff. There are two possible regularizations (in the sense of Section (5.1)), which correspond to eliminating the subspace where either all positively charged or all negatively charged coordinates vanish.

Looking at Eq. (5.1.1), it is easy to see that \( \mathbf{Z} \) “decouples”, in the sense that the \( GL(1, \mathbb{C}) = U(1) \mathbb{C} = \mathbb{C}^* \) acts trivially on it; indeed

\[ \left( \mathbf{Z}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}} \right) \sim \left( \mathbf{Z}, \lambda^{-1}\mathbf{\Psi}, \lambda\tilde{\mathbf{\Psi}} \right), \quad \lambda \in \mathbb{C}^*. \] (5.3.1)

Although we shall keep the same notation for the moduli spaces, they will be intended as the internal moduli spaces from now on, as the position moduli \( \mathbf{Z} \in \mathbb{C} \) always factorize. Given this, we identify \( \hat{\mathcal{M}}_{N_c,N_f;1} \) with \( \mathbb{WCP}^{N_f-1}[1^{N_c}, -1^{\tilde{N}_c}] \). We can regularize (in the sense of Section (5.1)) this space by insisting that \( \mathbf{\Psi} \neq 0 \). We indicate this space with the following notation:

\[ \mathcal{M}_{N_c,N_f;1} = \mathcal{O}(-1)^{\tilde{N}_c} \rightarrow \mathbb{CP}^{N_c-1} \equiv \mathbb{WCP}^{N_f-1}[1^{N_c}, -1^{\tilde{N}_c}] \] (5.3.2)
where $\mathcal{O}(-1)$ stands for the universal line bundle. Analogously, the dual regularization is obtained by imposing $\bar{\Psi} \neq 0$:

$$\mathcal{M}_{N_c,N_f;1} = \mathcal{O}(-1)^{1\otimes N_c} \to \mathbb{C}P^{N_c-1} \equiv W\mathbb{C}P^{N_c-1}[1\tilde{N}_c, -1^{N_c}].$$ (5.3.3)

Note that when $N_c = \tilde{N}_c$ these spaces become non-compact (local) Calabi-Yau manifolds, which corresponds to the fact that just in this case the conformal bound for a four-dimensional $U(N_c)$ with $\mathcal{N} = 2$ supersymmetry is saturated.

In the lump limit one must take $\Psi, \bar{\Psi} \neq 0$:

$$\mathcal{M}_{lump}^{N_c,N_f;1} = (\mathbb{C}^{N_c})^* \times \mathbb{C}P^{N_c-1} \simeq (\mathbb{C}^{N_c})^* \times \mathbb{C}P^{N_c-1} \simeq \{(\mathbb{C}^{N_c})^* \oplus (\mathbb{C}^{\tilde{N}_c})^*\}/\mathbb{C}^*, $$ (5.3.4)

with $F \times B$ denoting a fiber bundle with $F$ a fiber and $B$ a base. The $\mathbb{C}^*$ acts with charges $+1$ and $-1$ on $(\mathbb{C}^{N_c})^*$ and $(\mathbb{C}^{\tilde{N}_c})^*$ respectively. If we define

$$\mathcal{M}_{lump}^{N_c,N_f;1} \equiv W\mathbb{C}P^{N_f-1}[1^{N_c}, -1^{\tilde{N}_c}], $$ (5.3.5)

we can summarize the situation with the following diamond diagram:

Let us consider some concrete examples.

• $N_c = 1, N_f = 2$

This is a self-dual system. From $\hat{\mathcal{M}}_{1,2;1} = W\mathbb{C}P^1[1,-1]$ one finds $\hat{\mathcal{M}}_{1,2;1} = \mathbb{C}$ for both the original and the dual theory. The moduli space of lumps is obtained by removing the small lump singularity and it is $\mathcal{M}_{lump}^{1,2;1} = \mathbb{C}^*$. Explicitly, from the moduli matrix

$$H_0 = (z - z_0, b) \iff \{Z, \Psi, \bar{\Psi}\} = \{z_0, 1, b\}, $$ (5.3.6)

4The fiber of the universal line bundle at each point in $\mathbb{C}P^{n-1}$ is the line that it represents in $\mathbb{C}^n$. 65
one finds the solution \[ (5.2.4) \]
\[ \Omega = |z - z_0|^2 + |b|^2 \]  \hspace{1cm} \text{(5.3.7)}
and the non-vanishing condition is \( b \neq 0 \) (consider the point \( z = z_0 \)).
Removing the point \( b = 0 \) from the vortex moduli space \( C \) one obtains the lump moduli space \( C^* \). In summary:

\[
\begin{array}{c}
WC\mathbb{P}^1[1,-1] \\
\downarrow \mathbb{C} \quad \downarrow \text{Seiberg duality} \quad \downarrow \mathbb{C}^* \\
g^2 \rightarrow \infty \quad \rightarrow \quad g^2 \rightarrow \infty
\end{array}
\]

- \( N_c = 2, N_f = 3 \) dual to \( N_c = 1, N_f = 3 \)

We have now the “parent” moduli space, \( \tilde{M}_{2,3;1} = WC\mathbb{P}^2[1,1,-1] \).
The two dual regularizations are \( M_{2,3;1} = \mathbb{C}^2 \), namely the blow-up of the origin of \( \mathbb{C}^2 \) with an \( S^2 \simeq \mathbb{C} \mathbb{P}^1 \), and \( M_{1,3;1} = \mathbb{C}^2 \). The lump limit is \( M_{\text{lump}}^{2,3;1} = (\mathbb{C}^2)^* \), the two-dimensional complex vector space minus the origin.

All of the moduli spaces can be found using the moduli matrix. In the lump limit, the general solution for the original theory leads
\[
H_0 = \begin{pmatrix} 1 & b & 0 \\ 0 & z - z_0 & c \end{pmatrix} \leftrightarrow \left\{ Z, \Psi, \tilde{\Psi} \right\} = \left\{ z_0, \begin{pmatrix} -b \\ 1 \end{pmatrix}, c \right\},
\]
\hspace{1cm} \text{(5.3.8)}
\[
det \Omega|_{z = z_0} = |c|^2(1 + |b|^2)
\]
\hspace{1cm} \text{(5.3.9)}
and so the determinant vanishes, indicating a small lump singularity, on the blown-up 2-sphere \( c = 0 \). Removing this 2-sphere from the vortex moduli space \( \mathbb{C}^2 \) one is left with the lump moduli space \( (\mathbb{C}^2)^* \). In order to cover the whole moduli space \( \mathbb{C}^2 \), together with that in Eq. \( (5.3.8) \), one needs another patch for the moduli matrix. In the case of the dual theory
\[
H_0 = (z - \tilde{z}_0, \tilde{b}, \tilde{c}) \leftrightarrow \left\{ Z, \tilde{\Psi}, \tilde{\Psi} \right\} = \left\{ \tilde{z}_0, 1, \begin{pmatrix} \tilde{b} \\ \tilde{c} \end{pmatrix} \right\},
\]
\hspace{1cm} \text{(5.3.10)}
\[
\Omega|_{z = \tilde{z}_0} = |\tilde{b}|^2 + |\tilde{c}|^2
\]
\hspace{1cm} \text{(5.3.11)}

and so the determinant vanishes at the point \( \hat{b} = \hat{c} = 0 \) ∈ \( \mathbb{C}^2 \). The diamond diagram is:

\[
\begin{array}{c}
\mathbb{C}^2 \\
\text{Seiberg duality} \\
\tilde{\mathbb{C}}^2 \\
\mathbb{C}^2
\end{array}
\]

\[
\begin{array}{c}
WC\mathbb{P}^2[1, 1, -1] \\
\tilde{\mathbb{C}}^2 \\
\mathbb{C}^2 \\
(C^2)^*
\end{array}
\]

- \( N_c, N_f = N_c + 1 \) dual to \( N_c = 1, N_f \)

This is a generalization of the previous two examples. The parent space is \( \tilde{\mathcal{M}}_{N_c, N_c + 1; 1} = WC\mathbb{P}^N_c[1^{N_c}, -1] \). On one side we have \( \mathcal{M}_{N_c, N_c + 1; 1} = WC\mathbb{P}^N_c[1^{N_c}, -1] = \tilde{\mathbb{C}}^N_c \), which is \( \mathbb{C}^N_c \) with the origin blown up into a \( CP^{N_c - 1} \), while on the other side the dual moduli space is simply \( \mathcal{M}_{1, N_c + 1; 1} = WC\mathbb{P}^N_c[1^{N_c}, -1] = \mathbb{C}^{N_c} \). In the lump limit we are left with \( \mathcal{M}_{N_c, N_c + 1; 1} = (\mathbb{C}^{N_c})^* \) since in the original theory

\[
H_0 = \begin{pmatrix} 1_{N_c - 1} & b & 0 \\ 0 & z - z_0 & c \end{pmatrix} \Leftrightarrow \left\{ \mathbf{Z}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}} \right\} = \left\{ z_0, \begin{pmatrix} -b \\ 1 \end{pmatrix}, c \right\},
\]

(5.3.12)

\[
\det \Omega|_{z = z_0} = |c|^2 (1 + |b|^2)
\]

(5.3.13)

and so the small lump singularity is the blown-up \( CP^{N_c - 1} \) at \( c = 0 \) in the vortex moduli space. Here \( b \) is a column \((N_c - 1)\)-vector. For the dual theory

\[
H_0 = (z - \tilde{z}_0, \tilde{b}) \Leftrightarrow \left\{ \mathbf{Z}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}} \right\} = \left\{ \tilde{z}_0, 1, \tilde{b} \right\},
\]

(5.3.14)

\[
\Omega|_{z = \tilde{z}_0} = |\tilde{b}|^2
\]

(5.3.15)

which identifies the small lump singularity with the point \( |\tilde{b}| = 0 \) ∈ \( \mathbb{C}^{N_c} \). Here \( \tilde{b} \) is a row \( N_c \)-vector. These moduli spaces are summarized by the diamond diagram:

\[
\begin{array}{c}
\mathbb{C}^N_c \leftrightarrow \text{Seiberg duality} \leftrightarrow \mathbb{C}^N_c \\
\tilde{\mathbb{C}}^N_c \\
WC\mathbb{P}^N_c[1^{N_c}, -1] \\
(C^N_c)^*
\end{array}
\]
\( N_c = 2, N_f = 4 \)

This theory is again self-dual. The parent space is \( \hat{\mathcal{M}}_{2,4;1} = WC P^3[1, 1, -1, -1] \), which yields \( \mathcal{M}_{2,4;1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C} P^1 \), namely the resolved conifold \[104\]. The moduli space of lumps is \( \mathcal{M}_{2,4;1}^{\text{lump}} = (\mathbb{C}^2)^* \ltimes \mathbb{C} P^1 \).

Indeed

\[
H_0 = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & z - z_0 & c & d \end{pmatrix} \iff \left\{ Z, \Psi, \tilde{\Psi} \right\} = \left\{ z_0, \begin{pmatrix} -b \\ 1 \end{pmatrix}, (c, d) \right\},
\]

and from the non-vanishing condition

\[
\det \Omega|_{z = z_0} = (1 + |b|^2)(|c|^2 + |d|^2) = |

\Psi|^2 |\tilde{\Psi}|^2 \neq 0 \quad (5.3.16)
\]

we recognize \((c, d)\) as coordinates of \((\mathbb{C}^2)^*\) and \(b\) as the inhomogeneous coordinate of the base \(\mathbb{C} P^1\). Therefore one removes the \(\mathbb{C} P^1\) at \(c = d = 0\), that is \(\tilde{\Psi} = 0\). In the dual theory, on the other hand, the roles of \(\Psi\) and \(\tilde{\Psi}\) are interchanged and so one instead removes the \(\mathbb{C} P^1\) at \(\Psi = 0\), which is related by a flop transition to the \(\mathbb{C} P^1\) of the previous moduli space. In the end

\[
\begin{align*}
WC P^3[1, 1, -1, -1] & \quad \xrightarrow{\text{Seiberg duality}} \quad \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C} P^1 \\
\mathcal{O}(-1) \oplus \mathcal{O}(-1) & \quad \xrightarrow{\text{Seiberg duality}} \quad (\mathbb{C}^2)^* \ltimes \mathbb{C} P^1
\end{align*}
\]

It is suggestive to note that similar topological transitions of the type described above occur within the non-commutative vortex moduli space as the non-commutativity parameter is varied \[7, 8\].

### 5.4 Non-normalizable modes

The effective theory on the vortex worldsheet is obtained via the usual procedure \[96\] of promoting the moduli to slowly varying fields on the worldsheet \[16, 75, 6\]. It turns out to be a two dimensional sigma model whose Kähler potential can be calculated from the moduli matrix \[75, 82\] :

\[
K = \text{Tr} \int d^2 z \left( \nu^2 \log \Omega + \Omega^{-1} H_0 H_0^\dagger + \mathcal{O}(1/g^2) \right). \quad (5.4.1)
\]
This formula with explicit expression of the third term was first obtained after tedious calculation in terms of component fields [72], but the derivation has been drastically simplified by using superfields [82]. By virtue of translational symmetry, it is possible to show that the center-of-mass parameter is always decoupled [88] and, specifically, it appears with an ordinary kinetic term whose coefficient is proportional to the total tension. The center-of-mass is a free field.

Let us concentrate on the other moduli. Analyzing the divergences of the Kähler potential, one can establish which moduli among $2kN_f$ have an infinite kinetic term in the Lagrangian and are non-normalizable. Fluctuations of those moduli are frozen, as well as their motion in the geodesic approximation, while the evolution of the rest of the moduli will be allowed. Very recently it has been found that all modes become normalizable when we couple one semi-local vortex to gravity [105].

The divergent terms of the Kähler potential can come only from integrations around the boundary $|z| = L$ ($L$ is a suitable infra-red cutoff), since $\Omega$ is assumed to be invertible and smooth. Remembering that $\Omega \to v^{-2}H_0H_0^\dagger$ for large $z$, the divergent terms can be calculated keeping only the first term in Eq. (5.4.1):

$$v^2 \int_{|z|=L} d^2z \log \det (H_0H_0^\dagger) \sim v^2 \int_{|z|=L} d^2z \log \det \left( D^{-1}H_0(D^{-1}H_0)^\dagger \right)$$

$$= v^2 \int_{|z|=L} d^2z \log \det \left( 1_{N_C} + \frac{1}{z - \bar{Z}} \Psi \bar{\Psi} \right)^2$$

$$= v^2 \int_{|z|=L} d^2z \left[ \frac{1}{|z|^2} \Tr |\Psi \bar{\Psi}|^2 + O(|z|^{-3}) \right]$$

$$= 2\pi v^2 \log L \Tr |\Psi \bar{\Psi}|^2 + \text{const.} + O(L^{-1})$$

(5.4.2)

where we used a Kähler transformation $K \to K + f + f^*$, with $f = v^2 \int d^2z \log \det D^{-1}$. Equation (5.4.2) means that the elements of $\Psi \bar{\Psi}$ are non-normalizable and should be fixed. The number of non-normalizable parameters crucially depends on the rank of $\Psi \bar{\Psi}$:

$$r \equiv \text{rank} \left( \Psi \bar{\Psi} \right) \leq \min \left( k, N_C, \tilde{N}_C \right) \equiv j.$$  

(5.4.3)

In the following we calculate the number of non-normalizable moduli when the above inequality is saturated, $r = j$. This happens for generic points of the moduli space. It follows that for particular submanifolds of the moduli space when $r < j$ the number of normalizable parameters is enhanced.
Using the global symmetry $SU(N_C)_{C+F} \times SU(\tilde{N}_C)_F$, we can always fix $\tilde{\Psi} \tilde{\Psi}$ to have the following form:

$$\tilde{\Psi} \tilde{\Psi} = \left( \begin{array}{cc} \Lambda_r & 0 \\ 0 & 0 \end{array} \right)$$  \hspace{1cm} (5.4.4)$$

where $\Lambda_r = \text{diag}(\lambda_1, \cdots, \lambda_r)$ with positive real parameters $\lambda_i > 0$. Note that this symmetry of the vacuum is generally broken by the vortex and so it generates moduli for our solution. But the corresponding moduli are non-normalizable, so that we will not count them in the following.

To proceed further we have to distinguish two cases:

- $k \leq \min(N_C, \tilde{N}_C)$

  In this case the saturation of the inequality (5.4.3) means $r = k$, and the matrices $\Psi$ and $\tilde{\Psi}$ have the following block-wise form (suffixes indicate dimensions of blocks):

  $$\Psi = \left( \begin{array}{c|c} A_{[k \times k]} & B_{[(N_c - k) \times k]} \\ \hline \end{array} \right), \quad \tilde{\Psi} = \left( \begin{array}{c|c} C_{[k \times k]} & D_{[k \times (\tilde{N}_c - k)]} \end{array} \right),$$  \hspace{1cm} (5.4.5)$$

  from which we find $AC = \Lambda_k$. Because $\det \Lambda_k \neq 0 \Rightarrow \det A \neq 0$, we can completely fix $GL(k, \mathbb{C})$ by taking $A = 1_k$. Thus, we obtain:

  $$\Psi = \left( \begin{array}{c} 1_k \\ 0 \end{array} \right), \quad \tilde{\Psi} = (\Lambda_k, 0).$$  \hspace{1cm} (5.4.6)$$

The corresponding moduli matrix is:

$$H_0 = \left( \begin{array}{cccc} z1_k & -Z & 0 & \Lambda_k \\ 0 & 1_{N_c - k} & 0 & 0 \end{array} \right).$$  \hspace{1cm} (5.4.7)$$

From the above we find that the normalizable moduli are all contained in the $k \times k$ matrix $Z$, so that:

$$\dim \mathcal{M}_{\text{norm}}^{N_C, N_F, k} = 2k^2.$$  \hspace{1cm} (5.4.8)$$

From here it is easy to see that fundamental semi-local vortices, $k = 1$, always have only 2 real moduli, corresponding to the position on the plane. Orientation moduli are instead non-normalizable, independently of $N_c$ and $N_f$. This behavior is very different from the local case, $N_c = N_f$. 

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We assume $N_C \leq \tilde{N}_C$ without loss of generality (The results for $N_C \geq \tilde{N}_C$ are obtained using Seiberg duality $N_C \leftrightarrow \tilde{N}_C$). Thus the saturation of Eq. (5.4.3) leads $k = N_C$. For $\Psi$ and $\tilde{\Psi}$ we have the following block form:

$$\Psi = \left( A_{[N_c \times N_c]}, B_{[N_c \times (k-N_c)]} \right), \quad \tilde{\Psi} = \left( \begin{array}{cc} C_{[N_c \times N_c]} & D_{[N_c \times (\tilde{N}_C-N_c)]} \\ E_{[(k-N_c) \times N_c]} & F_{[(k-N_c) \times (\tilde{N}_C-N_c)]} \end{array} \right).$$

(5.4.9)

We see that $\Psi$ must have rank equal to $N_c$ so that can be always put in the following form:

$$\Psi = (1_{N_c}, 0) \quad (5.4.10)$$

via a $GL(k, C)$ transformation. Thus $\tilde{\Psi}$ and the remaining $GL(k, C)$ symmetry are

$$\tilde{\Psi} = \left( \begin{array}{c} \Lambda_{N_c} \\ E \\ F \end{array} \right), \quad \left( \begin{array}{cc} 1_{N_c} & 0 \\ G & H \end{array} \right) \in GL(k, C). \quad (5.4.11)$$

Here $E$ can be fixed to be zero using $G$. We obtain:

$$\Psi = (1_{N_c}, 0), \quad \tilde{\Psi} = \left( \begin{array}{c} \Lambda_{N_c} \\ 0 \\ T \end{array} \right), \quad Z = \left( \begin{array}{cc} X & Y \\ \tilde{Y} & W \end{array} \right), \quad (5.4.12)$$

with the remaining $u \in GL(k-N_C, C)$ action:

$$X \rightarrow X, \quad Y \rightarrow Y u, \quad \tilde{Y} \rightarrow u^{-1} \tilde{Y}, \quad W \rightarrow u^{-1} W u, \quad T \rightarrow u^{-1} T. \quad (5.4.13)$$

The normalizable parameters are contained in the $Z$ and $T$ matrices, from which we subtract the $(k-N_C)^2$ parameters of the remaining $GL$ action:

$$\dim \mathcal{M}_N^{\text{form}}_{N_C, N_F, k} = 2 \left( k^2 + (k-N_C)(\tilde{N}_C-N_C) - (k-N_C)^2 \right)$$

$$= 2 \left( (N_C + \tilde{N}_C)k - N_C \tilde{N}_C \right). \quad (5.4.14)$$

The formula obtained is clearly symmetric in $N_c$ and $\tilde{N}_C$. It is interesting to note, for instance, that $k = N_c$ vortices in the $U(N_c)$ theory always have $2N_c^2$ real, normalizable moduli for any $N_c$; they roughly correspond to the $2N_c$ positions in the plane and the $2N_c(N_c-1)$ relative sizes and orientations in the color-flavor space.
Figure 5.1: Wave functions of some normalizable moduli fields for $k = 2$ lumps in $N_c = 2$, $N_f = 4$ model. Position moduli look localized around the corresponding vortex, while relative orientation modulus lies in between.

5.5 Dynamics of the effective $k = 1$ vortex theory

The presence of non-normalizable modes has a remarkable consequence in the low-energy effective description of the vortex. As we have seen, these modes must be fixed, they are not dynamical. Even more remarkable are the consequences of the presence of non-normalizable modes with the physical dimension of a length, such as the size moduli. In this case the derivative expansion in the effective action will contain, generically, powers of $\lambda \partial$, where $\lambda$ is the size moduli and $\partial$ is a derivative with respect to a worldsheet coordinate. Furthermore we must consider $\lambda$ has an ultraviolet cut-off in the effective theory on the vortex [6].

It is practically impossible to evaluate expression (5.4.1), analytically or even numerically, in the general case, but one can hope to do it in some particular simple examples. In this Section we will derive the complete Kähler potential for a single non-Abelian semi-local vortex.

Using the set of coordinates defined by the moduli matrix formalism for a single non-Abelian semi-local vortex:

$$H_0 = \begin{pmatrix} 1_{N_c-1} & b & 0 \\ 0 & z - z_0 & c \end{pmatrix}, \quad (5.5.1)$$

we can determine the most general expression for the Kähler potential, compatible with the $SU(N_c)_{C+F} \times SU(\tilde{N}_c)_F \times U(1)$ isometry of the vacuum. Here $N_c - 1$ column vector $b$ and $\tilde{N}_c$ row vector $c$ are moduli parameters. To this end we have to find the transformation properties of $b$ and $c$ under this
symmetries. The moduli matrix (5.5.1) transforms as:
\[
\delta H_0 = -H_0 u + v(u, z) H_0, \quad \text{Tr} u = \text{Tr} v = 0, \quad u^\dagger = -u
\]
\[
u = \begin{pmatrix}
\Lambda_{N_c-1} + i \lambda 1_{N_c-1} & v & 0 \\
-v^\dagger & -i(N_c-1)\lambda & 0 \\
0 & 0 & \tilde{\Lambda}_{\tilde{N}_c}
\end{pmatrix}, \quad (5.5.2)
\]
where for simplicity \( u \) is an infinitesimal \( SU(N_c)_{C+F} \times SU(\tilde{N}_c)_F \times U(1) \) transformation, and \( v(u, z) \) is an infinitesimal \( V \)-transformation that pulls back the matrix \( H_0 \) into the standard form of Eq. (5.5.1). After some calculations we find the following transformation properties for the moduli parameters:
\[
\delta b = \Lambda_{N_c-1} \cdot b + i N_c \lambda b - v - (v^\dagger \cdot b) b,
\]
\[
\delta c = -i(N_c - 1) \lambda c + (v^\dagger \cdot b) c - c \cdot \tilde{\Lambda}_{\tilde{N}_c}, \quad (5.5.3)
\]
from which one can infer:
\[
\delta \log (1 + |b|^2) = -(v^\dagger \cdot b) + \text{c.c.}, \quad \delta \log |c|^2 = (v^\dagger \cdot b) + \text{c.c.},
\]
\[
\delta ((1 + |b|^2)|c|^2) = 0. \quad (5.5.4)
\]
These relations can be explained if we note that \((c_i, c_i b)\) (with arbitrary \( i \), \( i = \tilde{N}_c \) for instance) transforms like a fundamental of \( SU(N_c)_{C+F} \) while \( c \) as a fundamental of \( SU(\tilde{N}_c)_F \).

Since the moduli parameters are zero modes related to the symmetry breaking of \( SU(N_c)_{C+F} \times SU(\tilde{N}_c)_F \times U(1) \), the low energy action should be invariant under the symmetry. In other words, the Kähler potential should be written in terms of invariants under the transformation (up to Kähler transformation). The most general expression for the Kähler potential, up to Kähler transformations, is thus given by
\[
K(z_0, b, c) = A|z_0|^2 + F(|a|^2) + B \log(1 + |b|^2), \quad (5.5.5)
\]
where \( A \) and \( B \) are constants while \( F(|a|^2) \) is an unknown function of the invariant combination \( |a|^2 \equiv (1 + |b|^2)|c|^2 \). Note that a term \( \log |c|^2 \) would also be invariant, but can be absorbed by a redefinition of \( F(|a|^2) \) and \( B \). The constants and the function are determined as follows. First of all, \( z_0 \) is the center of mass, so it is decoupled from any other modulus and its coefficient \( A \) equals half of the vortex mass, \( A = \pi \nu^2 \). Next let us consider the function \( F(|a|^2) \). Now, if one fixes the orientational parameters to some constant, e.g.

\[\text{One can verify this property using the transformation laws of } b \text{ and } c. \text{ It is directly connected with the property of lump moduli spaces expressed by Eq. (5.3.4).}\]
\( b = 0 \) \(|a|^2 = |c|^2\), a non-Abelian vortex becomes simply an embedding of an abelian vortex into a larger gauge group, therefore the Kähler potential in (5.5.5) must reduce to that of an abelian semi-local vortex:

\[
K(z_0, 0, c) = K_{\text{abelian semi-local}}(z_0, c) = \pi v^2 |z_0|^2 + F(|c|^2) \tag{5.5.6}
\]

It is important that the function \( F(|a|^2) \) is independent of \( \tilde{N}_c \) \((\geq 1)\), because the solution for \( \tilde{N}_c = 1 \) can be embedded into those for \( \tilde{N}_c > 1 \). Furthermore, \( F(|a|^2) \), written in term of the moduli parameters defined by the moduli matrix and defined as an integral over the configurations, should be smooth everywhere. In particular, in the limit \(|a|^2 \to 0\) it must be unique and equal just to that of the ANO vortex. (A numerical result for \( F(|a|^2) \) with \( g = v^2 = 1 \) and \( L = 10^3 \) is shown in Fig. (5.2)). In this limit, which can be achieved letting \( c \to 0 \), the vortex reduces to a local vortex, and also the Kähler potential should reduce to that of a local vortex. \( B \) is thus the Kähler class of the non-Abelian vortex, \( B = 4\pi/g^2 \), as was found in Ref. [89]:

\[
K(z_0, 0, b) = K_{\text{non-Abelian local}}(z_0, b) = \pi v^2 |z_0|^2 + 4\pi g^2 \log(1 + |b|^2). \tag{5.5.7}
\]

This fixes the constants in (5.5.5). Therefore we find that the Kähler potential is determined uniquely in terms of that of an abelian semi-local vortex:

\[
K(z_0, b, c) = K_{\text{abelian semi-local}}(z_0, |a|) + 4\pi g^2 \log(1 + |b|^2) \tag{5.5.8}
\]

The function \( F(|a|^2) \), being the Kähler potential for a single abelian vortex, can be computed numerically. Furthermore it is possible to find analytically the following behavior:

\[
F(|a|^2) \sim \begin{cases} 
\pi v^2 \log (g^2 v^2 L^2 \alpha^{-1}) \times |a|^2 & \text{for } |a| \ll \frac{1}{g\sqrt{v^2}} \\
\pi v^2 |a|^2 \left( \log \frac{L^2}{|a|^2} + 1 \right) + \text{const.} & \text{for } |a| \gg \frac{1}{g v}. 
\end{cases} \tag{5.5.9}
\]

where \( \alpha \) is some unknown constant of \( \mathcal{O}(1) \). The behavior at small \( a \) is simply a consequence of the smoothness of the Kähler potential at \( a = 0 \) and the cut-off dependence (5.4.2). Since the coefficient of the \(|a|^2\) term is \( \mathcal{O}(L^2) \), this term is dominant even for a medium region of \(|a|\) as shown in Fig. (5.2)(a). The analytic form at large \( a \) can be related to the expression of the potential in the strong coupling limit as we will show in Eq. (5.5.19) below. It is very interesting to note that we used a very simple function to
Figure 5.2: (a) The red dots are numerical computations of $F(|a|^2)$, while the blue line is an interpolation done with the function shown in (5.5.10). (b) The red line is the Kähler metric $\partial_a \kappa_{\text{abelian semi-local}}(z_0, a)$ for an abelian semi-local vortex at finite gauge coupling and is obtained from the interpolation function in (a), while the blue dashed line is the metric in the infinite gauge coupling limit (lump limit). The cut-off has been set to a very big value, $L = 10^3$, and $g = v^2 = 1$.

interpolate the numerical results:

$$F(|a|)_{\text{interp}} = \text{const} + \pi v^2 \left( |a|^2 + \frac{\alpha}{g^2 v^2} \right) \left( \log \frac{L^2}{|a|^2 + \frac{\alpha}{g^2 v^2}} + 1 \right). \quad (5.5.10)$$

This function gives a very good interpolation with only one relevant free parameter, $\alpha$, which appears to be of order one. This interpolating function is nothing but the Kähler potential in the lump limit (Eq. (5.5.19)), regularized at $a = 0$ by the introduction of a sort of UV cut-off: $\rho_{\text{eff}} = \alpha/g^2 v^2$. This appears to be a very nice, though empirical, way to show that local vortices act as regularizers for small lump singularities.

Remarkably, in the case of a single vortex, the large-size limit is completely equivalent to the strong coupling limit. To see this, note that the relevant quantity that triggers both limit is the following ratio:

$$R = \frac{\rho}{\rho_{\text{loc}}}, \quad \rho_{\text{loc}} \equiv 1/(gv), \quad (5.5.11)$$

where $\rho$ is the physical size of the semi-local vortex, and $\rho_{\text{loc}}$ is the typical size of a local vortex. When the size moduli $\lambda$ vanish, the physical size $\rho$ shrink to $\rho_{\text{loc}}$, so that $R \geq 1$. In the strong coupling limit, $\rho_{\text{loc}} \to 0$, and $R \to \infty$. The lump limit is thus really defined by the limit $R \to \infty$, which can be achieved also at finite gauge coupling, just considering the limit in
which the physical size is very big: \( \rho \sim \lambda \to \infty \). The lump Kähler potential (see Eq. (5.5.12)) is thus also a good approximation at finite gauge coupling, provided that we restrict to solutions with a very big size. In fact one can see from Figure (5.2(b)) that this approximation is very good also for small size\(^6\).

In the lump limit the expression (5.4.1) can be calculated analytically:

\[
K \sim v^2 \text{Tr} \int d^2 z \log \Omega = v^2 \int d^2 z \log(\det \Omega) \sim v^2 \int d^2 z \log(\det H_0 H_0^\dagger).
\]

(5.5.12)

Let us consider two particular, dual, examples: \( k = 1, N_c = 2, \tilde{N}_c = 1 (N_f = 3) \). In this case all moduli are non-normalizable except for position moduli, so we find no nontrivial dynamics on the vortex. Nonetheless one may study the dynamics of these moduli by providing an infrared cut-off in (5.4.1). Furthermore, according to the above discussion, we can study the large size limit, in order to have an effective theory which can be derived analytically. The expression (5.5.12) is thus the Kähler potential for a \( \text{Gr}_{2,3} (\text{Gr}_{1,3}) \) lump with topological charge \( k = 1 \). In the abelian theory we find, from (5.3.10):

\[
K_{N_c=1, N_f=3} = v^2 \int_{|z| \leq L} d^2 z \log(|z - z_0|^2 + |\tilde{b}|^2 + |\tilde{c}|^2).
\]

(5.5.13)

If we set \( z_0 = 0 \) for simplicity, this integral is easily performed:

\[
K_{N_c=1, N_f=3} = v^2 \pi \left( |\tilde{b}|^2 + |\tilde{c}|^2 \right) \log \left( \frac{L^2}{|\tilde{b}|^2 + |\tilde{c}|^2} \right) + v^2 \pi \left( |\tilde{b}|^2 + |\tilde{c}|^2 \right) + O(L^{-1}),
\]

(5.5.14)

where we omit divergent terms that do not depend on the moduli. The corresponding metric is:

\[
L_{N_c=1, N_f=3} = v^2 \pi \left( |\partial_\mu \tilde{b}|^2 + |\partial_\mu \tilde{c}|^2 \right) \log \frac{L^2}{(|\tilde{b}|^2 + |\tilde{c}|^2)} + O(L^0).
\]

(5.5.15)

Note that we have obtained a conformally flat metric on \( \mathbb{C}^2 = \mathbb{R}^4 \), which might be expected given the \( U(1) \times SU(2)_F \) isometry that acts on the parameters \( \tilde{b} \) and \( \tilde{c} \).

\(^6\)Presumably Eq. (5.5.12) should give a good approximation to the Kähler potential also in the general case with several vortices, provided that we consider solutions with big typical sizes (to this end we should not consider, for example, configurations such that \( \det H_0 H_0^\dagger \) vanishes at some vortex point).
Now consider the non-Abelian theory with \( N_c = 2 \). The Kähler potential is given by:

\[
K_{N_c=2,N_f=3} = v^2 \pi |c|^2 (1 + |b|^2) \log \frac{L^2}{|c|^2 (1 + |b|^2)} + O(L^0), \tag{5.5.16}
\]

where we have used the moduli matrix coordinates defined in (5.3.8). Note that this potential is consistent with the general expression we gave in (5.5.8), in the lump limit, up to logarithmic accuracy. The \( SU(2)_{c+f} \times U(1) \) symmetry of the theory, which leaves the quantity \( |c|^2 (1 + |b|^2) \) invariant is again manifest. The metric that follows from this potential is:

\[
L_{N_c=2,N_f=3} = v^2 \pi \left[ |c|^2 |\partial_\mu b|^2 + (1 + |b|^2) |\partial_\mu c|^2 + \right.
\]
\[
+ \left. (c b^\dagger \partial_\mu c^\dagger \partial^\mu b + c.c.) \right] \log \frac{L^2}{|c|^2 (1 + |b|^2)}.
\tag{5.5.17}
\]

The expressions (5.5.15) and (5.5.17) are related by the following change of coordinates:

\[
\tilde{c} = c, \quad \tilde{b} = c b (|c|^2 (1 + |b|^2) = |\tilde{c}|^2 + |\tilde{b}|^2 \neq 0, c \neq 0). \tag{5.5.18}
\]

The regularized metric of a semi-local vortex is a conformally flat metric of \( \mathbb{C}^2 \) (modulo a change of coordinates). This is valid, in the large size limit, for both dual theories: \( N_c = 2, \tilde{N}_c = 1, N_f = 3 \). This is related to the fact that in both dual theories the semi-local vortex reduce to the same object, a \( Gr_{2,3} = Gr_{1,3} = \mathbb{C}P^2 \) lump.

The effective action in the lump limit, for generic \( N_c \) and \( \tilde{N}_c \) can be found from the following Kähler potential:

\[
K_{N_c,N_f} = v^2 \pi |c|^2 (1 + |b|^2) \log \frac{L^2}{|c|^2 (1 + |b|^2)} + v^2 \pi |c|^2 (1 + |b|^2), \tag{5.5.19}
\]

where \( b \) and \( c \) are vectors length \( N_c - 1 \) and \( \tilde{N}_c \) respectively.

5.6 Summary

We analyzed the moduli space of semi-local non-Abelian vortices both at the kinematical and at the dynamical level. First we provided an unambiguous smooth parametrization of the moduli space, and studied its topological

\(^7\)Alternatively, vectors \( \tilde{b} \) and \( \tilde{c} \) of dimensions \( \tilde{N}_c - 1 \) and \( N_c \) can be used, as already emphasized.
structure (without the metric). The moduli metric is defined only for normalizable moduli which parameterize a subspace inside the whole moduli space of dimension $2kN_F$. We used supersymmetry to derive an effective action for the system of $k$ vortices and showed that, even though a single semi-local vortex always has only non-normalizable moduli, higher winding configurations admit normalizable moduli, which roughly correspond to relative sizes and orientations in the internal space, as first noted in [88]. This means that, upon fixing the non-normalizable global moduli, the analysis of the geodesic motion on the moduli space becomes meaningful. Along the way we discovered an interesting relationship between the vortex moduli spaces of a Seiberg-like dual pair of theories at fixed winding number $k$: they both descend, by means of two alternative regularizations, from the same “parent space”, which is a non-Hausdorff space defined in terms of a certain holomorphic quotient. As a result they are guaranteed to be birationally equivalent and in fact they are related by a geometric transition. Moreover, in the limit of infinite gauge coupling we saw how they reduce to the same moduli space of $k$ Grassmannian sigma model lumps, as expected on general grounds. We also found the normalizable moduli enhancement on special submanifolds: the number of normalizable moduli can change depending on the point of moduli space we are dealing with. For instance, the orientational moduli $\mathbf{C}P^{N_c-1}$ of a $k = 1$ vortex are non-normalizable unless the size modulus vanishes. However they become normalizable in the limit of vanishing size modulus where the semi-local vortex shrinks to a local vortex (with physically non-zero size).
So far, all our discussions about non-Abelian vortices were focused on the BPS limit [20] (a single non-Abelian non-BPS vortex configuration is discussed in [106, 107, 108, 109, 110, 111]). No forces arise among BPS vortices, because there is a nice balance between the repulsive forces mediated by the vector particles and attractive forces mediated by the scalar particles. In this particular limit, the solutions to the equations of motion develop a full moduli space of solutions [96], which was the subject of our study in the previous Section. However, once the balance between the attractive force and the repulsive force is lost, the moduli space disappears. Alternatively we can think that an effective potential is generated on this moduli space. It is well known that ANO vortices in the Type I system feel an attractive force while those in the Type II model feel a repulsive force [20, 112, 113, 114, 115]. In condensed matter physics, it is also known that Type II vortices form the so-called Abrikosov lattice, [116] due to the repulsive force between them. Furthermore, lattice simulations give some evidence of the presence of a (marginal) Type II superconductivity in QCD [117, 118].

We are interested in studying interactions between non-Abelian vortices which are non-BPS. In non-supersymmetric theories, BPS configurations are obtained with fine-tuned values of the couplings. If supersymmetry exists in the real world, it is surely broken at a low energy scale; therefore non-BPS vortices are more natural than BPS ones.
6.1 Type I non-Abelian supersymmetric superconductors

A breakthrough in this context was the Seiberg-Witten solution \[3\] of \( \mathcal{N} = 2 \) super Yang-Mills theories; they found massless monopoles at strong coupling. Adding a small \( \mathcal{N} = 2 \) breaking mass term for the adjoint field, the monopoles condense creating dual vortex strings which carry a chromoelectric flux. The details of confinement in the Seiberg-Witten scenario are indeed quite different from QCD. The \( SU(N_c) \) gauge symmetry is spontaneously broken broken to \( U(1)^{N_c-1} \) by the expectation value of an adjoint field and the strings of the theory carry an Abelian \( U(1)^{N_c-1} \) charge. A careful examination shows that the “hadronic” spectrum is much richer than that of QCD \[119, 15\].

As we have already noticed, the very existence of the non-Abelian vortex is related to the presence of a color-flavor mixed global symmetry. As a consequence, non-Abelian vortices can be studied in many different theoretical settings; indeed it is possible to start with a \( \mathcal{N} = 1 \) theory \[120, 121\] or even with a non-supersymmetric theory \[21\]. The details of the effective \( 1 + 1 \) dimensional sigma model are different due to different number of fermions and various amount of supersymmetry. Also the number of quantum vacua is different, for example there are \( N \) vacua in the \( \mathcal{N} = 2 \) case \[122, 9, 7, 8\] and just one vacuum in the non-supersymmetric case \[21\].

In this Section, we study the impact on the vortices of the \( \mathcal{N} = 2 \) vortex model (1.1.3) of some mass terms \( \eta_0, \eta_3 \) for the adjoint fields, which break the extended supersymmetry. For concreteness, we will discuss the case \( N_c = N_f = 2 \). The vortex with winding number one is not anymore a BPS object, but still has a \( \mathbb{C}P^1 \) moduli space. On the other hand the physics for vortices with higher winding numbers is very different: almost all the flat directions in the moduli space are lifted by the parameters \( \eta_0, \eta_3 \). The force between two vortices is not as simple as in an Abelian superconductor, where we have attraction for Type I vortices and repulsion for Type II ones \[20, 113\]. There is a non-trivial dependence on the orientations of the two vortices in the internal space.

Even if the force between two vortices in our model is not attractive for all values of the vortex orientations \( \vec{n}_1, \vec{n}_2 \), we have a close resemblance with Type I Abelian vortices: we find that the scalars of the theory are lighter than the vector bosons. Hence if we consider two well separated vortices, we have that the prevailing part of the interaction is mediated by scalars and not by vectors. Moreover, for \( \vec{n}_1 = \vec{n}_2 \) the force is always attractive. We have also found evidences that the configurations which minimize the energy are always given by two coincident vortices, just as in the Type I Abelian case.
For these reasons we call these objects "non-Abelian Type I vortices".

6.1.1 Spectrum of the Theory

We start from our $\mathcal{N} = 2$ supersymmetric $U(2)$ gauge theory with $N_f = 2$ hypermultiplets (1.1.3), but we consider the following superpotential instead:

$$ W = \frac{1}{\sqrt{2}} \left[ \tilde{Q} f(a_0 + a^i \tau^i + \sqrt{2} m_f) Q_f + W_0(a_0) + W_3(a^i \tau^i) \right], \quad (6.1.1) $$

where the terms $W_{0,3}$ are of the form

$$ W_0 = \frac{N}{2} (-\xi a_0 + \eta_0 a_0^2), \quad W_3(a^i \tau^i) = \frac{N}{2} \eta_3 a^i a^i. \quad (6.1.2) $$

Here we have introduced two real positive mass parameters $\eta_0$ and $\eta_3$ for the adjoint scalars which break $\mathcal{N} = 2$ SUSY to $\mathcal{N} = 1$. These terms change only the $V_3$ piece of the F-term which we have already written in Eq. (1.1.6)

$$ V_3 = \frac{g^2}{2} \left| \text{Tr}_f[\tilde{Q} \tau^i Q] + N \eta_3 a^i a^i \right|^2 + \frac{e^2}{2} \left| \text{Tr}_f[\tilde{Q} Q] - \frac{N}{2} \xi + N \eta_0 a_0 \right|^2, \quad (6.1.3) $$

The vacuum of the theory is not changed by the parameters $\eta_{0,3}$:

$$ Q = \tilde{Q} = \sqrt{\frac{\xi}{2}} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad a = 0, \quad a^b = 0. \quad (6.1.4) $$

For $\eta_0 \neq 0$, the theory has also another classical vacuum:

$$ Q = \tilde{Q} = 0, \quad a = \frac{\xi}{2 \eta_0}, \quad a^k = 0, \quad (6.1.5) $$

which “runs away” at infinity for $\eta_0 = 0$. In what follows, we consider the vacuum (6.1.4) and the non-Abelian vortices therein.

The masses of the gauge bosons can easily be read of the Lagrangian:

$$ M_{U(1)}^2 = \xi e^2, \quad M_{SU(2)}^2 = \xi g^2. $$

The masses of the scalars are given by the eigenvalues $M_i^2$ of the mass matrix (calculated in the vacuum (6.1.4)):

$$ \mathcal{M} = \frac{1}{2} \frac{\partial^2 V}{\partial s_i \partial s_j}, \quad (6.1.6) $$

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where we denote by $s_k$ ($k = 1, 2, \cdots, 24$) the real scalar fields of the theory. The calculation is a bit tedious but quite straightforward (a very similar situation is discussed in Ref. [120], in the case of a D-term Fayet-Iliopoulos).

First of all, there are four zero eigenvalues, which correspond to the scalar particles eaten by the Higgs mechanism. There is one real scalar with mass $M_{S0} = M_{U(1)}$ which is in the same $\mathcal{N} = 1$ multiplet as the $U(1)$ massive photon and moreover there are also three scalars with a mass $M_{T0} = M_{SU(2)}$ in the same multiplet as the non-Abelian vector field. The other mass eigenvalues are given by

$$M_{S1,S2}^2 = \xi e^2 + e^4 \eta_0^2 \pm \sqrt{2\xi \eta_0^2 e^6 + e^8 \eta_0^4};$$

$$M_{T1,T2}^2 = \xi g^2 + g^4 \eta_3^2 \pm \sqrt{2\xi \eta_3^2 g^6 + g^8 \eta_3^4},$$

where the upper sign is for $M_{S1,T1}$ and the lower sign is for $M_{S2,T2}$. $M_{S1}$ and $M_{S2}$ have multiplicity 2; $M_{T1}$ and $M_{T2}$ have multiplicity 6. The mass of the fermions is obviously the same as the mass of the bosonic degrees of freedom, because of the unbroken $\mathcal{N} = 1$ supersymmetry.

Note that for $\eta_0 = \eta_3 = 0$ (which is also discussed in Ref. [123]), the mass degeneracy of the spectrum is bigger (the particles fit into $\mathcal{N} = 2$ hypermultiplets). The parameter $\eta_0$ affects only the masses of the particles which are in the same $\mathcal{N} = 2$ hypermultiplets as the $U(1)$ vector field; $\eta_3$ affects the mass of the particles which are in the non-Abelian vector hypermultiplet. The case of $\mathcal{N} = 2$ SQED was studied in Ref. [11]; the results are very similar to our $U(1)$ subsector.

In the limit $\eta_0 e \gg \sqrt{\xi}$, we find $M_{S1}^2 \approx 2e^4 \eta_0^2$ and $M_{S2}^2 \approx \xi^2/(4\eta_0^2)$. In a similar way, if $\eta_3 g \gg \sqrt{\xi}$ the masses become $M_{T1}^2 \approx 2g^4 \eta_3^2$ and $M_{T2}^2 \approx \xi^2/(4\eta_3^2)$. The particles with masses $M_{S1}$ and $M_{T1}$ (which in this limit correspond to the fields $a$ and $a^k$) become very massive and decouple from the low energy physics.

For $\eta_0 e \ll \sqrt{\xi}$ and $\eta_3 g \ll \sqrt{\xi}$ nonetheless we find

$$M_{S1,S2}^2 = \xi e^2 \pm \sqrt{\xi \eta_0 e^6}; \quad M_{T1,T2}^2 = \xi g^2 \pm \sqrt{\xi \eta_3 g^6}. \quad (6.1.8)$$

Some of the scalars become slightly heavier and some slightly lighter.

The mass eigenvectors take a quite complicated form for small $\eta_{0,3}$, with at non-trivial mixing between $Q, \bar{Q}$ and $a, a^k$. On the contrary they are quite simple for large $\eta_{0,3}$, because the fields $a, a^k$ decouple from the low energy physics. The effective Lagrangian for large $\eta_{0,3}$ is discussed in Appendix A.
6.1.2 (p, k) Coincident Vortices

Second order equations

In this Section, we will study some special solutions representing coincident vortices that live in an Abelian subgroup of the fields of the theory. These solutions are parameterized by two positive integers (p, k); the topological \( \mathbb{Z} \) winding number is given by \( w = p + k \). This kind of solution gives us the most general vortex with winding \( w = 1 \) up to a colour-flavour rotation\(^1\).

Due to the symmetry between \( Q \) and \( \tilde{Q}^\dagger \), for the vortex solution we can consistently set \( \tilde{Q} = Q^\dagger \)\(^2\). With this assumption the Euler-Lagrange equations of the theory are:

\[
\partial_\mu F^{\mu\nu} = -ie^2 \left( Q_f^\dagger D^\nu Q_f - (D^\nu Q_f)^\dagger Q_f \right),
\]

\[
\mathcal{D}_\mu F^{\mu\nu} = -ig^2 \left( Q_f^\dagger \tau_i D^\nu Q_f - (D^\nu Q_f)^\dagger \tau_i Q_f \right) - \epsilon_{ijk} (a_j (D^\nu a_k)^\dagger + \bar{a}_j D^\nu a_k),
\]

\[
\mathcal{D}^\mu \mathcal{D}_\mu Q = -\frac{1}{2} \frac{\delta V}{\delta Q^\dagger}, \quad \partial^\mu \partial_\mu a_0 = -e^2 \frac{\delta V}{\delta a_0}, \quad \mathcal{D}^\mu \mathcal{D}_\mu a_i = -g^2 \frac{\delta V}{\delta a_i}.
\]

(6.1.9)

We make the following axial symmetric ansatz (which in the BPS limit reduces to the one of Ref. \[16\]):

\[
Q = \begin{pmatrix}
\phi_0 e^{i\varphi} & 0 \\
0 & \phi_1 e^{i\varphi}
\end{pmatrix},
\]

\[
A^3_i = -\frac{\epsilon_{ij} x_j}{r^2} (p - k) - f_1(r), \quad A^0_i = -\frac{\epsilon_{ij} x_j}{r^2} (p + k) - f_0(r),
\]

\[
a_0 = \lambda_0(r), \quad a_3 = \lambda_3(r), \quad a_1 = a_2 = 0.
\]

(6.1.10)

Notice that the adjoint fields \( a, a^i \) are non-trivial in the non-BPS model, whereas they vanish everywhere when \( \eta_{0,3} \) are zero (BPS).

Let us introduce the \( S^2 \) coordinate \( n^i \), with \( i = 1, 2, 3 \) and \( |\vec{n}| = 1 \):

\[
n^i \tau^i = U \tau^3 U^\dagger,
\]

(6.1.11)

where \( U \in SU(2)_{c+f} \). Using this parametrization we can write down the expression for a \( w = 1 \) vortex with generic orientation \( n^i \):

\[
A^0_i = -\frac{\epsilon_{ij} x_j}{r^2} (1 - f_0), \quad A^i_i = -\frac{\epsilon_{ij} x_j}{r^2} (1 - f_1)n^i, \quad a^0 = \lambda_0, \quad a^i = n^i \lambda_3.
\]

(For higher winding, it follows from the moduli matrix formalism that it is not the most general solutions, at least in the BPS case.

\(^1\)This is equivalent to the assumption that \( \tilde{H} \equiv 0 \) in the D-term model \[3.1.2\].

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\[ Q = \tilde{Q}^1 = \frac{\phi_0 e^{i\varphi} + \phi_1}{2} + \frac{\phi_0 e^{i\varphi} - \phi_1}{2} \tau^i n^i. \]  

(6.1.12)

It is easy to see that the \((p, k)\) vortex, when \(p \neq k\) partially break the symmetry in Eq. (7.2.3); as a consequence, this object has some internal zero modes associated to this breaking. In fact, the vortex leaves a \(U(1)\) subgroup of \(SU(2)_{c+f}\) unbroken, so that zero modes parameterize a \(\mathbb{C}P^1 = SU(2)/U(1) = S^2\). If \(p = k\) the vortex is Abelian, it breaks completely \(SU(2)_{c+f}\), and no moduli space is generated.

The energy with respect to \(\phi_0, \phi_1, f_0, f_1, \lambda_0, \lambda_3\) is expressed as

\[ E = 2\pi \int r dr \left( \frac{f_0'^2}{2e^2r^2} + \frac{f_1'^2}{2g^2r^2} + \frac{\lambda_0'^2}{e^2} + \frac{\lambda_3'^2}{g^2} + 2(\phi_0'^2 + \phi_1'^2) + \frac{g^2}{2}(\phi_0'^2 + \phi_1'^2 - \xi + 2\eta_0\lambda_0)^2 + \frac{g^2}{2}(\phi_0'^2 - \phi_1'^2 + 2\eta_3\lambda_3)^2 + ((\lambda_0 + \lambda_3)\phi_0)^2 + ((\lambda_0 - \lambda_3)\phi_1)^2 \right). \]  

(6.1.13)

We have to minimize this expression with the appropriate boundary conditions for each \((p, k)\):

\[ f_1(0) = p - k, \quad f_0(0) = p + k, \quad f_1(\infty) = 0, \quad f_0(\infty) = 0, \quad \phi_0(\infty) = 1, \quad \phi_1(\infty) = 1, \quad \lambda_0(\infty) = 0, \quad \lambda_3(\infty) = 0. \]  

(6.1.14)

We also find for small \(r\):

\[ \phi_0 \propto \mathcal{O}(r^p), \quad \phi_1 \propto \mathcal{O}(r^k), \quad \lambda_0 \propto \mathcal{O}(1), \quad \lambda_3 \propto \mathcal{O}(1). \]  

(6.1.15)

The Euler-Lagrange equations obtained are:

\[ \frac{f_0''}{r} - \frac{f_0'}{r^2} = \frac{\phi_0'^2}{r}(f_1(\phi_0'^2 - \phi_1'^2) + f_0(\phi_0'^2 + \phi_1'^2)); \]

\[ \frac{f_1''}{r} - \frac{f_1'}{r^2} = \frac{\phi_1'^2}{r}(f_1(\phi_0'^2 + \phi_1'^2) + f_0(\phi_0'^2 - \phi_1'^2)); \]

\[ \frac{\phi_0''}{r} + \frac{\phi_0'}{r} - \frac{\phi_0(f_0 + f_1)^2}{4r^2} = \]

\[ = \phi_0((\lambda_0 + \lambda_3)^2 + \phi_0^2 + \phi_1^2 - \xi + 2\eta_0\lambda_0) + g^2(\phi_0^2 - \phi_1^2 + 2\eta_3\lambda_3)^2/2; \]

\[ \frac{\phi_1''}{r} + \frac{\phi_1'}{r} - \frac{\phi_1(f_0 - f_1)^2}{4r^2} = \]

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\[
\phi_1 \left( (\lambda_0 - \lambda_3)^2 + e^2 (\phi_0^2 + \phi_1^2 - \xi + 2\eta_0\lambda_0) - g^2 (\phi_0^2 - \phi_1^2 + 2\eta_3\lambda_3) \right);
\]
\[
\lambda''_0 + \frac{\lambda'_0}{r} = \frac{e^2 ((a_0 + a_3)\phi_0^2 + (a_0 - a_3)\phi_1^2 + e^4 \eta_0 (\phi_0^2 + \phi_1^2 - \xi + 2\eta_0\lambda_0))}{2};
\]
\[
\lambda''_3 + \frac{\lambda'_3}{r} = \frac{g^2 ((a_0 + a_3)\phi_0^2 - (a_0 - a_3)\phi_1^2 + g^4 \eta_3 (\phi_0^2 - \phi_1^2 + 2\eta_3\lambda_3))}{2} \tag{6.1.16}
\]

It is easy to check that these equations can be obtained substituting the ansatz (6.1.10) in Eqs. (6.1.9). This shows that the ansatz is consistent.

In the following, we will concentrate our effort on the study of the sectors with topological winding 1 and 2; in other words, we will discuss the (1, 0), the (1, 1) and the (2, 0) vortices. In the BPS limit \((\eta_0 = \eta_3 = 0)\) the tension is proportional to the topological winding number \((T_{(1,0)} = 2\pi \xi, T_{(1,1)} = T_{(2,0)} = 4\pi \xi)\). For non-BPS solutions, \(\eta_{0,3} \neq 0\), we find that the tension is always less than the BPS limit. This is because the non-BPS terms in the tension formula Eqs. (6.1.13) do not give any contribution if we put the BPS solutions into the expression \((\lambda_{0,3} \text{ are identically zero for the BPS solutions})\).

The non-BPS solutions will of course be a true minimum or saddle point of the energy functional, so that their energy will be smaller than that of the BPS configuration.\(^3\)

For fixed \(\xi\), the tension of the (1, 1) vortex is a function of only \(e, \eta_0\), because for this vortex \(f_1 = 0\) and \(\phi_0 = \phi_1\). This is clearly explained by the fact that the (1, 1) vortex is completely Abelian. On the other hand, the tension of the (1, 0) and of the (2, 0) vortex is a non-trivial function of all the parameters \(e, g, \eta_{0,3}\).

If we take \(g = e\) and \(\eta_3 = \eta_0\) the vortex becomes easier to study. In this case we can use a more convenient basis for the gauge field, which is just the sum and the difference of \(A_0^\mu\) and \(A_3^\mu\). The potential \(V\) also factorize, and takes the form \(V = V_1(\phi_0) + V_2(\phi_1)\). Each diagonal component of \(Q\) does not interact with the other ones, and can be treated as an Abelian vortex. For the (1, 0) vortex we can use the simple ansatz\(^4\)

\[
Q = \begin{pmatrix} \phi e^{i\varphi} & 0 \\ 0 & \sqrt{\xi/2} \end{pmatrix}, \tag{6.1.17}
\]

while for the (1, 1) and for the (2, 0) vortices we can use

\[
Q = \begin{pmatrix} \phi(r_1)e^{i\varphi_1} & 0 \\ 0 & \phi(r_2)e^{i\varphi_2} \end{pmatrix}, \quad Q = \begin{pmatrix} \phi e^{2i\varphi} & 0 \\ 0 & \sqrt{\xi/2} \end{pmatrix}. \tag{6.1.18}
\]

\(^3\)This also has a clear resemblance to the Abelian case, where Type I vortices have a smaller energy with respect to the BPS case.

\(^4\) Notice that we cannot impose \(\phi_1 = \sqrt{\xi/2}\) even for (1, 0) vortex in the generic models.

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The system reduces to the Abelian vortex studied in Ref. [11]. The tension of the (1, 1) vortex is exactly twice the tension of the (1, 0) one. In each of the $U(1)$ factors, we have Type I superconductivity. Since the two $U(1)$ subgroups are decoupled, the (1, 0) and (0, 1) vortices do not interact. Furthermore, the tension of the (2, 0) vortex is less than twice the tension of the (1, 0) vortex.

**Numerical Solutions**

At generic $e_0, e_3, \eta_0, \eta_3$ Eqs. (6.1.16) have been solved numerically. It is a little subtle to solve this system of ordinary differential equations directly. The difficulties basically arise because there are many equations; there are also subtleties in defining the boundary conditions at $\infty$, because, in general, the fields which appear in our ansatz do not correspond to mass eigenstates. In order to perform the numerics we found that the method of relaxation is very effective. We add an auxiliary time dependence to the profile functions $\vec{u} = (f_0, f_1, \phi_0, \phi_1, \lambda_0, \lambda_3)$. At $t = 0$ we start with some arbitrary functions $u_j(r, 0)$. The choice of these initial conditions is crucial to find convergence. The evolution in $t$ is then given by:

$$\frac{\partial u_j}{\partial t} = E_j.$$ (6.1.19)

If the solution converges with time to a static configuration, then at final time we have obtained a solution of the equations $E_j = 0$, which are equivalent to Eqs. (6.1.16). The results for $(p, k) = (1, 0), (2, 0), (1, 1)$ are shown in Fig. (7.1).

Figure 6.1: Profile functions $f_0$ (solid black), $f_1$ (solid blue), $\phi_0$ (long dashes, black), $\phi_1$ (long dashes, blue), $\lambda_0$ (short dashes, black), $\lambda_3$ (short dashed, blue) for the numerical values $\xi = 2, e = 1/4, g = 1/2, \eta_0 = \eta_3 = 1$. In the left panel are shown the profiles for the (1, 0) vortex, in the middle, the ones for the (2, 0) and in the right, the ones for the (1, 1). Note that in this last case $f_1 = \lambda_3 = 0$ and $\phi_0 = \phi_1$.

It is interesting to compare numerical result for the tension $2T_{1,0}, T_{2,0}$ and $T_{1,1}$ of the $2 \times (1, 0), (2, 0)$ and (1, 1) vortices, respectively (see Fig. (6.2)).
We have always found that $T_{2,0} < 2T_{1,0}$. This is consistent with the fact that at large separation distance, the force between two vortices with the same colour-flavour orientation is always attractive (we will discuss this aspect in Sect. 4). As can be checked in Fig. 6.2, three different regimes have been found for $T_{1,1}$: $T_{1,1} < T_{2,0} < 2T_{1,0}$ or $T_{2,0} < T_{1,1} < 2T_{1,0}$ or $T_{2,0} < 2T_{1,0} < T_{1,1}$.

If $\eta_0 = 0, \eta_3 \neq 0$, the tension of the $(1, 1)$ vortex is found to be the same as in the BPS case. The tensions of the $2 \times (1, 0)$ and of the $(2, 0)$ vortex are strictly less than that of the BPS vortices ($\eta_3 = 0$). Hence in this case $T_{2,0} < 2T_{1,0} < T_{1,1}$. On the contrary, if $\eta_3 = 0, \eta_0 \neq 0$, we have found $T_{1,1} < T_{2,0} < 2T_{1,0}$ for all the numerical values of the couplings that we have investigated.

### 6.1.3 Generic Coincident Vortices

#### The BPS case

The calculation of the number of dimensions of the $k$-vortex moduli space in $U(N_c)$ $\mathcal{N} = 2$ gauge theory with $N_f = N_c = N$ hypermultiplets has been reviewed in Section (3.1). The result is $2kN$. Thus for $\eta_0 = \eta_3 = 0$, the moduli space of a 2-vortex configuration is a manifold with eight real dimensions. Two of these dimensions are associated with the global position of the system; other 2 coordinates are associated with the relative position $R$ of the two elementary vortices. The remaining 4 coordinates are associated with the orientation of the system in the colour-flavour space. In this Section, we will write an ansatz for the case of coincident vortices ($R = 0$), and we
will show that it is non-trivially consistent with the second order equations of the theory.

Modulo a global $SU(2)$ rotation we can parameterize a subset of the moduli space with the angle $\alpha$ between $\vec{n}_1$ and $\vec{n}_2$. The expression for $Q = \tilde{Q}^\dagger$ is

$$Q = \begin{pmatrix}
-\cos \frac{\alpha}{2} e^{2i \varphi} \kappa_1 & \sin \frac{\alpha}{2} e^{i \varphi} \kappa_2 \\
-\sin \frac{\alpha}{2} e^{i \varphi} \kappa_3 & -\cos \frac{\alpha}{2} \kappa_4
\end{pmatrix}. \quad (6.1.20)$$

The ansatz for the gauge fields is:

$$A_{(i)}^0 = -\frac{\epsilon_{ij} x_j}{r^2} (2 - f_0), \quad A_{(i)}^3 = -\frac{\epsilon_{ij} x_j}{r^2} ((1 + \cos \alpha) - f_1),$$

$$A_{(i)}^1 = -\frac{\epsilon_{ij} x_j}{r^2} (\sin \alpha)(\cos \varphi)(1 - g), \quad A_{(i)}^2 = +\frac{\epsilon_{ij} x_j}{r^2} (\sin \alpha)(\sin \varphi)(1 - g). \quad (6.1.21)$$

We have introduced here the profile functions $\kappa_1(r), \kappa_2(r), \kappa_3(r), \kappa_4(r)$ for the squark scalars and $f_0(r), f_1(r), g(r)$ for the gauge field. For $r \to \infty$ all the gauge profile functions vanish and all the squark ones go to the value $\sqrt{\xi/2}$. The boundary conditions at $r \to 0$ are:

$$f(r) = 2 + \mathcal{O}(r^2), \quad f_1(r) = (1 + \cos \alpha) + \mathcal{O}(r^2), \quad g(r) = 1 + \mathcal{O}(r^3),$$

$$\kappa_1(r) \to \mathcal{O}(r^2), \quad \kappa_2(r) \to \mathcal{O}(r), \quad \kappa_3(r) \to \mathcal{O}(r), \quad \kappa_4(r) \to \mathcal{O}(1).$$

For the $(2,0)$ vortex we have:

$$\alpha = 0, \quad \phi_0 = \kappa_1, \quad \phi_1 = \kappa_4,$$

while for the $(1,1)$ vortex (after a simple diagonalization):

$$\alpha = \pi, \quad \phi_0 = \kappa_2 = \phi_1 = \kappa_3.$$

For the BPS vortex it is simpler to consider first order equations; but we are interested in understanding what is happening for $\eta_{0,3} \neq 0$. Thus we will write the equations in a form that can be easily generalized to a non-BPS setting. This will also give the possibility to check our equations and numerical results, just comparing the result for the tensions against the exact Bogomol’nyi bound; for completeness, we provide the first order BPS equations in Appendix B. The energy density due to the kinetic part of the gauge field is:

$$S_g = \frac{f_0'^2}{2 r^2 \epsilon^2} + \frac{f_1'^2}{2 r^2 g^2} + \frac{\sin^2 \alpha g'^2}{2 r^2 g^2}. \quad (6.1.22)$$
The part due to the kinetic energy of squark is:

\[ S_Q = \cos^2 \frac{\alpha}{2} (\kappa_1^2 + \kappa_4^2) + \sin^2 \frac{\alpha}{2} (\kappa_2^2 + \kappa_3^2) + \]
\[ + \cos^2 \frac{\alpha}{2} \left( \frac{(1 - \cos \alpha + f_0 + f_1)^2 \kappa_1^2}{4r^2} + \frac{(1 - \cos \alpha - f_0 + f_1)^2 \kappa_2^2}{4r^2} \right) + \]
\[ + \sin^2 \frac{\alpha}{2} \left( \frac{(1 + \cos \alpha - f_0 - f_1)^2 \kappa_2^2}{4r^2} + \frac{(1 + \cos \alpha + f_0 - f_1)^2 \kappa_3^2}{4r^2} \right) + \]
\[ + \frac{(1 - g)^2 \sin^2 \alpha}{4r^4} \left( \cos^2 \frac{\alpha}{2} (\kappa_1^2 + \kappa_4^2) + \sin^2 \frac{\alpha}{2} (\kappa_2^2 + \kappa_3^2) \right) - \]
\[ - \frac{(1 - g) \sin^2 \alpha}{2r^4} \left( (1 + f_0)\kappa_1 \kappa_3 + (1 - f_0)\kappa_2 \kappa_4 \right). \quad (6.1.23) \]

The part due to the potential reads:

\[ V_{BPS} = \frac{e^2}{2} \left( \cos^2 \frac{\alpha}{2} (\kappa_1^2 + \kappa_4^2) + \sin^2 \frac{\alpha}{2} (\kappa_2^2 + \kappa_3^2) - \xi \right)^2 + \]
\[ + \frac{g^2}{2} \left\{ \left( \cos^2 \frac{\alpha}{2} (\kappa_1^2 - \kappa_4^2) + \sin^2 \frac{\alpha}{2} (\kappa_2^2 - \kappa_3^2) \right)^2 + \sin^2 \alpha (\kappa_1 \kappa_3 - \kappa_2 \kappa_4)^2 \right\}. \quad (6.1.24) \]

The total energy is given by:

\[ E = 2\pi \int r dr (S_g + 2S_Q + V_{BPS}). \quad (6.1.25) \]

It is straightforward to write the Euler-Lagrange equations for this energy density, which are a system of seven second order equations, one for each profile function; for brevity we will not show them explicitly. We have solved this system numerically. In Fig. (6.3) it is shown an example of the solution. The tension is found to be equal to \( T_{BPS} = 4\pi \xi \) with an excellent precision for every \( \alpha \); this is a good numerical check for the solution obtained.

An analytical check of the ansatz can also be found substituting Eqs. (6.1.20) and (6.1.21) into the Euler-Lagrange equations (6.1.9). With this approach we find a system of the same seven second order equations with the following first order expression:

\[ K = g^2 r^2 (\kappa_3 \kappa_1' - \kappa_1 \kappa_3' + \kappa_2 \kappa_4' - \kappa_4 \kappa_2') - (1 - g) f_1' - (f_1 - \cos \alpha) g' = 0. \]
This seems to be a paradox, because this is a system of eight differential equation with seven unknown functions. Actually, everything is consistent, because using the seven second order equations we can show the following property:

\[
\frac{dK}{dr} = \frac{K}{r},
\]

which shows that \(K\) is linear in \(r\). From the boundary conditions of the profile functions, we find that the coefficient of this linear function has to be zero. This shows that our ansatz is consistent with the equations of motion.

**The non-BPS case:** \(\eta_0, \eta_3 \neq 0\)

For \(\eta_0, \eta_3 \neq 0\) the \((1, 1)\) and the \((2, 0)\) vortices are still solutions to the equations of motion; so these field configurations are extremal points of the energy which can be local minima or saddle points. For generic values of the parameters, we find \(T_{(1,1)} \neq T_{(2,0)}\), so the continuous moduli space interpolating between these two particular solutions disappears. For small values of \(\eta_0, \eta_3\) we expect that the low energy physics of these solitons is described by an effective potential of the moduli space. In this Section, we will estimate this potential numerically for generic values of \(\alpha\).

A constraint on this potential comes from the BPS limit at \(\eta_0, \eta_3 = 0\). In this case, a continuous family of degenerate solutions exists, with tension \(T = 4\pi\xi\). If we insert these solutions into the energy density for \(\eta_0, \eta_3 \neq 0\), the energy of these field configurations does not change. However, the solutions to the second order equations have energies which are less than this value.
This sets an upper bound:

\[ T(\alpha) \leq T_{BPS} = 4\pi\xi. \quad (6.1.27) \]

There is an obvious invariance of the equations:

\[ \alpha \rightarrow -\alpha. \]

Indeed, if we expand around \( \alpha = 0 + \delta \) or \( \alpha = \pi + \delta \) we find that the linear order in \( \delta \) is zero and that the first non-trivial correction to the tension is \( \mathcal{O}(\delta^2) \). This shows that solutions with \( \alpha = 0, \pi \) correspond to local minima or maxima of the tension. In order to find which of the two alternatives holds, an explicit calculation is needed.

In order to compute the potential of the vortex moduli space, we generalize the ansatz that we have used for the solutions in the BPS case, using the same expressions for the gauge fields, \( Q \) and the following expression for the adjoint fields:

\[ a_0 = \lambda_0(r), \quad a_3 = \lambda_3(r), \]

\[ a_1 = (\sin \alpha) \frac{x_1}{r} \lambda_{12}(r), \quad a_2 = (\sin \alpha) \frac{-x_2}{r} \lambda_{12}(r), \quad (6.1.28) \]

where we have introduced the profile functions \( \lambda_0, \lambda_3, \lambda_{12} \), with the following boundary conditions:

\[ \lambda_0(\infty) = 0, \quad \lambda_3(\infty) = 0, \quad \lambda_{12}(\infty) = 0, \quad (6.1.29) \]

and the following \( r \rightarrow 0 \) behaviour:

\[ \lambda_0 \propto \mathcal{O}(1), \quad \lambda_3 \propto \mathcal{O}(1), \quad \lambda_{12} \propto \mathcal{O}(r). \quad (6.1.30) \]

This ansatz is suggested by the expression we get for these adjoint fields in the limit of large \( \eta_0, \eta_3 \), where we can integrate these fields out (see Appendix (??)). In the following we replace these expressions in the action and find second order equations for the profile functions for generic \( \alpha \). These field configurations at \( \alpha \neq 0, \pi \) are not solutions to the full equations of motion, Eqs. \((6.1.9)\); they are just functional generalizations of the BPS solutions. We use these profiles as reasonable test functions to compute the effective moduli space potential.

The kinetic energy of the adjoint scalars is:

\[ S_a = \frac{\lambda_0^2}{e^2} + \frac{\lambda_3^2 + \sin(\alpha)^2 \lambda_{12}^2}{g^2} + \frac{\sin(\alpha)^2}{r^2 g^2} \left\{ (1 - g)^2 \lambda_3^2 + \right\} \]
Figure 6.4: Vortex profiles for $\alpha = \pi/3$ and $\xi = 2$, $e = 1/2$, $g = 1/4$, $\eta_0 = \eta_3 = 1$. In the left panel there are $\kappa_1$ (solid), $\kappa_2$ (long dashes), $\kappa_3$ (short dashes), $\kappa_4$ (dots); in the middle panel $f_0$ (solid), $f_1$ (long dashes), $g$ (short dashes); in the right panel $\lambda_0$ (solid), $\lambda_3$ (long dashes), $\lambda_{12}$ (short dashes).

Figure 6.5: Vortex tension as function of $\alpha$ in the topological winding 2 sector; the tension of the BPS 2-vortex is normalized to $T_{BPS} = 2$. In the left panel: $\xi = 2$, $e = 1/2$, $g = 1/2$, $\eta_0 = 0.1$, $\eta_3 = 1$. In the middle panel: $\xi = 2$, $e = 1/2$, $g = 1/2$, $\eta_0 = 1$, $\eta_3 = 0.1$. In the right panel: $\xi = 2$, $e = 1/2$, $g = 1/4$, $\eta_0 = \eta_3 = 1$.

\[
+ (f_1 - \cos(\alpha))^2 \lambda_{12}^2 + 2 \lambda_3 \lambda_{12} (\cos \alpha - f_1) (g - 1) \bigg) . \quad (6.1.31)
\]

The potential term is:

\[
V = \frac{\epsilon^2}{2} \left( \cos \left( \frac{\alpha}{2} \right)^2 (\kappa_1^2 + \kappa_4^2) + \sin \left( \frac{\alpha}{2} \right)^2 (\kappa_2^2 + \kappa_3^2) + 2 \eta_0 \lambda_0 - \xi \right)^2 + \\
+ \frac{g^2}{2} \left( \cos \left( \frac{\alpha}{2} \right)^2 (\kappa_1^2 - \kappa_4^2) + \sin \left( \frac{\alpha}{2} \right)^2 (\kappa_2^2 - \kappa_3^2) + 2 \eta_3 \lambda_3 \right)^2 + \\
+ \frac{g^2}{2} \sin(\alpha)^2 (\kappa_1 \kappa_3 - \kappa_2 \kappa_4 + 2 \eta_3 \lambda_{12})^2 + 2 \sin(\alpha)^2 \lambda_0 \lambda_{12} (\kappa_1 \kappa_3 - \kappa_2 \kappa_4) + \\
+ \left( \sin \left( \frac{\alpha}{2} \right)^2 \kappa_3^2 + \cos \left( \frac{\alpha}{2} \right)^2 \kappa_4^2 \right) \left( (\lambda_0 - \lambda_3)^2 + \sin(\alpha)^2 \lambda_{12}^2 \right) + \\
+ \left( \cos \left( \frac{\alpha}{2} \right)^2 \kappa_1^2 + \sin \left( \frac{\alpha}{2} \right)^2 \kappa_2^2 \right) \left( (\lambda_0 + \lambda_3)^2 + \sin(\alpha)^2 \lambda_{12}^2 \right) . \quad (6.1.32)
\]
The energy is the sum of all pieces:

$$\mathcal{E} = 2\pi \int r dr (S_g + 2S_Q + S_\alpha + V).$$ \hspace{1cm} (6.1.33)

The system of ten second order differential equations which we obtain is quite complicated, but can still be solved numerically. The qualitative plot of the profile functions is similar to the BPS case (see Fig. (6.3) and Fig. (6.4)). The tension is a non-trivial function on the coordinate $\alpha$, which gives us an effective potential of the moduli space. We solved this equations numerically for different values of the couplings $e, g, \eta_0, \eta_3$ and we have found three different regimes (see Fig. (6.5)). The tension can have a maximum at $\alpha = 0$ and a minimum at $\alpha = \pi$; we also find the opposite situation in which there is a minimum at $\alpha = 0$ and a maximum at $\alpha = \pi$. The third alternative is that both $\alpha = 0, \pi$ are local minima of the tension, with one of them a metastable minimum. We never obtain a minimum at $\alpha \neq 0, \pi$.

### 6.1.4 Vortex Interactions at Large Distance

**Vortex profiles at large distance**

At large distance $r$ from the center of the vortex, the equations can be linearized and solved analytically. Let us introduce the following notation:

$$\phi_0 = \sqrt{\xi/2} + \delta \phi_0, \quad \phi_1 = \sqrt{\xi/2} + \delta \phi_1, \quad \vec{v} = (\delta \phi_0, \delta \phi_1, \lambda_0, \lambda_3).$$ \hspace{1cm} (6.1.34)

The following linear differential equations can be written:

$$\vec{v}''(r) + \frac{\vec{v}'(r)}{r} - W \vec{v}(r) = 0,$$ \hspace{1cm} (6.1.35)

where the matrix $W$ is given by:

$$W = \begin{pmatrix}
\xi(e^2 + g^2)/2 & \xi(e^2 - g^2)/2 & \sqrt{\frac{\xi}{2}e^2}\eta_0 & \sqrt{\frac{\xi}{2}g^2}\eta_3 \\
\xi(e^2 - g^2)/2 & \xi(e^2 + g^2)/2 & \sqrt{\frac{\xi}{2}e^2}\eta_0 & -\sqrt{\frac{\xi}{2}g^2}\eta_3 \\
\sqrt{2\xi}e^4\eta_0 & \sqrt{2\xi}e^4\eta_0 & 2\eta_0^2e^4 + \xi e^2 & 0 \\
\sqrt{2\xi}g^4\eta_3 & -\sqrt{2\xi}g^4\eta_3 & 0 & 2\eta_3^2g^4 + \xi g^2
\end{pmatrix}.$$ \hspace{1cm} (6.1.36)

The eigenvalues of the matrix $W$ are in direct correspondence with some of the scalar spectrum of the theory (see Eq. (6.1.7)):

$$w_{1,2} = M_{S_1, S_2}^2 = \xi e^2 + e^4\eta_0^2 \pm \sqrt{2\xi\eta_0^2e^6 + e^8\eta_0^4},$$ \hspace{1cm} (6.1.37)
\[ w_{3,4} = M_{T_1,T_2}^2 = \xi g^2 + g^4\eta_3^2 \pm \sqrt{2}\xi^2\eta_3^2 g^6 + g^8\eta_3^4. \]

The corresponding eigenvectors are:

\[ \vec{v}_{1,2} = \left( \frac{-e^4\eta_0^2 \pm \sqrt{2}\xi e^4\eta_0^2 + e^8\eta_0^4}{2\sqrt{2}\xi e^4\eta_0}, \frac{-e^4\eta_0^2 \pm \sqrt{2}\xi e^4\eta_0^2 + e^8\eta_0^4}{2\sqrt{2}\xi e^4\eta_0}, 1, 0 \right), \]

\[ \vec{v}_{3,4} = \left( \frac{-g^4\eta_3^2 \pm \sqrt{2}\xi g^4\eta_3^2 + g^8\eta_3^4}{2\sqrt{2}\xi g^4\eta_3}, \frac{g^4\eta_3^2 \pm \sqrt{2}\xi g^4\eta_3^2 + g^8\eta_3^4}{2\sqrt{2}\xi g^4\eta_3}, 0, 1 \right). \]

Note that \( \vec{v}_k \) are also eigenvectors of the mass matrix defined in Eq. (6.1.6).

The solutions to these equations which are zero at infinity are given by the modified Bessel function:

\[ \vec{v}(r) = \sum_{k=1,\ldots,4} b_k \vec{v}_k K_0(\sqrt{w_k}r), \quad (6.1.38) \]

where \( b_k \) are appropriate constants which can be found solving the complete differential equation also at small \( r \). For large \( x \) we can use:

\[ K_0(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}. \quad (6.1.39) \]

The asymptotic solutions for the scalar profiles read:

\[ \vec{v}(r) \approx \sum_{k=1,\ldots,4} b_k \vec{v}_k \sqrt{\frac{\pi}{2\sqrt{w_k}r}} e^{-\sqrt{w_k}r}. \quad (6.1.40) \]

The large \( r \) equations for \( f_1 \) and \( f_0 \) are:

\[ f_0'' - \frac{f_0'}{r} - \xi e^2 f_0 = 0, \quad f_1'' - \frac{f_1'}{r} - \xi g^2 f_1 = 0. \quad (6.1.41) \]

This leads to the following asymptotic expression in terms of Bessel functions:

\[ f_{0,1} = c_{0,1} r K_1((e,g)\sqrt{\xi}r) \propto \sqrt{r} e^{-((e,g)\sqrt{\xi})r}, \quad (6.1.42) \]

where \( c_0, c_1 \) are constants which should be determined by the original 2nd order differential equations. Note that there is the identity: \( K_1(r) = -K_0'(r) \).

**Static vortex potential**

The next step is to reproduce the vortex asymptotic interactions in the effective linear theory by coupling the low-energy degrees of freedom to an
effective scalar density $\rho$ and an effective vector current $j_\mu$. We shall extend an approach used in Refs. [115, 17].

First of all, we have to discuss the bosonic particle spectrum of the theory. There is a massive $U(1)$ vector and a massive $SU(2)$ vector; then in principle there are 12 complex scalar fields ($Q, \tilde{Q}, A, A'$). For the vortex solution we have used the ansatz $Q = \tilde{Q}^\dagger$, so in order to discuss the vortex interactions we can neglect the modes that break this condition. There are 16 real fields, 4 of which are eaten by the Higgs mechanism; finally we have 12 physical scalars. We have already calculated the masses of these particles in Section (6.1.1); at low energy and at low coupling we can write a free theory which describes the infrared physics:

$$L_{\text{free}} = \frac{1}{4e^2} (F^0_{\mu\nu})^2 + \frac{\xi}{2} A^0_{\mu} A^0_{\mu} + \frac{1}{4g^2} (F^i_{\mu\nu})^2 + \frac{\xi}{2} A^i_{\mu} A^i_{\mu} + \frac{1}{2} (\partial_\mu S_l)^2 + \frac{1}{2} (\partial_\mu T^i_l)^2 + \frac{M^2_{S_l}}{2} S^2_l + \frac{M^2_{T^i_l}}{2} (T^i_l)^2. \quad (6.1.43)$$

This effective Lagrangian contains three real scalar fields, $S_l = 0, 1, 2$, which are $SU(2)$ singlets and three real scalar fields, $T^i_l = 0, 1, 2$, which are $SU(2)$ triplets. These scalars correspond to the appropriate eigenvectors of the mass matrix in Eq. (6.1.6). The index $i$ is an $SU(2)$ triplet index; this $SU(2)$ group corresponds to the $SU(2)_{c+f}$ in the full theory.

In order to include external sources (vortices) in this effective Lagrangian, we need the following effective terms:

$$L_{\text{source}} = \rho S_l S_l + \rho^i S_l T^i_l + j^0_{\mu} A^0_{\mu} + j^i_{\mu} A^i_{\mu}. \quad (6.1.44)$$

The corresponding wave equations are:

$$\Box + M^2_{S_l} S_l = \rho S_l, \quad \Box + M^2_{T^i_l} T^i_l = \rho^i T^i_l, \quad (6.1.45)$$

$$\Box + e^2 \xi A^0_{\mu} = j^\mu, \quad \Box + g^2 \xi A^i_{\mu} = j^i_{\mu}. \quad (6.1.45)$$

On the other hand, for the $(1, 0)$ vortex with orientation $n^i$, we have the following asymptotic profiles, converted into the singular real $Q$ gauge:

$$S_0 = 0, \quad S_1 = b_1 K_0(M_{S1} r), \quad S_2 = b_2 K_0(M_{S2} r), \quad (6.1.46)$$

$$T^0_1 = 0, \quad T^1_1 = b_3 n^i K_0(M_{T1} r), \quad T^2_1 = b_4 n^i K_0(M_{T2} r),$$

$$\vec{A}^0 = -c_0 (\vec{z} \wedge \nabla K_0(e \sqrt{\xi} r)), \quad \vec{A}^i = -c_3 n^i (\vec{z} \wedge \nabla K_0(g \sqrt{\xi} r)),$$

If we wish to include these extra modes in the low energy theory, we need only to promote the real fields $S_1, S_2, T^k_1, T^k_2$ in Eq. (6.1.43) to complex fields.
where $\nabla$ is only the ordinary gradient and not the covariant derivative as in the other Sections. We need the following mathematical identity for the 2 + 1 dimensional Laplacian of $K_0$ in term of Dirac’s $\delta$ function:

$$(-\Delta + M^2)K_0(Mr) = 2\pi \delta(\vec{r}).$$

(6.1.47)

The following expressions are found for the scalar densities corresponding to a vortex placed at the position $\vec{x}$ and having orientation $n^i$:

$$\rho_{S0} = 0, \quad \rho_{S1} = 2\pi b_1 \delta(\vec{x}), \quad \rho_{S2} = 2\pi b_2 \delta(\vec{x}),$$

(6.1.48)

$$\rho_{T0} = 0, \quad \rho_{T1} = 2\pi b_3 n^i \delta(\vec{x}), \quad \rho_{T2} = 2\pi b_4 n^i \delta(\vec{x}).$$

In a similar way we obtain the following expressions for the currents:

$$\vec{j} = -2\pi c_0 \hat{z} \wedge \nabla \delta(\vec{x}), \quad \vec{j}^i = -2\pi c_3 n^i \hat{z} \wedge \nabla \delta(\vec{x}).$$

(6.1.49)

Using these expressions, it is straightforward to compute the static intervortex potential between two vortices with orientations $\vec{n}_1$ and $\vec{n}_2$ at distance $R$:

$$U = 2\pi \left( c_0^2 K_0(e \sqrt{\xi} R) - b_1^2 K_0(M_{S1} R) - b_2^2 K_0(M_{S2} R) + (\vec{n}_1 \cdot \vec{n}_2)(c_3^2 K_0(g \sqrt{\xi} R) - b_3^2 K_0(M_{T1} R) - b_4^2 K_0(M_{T2} R)) \right).$$

(6.1.50)

In the BPS case ($\eta_0 = \eta_3 = 0$) this potential is exactly zero, because we have $M_{S1} = M_{S2} = e \sqrt{\xi}$, $M_{T1} = M_{T2} = g \sqrt{\xi}$ and $c_0^2 = b_1^2 + b_3^2$, $c_3^2 = b_3^2 + b_4^2$.

If $\eta_3, \eta_0 \neq 0$, we find that at large distance the particle with lowest mass is the one which dominates the interaction. We have always the following inequalities:

$$M_{S2} < e \sqrt{\xi} < M_{S1}, \quad M_{T2} < g \sqrt{\xi} < M_{T1}.$$  

(6.1.51)

Thus if $M_{S2} < M_{T2}$ then we have:

$$U \approx -2\pi b_2^2 K_0(M_{S2} R) \approx -2\pi b_4^2 \sqrt{\frac{\pi}{2M_{S2} R}} e^{-M_{S2} R},$$

(6.1.52)

which gives an always attractive force. On the other hand, if $M_{T2} < M_{S2}$:

$$U \approx -2\pi b_4^2 K_0(M_{T2} R)(\vec{n}_1 \cdot \vec{n}_2) \approx -2\pi b_4^2 \sqrt{\frac{\pi}{2M_{T2} R}} e^{-M_{T2} R (\vec{n}_1 \cdot \vec{n}_2)},$$

(6.1.53)

which gives attraction for $\vec{n}_1 = \vec{n}_2$ and repulsion for $\vec{n}_1 = -\vec{n}_2$. 

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A very peculiar thing happens for $e = g$ and $\eta_0 = \eta_3 \neq 0$. For these fine-tuned values of the couplings $M_{S2} = M_{T2} = M_2$ and $b_2 = b_4$, so the effective vortex potential has the form:

$$U \approx -2\pi b_2^2 K_0(M_2 R)(1 + \vec{n}_1 \cdot \vec{n}_2) \approx -2\pi b_2^2 \sqrt{\frac{\pi}{2M_2 R}} e^{-M_2 R}(1 + \vec{n}_1 \cdot \vec{n}_2)$$

which gives a flat potential for $\vec{n}_1 = -\vec{n}_2$. This is consistent with the fact that in this limit the $(1, 0)$ and the $(0, 1)$ vortices do not interact because they are completely decoupled (see the argument below Eq. (6.1.18)). This behaviour is similar to the one found in Refs. [124, 125] for global non-Abelian vortices.

### 6.1.5 Effective worldsheet theory

**Single vortex**

It is useful in the following to use the singular gauge in which the squarks fields at $r \to \infty$ tend to a fixed VEV and do not wind. In this gauge, the ansatz (6.1.12) for the single vortex reduces to

$$A_0^i = \frac{\epsilon_{ij}}{r^2} f_0, \quad A_i^i = \frac{\epsilon_{ij}}{r^2} f_1 n^i, \quad a^0 = \lambda_0, \quad a^i = n^i \lambda_3,$$

$$Q = \frac{\phi_0 + \phi_1}{2} 1 + \frac{\phi_0 - \phi_1}{2} \tau^i n^i.$$

We will assume that the orientational coordinates $\vec{n}$ are functions of the string worldsheet coordinates. $\vec{n}$ becomes a field of a 1+1 dimensional sigma model. This effective theory has no potential due to the fact that the $n^i$ parameterize some zero modes; in the following we will compute the kinetic term. For the gauge field components $A_{0,3}$, we will use the approach reviewed in Section 6.1.3. With the ansatz:

$$A_i = -\frac{1}{2} (\tau^a \epsilon^{abc} n_b \partial_c n_c) \rho(r), \quad i = 0, 3,$$

the field strength components $F_{ki}$ with $k = 0, 3$ and $i = 1, 2$ are not zero any more:

$$F_{ki} = \frac{1}{2} \partial_k n^a \tau^a \epsilon_{ij} \frac{x_j}{r^2} f_1 [1 - \rho(r)] + \frac{1}{2} (\tau^a \epsilon^{abc} n_b \partial_k n_c) \frac{x_j}{r} d \rho(r).$$

Substituting this expression into the kinetic term for the gauge field and for the squark fields, we obtain a simple generalization of the BPS case discussed in Refs. [16, 9, 82]:

$$S^{1+1} = \frac{\beta}{2} \int dt dz (\partial_i n^i)^2,$$
where:

$$\beta = \frac{2\pi}{g^2} \int r dr \left\{ \rho^2 + \frac{f_1^2(1-\rho)^2}{r^2} + \lambda_3^2(1-\rho)^2 + \right.$$  

$$+ g^2 \left\{ \phi_0^2 + \phi_1^2 \right\} \frac{\rho^2}{2} + (1-\rho)(\phi_0 - \phi_1)^2 \right\}.  \tag{6.1.59}$$

We have to solve the Euler-Lagrange equations for $\rho(r)$, with $\rho(0) = 1$ and $\rho(r \to \infty) = 0$. In the BPS case, where $\lambda_3$ is trivially 0, we can show from the equations of motion \[9\] that $\rho = 1 - \phi_0/\phi_1$ and that $\beta = 2\pi/g^2$ (see also \[82, 89\]); in the general case $\eta_5, \eta_0 \neq 0$ there is not such powerful analytical result, here we have to solve the equations for $\rho$ numerically and then calculate $\beta$.

In the BPS case we have additional fermionic zero modes, associated with the unbroken supercharges; at small $\eta_0, \eta_3 \neq 0$ these modes should be still present, but they will not be described anymore by the fermionic sector of a supersymmetric effective theory in 1+1 dimensions. We will not discuss this aspect in this paper and we will leave it as a problem for further investigation.

The color-flavor modes of the (2, 0) vortex are very similar to the (1, 0) ones: both the vortices have a $\mathbb{C}P^1$ moduli space and the value of $\beta$ can be determined using Eq. (6.1.59). For the (1, 1) vortex, nevertheless, these modes are just trivial because all the profile functions are proportional to the identity matrix.

Two well separated vortices

A proper description of the system has to take into account also the quantum aspects of the sigma model physics. Let us consider two vortices with internal orientations $\vec{n}_1, \vec{n}_2$. The relative distance between them can be promoted to a complex field $R$; the global position of the system, on the other hand, decouples from the other the degrees of freedom. If the distance of the two vortices is large ($|R| \to \infty$), we expect that the effective world-volume description of the bosonic degrees of freedom is:

$$S = \int dtdz \left\{ \frac{\beta}{2} (\partial_k n_1^a)^2 + \frac{\beta}{2} (\partial_k n_2^a)^2 + T |\partial_k R|^2 + v_s(|R|) + v_t(|R|) \vec{n}_1 \cdot \vec{n}_2 \right\},$$

where:

$$v_s(|R|) = -2\pi b_2^2 \sqrt{\frac{\pi}{2M_{S2}|R|}} e^{-M_{S2}|R|},$$

$$v_t(|R|) = -2\pi b_4^2 \sqrt{\frac{\pi}{2M_{T2}|R|}} e^{-M_{T2}|R|},$$

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where $T$ is the tension of a single vortex. This description is good only for large values of the VEV of the field $R$; at $R = 0$ the internal degrees of freedom are no longer described by $\mathbb{C}P^1 \times \mathbb{C}P^1$, but by the space $\mathbb{C}P^2/\mathbb{Z}_2$ (see Chapter 4). Moreover, the expression used for the potential is good only for large vortex separation.

If we keep the VEV of $R$ fixed (which physically corresponds to keep the distance of the two vortices fixed with some external device), the effective description is given by two $\mathbb{C}P^1$ sigma models with a small interaction term of the form $c \vec{n}_1 \cdot \vec{n}_2$.

6.1.6 Summary

For Abelian Type I superconductors, the force between two vortices with the same winding number is always attractive. This is true at large and at small distances, as shown by numerical calculations in Ref. [113]. In the model discussed here, for $\eta_0, \eta_3 > 0$, the masses of some of the scalars fields are always found to be less than the mass of the corresponding vector boson. In this sense we can think the system as a generalization of the Abelian Type I superconductor. However, here is an important difference: the force between two vortices is not always attractive; there is a non-trivial dependence on the coupling, the relative internal orientation and the distance.

We have studied the problem in two different limits: large vortices separation and coincident vortices. For large separations we have computed the leading potential analytically; the behavior at large distance is dominated by the particle with the lowest mass $M_{\text{low}}$. There are two main alternatives, which hold for different values of the couplings:

$$U(R) \propto \begin{cases} -\sqrt{\frac{1}{2M_{S2}^2}} e^{-M_{S2}R} & \text{for } M_{\text{low}} = M_{S2}, \text{ Type I} \\ -(\vec{n}_1 \cdot \vec{n}_2)\sqrt{\frac{1}{2M_{T2}^2}} e^{-M_{T2}R} & \text{for } M_{\text{low}} = M_{T2}, \text{ Type I}^* \end{cases} \quad (6.1.60)$$

where $M_{S2}, M_{T2}$ are the masses of the scalars in Eq. (6.1.7). In order to distinguish these regimes, we call them Type I and Type I*; for Type I* vortices the sign of the asymptotic force depend on $\vec{n}_1 \cdot \vec{n}_2$. For the fine-tuned values $e = g$ and $\eta_0 = \eta_3 \neq 0$, the relation $M_{S2} = M_{T2} = M_2$ holds, and the effective vortex potential has the form:

$$U(R) \propto -(1 + \vec{n}_1 \cdot \vec{n}_2)\sqrt{\frac{1}{2M_{T2}^2}} e^{-M_{T2}R}, \quad (6.1.61)$$

which gives a flat potential for $\vec{n}_1 = -\vec{n}_2$.

For coincident vortices we have found two stationary solutions of the equations of motion, the $(1, 1)$ and the $(2, 0)$ vortices, and we computed their
tensions numerically. The results are shown in Fig. (6.2); both the cases $T_{1,1} > T_{2,0}$ or $T_{2,0} > T_{1,1}$ are possible for different values of the coupling. The moduli space interpolating between these solutions at $\eta_0 = \eta_3 = 0$ disappears for non-zero values of one of these parameters (see Fig. (6.5)).

It is interesting to match the data of the two complementary approaches. Let us for simplicity consider the case of parallel ($\vec{n}_1 = \vec{n}_2$) and anti-parallel ($\vec{n}_1 = -\vec{n}_2$) vortices. In the case of parallel vortices at large separation distance, the force is always attractive; also from numerical calculations we find $T_{2,0} < 2T_{1,0}$ for all the values of the coupling that we have analyzed. We have not made the calculation for arbitrary distances, but we think that the above are a good evidence for the fact that the force between two parallel vortices is always attractive in our model.

For anti-parallel vortices, on the other hand, the situation is more complicated. At large distances there is attraction if $M_{T2} > M_{S2}$ and repulsion if $M_{T2} < M_{S2}$. For the choice $e = g$ and $\eta_0 = \eta_3$, the relation $M_{T2} = M_{S2}$ holds. For these particular values the two diagonal $U(1)$ factors decouple: there is no net classical force between the $(1, 0)$ and the $(0, 1)$ force for arbitrary distance (this configuration is not stable, because if we allow $\vec{n}_1$, $\vec{n}_2$ to vary, we have that $(2, 0)$ vortex has a lower energy). Indeed, if we keep $e = g$, we obtain $M_{T2} > M_{S2}$, $T_{1,1} < 2T_{1,0}$ for $\eta_0 > \eta_3$ and $M_{T2} < M_{S2}$, $T_{1,1} > 2T_{1,0}$ for $\eta_0 < \eta_3$. This is a good evidence that for $\eta_0 > \eta_3$ we have an attractive force and for $\eta_0 < \eta_3$ we have a repulsive one (see Fig. (6.6)).

If we relax the condition $e = g$, there are situations in which at large distance there is attraction (because $M_{T2} > M_{S2}$) and also we get $T_{1,1} > 2T_{1,0}$, as is shown in Fig. (6.7) to the left: this means that there is a critical distance, in which there is a minimum of the inter-vortex potential for anti-parallel vortices (if we allow the vortex orientation to flip, probably it will not

Figure 6.6: Qualitative plot of the vortex potential as function of the vortex distance for $\vec{n}_1 = -\vec{n}_2$ and $e = g$. For $\eta_0 > \eta_3$ we have attraction (left); for $\eta_0 < \eta_3$ we have repulsion (right). With the choice $\eta_0 = \eta_3$ there is no net classical force.
be a minimum any more, because we still have $T_{2,0} < 2T_{1,0}$). An example of this situation can be obtained with the couplings $\xi = 2, e = 1/4, g = 1/2, \eta_0 = 0, \eta_3 = 4$. Moreover we can obtain $M_{T2} < M_{S2}$ and $T_{1,1} < 2T_{1,0}$, which means that there is a critical distance at which there is a maximum of the inter-vortex potential (see Fig. (6.7) on the right). An example of this situation can be obtained with the couplings $\xi = 2, e = 1/2, g = 1/4, \eta_0 = 3.5, \eta_3 = 2$.

It is interesting that at $R = 0$ there are two different regimes, which depend on the values of the coupling, with very different properties. The physics of the $(2, 0)$ vortex is described by a bosonic $CP^1$ sigma model; the $(1, 1)$ vortex on the other hand is an Abelian vortex with no internal degrees of freedom. For some values of the coupling we have an evidence that both vortices are local minima of the tension (see Fig. (6.5)): one of the two is metastable (indeed for some fine tuned values of $\eta_{0,3}$ we have that both the vortices have the same tension).

A model with metastable vortices at weak coupling has already been studied in Ref. [78]. This behavior is reminiscent of $SU(N)$ Yang-Mills, where for each topological $n$-ality we can have different string tensions for each representation of the Wilson Loop. In each topological sector there is just one stable string, corresponding to the antisymmetric representation; there are evidences that the strings with other representations are metastable strings, at least in the large $N$ limit (see Ref. [126] for a discussion).

In this Section we have considered non-Abelian non-BPS vortices in an $\mathcal{N} = 1$ supersymmetric model. In the next Section, we will discuss a simpler, non-supersymmetric model, in which we can have both Type I and Type II non-Abelian superconductivity.
6.2 Static interactions

In the previous Section, we have discussed these aspects in an $\mathcal{N} = 2$ theory with an adjoint mass term which breaks the extended supersymmetry, and we have found a natural non-Abelian generalization of Type I superconductors. Even if the force between two non-Abelian vortices is not always attractive, we have found a close resemblance with Type I Abelian vortices: the lightest field of the theory is a scalar field. So if we put two vortices at large distance, the prevailing part of the interaction is mediated by the scalar particles and not by vector particles. Moreover, if the two vortices have the same orientation in the internal moduli space, the force is always attractive.

In this Section we study the same problem in another theoretical setting: an extension of the Abelian-Higgs model with arbitrary scalar couplings which is generically incompatible with the BPS limit. The simplest extension in this direction is a theory in which there are just two mass scales; the mass of the vector bosons and the mass of the scalars. There is one parameter $\lambda$ which controls the ratio of the two mass scales. We find that $\lambda < 1$ leads to an attractive force as a usual Abelian Type I, while for $\lambda > 1$ a repulsive force works, similarly to the usual Abelian Type II. There is no force between vortices with opposite $\mathbb{C}P^{N-1}$ orientation. For $\lambda < 1$ (Type I) this configuration is unstable and the true minimum of the potential corresponds to two coincident vortices with the same orientation. For $\lambda > 1$ (Type II) this configuration is stable; in other words a part of the moduli space corresponding to the relative distance between vortices with opposite orientations survives the non-BPS perturbation.

However, in more general theories where the masses of the Abelian and non-Abelian degrees of freedom are different, we find a more complicated picture. There are four mass scales, the masses of the $U(1)$ and of the $SU(N)$ vector bosons, the masses of the scalars in the adjoint representation and the singlet of $SU(N)_{C+F}$. At large distance, the interaction between two vortices is dominated by the particle with the lightest mass. So if we keep the four masses as generic parameters, at large vortex separation we find four different regimes that we call Type I, Type II, Type $I^*$ and Type $II^*$. In the last two categories repulsive and attractive interactions depend on the relative orientation. We study also numerically the interactions among two vortices at any separation with arbitrary orientations, and find that short distance forces also have rich qualitative features depending both on the relative orientations and the relative distance.
6.2.1 Theoretical set-up

A fine-tuned model

Our natural starting point is the D-term vortex theory of (3.1.2):

\[
\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{\lambda^2 g^2}{4} \left( v^2 1_N - HH^\dagger \right)^2 \right].
\]  

(6.2.1)

Here, for simplicity we take the same gauge coupling \( g \) for both the \( U(1) \) and \( SU(N) \) groups and we have introduced a new parameter \( \lambda \). We have only three couplings (\( g, \lambda, v \)). The vacuum of the model is the usual one:

\[
HH^\dagger = v^2 1_N.
\]  

(6.2.2)

As already stated, it breaks completely the \( U(N) \) gauge symmetry, but a global color-flavor locking symmetry \( SU(N)_{c+f} \) is preserved

\[
H \rightarrow U_c H U_f, \quad U_c = U_f^\dagger, \quad U_c \in SU(N)_c, \quad U_f \in SU(N)_f.
\]  

(6.2.3)

The trace part \( \text{Tr} H \) is a singlet under the color-flavor group and the traceless parts are in the adjoint representation.

We have two mass scales, one for the vector bosons and the other for the scalar bosons. The \( U(1) \) and the \( SU(N) \) gauge vector bosons have both the same mass

\[
M_{U(1)} = M_{SU(N)} = g v.
\]  

(6.2.4)

The masses of the scalars are given by the eigenvalues of the mass matrix. We start with \( 2N^2 \) real scalar fields in \( H \): \( N^2 \) of them are eaten by the gauge bosons (the Higgs mechanism) and the other \( N^2 \) (one singlet and the rest adjoint) have same masses

\[
M_s = M_{\text{ad}} = \lambda g v.
\]  

(6.2.5)

When we choose the critical coupling \( \lambda = 1 \) (BPS), the mass of these scalars is the same as the mass of the gauge bosons and the Lagrangian allows an \( \mathcal{N} = 2 \) supersymmetric extension. The BPS vortices saturating the BPS energy bound admit infinitely degenerate set of solutions.

Models with general couplings

Together with the fine-tuned model \( \text{[6.2.1]} \) we consider the more general model \( \text{[3.1.2]} \) keeping different gauge couplings, \( e \) for the \( U(1) \) part and \( g \)
for the $SU(N)$ part. The more general scalar potential is written here in a
different fashion:

\[
\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - \frac{1}{2e^2} f_{\mu\nu} f^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger \right] - V,
\]  

(6.2.6)

where we have defined $\hat{F}_{\mu\nu} = \sum_{A=1}^{N^2-1} F^A_{\mu\nu} t_A$ and $f_{\mu\nu} = F^0_{\mu\nu} t^0$ with $\text{Tr} (t^A t^B) = \delta^{AB}/2$ and $t^0 = 1/\sqrt{2N}$.

The scalar potential is:

\[
V = \frac{\lambda^2 g^2}{2} \sum_{A=1}^{N^2-1} (H^i t^A_H)^2 + \frac{\lambda^2 e^2}{4N} (H^i t^A_H - N v^2)^2
\]

\[
= \frac{\lambda^2 g^2}{4} \text{Tr} X^2 + \frac{\lambda^2 e^2}{4} \text{Tr} (X^0 t^0 - v^2 1_N)^2,
\]  

(6.2.7)

where

\[
X \equiv H H^\dagger = X^0 t^0 + \sum_{A=1}^{N^2-1} X^A t^A, \quad \tilde{X} \equiv \sum_{A=1}^{N^2-1} X^A t^A = 2 \sum_{A=1}^{N^2-1} (H^i t^A_H) t^A.
\]

The Lagrangian has the same symmetries as the previous fine-tuned model (6.2.1). The potential in Eq. (6.2.7) is the most general gauge invariant quartic potential which can be built with the matter content of the theory. The $U(1)$ and the $SU(N)$ vector bosons have different masses

\[
M_{U(1)} = e v, \quad M_{SU(N)} = g v.
\]  

(6.2.8)

Moreover, the singlet part of $H$ has a mass $M_s$ different from that of the adjoint part $M_{ad}$

\[
M_s = \lambda_e e v, \quad M_{ad} = \lambda_g g v.
\]  

(6.2.9)

When we take equal couplings, $g = e$ and $\lambda \equiv \lambda_e = \lambda_g$, the scalar potential reduces to the simple potential $V_{g=e} = \frac{\lambda^2 e^2}{4} \text{Tr} (X - v^2 1_N)^2$. For the critical values $\lambda_e = \lambda_g = 1$, the Lagrangian allows an $\mathcal{N} = 2$ supersymmetric extension and then the model admits BPS vortices which saturate the BPS energy bound.

### 6.2.2 Non-Abelian vortices in the fine-tuned model

**Vortex equations**

Let us consider the fine-tuned model (6.2.1). For convenience, we make the following rescaling of fields and coordinates:

\[
H \rightarrow v H, \quad W_\mu \rightarrow g v W_\mu, \quad x_\mu \rightarrow \frac{x_\mu}{g v}.
\]  

(6.2.10)

\footnote{In the case $g = e$ it is more compact to use $F_{\mu\nu} = \hat{F}_{\mu\nu} + f_{\mu\nu}$.}
The Lagrangian then in Eq. (6.2.1) takes the form

\[ \tilde{\mathcal{L}} = \frac{\mathcal{L}}{g^2 v^4} = \text{Tr} \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{\lambda^2}{4} \left( 1_N - HH^\dagger \right)^2 \right], \]

(6.2.11)

and the masses of vector and scalar bosons are rescaled to

\[ M_{U(1)} = M_{SU(N)} = 1, \quad M_s = M_{\text{ad}} = \lambda. \]

(6.2.12)

As explained in the introduction, the model with \( \lambda < 1 \) (\( \lambda > 1 \)) in the Abelian case (\( N = 1 \)) is called Type I (Type II) and the forces between vortices are attractive (repulsive). At the critical coupling \( \lambda = 1 \), there are no forces between vortices, so that multiple vortices stably coexist.

In order to construct non-BPS non-Abelian vortex solutions, we have to solve the following 2nd order differential equations, derived from the Lagrangian (6.2.1),

\[ \mathcal{D}_\mu F^{\mu\nu} - \frac{i}{2} \left[ H (\mathcal{D}^\nu H)^\dagger - (\mathcal{D}_\nu H)H^\dagger \right] = 0, \]

(6.2.13)

\[ \mathcal{D}_\mu \mathcal{D}^\mu H + \frac{\lambda^2}{4} (1 - HH^\dagger) H = 0. \]

(6.2.14)

From now on, we restrict ourselves to static configurations depending only on the coordinates \( x^1, x^2 \). Here we introduce a complex notation

\[ z = x^1 + ix^2, \quad \partial = \frac{\partial_1 - i\partial_2}{2}, \quad W = \frac{W_1 - iW_2}{2}, \quad \mathcal{D} = \frac{\mathcal{D}_1 - i\mathcal{D}_2}{2} = \partial + iW. \]

The equation of motions are of course not gauge invariant but covariant. It is possible to rewrite these equations in terms of gauge invariant quantities, instead of dealing with the original fields \( H \) and \( W_\mu \). For this purpose, we can use the same techniques which enabled us to write the master equation (3.1.15) within the moduli matrix formalism. Rewriting our fields as follows

\[ \tilde{W}(z, \bar{z}) = -i S^{-1}(z, \bar{z}) \partial S(z, \bar{z}), \quad H(z, \bar{z}) = S^{-1}(z, \bar{z}) \tilde{H}(z, \bar{z}), \]

(6.2.15)

where \( S \) takes values in \( GL(N, \mathbb{C}) \) and it is in the fundamental representation of \( U(N) \) while the gauge singlet \( \tilde{H} \) is an \( N \times N \) complex matrix. There is an equivalence relation \( (S, \tilde{H}) \sim (V(z)S, V(z)\tilde{H}) \), where \( V(z) \) is a holomorphic \( GL(N, \mathbb{C}) \) matrix with respect to \( z \), because different elements in the same equivalence class give us the same physical fields as in Eq. (6.2.15). In order to write down the equations of motion (6.2.13) and (6.2.14) in a gauge invariant fashion, we introduce a gauge invariant quantity

\[ \Omega(z, \bar{z}) \equiv S(z, \bar{z})S(z, \bar{z})^\dagger. \]

(6.2.16)
With respect to the gauge invariant objects $\Omega$ and $\tilde{H}$, the equations (6.2.13) and (6.2.14) are written in the following form

\[
4 \partial^2 \left( \Omega \partial \Omega^{-1} \right) - \tilde{H} \bar{\partial} \left( \tilde{H}^\dagger \Omega^{-1} \right) + \bar{\partial} \tilde{H} \tilde{H}^\dagger \Omega^{-1} = 0, \tag{6.2.17}
\]

\[
\bar{\partial} \left( \Omega^{-1} \partial \tilde{H} \right) + \partial \left( \Omega \partial \left( \Omega^{-1} \tilde{H} \right) \right) + \frac{\lambda^2}{4} \left( \Omega - \tilde{H} \tilde{H}^\dagger \right) \Omega^{-1} \tilde{H} = 0. \tag{6.2.18}
\]

Notice that Eq. (6.2.17) is a 3rd order differential equation. This is the price we have to pay in order to write down the equations of motion in terms of gauge invariant quantities. These equations must be solved with the following boundary conditions for $k$ vortices:

\[
\det \tilde{H} \to z^k, \quad \Omega \to \tilde{H} \tilde{H}^\dagger, \quad \text{as} \quad z \to \infty. \tag{6.2.19}
\]

The field strength is given by

\[
F_{12} = 2S^{-1} \bar{\partial} \left( \Omega \partial \Omega^{-1} \right) S. \tag{6.2.20}
\]

Notice that Eq. (6.2.17) is invariant under the $SU(N)$ flavor symmetry while Eq. (6.2.18) is covariant. This leads to Nambu-Goldstone zero modes for vortex solutions.

**BPS Limit**

To see the relation with the BPS equations, let us take a holomorphic function $\tilde{H}$ with respect to $z$ as

\[
\tilde{H} = H_0(z). \tag{6.2.21}
\]

Then the equations (6.2.17) and (6.2.18) reduce to

\[
\bar{\partial} \left[ 4 \bar{\partial} \left( \Omega \partial \Omega^{-1} \right) - H_0 H_0^\dagger \Omega^{-1} \right] = 0, \tag{6.2.22}
\]

\[
\bar{\partial} \left( \Omega \partial \Omega^{-1} \right) + \frac{\lambda^2}{4} \left( 1 - H_0 H_0^\dagger \Omega^{-1} \right) = 0. \tag{6.2.23}
\]

These two equations are consistent only in the BPS limit $\lambda = 1$. The equation (6.2.23) is the master equation for the BPS non-Abelian vortex and the holomorphic matrix $H_0(z)$ is the moduli matrix for BPS vortices. For any given moduli matrix $H_0(z)$, given the corresponding solution to the master equation, the physical fields $W_\mu$ and $H$ are obtained via Eq. (6.2.15). All the complex parameters contained in the moduli matrix are moduli of the BPS vortices. For example, the position of the vortices can be read from the moduli matrix as zeros of its determinant $\det H_0(z_i) = 0$. Furthermore, the number of vortices (the units of magnetic flux of the configuration) corresponds to
the degree of $\det H_0(z)$ as a polynomial with respect to $z$. The classification of the moduli matrix for the BPS vortices is given in Ref. [10] [75].

From now on, we consider a $U(2)$ gauge theory ($N = 2$), which is the minimal model for non-Abelian vortices. As described in Section (3.3), the minimal winding BPS vortex is described by two moduli matrices

$$H_0^{(1,0)} = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \quad H_0^{(0,1)} = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix},$$

(6.2.24)

and the orientational vectors are

$$\vec{\phi}^{(1,0)} = \begin{pmatrix} 1 \\ b' \end{pmatrix}, \quad \vec{\phi}^{(0,1)} = \begin{pmatrix} b \\ 1 \end{pmatrix}.$$  

(6.2.25)

Here "\sim" stands for an identification up to complex non zero factors: $\vec{\phi} \sim \lambda \vec{\phi}$, $\lambda \in \mathbb{C}^*$. If we start with $\vec{\phi} = (1, 0)^T$, we can recover $\vec{\phi} = (1, b')^T$ by use of the color-flavor rotation as

$$\vec{\phi} = U_T \vec{\phi} \quad \Leftrightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ \beta^*/\alpha^* \end{pmatrix}$$

(6.2.26)

with $|\alpha|^2 + |\beta|^2 = 1$. Thus we identify $b'$ and $\beta^*/\alpha^*$. In what follows, we will call two non-Abelian vortices with equal orientational vectors parallel, while when they have orthogonal orientational vectors we will call them anti-parallel. The reader must keep in mind that vortices are always parallel in real space. Throughout this paper, we use the words parallel and anti-parallel only referring to the internal orientational vectors.

Generic configurations of two vortices at arbitrary positions and with arbitrary orientations are described by the moduli matrices introduced in Chapter (4)

$$H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ z - \phi' \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 - \alpha'z - \beta' & 0 \\ -\alpha'z - b' \end{pmatrix}.$$  

(6.2.27)

The superscripts label patches covering the moduli space. One more patch similar to (2, 0) is needed to complete the full moduli space. The positions of the vortices are the roots of $z^2 - (\phi + \tilde{\phi})z + \phi\tilde{\phi} - \eta\tilde{\eta} = z^2 - \alpha'z - \beta' = 0$. By using translational symmetry we can set $z_1 + z_2 = 0$ ($\phi + \tilde{\phi} = \alpha' = 0$) without loss of generality. The orientation vectors are $\vec{\phi}^{(1,1)}_1 = (\eta, z_1 - \phi)^T$ and $\vec{\phi}^{(1,1)}_2 = (\eta, z_2 - \phi)^T$ for the (1, 1) patch, while they are $\vec{\phi}^{(2,0)}_1 = (1, a'z_1 + b')^T$ and $\vec{\phi}^{(2,0)}_2 = (1, a'z_2 + b')^T$ for the (2, 0) patch. Overall complex factor does not have physical meaning, so that each vector takes value on $\mathbb{C}P^1$. We can describe anti-parallel vortices only in the (1,1) patch when $\eta = \tilde{\eta}$, because of
\[\vec{\phi}_1 \vec{\phi}_1 = 0.\] On the other hand, we can describe parallel vortices only in the (2, 0) patch when \(a'z_1 + b' = a'z_2 + b'.\)

For convenience, let us take a special subspace where \(\tilde{\eta} = 0\) in the (1, 1) patch:

\[H_{0 \text{ red}}^{(1,1)} \equiv \begin{pmatrix} z - z_0 & -\eta \\ 0 & z + z_0 \end{pmatrix}, \quad z_0 = z_1 = -z_2 = \phi = -\tilde{\phi}. \quad (6.2.28)\]

One can always recover generic points in Eq. (6.2.27) using flavor rotations. The parameters \((z_0, \eta)\) and \((\beta, a', b')\) are related by the following relations: \(\beta' = z_0^2, \quad a' = 1/\eta\) and \(b' = -z_0/\eta\). The orientational vectors are then of the form

\[\vec{\phi}_1^{(1,1)}|_{z = z_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\phi}_2^{(1,1)}|_{z = -z_0} = \begin{pmatrix} \eta \\ -2z_0 \end{pmatrix}. \quad (6.2.29)\]

When the vortices are coincident, however, the rank of the moduli matrix at the vortex position reduces. In this case we can no longer define two independent orientations for each vortex but we can only define an overall orientation. In fact, one can see that when \(z_0 = 0\), the two orientations in Eq. (6.2.29) are both equal to \(\vec{\phi}^{(1,1)} = (1, 0)^T\). We cannot really give to the parameter \(\eta\) an exact physical meaning of a relative orientation between two coincident vortices. It is better, in this case, to consider this parameter merely as an internal degree of freedom of the composite vortex. When we take correctly into account both the parameter \(\eta\) and the global flavour rotations that we previously factorized out, we recover the full moduli space for coincident vortices, which is \(\tilde{W}_{\mathbb{C}P^2_{(2,1,1)}} \cong \mathbb{C}P^2/Z_2\) \([90, 91, 127]\).

The definitions of the position and orientation of a single vortex can be rigorously extended to the non-BPS case by merely replacing \(H_0(z)\) with \(\tilde{H}_0(z, \bar{z})\). For configurations with several vortices, all the flat directions that are not related to Goldstone modes or translational symmetries will disappear. It is possible to use these definitions as constraints on the \(\tilde{H}_0(z, \bar{z})\) matrix to fix positions and orientations. Then our formalism allows us to study the static interactions of non-BPS configurations.

### 6.2.3 Vortex interaction in the fine-tuned model

We now concentrate on the fine-tuned model \([6.2.11]\). We will first calculate the masses of a special class of non-BPS coincident vortices. Then we will derive an effective potential for coincident almost BPS vortices but with generic value of the internal modulus parameter. Finally, we will compute an effective potential for two almost BPS vortices at any distance and with any relative orientations.
$(k_1, k_2)$ coincident vortices

The minimal winding solution in the non-Abelian gauge theory is a mere embedding of the ANO solution into the non-Abelian theory. This is obvious also from the moduli matrix view point. In fact, in the non-Abelian moduli matrix (6.2.24) we can put $b$ (or $b'$) to zero with a global flavour rotation. Henceforth we can recognize the moduli matrix for the single ANO vortex: $H_0^{\text{ANO}}(z) = z - z_0$ as the only non-trivial element of the moduli matrix.

This kind of embedding is also useful to investigate a simple non-BPS configuration. Let us start with the moduli matrix for a configuration of $k$ coincident vortices. Since we have an axial symmetry around the $k$ coincident vortices, we can make the following reasonable ansatz for $\Omega$ and $\tilde{H}$:

$$\Omega^{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & w(r) \end{pmatrix}, \quad \tilde{H}^{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & f(r)z^k \end{pmatrix}. \quad (6.2.30)$$

Note that $f(r) = 1$ means $\tilde{H}^{(0,1)} = H_0^{(0,1)}(z)$ which is nothing but the BPS solution. We will call the multiple vortex which is generated by the ansatz in Eq. (6.2.30) “$(0,k)$-vortex”. In terms of the two fields $w(r) = e^Y(r)$ and $f(r)$ the equations (6.2.17) and (6.2.18) reduce to the following form:

$$Y'' + \frac{1}{r}Y' - \frac{2}{f} \left( f'' + \frac{1 + 2k}{r} f' - Y'' f' \right) - \lambda^2 \left( 1 - r^{2k} f^2 e^{-Y} \right) = 0, \quad (6.2.31)$$

$$Y''' + \frac{1}{r}Y'' - \frac{1}{r^2} Y' + e^{-Y} r^{2k-1} f^2 (2k - rY') = 0. \quad (6.2.31)$$

The boundary conditions are

$$Y \to 2k \log r, \quad Y' \to 2k/r, \quad f \to 0 \quad (r \to \infty); \quad Y' \to 0, \quad f' \to 0 \quad (r \to 0). \quad (6.2.32)$$

Although it is quite impossible to solve these differential equation analytically, we can solve them numerically. The results for the single vortex ($k = 1$) are shown in Fig. (6.8), where we used several different values of $\lambda$.

When $k \geq 2$ it is possible that the ansatz (6.2.30) does not give the true solution (minimum of the energy) of the equations of motion (6.2.17) and (6.2.18). This is because there could be repulsive forces between the vortices. With ansatz (6.2.30) we fix the positions of all the vortices at the origin by hand. The reduced equations (6.2.31) are nevertheless still useful to investigate the interactions between two vortices. The results are listed in Table (6.1). For $\lambda = 1$, the masses are identical to integer values, up to $10^{-5}$ order, which are nothing but the winding number of the vortices. Furthermore, our numerical results for generic $\lambda$ are in perfect agreement.
Figure 6.8: Numerical plots of $Y$ (left) and $f$ (right) for the single vortex: black for $\lambda = 1$, red for $\lambda = 1.7$, blue for $\lambda = 0.5$. The broken line is $2 \log r$.

<table>
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</tr>
</tbody>
</table>

Table 6.1: Numerical value for the masses of coincident vortices.

with the numerical value for ANO vortices obtained about 30 years ago by Jacobs and Rebbi \[113\]. As mentioned, this happens because the $(0, k)$-vortex is obtained by embedding of the $k$ ANO vortices.

There is another type of composite configuration which can easily be analyzed numerically. These configurations are generated by the following ansatz for $\Omega$ and $\tilde{H}$

$$
\Omega^{(1,1)} = \begin{pmatrix}
w_1(r) & 0 \\
0 & w_2(r)
\end{pmatrix}, \quad \tilde{H}^{(1,1)} = \begin{pmatrix}
f_1(r)z^{k_1} & 0 \\
0 & f_2(r)z^{k_2}
\end{pmatrix}.
$$

(6.2.33)

This ansatz corresponds to a configuration with $k_1$ composite vortices which wind in the first diagonal $U(1)$ subgroup of $U(2)$ and with $k_2$ coincident vortices that wind the second diagonal $U(1)$ subgroup. The two sets of vortices
can be considered each as embedded ANO vortices for the two decoupled Abelian subgroups. We refer to these decoupled non-Abelian vortices as a “($k_1, k_2$)-vortex”. The mass of a ($k_1, k_2$)-vortex is thus the sum of the mass of the ($k_1, 0$)-vortex and that of the ($0, k_2$)-vortex. For example, the mass of ($1, 1$)-vortex is double of the mass of the ($0, 1$)-vortex listed in the first column of Table (6.1). As in the previous case, we get the minima of the energy under the constraint that the vortices are coincident.

![Figure 6.9: Spectrum of the (0, 2) and (1, 1) coincident vortices.](image)

Because our fine-tuned non-Abelian model is a simple extension of the Abelian-Higgs model, we expect similar behavior for the interactions. Actually we have only one parameter $\lambda$. Thus we will call the non-Abelian vortices for $\lambda < 1$ Type I, while they will be called Type II for $\lambda > 1$. From Fig. (6.9), in which is summarized the relevant data of Table (6.1), we can argue which kind of interaction appears between two non-Abelian vortices. In the Type I case, the ($0, 2$)-vortex is energetically preferred to the ($1, 1$)-vortex, while in Type II case the ($1, 1$)-vortex is preferred. If the two vortices are separated sufficiently, we can ignore any interaction between them. Regardless of their orientations, the mass of two well separated vortices is twice that of the single vortex. This mass is equal to the mass of the ($1, 1$)-vortex. Furthermore, it seems that the two separated anti-parallel vortices do not interact, and the energy does not depend on the relative distance. For the Type I case, Fig. (6.9) suggests that the configuration with ($2, 0$)-vortices is the true minimum of the system. This means that attractive forces appears between vortices with different orientations. An attractive force also works in the internal space, which aligns the orientations. In the Type II case, it seems that we do not have an isolated minimum of the energy. In fact, all the anti-parallel configurations with arbitrary distance have the same value of the energy. In the following Sections we will confirm the picture we have
Effective potential for coincident vortices

The dynamics of BPS solitons can be investigated by the so-called moduli approximation [96]. The effective action is a massless non-linear sigma model whose target space is the moduli space. The sigma model is obtained by plugging a BPS solution into the original Lagrangian and promoting the moduli parameters to massless fields, then picking up quadratic terms in the derivatives with respect to the vortex world-volume coordinates

\[ L = \int dx^1 dx^2 \mathcal{L} [H_{\text{sol}}(\varphi_i(t, x^3)), W_{\text{sol}}^\mu(\varphi_i(t, x^3))]_{\lambda=1}, \quad (6.2.34) \]

where \( \varphi_i \) represents the set of moduli parameters \( (\eta, \tilde{\eta}, \phi, \tilde{\phi}) \) or \( (\alpha', \beta', a', b') \) contained in the moduli matrix [6.2.27].

If the coupling constant \( \lambda \) is close to the BPS limit \( \lambda = 1 \), we can still use the moduli approximation, to investigate dynamics of the non-BPS non-Abelian vortices by adding a potential of order \(|1 - \lambda^2| \ll 1\) to the massless sigma model

\[ L = \int dx^1 dx^2 \mathcal{L} [H_{\text{sol}}(\varphi_i(t, x^3)), W_{\text{sol}}^\mu(\varphi_i(t, x^3))]_{\lambda=1} - V(\varphi_i). \quad (6.2.35) \]

We shall now calculate the effective potential \( V(\varphi_i) \) using the method suggested by Hindmarsh, who calculated this effective potential for non-BPS semi-local vortex in the Abelian-Higgs model [128].

First we write the Lagrangian (6.2.11) in the following way

\[ \tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\text{BPS}} + \frac{\lambda^2 - 1}{4} (1_N - HH^\dagger)^2. \quad (6.2.36) \]

We get non-BPS corrections of order \( O(\lambda^2 - 1) \) by putting BPS solutions into Eq. (6.2.36). This is because the first term is minimized by the BPS solution, while the second one is already a term of order \( O(\lambda^2 - 1) \). The energy functional thus takes the following form

\[ \mathcal{E} = \frac{E}{2\pi v^2} = 2 + \frac{(\lambda^2 - 1)}{8\pi} \int dx^1 dx^2 \text{Tr} \left( 1 - H_{\text{BPS}}(\varphi_i) H_{\text{BPS}}^\dagger(\varphi_i) \right)^2 \quad (6.2.37) \]

where \( H_{\text{BPS}}(\varphi_i) \) stands for the BPS solution generated by the moduli matrices in Eq. (6.2.27). The first term corresponds to the mass of two BPS vortices.
and the second term is the deviation from the BPS solutions which is nothing but the effective potential we want.

In this Section we consider the effective potential on the moduli space of coincident vortices. To this end, it suffices to consider only the following matrices

\[ H_0^{(1,1)} = \begin{pmatrix} z & -\eta \\ 0 & z \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a'z & 1 \end{pmatrix}. \]

(6.2.38)

The parameters \( \eta \) and \( a' \) are related by \( \eta = 1/a' \). The effective potential on the moduli space for two coincident vortices is thus

\[
\frac{V(|\eta|)}{2\pi v^2} = \frac{(\lambda^2 - 1)}{8\pi} \int dx^1 dx^2 \text{Tr} \left( 1 - H_{\text{BPS}}(|\eta|) H_{\text{BPS}}^\dagger(|\eta|) \right)^2 \equiv (\lambda^2 - 1) \mathcal{V}(|\eta|),
\]

(6.2.39)

where we have defined a reduced effective potential \( \mathcal{V} \) which is independent of \( \lambda \). To evaluate this effective potential, we need to solve the BPS equations for a composite state of two non-Abelian vortices with an intermediate value of \( \eta \). Such numerical solutions found in [91]. We propose here another reliable technique for the numerics, which needs much less algebraic efforts.

In the moduli matrix formalism, what we should solve is only the master equation (6.2.23) for \( \Omega \) with \( \lambda = 1 \) and \( \hat{H}(z, \bar{z}) = H_0(z) \). Because of the axial symmetry of the composite vortex and the boundary condition at infinity:

\[
\Omega \to H_0(z) H_0^\dagger(\bar{z}),
\]

(6.2.40)

\footnote{The potential depends only on \( |\eta| \) because the phase of \( \eta \) can be absorbed by the flavor symmetry. The same holds for the coordinate \( a' \).}

Figure 6.10: Left: Numerical plots of \( Y_1 \) (red), \( Y_2 \) (green) and \( Y_3 \) (blue) for \( \eta = 3 \). The black line is \( Y_1 = Y_3 \) for \( \eta = 0 \). Right: Magnetic flux \( \text{Tr} F_{12} \) with \( \eta = 0 \) (red), \( \eta = 3 \) (green) and \( \eta = \infty \) (\( a' = 0 \)) (blue).
we can make a simple ansatz for $\Omega$. For example in the patch $(1, 1)$ we can write

$$
\Omega^{(1,1)} = \left( \begin{array}{ccc}
    w_1(r) & -\eta e^{-i\theta} w_2(r) \\
    -\eta e^{i\theta} w_2(r) & w_3(r)
\end{array} \right) \tag{6.2.41}
$$

The advantage of the moduli matrix formalism is that only three functions $w_i(r)$ are needed and the formalism itself is gauge invariant. Plugging the ansatz into Eq. (6.2.23), after some algebra we get the following differential equations

\begin{align*}
Y_1'' + \frac{1}{r} Y_1' + \frac{|\eta|^2 (1 + r Y_1' - r Y_2')^2}{|\eta|^2 - e^{Y_1+Y_3-2Y_2}} &= 1 - e^{-Y_1}(r^2 + |\eta|^2); \\
Y_2'' + \frac{1}{r} Y_2' - \frac{1}{r^2} \frac{(1 + r Y_1' - r Y_2')(1 + r Y_2' - r Y_3')}{1 - |\eta|^2 e^{-Y_1-Y_3+2Y_2}} &= 1 - e^{-Y_2} r; \\
Y_3'' + \frac{1}{r} Y_3' + \frac{|\eta|^2 (1 + r Y_2' - r Y_3')^2}{r^2 |\eta|^2 - e^{Y_1+Y_3-2Y_2}} &= 1 - e^{-Y_3} r^2, \tag{6.2.42}
\end{align*}

where we have redefined the fields as $w_i(r) = e^{Y_i(r)}$ with $i = 1, 2, 3$. We solve numerically these differential equations using a simple relaxation method, see Fig. (6.10).

The effective potential can be obtained by plugging numerical solutions into Eq. (6.2.39). The result is shown in Fig. (6.11). When we consider configurations with big values of $|\eta|$, it is better to switch to the other patch and use the variable $a'$. We can still make a similar ansatz for $\Omega$, leading to simple differential equations like those in Eqs. (6.2.42). In the left of Fig. (6.11) we show a numerical plot of the reduced effective potential $V(|\eta|)$ from $|\eta| = 0$ ($|a'| = \infty$) to $|\eta| = 20$. In the right of Fig. (6.11) we show another

![Figure 6.11: Numerical plots of the effective reduced potential $V(|\eta|)$. In the second plot $a' = 1/\eta$.](image)
plot from $|a'| = 0$ ($|\eta| = \infty$) to $|a'| = 1/20$. To give an estimate of the range of validity of our approximation we can compare the results obtained in this Section with the numerical integrations obtained for $(2,0)$-vortices and also $(1,1)$-vortices. The comparison is made in Table (6.2) from which we can argue that the effective potential approximation gives result with an accuracy around 10% for the range of values $0.7 < \lambda < 1.15$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
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<th>$(2,0)_{\text{eff}}$</th>
<th>$(1,1)_{\text{num}}$</th>
<th>$(1,1)_{\text{eff}}$</th>
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<td>1.92115</td>
</tr>
<tr>
<td>0.95</td>
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</tr>
<tr>
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<td>2.08106</td>
<td>2.08715</td>
</tr>
<tr>
<td>1.15</td>
<td>2.15843</td>
<td>2.17496</td>
<td>2.12018</td>
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</tr>
<tr>
<td>1.2</td>
<td>2.20944</td>
<td>2.23870</td>
<td>2.15843</td>
<td>2.18260</td>
</tr>
</tbody>
</table>

Table 6.2: Numerical value for the masses of coincident vortices. $(2,0)_{\text{num}}$ is for the numerical results while $(2,0)_{\text{eff}}$ is for our approximation using the effective potential.

In the Type II case ($(\lambda^2 - 1) > 0$) the effective potential has the same qualitative behavior as showed in the figure. As we expected, it has a minimum at $|\eta| = 0$. This matches the previous result that the $(1,1)$-vortex is energetically preferred to the $(2,0)$-vortex. In the Type I case ($(\lambda^2 - 1) < 0$) the shape of the effective potential can be obtained just by flipping the overall sign of the effective potential of the Type II case. Then the effective potential always takes a negative value, which is consistent with the fact that the masses of the Type I vortices are less than that of the BPS vortices, see the middle column in Table (6.1). Contrary to the Type II case, the Type I potential has a minimum at $|a'| = 0$ ($|\eta| = \infty$). This means that the $(2,0)$-vortex is preferred with respect to the $(1,1)$ vortex.

**Interaction at generic vortex separation**

In this subsection we go on investigating the interactions of non-Abelian vortices in the $U(2)$ gauge group at generic distances. As in Sect. (6.2.3) we will use the moduli space approximation, considering only small deviations from the BPS case. The generic configurations are described by the moduli matrices in Eq. (6.2.27). We will consider here only the reduced $(1,1)$ patch
defined in Eq. (6.2.28). By putting the two vortices on the real axis we can reduce \( z_0 \) to a real parameter \( d \). Furthermore, by the flavor symmetry, we can freely put \( \tilde{\eta} = 0 \) and suppress the phase of \( \eta \). The relevant configurations will be described by the following moduli matrix:

\[
H_0^{(1,1)}_{\text{red}} = \begin{pmatrix}
  z - d & -\eta \\
  0 & z + d
\end{pmatrix}, \tag{6.2.43}
\]

where \( 2d \) is the relative distance and \( \eta \) the relative orientation.

Figure 6.12: The magnetic flux \( \text{Tr} \ F_{12} \) of a configuration of 2 BPS vortices. The figure shows slices including the centers of the vortex. (red, green, blue) correspond to \( \eta = (0, 4, \infty) \) while (solid, small broken, wide broken) lines to \( d = (0, 1, 4) \). The left panel shows the fine-tuned model with \( e = g \) and the right shows the model with \( e = 2g \).

Now let us study the effective potential as function of \( \eta \) and \( d \). We first need the numerical solution to the BPS master equation for two vortices with any relative distance and orientation. Unlike the computation for the coincident vortices, we do not have an axial symmetry. We can no longer reduce the problem to one spatial dimension by making an appropriate ansatz. Nevertheless the moduli matrix formalism is a powerful tool also for the numerical calculations. The master equation is a 2 by 2 hermitian matrix, so it includes four real 2nd order partial differential equations. Despite the great complexity of this system of coupled equations, the relaxation method is very effective to solve the problem. We show several numerical solutions in Fig. (6.12). As before, once we get the numerical solution to the BPS equations, the effective potential is obtained by plugging them into Eq. (6.2.37).

The numerical plot of the reduced effective potential \( V \) is shown in Fig. (6.13). The effective potential for the Type II case has the same shape, up to a small positive factor \((\lambda^2 - 1)\). The potential forms a hill whose top is at
((d, |η|) = (0, ∞). It clearly shows that two vortices feel repulsive forces, in both the real and internal space, for every distance and relative orientation. The minima of the potential has a flat direction along the d-axis where the orientations are anti-parallel (η = 0) and along the η axis at infinite distance (d = ∞). Therefore the anti-parallel vortices do not interact.

![Figure 6.13: Numerical plot of the reduced effective potential V(η, d).](image)

In the Type I case (λ < 1) the effective potential is upside-down of that of the Type II case. There is unique minimum of the potential at (d, |η|) = (0, ∞). This means that attractive force works not only for the distance in real space but also among the internal orientations. Configurations with anti-parallel orientations do not interact, but these configurations represent unstable points of equilibrium. Type I vortices always stick together.

### 6.2.4 Vortices with generic couplings

In this Section we will shift from the fine-tuned U(N) model \((6.2.1)\) to the more general model defined in Eqs. \((6.2.6)\) and \((6.2.7)\). This will lead to more complicated algebra, but we will also clarify interactions with different qualitative behaviors. There are 5 parameters \((g, e, λ_g, λ_e, v)\) in the model, but we can reduce their number with the following rescaling

\[
H \rightarrow vH, \quad W_\mu \rightarrow evW_\mu, \quad x_\mu \rightarrow \frac{x_\mu}{ev}.
\] (6.2.44)
Then the Lagrangian (6.2.6) is expressed as follows

$$
\mathcal{L} = \text{Tr} \left[ -\frac{1}{2\gamma^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - \frac{1}{2} f_{\mu\nu} f^{\mu\nu} + D_\mu (D^\mu H) \right] - \frac{\gamma^2 \lambda^2}{4} \text{Tr} \hat{X}^2 - \frac{\lambda^2}{4} \text{Tr} (X^0 t^0 - 1_N)^2, \tag{6.2.45}
$$

where $\mathcal{L} = \mathcal{L}/e^2 v^4$ and we have introduced the ratio of the two gauge couplings

$$
\gamma \equiv \frac{g}{e}. \tag{6.2.46}
$$

We have three effective couplings $\gamma, \lambda_e, \lambda_g$ and the rescaled masses of particles are

$$
M_{U(1)} = 1, \quad M_{SU(N)} = \gamma, \quad M_s = \lambda_e, \quad M_{ad} = \gamma \lambda_g. \tag{6.2.47}
$$

The Lagrangian can be thought of as the bosonic part of a supersymmetric model only when both the parameters $\lambda_e$ and $\lambda_g$ are unity. In the BPS case we can easily find the BPS equations

$$
\bar{D} H = 0, \quad \hat{F}_{12} = \frac{\gamma^2}{2} \hat{X}, \quad F_{12}^0 t^0 = \frac{1}{2} (X^0 t^0 - 1_N). \tag{6.2.48}
$$

The last two equations can be rewritten in a compact way

$$
F_{12} = \frac{1}{2} (X - 1_N) + \frac{\gamma^2 - 1}{2} \hat{X}. \tag{6.2.49}
$$

The first BPS equation in Eq. (6.2.48) can be solved using the moduli matrix in the usual way

$$
H = S^{-1}(z, \bar{z}) H_0(z), \quad \bar{W} = -i S^{-1} \partial S, \tag{6.2.50}
$$

where $S$ is a $GL(N, \mathbb{C})$ matrix. We would like to stress that this solution does not depend on $\gamma$ so that the moduli space of the BPS vortices is the same as that of the well investigated vortices in the equal gauge coupling theory $g = e$. The Eq. (6.2.49) can be rewritten in a gauge invariant fashion as

$$
\partial (\Omega \partial^{-1} \Omega^{-1}) = \frac{1}{4} (\Omega_0 \Omega^{-1} - 1_N) + \frac{\gamma^2 - 1}{4} \left( \Omega_0 \Omega^{-1} - \frac{\text{Tr} (\Omega_0 \Omega^{-1})}{N} 1_N \right),
$$

where $\Omega = SS^\dagger$ is same as before and $\Omega_0 \equiv H_0 H_0^\dagger$. 

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Now we are ready to investigate interactions between two almost BPS vortices by using the same strategy as we used in Sec. 4. An effective action of the moduli dynamics for appropriately small $|1 - \lambda^2| \ll 1$ is obtained by plugging BPS solutions into the action. Then we get

$$V(\eta, d; \gamma, \lambda_e, \lambda_g) = \frac{1}{2} \int d\tilde{x}^2 \left( \frac{\gamma^2(\lambda^2 - 1)}{8\pi} \text{Tr} \tilde{X}^2 + \frac{\lambda^2 - 1}{8\pi} \text{Tr} \left( X^0t^0 - 1_N \right)^2 \right)$$

where we have used the BPS equations in the second line. Let us define the Abelian and the non-Abelian potentials as

$$V_e(\eta, d; \gamma) = \int d\tilde{x}^2 \text{Tr} (F_1 t_1)^2, \quad V_g(\eta, d; \gamma) = \int d\tilde{x}^2 \text{Tr} (\hat{F}_1)^2. \quad (6.2.52)$$

The true potential is a linear combination of them

$$V(\eta, d; \gamma, \lambda_e, \lambda_g) = (\lambda^2 - 1)V_e(\eta, d; \gamma) + \frac{\lambda^2 - 1}{\gamma^2} V_g(\eta, d; \gamma). \quad (6.2.53)$$

Notice that $V_{e,g}$ is determined by the BPS solutions, so it does not depend on $\lambda_{e,g}$.

**Equal gauge coupling $\gamma = 1$ revisited**

In Sec. 4 we have discussed the effective potential for two vortices for any separation and with any relative orientation in a model with $\gamma = 1$ and $\lambda = \lambda_g = \lambda_e$. The potential is shown in Fig. (6.13). The effective potential comes from two pieces: the Abelian $V_e$ and the non-Abelian $V_g$ potential given in Eq. (7.2). In Figs. (6.14) and (6.15) we show $V_e$ and $V_g$ taking several slices of Fig. (6.13).

Let us consider the case with $\lambda^2 e^2 - 1 > 0$ and $\lambda^2 g - 1 > 0$. In this case the effective potential will have the same qualitative behaviors like the reduced potentials in the Figs. (6.14), (6.15) and (6.16). The figures show how $V_e$ and $V_g$ behaves very differently. In particular, the Abelian potential is always repulsive, both in the real and internal space (see the red dots in Figs. (6.14) and (6.15)). The non-Abelian potential is on the contrary sensitive on the orientations. In particular, Fig. (6.14) shows that it is repulsive for parallel

---

8. Attraction in the internal space here just means that orientations tend to become the same.
vortices while it is attractive for anti-parallel ones. Furthermore, the non-Abelian potential becomes almost flat (along the spatial coordinate $d$) with the orientation $\eta \sim 4$, see the middle of Fig. (6.14). The blue dots in Fig. (6.15) reveal that the non-Abelian potential always gives attractive forces in the internal space. When the two scalar couplings are equal, $\lambda^2_e = \lambda^2_g$, the left picture in Fig. (6.14) clearly shows how the two potentials exactly cancel for anti-parallel vortices, recovering the result of the previous Section.

Of course, the true effective potential depends on $\lambda_e$ and $\lambda_g$ through the combination in Eq. (6.2.53). This indicates the interaction between non-Abelian vortices is quite rich in comparison with that of the ANO vortices. Let us make a further example. The rightmost panel in Fig. (6.14) shows that $V_e$ and $V_g$ for parallel vortices ($\eta = \infty$) are identical. Thus if we take $\lambda^2_e + \lambda^2_g = 2$, the interactions between parallel vortices vanishes while between anti-parallel vortices they do not canceled out. This is just opposite to what we found in the case $\lambda^2_e = \lambda^2_g$. We summarize the possible behaviors of the Abelian and non-Abelian forces, when the gauge couplings are equal $\epsilon = g$

\footnote{This statement can be proved analytically.}
Figure 6.16: The Abelian potential $V_e$ (left) and the non-Abelian potential $V_g$ (right) for $\gamma = 1$.

<table>
<thead>
<tr>
<th>$\lambda^2_g &gt; 1$</th>
<th>$\lambda^2_g = 1$</th>
<th>$\lambda^2_g &lt; 1$</th>
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<tr>
<td>$\lambda^2_e &gt; 1$</td>
<td>$N=0, A=+$</td>
<td>$N=0, A=+$</td>
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<tr>
<td>$\lambda^2_e &lt; 1$</td>
<td>$N=0, A=+$</td>
<td>$N=0, A=+$</td>
</tr>
</tbody>
</table>

Table 6.3: The forces between two vortices in the $\gamma = 1$ case. $N$ and $A$ stand for non-Abelian force and Abelian force, respectively. $0$ means no force, $+$ means repulsive and $-$ means attractive. $N=(-, +)$ means that there is an attractive force for anti-parallel vortices and a repulsive force for parallel vortices.

$(\gamma = 1)$, in Table 6.3).

**Different gauge coupling $\gamma \neq 1$**

We now consider interactions between non-Abelian vortices with different gauge coupling $e \neq g$ ($\gamma \neq 1$). Several numerical solutions are given in the left panel of Fig. (6.12). In Figs. (6.17) and (6.18) we show two numerical examples for the reduced effective potentials $V_e, V_g$ given in Eq. (7.2).

The plots show that the qualitative features of $V_e$ and $V_g$ are basically the same as what is discussed in the equal gauge coupling case ($\gamma = 1$). Therefore, the qualitative classification of the forces given in Table 6.3 is still valid for $\gamma \neq 1$. We observe that the Abelian potential tends, at large distances, to a value smaller than the non-Abelian one for $\gamma < 1$, while opposite happens for $\gamma > 1$. The only things that have non dependence
Figure 6.17: Effective potential with $\gamma = 1/2$ vs. separation. (red,blue)=$(V_e,V_g)$.

Figure 6.18: Effective potential with $\gamma = 1.3$ vs. separation. (red,blue)=$(V_e,V_g)$.

on $\gamma$ are the values of the potentials at $(1,1)$-vortices with $(d,\eta) = (0,0)$. Regardless of the gauge couplings $V_g(\eta = 0, d = 0) = 0$ while $V_e(\eta = 0, d = 0) \approx 0.41$. This is because the corresponding solution is proportional to the unit matrix, so that there are no contributions from the non-Abelian part.

The effective potential is obtained from the linear combination in Eq. (6.2.53) and also depends on three parameters $\gamma$, $\lambda_e$ and $\lambda_g$. With this big freedom we can obtain a lot of interesting interactions. For example, we can have potentials which develop a global minimum at some finite non zero distance. In such cases two vortices may be bounded at that distance. We can show a concrete potential in Fig. (6.19). The figure shows the presence of a minimum around $d \sim 2^{10}$. This kind of behavior have not been found for the ANO Type I/II vortices and the possibility of bounded vortices really results from the non-Abelian symmetry$^{11}$.

$^{10}$These kind of potentials are really generic. In fact, the presence of this minima does not require strong constraints on the couplings.

$^{11}$Similar behaviors in the static intervortex potential were also observed for $\mathbb{Z}_3$ vortices in $^{129}$. 

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Figure 6.19: $\gamma = 1/2$, $\lambda_e = 1.2$, $\lambda_g = 1.06$: From $\eta = 0$ (green) to $\eta = 7$ (blue) with $d = 0 \sim 5$ for each $\eta$.

### 6.2.5 Interaction at large vortex separation

**Vortices in fine-tuned models** $e = g$ and $\lambda_e = \lambda_g$

In this subsection we will obtain an analytic formula for the asymptotic forces between vortices at large separation. We follow the technique developed in Refs. 115, 17. First of all, we need to find asymptotic behaviors of the scalar and the gauge fields. We again consider the $(1, 0)$-vortex

$$H_0(z)^{(1,0)} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{\phi}_1^{(1,0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (6.2.54)$$

When we are sufficiently far from the core of the vortex, $Y(r)$ and $f(r)$ in Eqs. (6.2.31) can be written as

$$Y = 2 \log r + \delta Y, \quad f = 1 + \delta f \quad (6.2.55)$$

where $\delta Y$ and $\delta f$ are small quantities. Plugging these into Eqs. (6.2.31) and taking only linear terms in $\delta Y$ and $\delta f$, we obtain the following linearized equations

$$\delta Y'' + \frac{1}{r} \delta Y' - \lambda^2 \delta Y - 2 \left( \delta f'' + \frac{1}{r} \delta f' - \lambda^2 \delta f \right) = 0, \quad (6.2.56)$$

$$\delta Y''' + \frac{1}{r} \delta Y'' - \frac{1}{r^2} \delta Y' - \delta Y' = 0. \quad (6.2.57)$$

Solutions to these equations are analytically obtained to be

$$\delta Y - 2 \delta f = -\frac{q}{\pi} K_0(\lambda r), \quad \delta Y' = \frac{m}{\pi} K_0(r) - C, \quad (6.2.58)$$
where \( K_0(r) \) is the modified Bessel’s function of zeroth order and \( q, m \) and \( C \) are integration constants. \( C \) must be 0 because \( \delta Y \to 0 \) as \( r \to \infty \) while \( q, m \) should be determined by the original equation of motion. In the BPS case, \( \delta f \equiv 0 \) \((f \equiv 1)\), thus \( q = -m \). Eq. (6.2.58) leads to the well known asymptotic behavior of the ANO vortex

\[
H_{[1,1]} = f e^{-\frac{Y}{2}z} \simeq \left(1 + \delta f - \frac{1}{2} \delta Y\right) e^{i\theta} = \left(1 + \frac{q}{2\pi} K_0(\lambda r)\right) e^{i\theta}, \quad (6.2.59)
\]

\[
\bar{W}_{[1,1]} = -\frac{i}{2} \bar{\partial}Y \simeq -\frac{i}{4} e^{i\theta} \frac{d}{dr} \left(2 \log r + \delta Y\right) = -\frac{i}{2} \left(\frac{1}{r} - \frac{m}{2\pi} K_1(\lambda r)\right) e^{i\theta}, \quad (6.2.60)
\]

where \( K_1 \equiv -K'_0 \) and we have defined \( H_{[1,1]} \) and \( \bar{W}_{[1,1]} \) as \([1,1]\) elements of \( H \) and \( \bar{W} \) in Eq. (6.2.15) with the \( k = 1 \) ansatz (6.2.30).

Next we treat the vortices as point particles in a linear field theory coupled with a scalar source \( \rho \) and a vector current \( j^{\mu} \). To linearize the Yang-Mills-Higgs Lagrangian, we choose a gauge such that the Higgs fields is given by the following hermitian matrix

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} h^0 + h^3 & h^1 - ih^2 \\ h^1 + ih^2 & h^0 - h^3 \end{pmatrix}, \quad W^{\mu} = \frac{1}{2} \begin{pmatrix} w^0 + w^3 & w^1 - iw^2 \\ w^1 + iw^2 & w^0 - w^3 \end{pmatrix}. \quad (6.2.61)
\]

Here all the fields \( h^a, w^{a\mu} \) are real. Then the quadratic part of the Lagrangian (6.2.11) is of the form

\[
\mathcal{L}^{(2)}_{\text{free}} = \sum_{a=0}^{3} \left[-\frac{1}{4} f^{a}_{\mu\nu} f^{a \mu\nu} + \frac{1}{2} w^a_{\mu} w^{a\mu} + \frac{1}{2} \bar{\partial}^a h^a \bar{\partial}^\mu h^a - \frac{\lambda^2}{2} (h^a)^2 \right] \quad (6.2.62)
\]

where we have defined the Abelian field strength \( f^{a}_{\mu\nu} \equiv \bar{\partial}_\mu w^a_{\nu} - \bar{\partial}_\nu w^a_{\mu} \). We also take into account the external source terms to realize the point vortex

\[
\mathcal{L}_{\text{source}} = \sum_{a=0}^{3} \left[\rho^a h^a - j_{\mu}^a w^{a\mu}\right]. \quad (6.2.63)
\]

The scalar and the vector sources should be determined so that the asymptotic behavior of the fields in Eqs. (6.2.59) and (6.2.60) are replicated. Equations of motions are of the form

\[
\left(\Box + \lambda^2\right) h^a = \rho^a, \quad \left(\Box + 1\right) w^a_{\mu} = j^a_{\mu}. \quad (6.2.64)
\]

In order to replicate the \((1,0)\)-vortex corresponding to the \( k = 1 \) ansatz (6.2.30), we just need to mimic the result of Refs. 115, 17 because the single
non-Abelian vortex is a mere embedding of the ANO vortex as mentioned earlier. In fact, only \((h^0, w^0, \rho^0, j^0_\mu) = (h^3, w^3, \rho^3, j^3_\mu)\) are relevant and all the others are zero:

\[
\begin{align*}
h^0 &= h^3 = \frac{q}{2\pi} K_0(\lambda r), \\
w^0 &= w^3 = -\frac{m}{2\pi} \hat{k} \times \nabla K_0(r), \\
\rho^0 &= \rho^3 = q \delta(r), \\
j^0 &= j^3 = -m \hat{k} \times \nabla \delta(r),
\end{align*}
\]

where \(\hat{k}\) is a spatial fictitious unit vector along the vortex world-volume. The vortex configuration with general orientation is also treated easily, since the origin of the orientation is the Nambu-Goldstone mode associated with the broken SU(2) color-flavor symmetry

\[
H_0 \rightarrow H_0(z)^{(1,0)} U_F, \quad \phi_2 = U_F^\dagger \phi_1^{(1,0)} = \left( \begin{array}{c} \alpha^* \\ \beta^* \end{array} \right), \quad U_F \equiv \left( \begin{array}{cc} \alpha & \beta \\ -\beta^* & \alpha^* \end{array} \right),
\]

where \(|\alpha|^2 + |\beta|^2 = 1\). The fields \(H\) and \(W_\mu\) receive the following transformations, keeping the hermitian form of (6.2.61).

\[
\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \rightarrow U_F^\dagger \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U_F = \begin{pmatrix} |\alpha|^2 & \alpha^* \beta \\ \alpha \beta^* & |\beta|^2 \end{pmatrix} X.
\]

The scalar interaction between a vortex at \(x = x_1\) with the orientation \(\vec{\phi}_1\) and another vortex at \(x = x_2\) with the orientation \(\vec{\phi}_2\) can be obtained by

\[
L_h = \int dx^2 \text{Tr} \left[ \left( \begin{array}{cc} h^0(x - x_1) & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} |\alpha|^2 & \alpha^* \beta \\ \alpha \beta^* & |\beta|^2 \end{array} \right) \rho^0(x - x_2) \right] = |\alpha|^2 \frac{q^2}{2\pi} K_0(\lambda |x_1 - x_2|).
\]

The gauge interaction is also obtained by similar way

\[
L_w = -\int dx^2 \text{Tr} \left[ \left( \begin{array}{cc} w^0(x - x_1) & 0 \\ 0 & 0 \end{array} \right) \cdot \left( \begin{array}{cc} |\alpha|^2 & \alpha^* \beta \\ \alpha \beta^* & |\beta|^2 \end{array} \right) j^0(x - x_2) \right] = -|\alpha|^2 \frac{m^2}{2\pi} K_0(|x_1 - x_2|).
\]

Then total potential is \(V_{\text{int}} = -L_h - L_w\)

\[
V_{\text{int}} = -\frac{|\vec{\phi}_1 \vec{\phi}_2|^2}{|\vec{\phi}_1|^2 |\vec{\phi}_2|^2} \left( \frac{q^2}{2\pi} K_0(\lambda |x_1 - x_2|) - \frac{m^2}{2\pi} K_0(|x_1 - x_2|) \right),
\]

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where \( |\alpha|^2 = \frac{|\phi_1 \phi_2|^2}{|\phi_1|^2 |\phi_2|^2} \) is invariant under the global color-flavor rotation.

When two vortices have parallel orientations, this potential becomes that of two ANO vortices \([115]\). On the other hand, the potential vanishes when their orientations are anti-parallel. This agrees with the numerical result found in the previous Sections. In the BPS limit \( \lambda = 1 \) (\( q = m \)), the interaction becomes precisely zero.

Since \( K_0(\lambda r) \sim \sqrt{\pi/2} \lambda r e^{-\lambda r} \), the potential asymptotically reduces to

\[
V_{\text{int}} \simeq \begin{cases} 
- \frac{|\phi_1 \phi_2|^2}{|\phi_1|^2 |\phi_2|^2} \frac{q^2}{2\pi} \sqrt{\frac{\pi}{2\lambda r}} e^{-\lambda r} & \text{for } \lambda < 1 \text{ Type I} \\
\frac{|\phi_1 \phi_2|^2}{|\phi_1|^2 |\phi_2|^2} \frac{m^2}{2\pi} \sqrt{\frac{\pi}{2r}} e^{-r} & \text{for } \lambda > 1 \text{ Type II}
\end{cases}
\]

where \( r \equiv |x_1 - x_2| \gg 1 \). If we fix the relative orientation being some finite value, the force \( F_r = -\partial_r V_{\text{int}} \) between two vortices is attractive for \( \lambda < 1 \) and repulsive for \( \lambda > 1 \) similar to the force between ANO vortices. The force vanishes when the relative orientation becomes anti-parallel. If we fix the distance by hand, the orientations tend to be anti-parallel for the Type II while the parallel configuration are preferred for the Type I case.

### Vortices with general couplings

It is quite straightforward to generalize the results of the previous Section to the case of generic couplings. We can of course use the same gauge as in Eq. (6.2.61). The quadratic Lagrangian (6.2.6) is of the form

\[
L^{(2)}_{\text{free}} = \sum_{a=1}^{3} \left[ -\frac{1}{4\gamma^2} f^{a\mu} f_{a\mu} + \frac{1}{2} w^{a}_{\mu} w^{a\mu} + \frac{1}{2} \partial_{\mu} h^{a} \partial^{\mu} h^{a} - \frac{\lambda^2 g^2}{2} (h^{a})^2 \right] + \left[ -\frac{1}{4} f^{0\mu} f^{0\mu} + \frac{1}{2} w^{0}_{\mu} w^{0\mu} + \frac{1}{2} \partial_{\mu} h^{0} \partial^{\mu} h^{0} - \frac{\lambda^2 e^2}{2} (h^{0})^2 \right].
\]

The external sources can be still reproduced by source terms as in Eqs. (6.2.63), (6.2.64). The linearized equations following from the above Lagrangian are of the form

\[
\left( \frac{1}{\gamma^2} \Box + 1 \right) w^{a}_{\mu} = j^{a}_{\mu}, \quad \left( \Box + \lambda^2 g^2 \gamma^2 \right) h^{a} = \rho^{a},
\]

\[
(\Box + 1) w^{0}_{\mu} = j^{0}_{\mu}, \quad \left( \Box + \lambda^2 e^2 \right) h^{0} = \rho^{0}.
\]

For the \((1,0)\)-vortex the only non-zero profile functions are \((h^0, w^0_\mu, \rho^0, j^0_\mu)\) and \((h^3, w^3_\mu, \rho^3, j^3_\mu)\). But these profiles are no longer equal and we need to
deal with them independently. The only difference from the similar equations (6.2.64) is for the masses of the particles. The masses are directly related to asymptotic tails of vector and scalar fields. We can easily find the solutions by doubling Eqs. (6.2.65)

\[
\begin{align*}
h^0 &= q^0 K_0(\lambda_e r), \\
\rho^0 &= q^0 \delta(r), \\
h^3 &= q^3 K_0(\lambda_g \gamma r), \\
\rho^3 &= q^3 \delta(r),
\end{align*}
\]

(6.2.74)

The vortex with the orientation \( \vec{\phi}_1 \) in Eq. (6.2.66) can be obtained by performing an SU(2)C+\( F \) rotation like Eq. (6.2.67)

\[
\left( \begin{array}{c} X^0 + X^3 \\ 0 \\ X^0 - X^3 \end{array} \right) \rightarrow \left( \begin{array}{c} X^0 \\ (|\alpha|^2 - |\beta|^2) \frac{X^3}{2} \\ (\alpha^* \beta - \beta^* \alpha) \frac{X^0}{2} + (\alpha^* \beta + \beta^* \alpha^*) \frac{X^3}{2} \\ (\alpha^* \beta^* - \beta^* \alpha^*) \frac{X^0}{2} + (\alpha^* \beta + \beta^* \alpha^*) \frac{X^3}{2} \\ -(|\alpha|^2 - |\beta|^2) \frac{X^3}{2} \end{array} \right).
\]

Similar to Eqs. (6.2.68) and (6.2.69), we find the total potential \( V_{\text{int}} \)

\[
V_{\text{int}} = \frac{1}{2} \left( -\frac{(q^0)^2}{2\pi} K_0(\lambda_e |x_1 - x_2|) + \frac{(m^0)^2}{2\pi} K_0(|x_1 - x_2|) \right) + \\
+ \left( \frac{1}{2} - \frac{(q^3)^2}{2\pi} K_0(\lambda_g \gamma |x_1 - x_2|) + \frac{(m^3)^2}{2\pi} K_0(\gamma |x_1 - x_2|) \right).
\]

When we tune the parameters to \( \gamma = 1 \) and \( \lambda_g = \lambda_e \) (\( q^0 = q^3 \), \( m^0 = m^3 \)), this effective potential is exactly identical to that of Eq. (6.2.70). In the BPS limit \( \lambda_e = \lambda_g = 1 \), the interaction becomes precisely zero because \( q^0 = m^0 \) and \( q^3 = m^3 \).

At large distance, the interactions between vortices are dominated by the
particles with the lowest mass $M_{\text{low}}$. There are four possible regimes

$$V_{\text{int}} = \begin{cases} 
-\frac{(q^0)^2}{4\pi} \sqrt{\frac{\pi}{2\lambda_e r}} e^{-\lambda_e r} & \text{for } M_{\text{low}} = M_s, \quad \text{Type I} \\
- \left( \frac{\phi_1^* \phi_2}{|\phi_1| |\phi_2|} \right)^2 - \frac{1}{2} \left( \frac{(q^0)^2}{4\pi} \sqrt{\frac{\pi}{2\lambda_e \gamma r}} e^{-\lambda_e \gamma r} \right) & \text{for } M_{\text{low}} = M_{\text{ad}}, \quad \text{Type I}^* \\
\frac{(m_0)^2}{4\pi} \sqrt{\frac{\pi}{2\gamma r}} e^{-\gamma r} & \text{for } M_{\text{low}} = M_{\text{U}(1)}, \quad \text{Type II} \\
\left( \frac{\phi_1^* \phi_2}{|\phi_1| |\phi_2|} \right)^2 - \frac{1}{2} \left( \frac{(m_3)^2}{4\pi} \sqrt{\frac{\pi}{2\gamma r}} e^{-\gamma r} \right) & \text{for } M_{\text{low}} = M_{\text{SU}(2)}, \quad \text{Type II}^* 
\end{cases}$$

(6.2.75)

which exhaust all the possible kinds of asymptotic potentials of this system. This generalizes the Type I/II classification of Abelian superconductors. We have found two new categories, called Type I$^*$ and Type II$^*$, in which the force can be attractive or repulsive depending on the relative orientation. The Type I force is always attractive and the Type II force is repulsive regardless of the relative orientation. On the other hand, Type I$^*$ and Type II$^*$ depend on the relative orientation. In the Type I$^*$ case the forces between parallel vortices are attractive while anti-parallel vortices repel each other. The Type II$^*$ vortices feel opposite forces to the type I$^{12}$. Note that we have used the same terms Type I/II for the fine-tuned model in sect. (6.2.5). In the perspective of this Section, they should be called Type I$^*$/I$^*$ because of the degeneracy of some masses. It is interesting to compare these results with the recently studied asymptotic interactions between non-BPS non-Abelian global vortices $^{130, 124, 125}$. A very different feature of global vortices is that the interactions are always repulsive. This is because they are mediated by Nambu-Goldstone zero modes, whereas in our model these particles are all eaten by the gauge bosons thanks to the Higgs mechanism.

We find a nice matching of qualitative features between the numerical results of the previous Sections and the semi-analytical results in this Section. Let us look at Figs. (6.14), (6.17) and (6.18). In all the cases, we found that the Abelian potentials are attractive regardless of the orientations while the non-Abelian potentials are sensitive to those. These properties are well shown also in the semi-analytical results in (6.2.75). The Type I/II interactions originated by the $U(1)$ part are independent of the orientations whereas the Type I$^*$/II$^*$ which are coming from the $SU(N)$ part do depend on them.

$^{12}$In the previous Section we have found a similar result in a supersymmetric theory. The relation between the two notations is explained in the Appendix. The supersymmetric theory shows only type I/I$^*$ behaviors.
The result in Eq. (6.2.75) is easily extended to the general case of $U(1) \times SU(N)$. This can be done by just thinking of the orientation vectors $\phi$ as taking values in $CP^{N-1}$.

6.2.6 Summary

In this Section we have studied static interactions between non-BPS vortices in $SU(N) \times U(1)$ gauge theories with Higgs fields in the fundamental representation. We have discussed models with arbitrary gauge and scalar couplings. We have numerically computed the effective potential for almost BPS configurations for arbitrary separations and any internal orientations. We have also obtained analytic expressions for the static forces between well separated non-BPS vortices. This expression is valid also for models far from BPS limit.

For the fine-tuned model we found interaction pattern similar to that of the ANO vortices in the Abelian-Higgs model. The numerical effective potential is given in Fig. (6.13), it depends on both the relative distance and the orientations. The asymptotic potential between two vortices is given by Eq. (6.2.71). In this model the mass of the $U(1)$ and of the $SU(N)$ vector bosons are same $M_{SU(N)} = M_{U(1)}$, and also all the scalars have the same masses $M_s = M_{ad}$. We thus have only two mass scales, which corresponds to two different asymptotic regimes. For $\lambda < 1$ ($M_s < M_{U(1)}$) there is universal attraction (Type I) and for $\lambda > 1$ ($M_s > M_{U(1)}$) universal repulsion (Type II). Both the numerical and the analytical result show that the interactions between two anti-parallel vortices vanish; this configuration is unstable for Type I vortices and stable for Type II. So in this last case the part of the moduli space which corresponds to vortices with opposite $CP^1$ orientations at arbitrary distance survives the non-BPS perturbation.

In models with arbitrary couplings, on the other hand, the pattern of interactions becomes richer. In this case we considered separately the Abelian and non-Abelian contributions $V_a$ and $V_g$ to the effective potential. The two show very different qualitative behavior. While the Abelian contribution is always attractive (or repulsive) for a given choice of the parameters, the non-Abelian one can be attractive or repulsive depending on the relative internal orientation of the two vortices. These properties, combined with the fact that the full effective potential is the linear combination given in Eq. (6.2.53), give rise to many qualitatively different type of interactions depending on the choice of the parameters of the theory. Such a variety also appears in the possible asymptotic behavior of the interactions. In the theory there are four different mass scales, the masses of the $SU(N)$ vector bosons $M_{SU(N)}$, the $U(1)$ vector boson $M_{U(1)}$, the adjoint scalars $M_{ad}$ and the singlet scalar
Ms under the color-flavor symmetry. This leads to four different asymptotic regimes, classified in Eq. (6.2.75). We found, in addition to Type I and Type II, new types of interaction mediated by the non-Abelian particles, which we call Type I* and Type II*. The Type I (Type II) force is attractive (repulsive), and occurs when the singlet scalar (U(1) vector boson) has the smallest mass. These forces do not depend on the relative orientation. In the Type I* case there is an attractive force for parallel orientations and repulsive for anti-parallel ones where the asymptotic force is mediated by the lightest non-Abelian scalar fields. On the other hand for Type II* there is repulsion for parallel orientation and attraction for anti-parallel ones and the force at large distance is mediated by the lightest non-Abelian vector field.

The dynamics of the interactions of the non-BPS non-Abelian vortices is quite rich. Let us give comments on some possible further directions:

- **Non-Abelian vortices and Abrikosov lattice:** In the usual Type II superconductor (the Abelian-Higgs model), if a large number of vortices penetrate a region of given area A, they will form a hexagonal lattice (Abrikosov lattice) rather than forming a square one. This is verified experimentally. If we consider the same for non-Abelian vortices, we expect that the property of the lattice can be quite different, say, lattice spacing and/or form can change. An interesting possibility is the appearance of phase transitions due to the change of the lattice structure when the density of vortices change. This eventuality is suggested by the possible presence of interactions which change with distances. For example in the case of Fig. (6.19) there are repulsive forces at short distances while at large distances they are attractive.

- **Quantum aspects:** In this paper we focused on the classical aspects of the interactions between non-Abelian vortices. In the theoretical set-up that we have discussed, the quantum aspects of the infrared physics of a single vortex are described by an effective bosonic \( \mathbb{C}P^{N-1} \) sigma model (the theoretical setting is in this sense similar to the one discussed Ref. [21]). Let us consider two vortices at large distance; in the Type I* and in the Type II* regimes the quantum physics will be described by two \( \mathbb{C}P^{N-1} \) sigma models with an interaction potential given by Eq. (6.2.75). It would be interesting to study the effect of this term in the sigma model physics. Another interesting problem is the numerical determination of the effective theory for vortices at generic separations. To do this one has to determine the Manton metric on the full moduli space.
Chapter 7

Stability of non-Abelian semi-local vortices

In this Chapter we study the stability of semi-local vortices in the presence of the $\mathcal{N} = 1$ perturbation which makes the vortices non-BPS. In the case of a theory with a mass gap ($N_f = N_c$), the ’t Hooft standard classification of possible massive phases [131] is applicable; as a result, there is a very clear relation between the Higgs mechanism and magnetic confinement. In the case with several additional flavors ($N_f > N_c$), this is not a priori clear due to the presence of massless degrees of freedom (see Ref. [132] for some of the subtleties associated with the massless case).

The question is also important from the point of view that the vortex system (with gauge group $H$) being studied is actually a low-energy approximation of an underlying system with a larger gauge group, $G$ (See Chapter 2). The homotopy-map argument shows that the regular monopoles arising from the symmetry breaking $G \rightarrow H$ are actually unstable in the full theory, when the low-energy VEVs breaking completely the gauge group are taken into account. In other words, such monopoles are actually confined by the vortices developed in the low-energy $H$ system: this allows us to relate the continuous moduli and group transformation properties of the vortices to those of the monopoles appearing at the ends, thus explaining the origin of the dual gauge groups, such as the Goddard-Nuyts-Olive-Weinberg (GNOW) duals [10]. The stability problem on non-BPS non-Abelian strings have also been studied in another system: the Seiberg-dual theory of the $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ QCD [133].

In these considerations a subtle but crucial point is that when small terms arising from the symmetry breaking $G \rightarrow H$ are taken into account, neither the high energy system (describing the symmetry breaking $G \rightarrow H$ and regular monopoles) nor the low-energy system (in the Higgs phase $H \rightarrow$
describing the vortices) is BPS-saturated any longer. This on the one hand allows the monopoles and vortices to be related in a one-to-one map (each vortex ends up on a monopole); on the other hand, the system is no longer BPS and we must carefully check the fate of the moduli of the BPS vortices which do not survive the perturbation. In fact, zero modes related to global symmetries (orientational modes) still survive, because of their origin as Goldstone bosons, while other modes are no more protected by supersymmetry. These issues are the subject for investigation in this Chapter.

### 7.1 Abelian Vortices

The question of stability of the semi-local vortices \[73, 74, 94\] for Abelian systems has been investigated by Hindmarsh \[74\] and other authors \[134, 109, 110\], some time ago. The model which has been studied is an Abelian Higgs system with more than one flavor (\(N_f > 1\)),

\[
L = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi (D^\mu \phi) - \frac{\lambda}{2} (\phi^\dagger \phi - \xi^2),
\]

(7.1.1)

where \(D_\mu = \partial_\mu - i A_\mu\) is the standard covariant derivative, \(\phi = (\phi_1, \phi_2, \ldots, \phi_{N_f})\) represents a set of complex scalar matter fields of the same charge. This model is sometimes called the semi-local model since not all global symmetries, i.e. \(U(N_f)\) here, are gauged. As a consequence, the vacuum manifold \(M\) is \(\mathbb{CP}^{N_f-1} = SU(N_f)/[SU(N_f - 1) \times U(1)]\). Since the first homotopy group of \(M\) is trivial for \(N_f > 1\), vortex solutions are not necessarily stable. For \(\beta \equiv \lambda/e^2 < 1\) (i.e. Type I superconductors) the vortex of ANO-type \[116, 135\] is found to be stable. In the interesting special (BPS) case, \(\beta = 1\), there is a family of vortex solutions with the same tension, \(T = 2\pi \xi\). Except for the special values of the moduli (in the space of solutions), which represents the ANO vortex (sometimes called a “local vortex”), the vortex has a power-like tail in the profile function, hence the vortex width (thickness of the string) can be of an arbitrary size. In the limit of large size, the vortex essentially reduces to the \(\mathbb{CP}^{N_f-1}\) sigma-model lump (or two-dimensional skyrmion), characterized by \(\pi_2(\mathbb{CP}^{N_f-1}) = \mathbb{Z}\).

For \(\beta > 1\) (i.e. Type II superconductors), vortices are found to be unstable against fluctuations of the extra fields (flavors) which increase the size (and spread out the flux).
7.2 Non-Abelian Vortices

We consider the same $\mathcal{N} = 2$ supersymmetric ($U(N_c)$ gauge theory of Section 1.1. The only difference here is the we consider a generic number ($N_f > N_c$) of hypermultiplets $Q_f, \bar{Q}_f$ ($f = 1, \ldots, N_f$). Since all the essential features are already present in the minimal case, $N_c = 2$ and $N_f = 3$, we will concentrate in the $U(2)$ mode. The extension to more general cases, $N_c > 2$, is straightforward.

If the squark multiplets are kept massless, as usual, the theory has a degenerate set of vacua in the Higgs phase where the gauge symmetry is completely broken, which is the cotangent bundle over the complex Grassmannian manifold

$$\mathcal{M}_{\text{vac}} = \text{Gr}_{N_f, N_c} \simeq SU(N_f)/[SU(N_c) \times SU(N_f - N_c) \times U(1)]$$

for general $N_c$. It is important to notice that the first homotopy group of the Grassmannian manifold is trivial (for $N_f > N_c$). Up to gauge and flavor rotations, we can choose the following VEV for the scalar fields

$$Q = \hat{Q}^i = \sqrt{\frac{\xi}{2}} \left( \begin{array}{ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array} \right), \quad a_0 = 0, \quad a = 0,$$

where $a \equiv a_i \tau^i/2$. The vacuum is invariant under the following global color-flavor locked rotations: $(U_c \in SU(2), U_l \in SU(2) \subset SU(N_f))$

$$Q \rightarrow U_c Q U_l^\dagger, \quad \hat{Q} \rightarrow U_l^\dagger \hat{Q} U_c, \quad a \rightarrow U_c a U_c^\dagger, \quad F_{\mu\nu} \rightarrow U_c F_{\mu\nu} U_c^\dagger.$$

In addition to the case with $N_c = N_f$, the vacuum is also invariant under pure $SU(N_f - 2)$ flavor rotations which act on the last $N_f - 2$ columns.

When the bare masses for the squarks are tuned to some special values, there exist quantum vacua in which the non-Abelian gauge symmetry $SU(2) \times U(1)$ is preserved (the so-called $r = 2$ vacua) [64]. The low energy effective theory in these vacua is exactly the theory we are studying here. As the quantum vacua with an $SU(2)$ magnetic gauge group require the presence of a sufficient number of flavors, $N_f \geq 2r = 4$, we must necessarily deal with a system with an excess number of flavors ($N_f > N_c$). As these systems in the BPS approximation ($\eta, \mu = 0$) contain semi-local vortices with arbitrarily large widths, the necessity of studying the fate of these vortices in the presence of the perturbations $\eta, \mu$ presents itself quite naturally.

We will start by simply embedding the well-known solution for $N_f = 2$
flavors (6.1.10) in the present model with additional flavors:

\[ Q = \begin{pmatrix} \phi_0(r)e^{i\theta} & 0 & 0 & \cdots & 0 \\ 0 & \phi_1(r) & 0 & \cdots & 0 \end{pmatrix}, \quad a_0 = \lambda_0(r), \quad a = \lambda_1(r) \frac{r^3}{2}, \]

\[ A_i = A_i^a \frac{r^0}{2} = -\epsilon_{ij} x^j [1 - f_1(r)] \frac{r^3}{2}, \quad A_i^0 = -\epsilon_{ij} x^j [1 - f_0(r)], \quad (7.2.4) \]

The six functions \{\phi_0, \phi_1, f_0, f_1, \lambda_0, \lambda_1\} are found numerically by using the same techniques of the previous Section. A numerical result is shown in Fig. (7.1).

In the BPS limit \((\eta = \mu = 0)\), \(\lambda_0(x) = \lambda_1(x) = 0\), the substitution of this configuration into the Lagrangian (1.1.3) gives the BPS value, \(T_{\text{BPS}} = 2\pi \xi\), even if it is no longer a solution to the vortex equations (6.1.9), for \(\eta, \mu \neq 0\). This means that the non-BPS vortex tension derived from Eq. (6.1.9) is necessarily less than the BPS value. In other words, the model we are considering always yields type I superconductivity [136]. Our result below – the stability of the ANO-like vortices – thus generalizes naturally the known correlation between the type of superconductor and the kinds of stable vortices, found in the Abelian Higgs models [74, 137, 134]. Also, in so much as we study a softly-broken \(\mathcal{N} = 2\) supersymmetric gauge theory and its low-energy vortex system, the present work can be regarded as an extension of the one in \(SU(2)\) gauge theory [11], even though in the latter the low-energy system was naturally Abelian (\(U(1)\)). We have verified that the same qualitative conclusion

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holds in that case. In the next Section the stability of this solution under perturbations of the additional flavors will be studied.

7.2.1 Fluctuation Analysis

In this Section, we will generalize the fluctuation analysis of the stability of the Abelian vortex of Ref. [74] to the non-Abelian vortex. As we have mentioned, the first homotopy group of the vacuum manifold is trivial, thus the local vortex found in the previous Section can be unstable. To study the stability of non-BPS local vortices, we must consider the quadratic variations of the Lagrangian due to small perturbations of the background fields. It turns out that the quadratic variation of the Lagrangian can be written as the sum of two pieces

$$\delta^2 L = \delta^2 L\bigg|_{\text{local fields}} + \delta^2 L\bigg|_{\text{semi-local fields}},$$

(7.2.5)

where the first term denotes the variation with respect to the fields describing a non-trivial background vortex configuration (i.e. the “local” fields), while the second term is the variation with respect to the “semi-local” fields ($Q_f, \tilde{Q}_f$ with $N_f \geq f > N_c = 2$). The key point is that there are no mixed terms (at the second order) between the variations of the background fields and the “semi-local” fields. The first term cannot give rise to instabilities (i.e. the local vortices with $N_f = 2$ are topologically stable). Hence, it suffices for our purpose to study only the second term

$$\delta^2 L\bigg|_{\text{semi-local fields}} = (D_b \delta Q_3)^\dagger D_b \delta Q_3 + (D_b \delta \tilde{Q}_3)(D_b \delta \tilde{Q}_3)^\dagger - \delta^2 V|_{Q_3, \tilde{Q}_3},$$

(7.2.6)

where we have taken, for simplicity and without loss of generality, $N_f = 3$. The subscript ‘b’ denotes background fields which we fix to be given by the ansatz (6.1.10) in the following. Let us work out this variation explicitly. Even if $\tilde{Q}_b^\dagger = Q_b$ in the background fields we keep their variations independent.

Let us first consider the variation of the potential (1.1.6), piece by piece:

**Variation of $V_1$**

Remember that we are only interested in the variations that involve the semi-local fields, i.e. the variation with respect to the third flavor. The crucial observation is that the background value for this field is zero: $\tilde{Q}_3^\dagger = Q_3 = 0$. This means that the variation of terms such as $\text{Tr}[Q^\dagger \tau_i Q]$ are already quadratic in the perturbation. Furthermore, the term $\epsilon_{ijk} \bar{a}_j a_k$ evaluated on
the background fields is zero, due to $a_3$ being the only non-zero non-Abelian adjoint field. Thus, the variation of $V_1$ with respect to the semi-local fields is at least cubic, coming from the cross terms involving the adjoint field. Hence, the quadratic variation vanishes

$$\delta^2 V_1 |_{Q_3, \tilde{Q}_3} = 0 . \quad (7.2.7)$$

**Variation of $V_2$**

$V_2$ contains no adjoint fields. The same argument used for $V_1$ goes through: the variations of $V_2$ are at least quartic

$$\delta^2 V_2 |_{Q_3, \tilde{Q}_3} = 0 . \quad (7.2.8)$$

**Variation of $V_3$**

The variation of $V_3$ gives us the first non-trivial contribution. In this case the variation with respect to the semi-local fields is at least quadratic, and it is given by

$$\delta^2 V_3 |_{Q_3, \tilde{Q}_3} = \frac{g^2}{2} \left( \text{Tr}_f(Q^\dagger_3 r^3 Q_3) + 2 \mu a_{3b} \right) \left( \delta \tilde{Q}_3 r^3 \delta Q_3 + \text{c.c.} \right)$$

$$+ \frac{\epsilon^2}{2} \left( \text{Tr}_f(Q^\dagger_3 Q_3) - \xi + 2 \eta a_{0b} \right) \left( \delta \tilde{Q}_3 \delta Q_3 + \text{c.c.} \right) . \quad (7.2.9)$$

**Variation of $V_4$**

The variation of $V_4$ is also quadratic and is simply

$$\delta^2 V_4 |_{Q_3, \tilde{Q}_3} = \frac{1}{2} \delta Q_3^\dagger \left( a_{0b} + r^3 a_{3b} \right) \delta Q_3 + \frac{1}{2} \delta \tilde{Q}_3 \left( a_{0b} + r^3 a_{3b} \right)^2 \delta \tilde{Q}_3^\dagger . \quad (7.2.10)$$

The variations are not diagonal, thus involve mixed terms like $\delta \tilde{Q}_3 \delta Q_3$. We can easily diagonalize them by the following change of coordinates (keeping the kinetic terms canonical)

$$\delta Q_3 = \frac{1}{\sqrt{2}} (q + \tilde{q}^\dagger) , \quad \delta \tilde{Q}_3 = \frac{1}{\sqrt{2}} (q^\dagger - \tilde{q}) , \quad (7.2.11)$$

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which yields the following variation of the potential to second order

\[
\begin{align*}
\delta^2 V|_{Q_3, \tilde{Q}_3} &= \frac{g^2}{2} \left( \text{Tr}(Q_b^3 \rho^3 Q_b) + 2 \mu a_{3b} \right) \left( |q_I|^2 - |\tilde{q}_I|^2 - |q_{II}|^2 + |\tilde{q}_{II}|^2 \right) \\
&\quad + \frac{e^2}{2} \left( \text{Tr}(\tilde{Q}_b Q_b) - \xi + 2 \eta a_{ob} \right) \left( |q_I|^2 - |\tilde{q}_I|^2 + |q_{II}|^2 - |\tilde{q}_{II}|^2 \right) \\
&\quad + \frac{1}{2} \left( a_{ob} + a_{3b} \right)^2 \left( |q_I|^2 + |\tilde{q}_I|^2 \right) + \frac{1}{2} \left( a_{ob} - a_{3b} \right)^2 \left( |q_{II}|^2 + |\tilde{q}_{II}|^2 \right),
\end{align*}
\]

(7.2.12)

where the capital indices \( I \) and \( II \) label the color components. The problem of stability is now reduced to studying four decoupled Schrödinger equations.

We expand the fluctuations as

\[
q_{I,II} = \sum_k \psi^{(k)}_{I,II} e^{ik\theta}, \quad \tilde{q}_{I,II} = \sum_k \tilde{\psi}^{(k)}_{I,II} e^{-ik\theta}.
\]

(7.2.13)

Using these expansions, the quadratic variations of the energy density (7.2.6) give rise to the following Schrödinger equations

\[
\begin{align*}
\left[ -\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + V^{(k)}_{I,II} \right] \psi^{(k)}_{I,II} &= M^{(k)}_{I,II} \psi^{(k)}_{I,II}, \\
\left[ -\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + \tilde{V}^{(k)}_{I,II} \right] \tilde{\psi}^{(k)}_{I,II} &= \tilde{M}^{(k)}_{I,II} \tilde{\psi}^{(k)}_{I,II},
\end{align*}
\]

(7.2.14)

which must be solved with the following boundary conditions at small \( r \):

\[
\psi^{(k)}_{I,II}(r), \tilde{\psi}^{(k)}_{I,II}(r) = r^k + O(r^{k+1}).
\]

(7.2.15)

Figure 7.2: Potentials as functions of \( r \): (a) \( V_I^{(0)} \) and \( V_{II}^{(0)} \). (b) \( \tilde{V}_I^{(0)} \) and \( \tilde{V}_{II}^{(0)} \). (c) \( V_I^{(1)} \) and \( \tilde{V}_I^{(1)} \). The results are plotted for \( e = 2, g = 1, \xi = 2, \eta = 0.2 \) and \( \mu = 0.4 \).
The effective potentials are

\[ V_{I,II}^{(k)} = \frac{1}{4r^2} [f_0 - 1 \pm (f_1 - 1) + 2k]^2 + \frac{1}{2}(\lambda_0 \pm \lambda_1)^2 \]

\[ + e^2 \left( \phi_0^2 + \phi_1^2 - \xi + 2 \eta \lambda_0 \right) \pm g^2 \left( \phi_0^2 - \phi_1^2 + 2 \mu \lambda_1 \right) \]

(7.2.16)

\[ \tilde{V}_{I,II}^{(k)} = \frac{1}{4r^2} [f_0 - 1 \pm (f_1 - 1) + 2k]^2 + \frac{1}{2}(\lambda_0 \pm \lambda_1)^2 \]

\[ - e^2 \left( \phi_0^2 + \phi_1^2 - \xi + 2 \eta \lambda_0 \right) \mp g^2 \left( \phi_0^2 - \phi_1^2 + 2 \mu \lambda_1 \right). \]

(7.2.17)

(7.2.18)

The upper signs refer to the color-index \( I \), while the lower to index \( II \). Since only the first terms are dependent on \( k \), it is easy to check that

\[ \left( V_{I}^{(k)}, V_{II}^{(k)} \right) \geq \left( \min \left[ V_I^{(0)}, V_I^{(1)} \right], V_{II}^{(0)} \right), \]

\[ \left( \tilde{V}_{I}^{(k)}, \tilde{V}_{II}^{(k)} \right) \geq \left( \min \left[ \tilde{V}_I^{(0)}, \tilde{V}_I^{(1)} \right], \tilde{V}_{II}^{(0)} \right), \]

(7.2.19)

for all \( k \). To prove the above inequalities, it is sufficient to recall that the profile functions for the gauge fields satisfy: \( 0 \leq f_{0,1} \leq 1 \).

Figure 7.3: Left panel: the value of the wave function \( \psi_I^{(0)} \) at large distance \( r = 20 \), in the unit of the vortex size, as a function of \( M \) (see Eqs. (7.2.14)). There are no zeros at negative values of \( M \): this implies the absence of normalizable tachyonic modes. Right panel: the wave function for the zero energy fluctuation: \( M = 0 \). The linear behavior is a common feature for zero energy wave functions. The numerical values of the parameters are chosen as in Fig. (7.2).

From a well-known theorem in one-dimensional quantum mechanics, it follows that (valid for the lowest eigenvalues)

\[ \left( M_I^{(k)}, M_{II}^{(k)} \right) \geq \left( \min \left[ M_I^{(0)}, M_I^{(1)} \right], M_{II}^{(0)} \right), \]

\[ \left( \tilde{M}_I^{(k)}, \tilde{M}_{II}^{(k)} \right) \geq \left( \min \left[ \tilde{M}_I^{(0)}, \tilde{M}_I^{(1)} \right], \tilde{M}_{II}^{(0)} \right). \]

(7.2.20)
Hence, in order to exclude the existence of negative eigenvalues, it suffices to study the six potentials: $V_I^{0(1)}$, $\tilde{V}_I^{0(1)}$, $V_{II}^{0}$ and $\tilde{V}_{II}^{0}$. Fig. (7.2) shows that only the potentials $V_I^{0}$ and $V_{II}^{0}$ can be sufficiently negative, such that one could expect negative eigenvalues. The potentials $V_I^{1(1)}$ and $\tilde{V}_I^{1(1)}$ are even divergent at the core of the vortex. Indeed, the results of the numerical analysis show (up to the numerical precision) that there are no negative eigenvalues for any of the potentials, $V_I^{0,1}$, $\tilde{V}_I^{0,1}$, $V_{II}^{0}$ and $\tilde{V}_{II}^{0}$. We have checked this statement in a very wide range of values of the couplings of the theory. This strongly suggests the absence of negative eigenvalues for all values of the couplings. The result is shown in Fig. (7.3) for a particular choice of parameters. Note that all the potentials go to zero at large distance. This means that there will be a set of positive eigenvalues that represent the continuum states, which can be interpreted as interaction modes with the massless Goldstone bosons of the bulk theory.

The status of the zero energy wave functions however is subtler. The wave functions for the non-BPS (Fig. (7.3)) and the BPS cases (Fig. (7.4)) are both non-normalizable, hence they should both be interpreted as a part of the continuum. However, there is an important difference between the two cases. In the BPS case, the wave function is limited and goes to zero at infinity; its squared norm diverges only logarithmically with the transverse volume ($\sim \log L^2$). In the non-BPS case, the wave function is unlimited, and it diverges linearly (Fig. (7.3)). This implies a divergence of its squared norm being quadratic in the transverse volume ($\sim L^4$). Such a fluctuation changes the vacuum expectation values of the scalars at infinity, thus does not correspond to a size zero mode (collective coordinate). The situation is clearer when our vortex system is put into a finite volume. In the BPS case, the wave function of the semi-local mode asymptotically approaches zero and is an (approximately) acceptable wave function. In a box, it represents a zero
energy bound state. On the contrary, the wave function for the non-BPS case is non-normalizable. The corresponding zero energy state is eliminated and the semi-local excitations have a mass gap. This observation strongly suggests that in the non-BPS case the size moduli disappears.

As a further check, we note that the potential which gives rise to the size zero modes in the BPS limit (which we know exist in that case) is precisely $V_I^{(0)}$. The zero mode is thus given by a fluctuation of the field $\psi_I^{(0)}$. Using Eq. (7.2.11) we see that $\delta Q_3 = \delta \tilde{Q}_3 = \sqrt{2} \psi_I^{(0)}$, while all other components of $Q_I$ and $\tilde{Q}_I$ vanish. This is exactly the fluctuation we expect for the semi-local vortex, which in the BPS limit is known to have the following form [6, 138]:

$$Q = \begin{pmatrix} \phi_0(r) e^{i\theta} & 0 & \chi(r) \\ 0 & \phi_1(r) & 0 \end{pmatrix}. \quad (7.2.21)$$

7.3 Meta-stability versus Absolute Stability

The results of the previous Section indicate that the local (ANO-like) vortex embedded in a model with additional flavors is a local minimum of the action. The problem whether or not this solution is a global minimum cannot be addressed with a fluctuation analysis only. In principle, the existence of an instability related to large fluctuations of the fields is still not excluded. However, we have a reason to believe that the local vortex is truly stable, being the global minimum of the action. Note that a vortex cannot dilute in the whole space. In fact, such diluted configuration would have an energy that is equal to the BPS bound, $T_\infty = 2\pi \xi = T_{BPS}$, while the tension of the local vortices, in the class of theories we are considering, is always less: $T < T_{BPS}$, as already noted. To see this, we will first show that the semi-local vortex solution, obtained in the BPS limit, is an approximate solution to the non-BPS equations, in the limit of an infinite size of the vortex. Then, it is easy to check that the tension of such approximate configuration converges to the BPS value. Let us write the equations of motion for the semi-local
configuration \[\text{(7.2.21)}\]

\[
\begin{align*}
\phi_0'' + \phi_0' - \frac{(f_0 + f_1)^2 \phi_0}{4r^2} &= \frac{\phi_0}{2} (\lambda_0 + \lambda_1)^2 + e^2A + g^2B, \\
\phi_1'' + \phi_1' - \frac{(f_0 - f_1)^2 \phi_1}{4r^2} &= \frac{\phi_1}{2} (\lambda_0 - \lambda_1)^2 + e^2A - g^2B, \\
\chi'' + \chi' - \frac{(f_0 + f_1 - 2)^2 \chi}{4r^2} &= \chi \frac{\lambda_0 + \lambda_1}{2} (\lambda_0 + \lambda_1)^2 + e^2A + g^2B, \\
\lambda_0'' + \lambda_0' &= e^2 (\lambda_0 + \lambda_1)(\phi_0' + \chi') + (\lambda_0 - \lambda_1)\phi_0^2 + e^2\eta A, \\
\lambda_1'' + \lambda_1' &= g^2 (\lambda_0 + \lambda_1)(\phi_1' + \chi') - (\lambda_0 - \lambda_1)\phi_1^2 + g^2\mu B.
\end{align*}
\]

(7.3.1)

where we have defined

\[
A \equiv \phi_0^2 + \phi_1^2 + \chi^2 - \xi + 2\eta \lambda_0, \quad B \equiv \phi_0^2 - \phi_1^2 + \chi^2 + 2\mu \lambda_1.
\]

(7.3.2)

For simplicity, we take the gauge couplings to be equal: \(e = g\). Let us consider the following set of approximate solutions corresponding to a semi-local BPS vortex in the limit of large size (lump limit): \(a \gg 1/e\sqrt{\xi}\):

\[
(\phi_0, \phi_1, \chi) = \sqrt{\frac{\xi}{2}} \left( \frac{r}{\sqrt{r^2 + |a|^2}}, 1, \frac{a}{\sqrt{r^2 + |a|^2}} \right),
\]

\[
f_0 = f_1 = \frac{|a|^2}{r^2 + |a|^2}, \quad \lambda_0 = \lambda_1 = 0.
\]

(7.3.3)

The right hand sides of Eqs. \[\text{(7.3.1)}\] calculated with this configuration vanish obviously. The left hand sides are derivative terms which disappear in the large size limit: the set of fields above, trivially satisfy the fourth and the last two equations, while they satisfy the other equations up to terms of order \(O(1/|a|^2)\)

\[
\begin{align*}
\left| f_0'' - \frac{f_0'}{r} \right| &= \frac{8|a|^2r^2}{(r^2 + |a|^2)^3} \leq \frac{32}{27|a|^2}, \\
\left| \phi_0'' + \phi_0' - \frac{\phi_0(f_0 + f_1)^2}{4r^2} \right| &= \sqrt{\frac{\xi}{2}} \left( \frac{2|a|^2r^3}{(|a|^2 + r^2)^{5/2}} \right) \leq \sqrt{\frac{\xi}{2}} \left( \frac{4}{5} \right)^{3/2} \frac{1}{|a|^2}; \\
\left| \chi'' + \chi' - \frac{\chi(-2 + f_0 + f_1)^2}{4r^2} \right| &= \sqrt{\frac{\xi}{2}} \left( \frac{2|a|^3}{(|a|^2 + r^2)^{5/2}} \right) \leq \sqrt{\frac{\xi}{2}} \frac{2}{|a|^2}.
\end{align*}
\]

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Thus, we see that the configuration of Eq. (7.3.3), in the limit $|a| \rightarrow \infty$, is a solution to the non-BPS equations of motion: it represents an infinitely diluted vortex, having the tension equal to the BPS value.

We thus see that an infinitely wide vortex is not favored energetically. The only remaining possibility is the existence of a true minimum for a vortex with some intermediate size, which involves also a non-trivial configuration for the semi-local fields. This case is very unlikely. In fact, we could not find any such solution in our numerical surveys. The relaxation method should have detected such an unexpected stable vortex.

We consider our results as a strong evidence for the fact that the local vortex is a true minimum, absolutely stable against becoming a semi-local vortex.

7.4 Low-energy Effective Theory

In this short Section we only make some considerations on the low-energy effective theory of the vortex which follow from the results of the previous Sections.

In the semi-local case, the vacuum has no mass gap, thus there are massless Nambu-Goldstone bosons in the bulk. This fact gives rise to subtleties (which we encountered in Sec. (7.2.1)) when defining an effective theory for the vortex only. This is because both the vortex excitations and the massless particles in the bulk appear as light modes. As previously mentioned, semi-local excitations are non-normalizable and the corresponding states must be considered as bulk excitations. In a rigorous effective theory, we should take into account only localizable and normalizable modes, which discards the bulk excitations. This holds both in the BPS and non-BPS cases. In fact, effective actions which include semi-local excitations have already been proposed in the literature. The effective actions for BPS semi-local vortices derived in Ref. [6, 75], however, are valid only in a finite volume.

It seems to us that the following point has not been stressed clearly enough in the literature. We consider all the excitations related to non-normalizable zero energy wave functions as part of the massless states of the bulk theory. In the BPS case, the surviving supersymmetry enables us to describe these fluctuations as collective coordinates of the vortex. It is known, for example, that a single non-Abelian BPS vortex has non-normalizable size parameters (in the case $N_f > N_c$). From the point of view that we propose here, however,

1Similar issues have been already discussed for monopoles and vortices, see for example Refs. [139, 96, 140].
the effective action on the local vortex is given by the orientational part only
(normalizable states). The other collective coordinates (size) give rise to zero
energy fluctuations which represent bulk excitations.

While these considerations might just be a matter of interpretation in the
BPS case, they could be important if one wants to push further the study of
the non-BPS vortices we started in this work.

7.5 Summary

We have examined the stability of the non-Abelian vortices in the context
of an \( \mathcal{N} = 2 \), \( H = U(N_c) \) model with \( N_f > N_c \) flavors and an \( \mathcal{N} = 1 \)
perturbation. The particular perturbation chosen (the adjoint scalar mass
terms) makes our vortices non-BPS. Local (ANO-like) vortices are found to
be stable; the vortex moduli corresponding to the semi-local vortices (present
in the BPS case) disappear, leaving intact the orientational moduli \(-\mathbb{C}P^{N_c-1}\)
in this particular model – related to the exact symmetry of the system.

This conclusion seems to be very reasonable, as the narrow, ANO-like
vortices are needed to eliminate the regular monopoles from the spectrum
of the full system, in the case our \( H \) gauge theory arises as a low-energy
approximation of an underlying \( G \) theory, after the symmetry breaking \( G \rightarrow H \). If the semi-local vortex would have survived the non-BPS perturbations,
the magnetic monopoles would be deconfined \([6]\). Happily, this is not the
case; the Higgs phase of the (electric) low-energy theory is actually a magnetic
confinement phase.
Chapter 8

A Dual Model of Confinement
for $SO(N)$ Gauge Theories

We will apply the results on the moduli space of composite vortices to test our ideas about duality transformations, introduced in Chapter 2, against another class of theories,

$$SO(2N + 1) \xrightarrow{\langle \phi_1 \rangle \neq 0} SU(r) \times U(1)^{N-r+1} \xrightarrow{\langle \phi_2 \rangle \neq 0} 1.$$  \hspace{1cm} (8.0.1)

One of the reasons why this case is interesting is that the semiclassical monopoles arising from the symmetry breaking $SO(2N + 1) \xrightarrow{\langle \phi_1 \rangle \neq 0} U(N)$ appear to belong to the second-rank symmetric tensor representation of $SU(N)$ [51, 37]. Another, related reason is the fact that since $\pi_1(G) = \pi_1(SO(2N + 1)) = \mathbb{Z}_2$, the homotopy map Eq. (2.2.1) is less trivial in this case. Thirdly, according to the detailed analysis of the softly-broken $\mathcal{N} = 2$ theories with $SO(N)$ gauge group [141] the quantum mechanical behavior of the monopoles is different for $r = N$ and for $r < N$. Non-Abelian monopoles belonging to the fundamental representation of the dual $SU(r)$ group appears only for $r \leq N_f/2$, and because of the requirement of asymptotic freedom of the original theory ($N_f < 2N - 1$), this is possible only for $r < N$. It is very encouraging that such a difference in the behavior of non-Abelian monopoles indeed follows, as we shall see, from the way we define the dual group though the transformation properties of mixed monopole-vortex configurations and homotopy map.
8.1 Maximal $SU$ factor; $SO(5) \rightarrow U(2) \rightarrow 1$

Let us first consider the case the $SU(N)$ factor has the maximum rank,

$$SO(2N + 1) \stackrel{\langle \phi \rangle \neq 0}{\rightarrow} U(N).$$

To be concrete, let us consider the case of an $SO(5)$ theory, where a scalar VEV of the form

$$\langle \Phi \rangle = \begin{pmatrix}
0 & iv & 0 & 0 & 0 \\
-iv & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & iv & 0 \\
0 & 0 & -iv & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

breaking the gauge group as $SO(5) \rightarrow H = SU(2) \times U(1)/\mathbb{Z}_2 = U(2)$. We assume that at lower energies some other scalar VEVs break $H$ completely, leaving however a color-flavor diagonal $SU(2)$ group unbroken. This model arises semiclassically in softly broken $\mathcal{N} = 2$ supersymmetric $SO(5)$ gauge theory with large, equal bare quark masses, $m$, and with a small adjoint scalar mass $\mu$, with scalar VEVs given by $v = m/\sqrt{2}$ in Eq. (8.1.1) and

$$Q = \dot{Q}^i = \sqrt{\frac{\mu m}{2}} \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
i & 0 & 0 & \cdots \\
0 & i & 0 & \cdots \\
0 & 0 & 0 & \cdots
\end{pmatrix}.$$ (8.1.2)

(See Appendix (??), also the Section 2 of [141], for more details).

The $SO(4) \sim SU(2) \times SU(2)$ subgroup living on the upper-left corner is broken to $SU(2) \times U(1)$, giving rise to a single ’t Hooft-Polyakov monopole. On the other hand, by embedding the ’t Hooft-Polyakov monopole in the two $SO(3)$ subgroups (in the (125) and (345) subspaces), one finds two more monopoles. All three of them are degenerate. Actually, E. Weinberg [142, 129] has found a continuous set of degenerate monopole solutions interpolating these, and noted that the transformations among them are not simply related to the unbroken $SU(2)$ group.

From the point of view of stability argument, Eq. (2.2.1), this case is very similar to the case considered by ’t Hooft, as $\pi_1(SO(5)) = \mathbb{Z}_2$: a singular $\mathbb{Z}_2$ Dirac monopole can be introduced in the theory. The minimal vortex of

$^1$ This and similar cases are sometimes referred to as “accidentally degenerate case” in the literature.
the low-energy theory is truly stable in this case, as a minimal non-trivial element of \( \pi_1(H) \) represents also a non-trivial element of \( \pi_1(G) \). This can be seen as follows. A minimum element of \( \pi_1(H) = \pi_1(U(2)) \sim \mathbb{Z} \) corresponds to simultaneous rotations of angle \( \pi \) in the (12) and (34) planes (which is a half circle of \( U(1) \)), which brings the origin to the \( \mathbb{Z}_2 \) element of \( SU(2) \), \( \text{diag}(-1, -1, -1, -1, 1) \), followed by an \( SU(2) \) transformation back to the origin, an angle \( -\pi \) rotation in the (12) plane and an angle \( \pi \) rotation around (34) plane. The net effect is a \( 2\pi \) rotation in the (34) plane, which is indeed a non-trivial element of \( \pi_1(SO(5)) = \mathbb{Z}_2 \). Such a vortex would confine the singular Dirac monopole, if introduced into the theory (See Fig. (2.1)).

On the other hand, there are classes of vortices which appear to be stable in the low-energy approximation, but are not so in the full theory. In fact non-minimal \( k = 2 \) elements of \( \pi_1(H) = \pi_1(SU(2) \times U(1) / \mathbb{Z}_2) \sim \mathbb{Z} \) are actually trivial in the full theory. This means that the \( k = 2 \) vortices must end at a regular monopole. Vice versa, as \( \pi_2(SO(5)) = 1 \), the regular 't Hooft Polyakov monopoles of high-energy theory must be confined by these non-minimal vortices and disappear from the spectrum.

The transformation property of \( k = 2 \) vortices has been studied recently in \cite{90, 91}, and in particular, in Section (4.1.1). It turns out that the moduli space of the \( k = 2 \) vortices is a \( \mathbb{C}P^2 \) with a conic singularity. It was shown that the generic \( k = 2 \) vortices transform under the \( SU(2)_{c+f} \) group as a triplet. At a particular point of the moduli - an orbifold singularity - the vortex is Abelian: it is a singlet of \( SU(2)_{c+f} \).

As the full theory has an exact, unbroken \( SU(2)_{c+f} \) symmetry, it follows from the homotopy-group argument of Section (2.2) that the monopoles in the high-energy \( SO(5) \rightarrow U(2) \) theory have components transforming as a triplet and a singlet of \( SU(2)_{c+f} \).

Note that it is not easy to see this result – and is somewhat misleading to attempt to do so – based solely on the semi-classical construction of the monopoles or on the zero-mode analysis around such solutions, where the unbroken color-flavor symmetry is not appropriately taken into account. Generically, the “unbroken” color \( SU(2) \) group suffers from the topological obstruction or, perturbatively, from the pathology of non-normalizable gauge zero-modes \cite{17, 48, 49, 50, 51}, as we noted already.

Nevertheless, there are indications that the findings by E. Weinberg \cite{142, 129} are consistent with the properties of the \( k = 2 \) vortices. In the standard

\[2\] In another complex codimension-one subspace, they appear to transform as a doublet. However quantum states of any triplet of \( SU(2) \) contains such an orbit. The state of maximum \( S_z \) \( |1, 1\rangle \), transforms under \( SU(2) \) as an \( SO(3) \) vector, staying on a subspace \( S^2 \sim CP^1 \subset CP^2 \).
way to embed $SU(2)$ subgroups through the Cartan decomposition (we follow here the notation of [142, 129]),

$$t_1(\nu) = \frac{1}{(2\nu^2)^{-1/2}} (E_\nu + E_{-\nu}); \quad t_2(\nu) = \frac{-i}{(2\nu^2)^{-1/2}} (E_\nu - E_{-\nu});
$$

$$t_3 = (\nu^2)^{-1} \nu_j T_j, \quad (8.1.3)$$

where $\nu$ denotes the non-vanishing root vectors of $SO(5)$ Fig. (8.1), the unbroken $SU(2)$ group is generated by $\gamma$. The monopole associated with the root vector $\beta$ and the (equivalent) one given by $\mu$ naturally form a doublet of the “unbroken” $SU(2)$, while the monopole with the $\alpha$ charges is a singlet. The continuous set of monopoles interpolating among these monopoles found by Weinberg are analogous to the continuous set of vortices we found, which form the points of the $\mathbb{C}P^2$, which transform as a triplet. (See the Fig. (4.1)).

An even more concrete hint of consistency comes from the structure of the moduli space of the monopoles. The moduli metric found in [142, 129] is

$$ds^2 = M d\mathbf{x}^2 + \frac{16\pi^2}{M} d\chi^2 +$$

$$+ k \left[ \frac{d b^2}{b} + b (d\alpha^2 + \sin^2 \alpha d\beta^2 + (d\gamma + \cos \alpha d\beta)^2) \right]. \quad (8.1.4)$$

By performing a simple change of coordinate, $B \equiv 2\sqrt{b}$, it becomes evident
that the moduli space has the structure

\[ \mathbb{C}^2/\mathbb{Z}_2, \] (8.1.5)

apart from the irrelevant factor \( \mathbb{R}^3 \) (the position of the monopole) and \( S^1 \) \( (U(1) \) phase\). \(^3\) Eq. (8.1.5) coincides with the moduli space of the \( k = 2 \) co-axial vortices, seen in the central \((1,1)\) patch.

These considerations strengthen our conclusion that the continuous set of monopoles found in \([142,129]\) belongs to a singlet and a triplet representations of the dual \( SU(2) \) group. Although the detailed properties of the moduli spaces for monopoles and vortices are different\(^4\) this could be related to the fact that one should ultimately consider a smooth monopole-vortex mixed configurations in the full theory, not each of them separately. Also, related to this point, there remains the fact that the dual group which is exact and under which monopoles transform, is not the original \( SU(2) \) subgroup but involves the flavor group essentially.

Note that our conclusion is based on the exact symmetry, and should be reliable. However, the degeneracy among all the vortices (or the monopoles) lying in the entire moduli space \( \mathbb{C}P^2/\mathbb{Z}_2 \) found in the BPS limits, is an artifact of the lowest-order approximation. Only the degeneracy among the vortices (or among the monopoles) belonging to the same multiplet is expected to survive quantum mechanically. \(^1\) and \(^3\) vortex tensions (monopole masses) will split. Which of the multiplets (1 or 3) will remain stable, after quantum corrections are taken into account, is a question just lying beyond the power of semiclassical considerations.

In the context of asymptotically-free \( \mathcal{N} = 2 \) supersymmetric models, there are no indications that the triplet monopoles of \( SO(5) \rightarrow U(2) \) theory survive quantum mechanically. This result can be actually understood by a simple renormalization-group argument:

- In a \( SO(2N + 1) \) theories with \( \mathcal{N} = 2, 1 \) supersymmetries, the condition for the original theory to be asymptotic-free (\( N_f \) less than \( 2N - 1, \frac{3(2N - 2)}{2} \), respectively)\(^5\) is not compatible with the low-energy \( SU(N) \) theory being non-asymptotic-free (\( N_f \geq 2N \) and \( N_f \geq 3N \), respectively.)

\(^3\) The monopole modulus due to the unbroken \( U(1) \subset U(2) \) is not present in the full system, where the gauge group is completely broken.

\(^4\) The first is known to be hyper-Kähler and the second Kähler – indeed \( \mathbb{C}P^2/\mathbb{Z}_2 \) does not admit hyper-Kähler structure.

\(^5\) The counting is made for the appropriate supersymmetry multiplets, \( N_f \) hypermultiplets for \( \mathcal{N} = 2; N_f \) chiral multiplets for \( \mathcal{N} = 1 \) supersymmetric \( SO(N) \) theory.

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The problem would not arise if the rank of the unbroken $SU(r)$ were smaller. That such a “sign-flip” of the beta function is a necessary condition for the emergence of low-energy non-Abelian monopoles has been pointed out some time ago by one of the authors [143], even though the validity of such an argument for non-supersymmetric theories is perhaps not obvious.

If the condition of asymptotic freedom of the ultraviolet theory is dropped, then there are no such constraints, and it makes sense to consider symmetry breaking patterns such as $SO(2N + 1) \rightarrow U(N)$. Our conclusion that the monopoles of $SO(5) \rightarrow U(2)$ system transform as a triplet or a singlet would apply under such conditions. Analogously, we expect the monopoles in the system $SO(2N + 1) \rightarrow U(N)$ to transform as a second-rank symmetric or antisymmetric representation.

8.2 $SO(2N+1) \rightarrow SU(r) \times U(1)^{N-r-1} \rightarrow 1 \ (r < N)$

Consider now the cases in which the unbroken $SU(r)$ factor has a smaller rank, $SO(2N + 1) \rightarrow SU(r) \times U(1)^{N-r+1} \rightarrow 1$, where $r < N$. For concreteness, let us discuss the case of an $SO(7)$ theory,

$$SO(7) \xrightarrow{\langle \phi_1 \rangle \neq 0} U(2) \times U(1) \xrightarrow{\langle \phi_2 \rangle \neq 0} 1.$$  \hspace{1cm} (8.2.1)

As we are interested in a concrete dynamical realization of this, we consider the softly broken $\mathcal{N} = 2$ theory, with $N_f = 4$ quark hypermultiplets. Such a number of flavors ensures both the original $SO(7)$ theory being asymptotically free and the $SU(2)$ subgroup being non-asymptotically free. The low-energy gauge group $U(2) \times U(1)$ is completely broken by the squark VEV’s similar to Eq. (8.1.2). The large VEV $\langle \phi_1 \rangle$ has the form:

$$\langle \phi_1 \rangle = \begin{pmatrix} 0 & iv_0 & 0 & 0 & 0 & 0 & 0 \\ -iv_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & iv_0 & 0 & 0 & 0 \\ 0 & 0 & -iv_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & iv_1 & 0 \\ 0 & 0 & 0 & 0 & -iv_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_1 \neq v_0. \hspace{1cm} (8.2.2)$$

The “unbroken” $U(2)$ lies in $SO(4)_{1234} \sim SU(2) \times SU(2)$ while the $U(1)$ factor corresponds to the rotations in the 56 plane (see Appendix (??)). The semiclassical monopoles of high-energy theory are\(^6\)

\(^6\)Within the softly broken $\mathcal{N} = 2$ theory, the quantum mechanical vacua with $SU(2) \times U(1)^2$ gauge symmetry, in the limit $m_1 = m \simeq \Lambda$, appears to arise from
(i) a triplet of degenerate monopoles of mass $2|v_0|/g$ (they arise as in the $SO(5)$ theory discussed above);

(ii) a doublet of degenerate monopoles of mass $|v_0 - v_1|/g$: they arise from the breaking of $SU_+(2) \subset SO(4)_{1256}$ and $SU_+(2) \subset SO(4)_{3456}$ (see Appendix (??));

(iii) a doublet of degenerate monopoles of mass $|v_0 + v_1|/g$: they also arise from the breaking of $SU_-(2) \subset SO(4)_{1256}$ and $SU_-(2) \subset SO(4)_{3456}$;

(iv) a singlet monopole of mass $2|v_1|/g$ arising from the breaking of $SO(3)_{567}$.

Which of these semiclassical monopoles are the lightest and which of them are stable against decay into lighter monopole pairs, depend on the various VEVs. It is possible that the monopoles (ii) or (iii) are the lightest of all. Of course more detailed issues such as which of the degeneracies survives quantum effects, are questions which go beyond the semiclassical approximations.

In fact, when $v_0, v_1 \sim \Lambda$ the standard semi-classical reasoning fails to give any reliable answer: a fully quantum-mechanical analysis is needed. Fortunately, in the softly broken $\mathcal{N} = 2$ theory such analyses have been performed \[141\] and we do know that the light monopoles in the fundamental representation $(2)$ of $SU(2)$ appear in an appropriate vacuum.

Knowing this, we might try to understand how such a result may follow from our definition of the dual group. At low energies the gauge group $U(2) \times U(1)$ is completely broken, leaving a color-flavor diagonal $SU(2)_{c+f}$ symmetry unbroken. The theory possesses vortices of

$$\pi_1(U(2) \times U(1)) = \mathbb{Z} \times \mathbb{Z}. \tag{8.2.3}$$

The minimal vortices corresponding to $\pi_1(U(2)) = \mathbb{Z}$ transform as a 2 of $SU(2)_{c+f}$.

A minimum element of $\pi_1(U(2) \times U(1))$ such as an angle $2\pi$ rotation in the $U(1)_{56}$ factor, or the minimal $U(2)$ loop, corresponds to vortices stable the semiclassical vacua of the form of Eq. (8.2.2), with $v_0 = m/\sqrt{2} \gg \Lambda$, $v_1 = 0$, with classical symmetry $SU(2) \times U(1) \times SO(3)_{567}$. The $SO(3)_{567}$ gauge sector (pure $\mathcal{N} = 2$ theory) becomes strongly-coupled at low energies and breaks itself to $U(1)$. Thus it would be more correct to say $v_1 \sim \Lambda$, but then the discussion about semiclassical monopole masses $\sim v_1/g$, etc., should not be taken too literally. If one wishes, one could consider a larger gauge group, e.g., $SO(9)$, to do a straightforward semiclassical analysis for an unbroken $SU(2)$ group. In general, the relation between the classical vacua and the fully quantum mechanical vacua is a rather subtle issue. See for instance the discussions in \[62\].
in the full theory. They would confine Dirac monopoles associated with 
\( \pi_1(SO(7)) = \mathbb{Z}_2 \), if the latter were introduced in the theory.

The regular monopoles in which we are interested in, are instead confined 
by some non-minimal \( (k = 2) \) vortices of the low-energy theory. However, 
in contrast to the \( SO(5) \) theory discussed in the preceding subsection, this 
\textit{does not} necessarily imply a second-rank tensor representation of \( SU(2)_{c+f} \) 
of these monopoles. In fact, the monopoles of the (ii) group, for instance, 
carry the minimum charge of \( U(2) \) and an unit charge of \( U(1) \). Therefore, the 
relevant \( k = 2 \) vortex corresponds to the minimum element both of \( \pi_1(U(2)) \) 
and of \( \pi_1(U(1)) \), generated by a \( 2\pi \) rotation in the 56 plane together with a 
minimal loop of \( \pi_1(U(2)) \), analogous to the one discussed in the preceding 
subsection. As a consequence the monopoles confined by such vortices, by 
our discussion of Section 3, transform as a \textit{doublet} of the dual group \( \tilde{SU}(2) \sim SU(2)_{c+f} \).

This discussion naturally generalizes to all other cases with symmetry 
breaking, \( SO(2N+1) \rightarrow SU(r) \times U(1)^{N-r+1} \rightarrow 1 \), \( r < N \). The dual magnetic 
\( SU(r) \) group observed in the low-energy effective theory [141], under which 
the light monopoles transform as a fundamental multiplet, thus matches 
nicely with the properties of the dual \( \tilde{SU}(2) \sim SU(2)_{c+f} \) group.

The cases of \( SO(2N) \rightarrow SU(r) \times U(1)^{N-r+1} \rightarrow 1 \), \( r < N - 1 \) are similar. 
We expect that there is a qualitative difference between the breaking with 
the maximum (or next to the maximum) rank \( SU \) factor and smaller \( SU(r) \) 
unbroken groups. Such a difference is indeed observed in the fully quantum 
mechanical analysis of \( SO(N) \) theory [141].

The behavior of monopoles in asymptotic-free \( USp(2N) \) theories \( (N_f < 2N+2) \) is more similar to those appearing in the \( SU(N) \) theories, because of 
the property, \( \pi_1(USp(2N)) = 1 \). All monopoles are regular monopoles due 
to the partial symmetry breaking, \( USp(2N) \rightarrow SU(r) \times U(1)^{N-r+1} \), \( r \leq N \). 
The transformation property of these monopoles, in the theory with exact 
unbroken \( SU(r)_{c+f} \) global symmetry, is deduced from the transformation 
properties among the non-Abelian vortices of the low-energy system \( SU(r) \times 
U(1)^{N-r+1} \rightarrow 1 \) : they transform as \( r \) of \( SU(r)_{c+f} \). Such a result is consistent 
dynamically, as long as \( r \leq N_f/2 \). It is comfortable that these are precisely 
what is found from the quantum mechanical analysis [64].
8.3 Other symmetry breaking patterns and GNOW duality

Before concluding this Section, let us add a few remarks on other symmetry breaking patterns such as $SO(2N + 3) \rightarrow SO(2N + 1) \times U(1)$ and $USp(2N + 2) \rightarrow USp(2N) \times U(1)$, and the resulting GNOW monopoles. These cases might be interesting as the GNOW dual groups are different from the original one: the dual of $SO(2N + 1)$ is $USp(2N)$ and vice versa. It is possible to analyze these systems, again setting up models so that the “unbroken group” is completely broken at a much lower mass scales by the set of squark VEVs. Indeed such a preliminary study has been made in [144], where the emergence of the GNOW dual group has been found.

However, the quantum fate of these GNOW dual monopoles is unclear. More precisely, within the concrete $\mathcal{N} = 2$ models we are working on where the exact quantum fate of the semiclassical monopoles is known from the analyses made at small $m, \mu$ [141], we know that these GNOW monopoles do not survive quantum effects. Only the monopoles carrying the quantum numbers of the $SU(r)$ subgroups discussed in the previous subsection appear. On the other hand, there is clearly a reason why the GNOW monopoles cannot appear at low energies in these cases: the low-energy effective action would have a wrong global symmetry. GNOW monopoles are not always relevant quantum mechanically\footnote{Seiberg duals of $\mathcal{N} = 1$ supersymmetric theories with various matter contents, provide us with more than enough evidence for it.}. These and other peculiar (but consistent) quantum properties of non-Abelian monopoles have been recently discussed in [62].

8.4 Summary

In this Chapter we have examined an idea about the “non-Abelian monopoles”, put forward some time ago [36], more systematically and by using some recent results on the non-Abelian vortices. According to this idea, the dual transformation of non-Abelian monopoles occurring in a system with gauge symmetry breaking $G \rightarrow H$ is to be defined by setting the low-energy $H$ system in Higgs phase, so that the dual system is in confinement phase. The transformation law of the monopoles follows from that of monopole-vortex mixed configurations in the system

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1, \quad (v_1 \gg v_2)$$
under an unbroken, exact color-flavor diagonal symmetry \( H_{c+f} \sim \tilde{H} \). The transformation properties of the regular monopoles (classified by \( \pi_2(G/H) \)) follow from those among the non-Abelian vortices (classified by \( \pi_1(H) \)), via the isomorphism \( \pi_1(G) \sim \pi_1(H)/\pi_2(G/H) \). Our results, obtained in the semiclassical approximation (reliable at \( v_1 \gg v_2 \gg \Lambda \)) of softly-broken \( \mathcal{N} = 2 \) supersymmetric \( SU(N) \) and \( SO(N) \) theories, are -- very non-trivially -- found to be consistent with the fully quantum-mechanical low-energy effective action description (valid at \( v_1, v_2 \sim \Lambda \)), available in these theories.

For \( G = SU(N+1), H = U(N), G_F = SU(N_f), N_f \geq 2N \), this argument proves that the monopoles induced by the \( G/H \) breaking transform as \( N \) of \( \tilde{H} = SU(N) \). Analogous result holds for \( G = SU(N+1), H = U(r), G_F = SU(N_f), r \leq N_f/2 \), where the semi-classical monopoles transform as in the fundamental multiplets \( (r) \) (as well as some singlets) of \( SU(r) \). These results are in agreement with what was found in the fully quantum mechanical treatment of the system. [59, 60, 64].

For \( G = SO(2N+1), H = U(r) \times U(1)^{N-r}, G_F = SU(N_f) \) (with \( r \leq N_f/2, r < N \)) we find monopoles which transform in the fundamental representation of the dual \( \tilde{SU}(r) = SU(r)_{c+f} \) group. This result is again consistent with the fully quantum mechanical analysis of \( \mathcal{N} = 2 \) supersymmetric \( SO(N) \) models [141] and in agreement with the universality of certain superconformal theories discovered in this context by Eguchi et. al. [145].

In the case of maximal-rank \( SU \) subgroup, such as \( G = SO(5), H = U(2) \), there is a qualitative difference both in our duality argument and in the full quantum results. For instance the set of monopoles found earlier by E. Weinberg is shown to belong to a singlet and a triplet representations of the dual \( SU(2) \) group, but their quantum fate is not known. In supersymmetric models a renormalization-group argument suggests (and the explicit analysis of softly broken \( \mathcal{N} = 2 \) theory shows) that the triplet does not survive the quantum effects, as long as the underlying \( SO(5) \) theory is asymptotically free.

For \( G = SO(2N), H = U(r) \times U(1)^{N-r}, G_F = SU(N_f) \) the situation is similar. When \( r < N - 1, r \leq N_f/2 \) we find monopoles transforming in the representation of the dual \( \tilde{SU}(r) = SU(r)_{c+f} \), whereas the maximal and next-to-maximal cases, \( r = N, N - 1 \), encounter the same renormalization-group constraint as in \( SO(2N+1) \).

Finally for \( G = USp(2N), H = U(r) \times U(1)^{N-r}, G_F = SU(N_f) \) the picture is very much like in \( SU(N + 1) \). We have monopoles in the fundamental representation of the dual \( \tilde{SU}(r) = SU(r)_{c+f} \) as long as \( N_f \geq 2r \).

Summarizing, in the context of softly-broken \( \mathcal{N} = 2 \) supersymmetric gauge theories with \( SU, SO \) and \( USp \) groups, where fully quantum me-
chanical results are available by combining the various knowledges such as the Seiberg-Witten curves, decoupling theorem, Nambu-Goldstone theorem, non-renormalization of Higgs branches, $\mathcal{N} = 1$ ADS instanton superpotential, vacuum counting, universality of conformal theories, etc., our idea on non-Abelian monopoles is in agreement with these known exact results.
Chapter 9

Universal Reconnection of non-Abelian Cosmic Strings

The issue of reconnection (intercommutation, recombination) of colliding cosmic strings attracts much interest recently (see [146] for reviews), owing to the fact that the reconnection probability is related to the number density of the cosmic strings, which is strongly correlated with possible observation of them. However, solitonic strings may appear in numerous varieties of field theories, which certainly makes any prediction complicated. In this Chapter, we employ the moduli matrix formalism to show that, in a wide variety of field theories admitting supersymmetric generalization, inevitable reconnection of colliding solitonic strings (i.e. reconnection probability is unity) is universal. The inevitable reconnection of local strings in Abelian Higgs model [148] (see also [150]) has been known for decades, and this universality was found in [90] for non-Abelian local strings in $N_C = N_F$ gauge theories. Via a different logic and explicit computations, the results of this Chapter extends the universality to semi-local strings [73, 74] with $N_C < N_F$, which is consistent with recent numerical simulations [151, 152].

The reconnection of the vortex strings can be understood [148, 149] as right-angle scattering of vortices in head-on collisions [153, 154, 155, 156, 157, 158, 159, 160] appearing in a spatial slice. We use moduli space approximation where the motion of the strings is slow enough, to find universal right-angle scattering of vortices on two spatial dimensions. The moduli matrix formalism gives a well-defined set of moduli coordinates, and with that the analysis of the motion is quite simple and robust.
9.1 Reconnection of non-Abelian local strings

We deal with the local strings \((N_C = N_F)\), followed by the semi-local strings \((N_C < N_F)\). We will find that essential feature can be captured in the case \(N_C = N_f = 2\). Single vortex \((k = 1)\) moduli space is \(\mathbb{C} \times \mathbb{C}P^1\) with \(\mathbb{C}\) the position of the vortex string in \(z\)-plane and \(\mathbb{C}P^1\) the orientational moduli concerning the internal color-flavor space \([7, 8, 16]\), while the moduli space of separated two \((k = 2)\) vortices is a symmetric product \((\mathbb{C} \times \mathbb{C}P^1)^2/\mathbb{S}_2\). The reconnection problem is related to how they collide in the full \(k = 2\) moduli space, parameterized by the moduli matrices (see Chapter 2):

\[
H_0^{(0,2)} = \begin{pmatrix}
1 & -az - b \\
0 & z^2 - az - \beta
\end{pmatrix},
H_0^{(1,1)} = \begin{pmatrix}
z - \phi & -\eta \\
-\tilde{\eta} & z - \tilde{\phi}
\end{pmatrix}.
\]

The superscripts label patches covering the moduli space, but one more patch \((2,0)\) is needed to cover the whole manifold. Since the \((0,2)\) patch covers all the moduli space except lower-dimensional submanifolds, this is sufficient for computing the reconnection probability. The moduli space of the two coincident vortices in this theory has been studied in Chapter 2 and found to be \(\mathbb{C} \times W\mathbb{C}P^2_{(2,1,1)} \simeq \mathbb{C} \times \mathbb{C}P^2/\mathbb{Z}_2\), which any collision of strings goes through. The locations \(z_1\) and \(z_2\) of the vortices and the orientation vectors \(\vec{\phi}_1\) and \(\vec{\phi}_2\) of the internal moduli for each vortex are determined by

\[
\det H_0 = (z - z_1)(z - z_2), \quad H_0(z = z_i)\vec{\phi}_i = 0.
\]

We parameterize the vectors as \(\vec{\phi}_i = (b_i, 1)^T\) with \(b_i = az_i + b\), and the relations to the original parameters are

\[
a = \frac{b_1 - b_2}{z_1 - z_2}, \quad b = \frac{b_2z_1 - b_1z_2}{z_1 - z_2}, \quad \alpha = z_1 + z_2, \quad \beta = -z_1z_2.
\]

Physical meaning of the parameters \((z_i, b_i)\) is clear, but they can cover only the subspace \(z_1 \neq z_2\) because the relations \((9.1.3)\) are not defined at \(z_1 = z_2\).

Let us consider slow motion of the moduli parameters, à la Manton [96], to show the universal right-angle scattering in the vortex collision. We have to use the parameters \((a, b, \alpha, \beta)\), not \((z_i, b_i)\), because, as we have shown, the moduli space metric with respect to the former parameters (which appear linearly in the moduli matrix \(H_0\)) is smooth and non-vanishing. With these “well-defined” parameters of the moduli space, at least for a certain period of time around the collision moment, one can approximate the moduli motion as linear functions of \(t\) (since the coordinates are subject to free motion):

\[
a = a_0 + \epsilon_1 t + \mathcal{O}(t^2), \quad b = b_0 + \epsilon_2 t + \mathcal{O}(t^2),
\]

\[
\alpha = 0 + \mathcal{O}(t^2), \quad \beta = \epsilon_3 t + \mathcal{O}(t^2).
\]
where $\epsilon_i$, $a_0$ and $b_0$ are constant. Here $\alpha$ is the center of mass of the vortices (see the later discussion for identifying the decoupled center-of-mass parameter), and thus set to be zero around $t = 0$. We have used a time translation so that a constant term in $\beta(t)$ vanishes. This is equivalent to choose the collision moment as $t = 0$.

Physical interpretation of the motion (9.1.4) and (9.1.5) can be extracted by looking at the solution in terms of $z_i$ and $b_i$. From (9.1.3), we obtain

$$z_1 = -z_2 = \sqrt{\epsilon_3 t} + O(t^{3/2}),$$  \hspace{1cm} (9.1.6)

$$b_i = b_0 + (-1)^{i-1}a_0\sqrt{\epsilon_3 t} + O(t).$$  \hspace{1cm} (9.1.7)

The first equation shows that the vortices are scattered by the right angle; since the time dependence is $\sqrt{t}$, when time varies from negative to positive, the vortex moves from the imaginary axis to the real axis. As stressed before, this right-angle scattering means that the vortex strings are reconnected. So, generic collision results always in reconnection.

When $a_0 = 0$ in (9.1.7), the orientational moduli for each vortex coincide, which corresponds to a reduction to the case of the Abelian-Higgs model. Here we have shown that even when $a_0 \neq 0$ and the non-Abelian strings have different orientational moduli at the initial time, as they approach each other in the real space, the internal moduli approach each other; in particular, $b_i$ experiences the right-angle scattering, too. This is the only consistent solution to the moduli equations of motion, with generic initial conditions. Note that this understanding comes from the re-description in terms of $b_i$ and $z_i$, while the true and correct motion in the moduli space is determined by the moduli parameters ($a, b, \alpha, \beta$), which have linear dependence in $t$.

Although we have shown (by using the (0,2) patch) that the reconnection probability is unity, it is instructive to look at the other patches to see what happens in the submanifold(s) of the moduli space which cannot be described by the (0,2) patch. In fact the submanifold includes the $\mathbb{Z}_2$ singularity of the $\mathbb{C}P^2/\mathbb{Z}_2$. This corresponds to the situation where the vortices sit in two decoupled $U(1)$ sub-sectors of the $U(2)$ in the original field theory and where strings should pass through each other in collision in that special case. In the (1,1) patch, the condition for coincident vortices, namely $\det H_0 = z^2$, reads

$$\tilde{\phi} = -\phi, \hspace{0.5cm} \phi\tilde{\phi} - \eta\tilde{\eta} = 0,$$  \hspace{1cm} (9.1.8)

which can be parameterized by $X$ and $Y$ through $XY = -\phi = \tilde{\phi}, X^2 \equiv \eta, Y^2 \equiv -\tilde{\eta}$. The $\mathbb{Z}_2$ symmetry $(X, Y) \sim (-X, -Y)$ is manifest. Note that the orbifold singularity $X = Y = 0$ $(\eta = \phi = \tilde{\eta} = \phi = 0)$ is present only in the submanifold $z_1 = z_2$, while the full moduli space is smooth. One can confirm
this by computing the Kähler potential explicitly around the origin of the
(1,1) patch, \( K = 2\pi c(|\phi|^2 + |\hat{\phi}|^2 + |\eta|^2 + |\hat{\eta}|^2) + \text{higher} \), which shows that there
the metric is smooth and non-vanishing. Going to the \((X,Y)\) coordinates on
the submanifold, we obtain a metric of a \( \mathbb{Z}_2 \) orbifold, \( K \propto (|X|^2 + |Y|^2)^2 \).

Let us study geodesic motion on the moduli space to see the reconnection.
After imposing the center-of-mass condition \( z_1 = -z_2 \), we obtain the motion
of the moduli parameters

\[
\phi = -\tilde{\phi} = -XY + s_1 t + O(t^2), \quad \eta = X^2 + s_2 t + O(t^2), \quad \tilde{\eta} = -Y^2 + s_3 t + O(t^2),
\]

where \( X, Y \) and \( s_{1,2,3} \) are constant. We have chosen the collision moment
to be \( t = 0 \), so that the constant terms in the above satisfy the constraint
\( (9.1.8) \). The orientational moduli \( b_i \) are obtained as \( b_i = \eta/(z_i - \phi) \).

From this generic solution of the equations of motion, we compute (for
\( |X|^2 + |Y|^2 \neq 0 \))

\[
z_1 = -z_2 = \sqrt{\phi^2 + \eta \tilde{\eta}} = \sqrt{s_1 t + O(t^{3/2})}, \quad (9.1.11)
\]

\[
b_i = X Y^{-1} + (-1)^i Y^{-2} \sqrt{s_1 t + O(t)}, \quad (9.1.12)
\]

where \( s = -2s_1 XY + s_3 X^2 - s_2 Y^2 \). Therefore, we confirm the generic re-
connection for \( s \neq 0 \). The condition \( s = 0 \) is equivalent to \( \epsilon_3 = 0 \) in the
analysis of the \( (0,2) \) patch, because among the patches we have a relation
\( \beta = \eta \tilde{\eta} - \phi \tilde{\phi} = s t \). \( s = \epsilon_3 = 0 \) can be achieved only by finely tuned initial
conditions, so we are not interested in it.

When \( X = Y = 0 \) (this point is not covered by the \( (0,2) \) patch, so the
identification \( s = \epsilon_3 \) fails), we obtain

\[
z_1 = -z_2 = \sqrt{s_1^2 + s_2 s_3} t + O(t^{3/2}), \quad (9.1.13)
\]

\[
b_i = s_1 s_3^{-1} + (-1)^i s_3^{-1} \sqrt{s_1^2 + s_2 s_3} + O(t^{1/2}), \quad (9.1.14)
\]

which shows no reconnection. Note that this finely tuned collision allows
constant non-parallel orientations \( b_1 \neq b_2 \) at the collision, in contrast to the
general case \( (9.1.7, 9.1.12) \) where \( b_1 = b_2 \) at \( t = 0 \). One observes that
the reconnection is intimately related to the parallelism of the orientation
vectors \( b_i \), as is along the intuition. But the significant is that parallel \( b_i \)
at the collision moment follows from generic initial conditions, which is clarified
here in the explicit computations in the moduli matrix formalism.

For \( N_C = N_F > 2 \) (the orientational moduli space is \( \mathbb{C}P^{N_C - 1} \)), the same
argument finds that the probability is unity. The moduli matrix of \((0, \cdots, 0, 2)\)
patch is
\[ H_0^{(0,\cdots,0.2)} = \begin{pmatrix} 1_{N_C-1} & \vec{a}z - \vec{b} \\ 0^T & z^2 - \alpha z - \beta \end{pmatrix}. \] (9.1.15)

The center-of-mass parameter is identified with \( \alpha \) and we put it zero. Then, we have \( \beta = z_1^2 \), and the solution of the equation of motion for \( \beta \) is the same as (9.1.5), after the time translation. Finally we have (9.1.6), therefore we conclude that reconnection occurs, irrespective of the other moduli parameters \( \vec{a} \) and \( \vec{b} \). Because the (0,0,\cdots,2) patch covers generic points of the moduli space, the reconnection probability is unity. The results are completely consistent with [90] which used a different logic though.

### 9.2 Reconnection of semi-local strings

We shall show that the reconnection probability is unity also for the semi-local strings, \( N_C < N_F \). We follow the same logic and find that it applies to rather generic theories, showing universality of reconnection. Since non-Abelian case can be examined straightforwardly, we concentrate on Abelian case with \( N_F = 2 \). The moduli matrix is
\[ H_0 = (z^2 - \alpha z - \beta, az + b). \] (9.2.1)

In the following, we shall show that (i) even in this semi-local case the center-of-mass coordinate is \( \alpha \) and thus put to be zero, and (ii) the parameter \( a \) (which is associated with the size of the vortex) is non-normalizable and put to be constant. Using these facts, the logic leading to the reconnection is the same for the remaining normalizable parameters: \( z_1 = \sqrt{\beta} = \sqrt{\epsilon_3} t \). We find the universality in reconnection. Note that the additional moduli parameters appearing from the extra flavors, \( a \) and \( b \), does not play any role in showing the reconnection. This is clearly the same even for non-Abelian semi-local strings. With the help of the moduli matrix, one can also show that the reconnected semi-local strings have the same width, which is expected from a geometrical viewpoint.

Let us identify the non-normalizable mode by studying possible infrared divergence in the Kähler potential (5.4.1). The asymptotic boundary condition for the master equation (3.1.15) is \( \Omega \to (1/c)H_0H_0^\dagger \), and using the expression of \( H_0 \) (9.2.1), we find only the first term in (5.4.1) is relevant. By the Kähler transformation \( K \to K + f + f^* \) with \( f \equiv -c \int d^2z \log(z^2 - \alpha z - \beta) \), we obtain for large \(|z|\)
\[ K \sim c \int d^2z \log \left[ 1 + \frac{|a|^2}{|z|^2} \right] \sim c \int d^2z \frac{|a|^2}{|z|^2} = 2\pi c |a|^2 \log L \]

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where in the last expression we introduced a cut-off radius $L(\rightarrow \infty)$. This divergence shows that the parameter $a$ is non-normalizable. We have to fix this mode to be constant, so that the effective Lagrangian is finite. In other words, motion of the parameter $a$ is frozen because the kinetic term of $a$ diverges and any motion costs infinite energy.

Next, we provide a method to determine the center-of-mass parameter, which is decoupled from the others. We write the moduli matrix in the following form,

$$H_0 = \begin{pmatrix} (z - z_1)(z - z_2), a(z - z_3) \end{pmatrix}.$$  \hfill (9.2.2)

in which the parameters are not the “well-defined” parameters. In this form, there is a translation symmetry $z \rightarrow z + \delta, z_i \rightarrow z_i + \delta$. Let us assume that $z_0$, which is a linear combination of $z_1, z_2$ and $z_3$, is the center-of-mass parameter. The other two parameters independent of $z_0$ should be selected properly from the three $z_i' \equiv z_i - z_0 (i = 1, 2, 3)$. We compute the metric from the Kähler potential, for this set of independent coordinates. The complete decoupling of $z_0$ from the remaining parameters is ensured if the metric component $g_{\bar{0}0} \equiv \delta^2 K / \delta z_i' \delta \bar{z}_0$ vanishes. We can compute it as

$$g_{\bar{0}0} = -\frac{\delta}{\delta z_i'} \int d^2 z \frac{\delta}{\delta \bar{z}} \tilde{K}(z, z_0, z_i') = -\frac{\delta}{\delta z_i'} \oint dz \tilde{K},$$  \hfill (9.2.3)

where $\tilde{K}$ is the integrand of the Kähler potential, and we used the fact that $z_0$ dependence in $\tilde{K}$ is always through the combination $z - z_0$. The explicit expression \[^{9.2.2}\] gives, after an appropriate Kähler transformation, for large $|z|$,

$$-\frac{\delta}{\delta z_i'} \oint dz c \log \left(1 - \frac{z_1 + z_2}{z} - \frac{\bar{z}_1 + \bar{z}_2}{\bar{z}} + \cdots\right) = 4\pi c \frac{\delta}{\delta z_i'} \frac{z_1 + z_2}{2}.$$  

Vanishing of this means that $z_i'$ is orthogonal to the combination $z_1 + z_2$, which shows that the center-of-mass parameter is $z_0 = (z_1 + z_2)/2 = \alpha/2$. This result is non-trivial, because there is another dimensionful parameter $z_3$ which might have been involved with the definition of the center-of-mass.

### 9.3 Summary

While we studied the critical coupling, non-critical region (which can be smoothly deformed from the critical coupling) has the same universality, since in the moduli space it is described by introduction of potential terms along relative position moduli induced by attractive/repulsive force between Type
I/II strings. Even for the repulsive case two strings must collide, because parts of two strings far from the collision point do not feel a force and the potential induced around the collision point is negligible compared with the total string energy. Adding small mass terms breaking flavor symmetry can be treated similarly (see for example [90]).

The universal reconnection we found is valid in a low energy regime where the moduli space approximation is valid classically. However, as in the case of Abelian-Higgs model, numerical simulations [151] showed robustness of the reconnection even for high energy collisions. We hope that, in the future observation, this universality may help for distinguishing solitonic strings from cosmic superstrings/D-branes which have lower reconnection probabilities [159] [150]. The moduli matrix formalism has opened up new paths to analyze BPS solitons. It would be intriguing to apply it further to more involved/realistic situations, such as cosmic string webs and thermal phase transitions.
Part III

Vortices in Arbitrary Gauge Theories
Chapter 10

Vortices and Lump

In Chapter 5 we discussed semi-local vortices and their relations with the sigma model lumps which arise in the theory at strong coupling. In this Chapter, we will deepen these relations. We will state the existence of a precise correspondence between vortices and lumps in a very wide class of theory, including and generalizing the discussions of Chapter 5. The existence of these links, enable us to consider a bottom-up approach to the construction of vortices. First, we will introduce a generalized construction for lumps. This construction will straightforwardly give us a construction for vortices, as soon as the gauge couplings are restored to finite values.

10.1 Model and BPS Vortex Equations

Let us start with presenting the general class of models in which we will construct vortices. We consider non-Abelian gauge theories with gauge group $G = G' \times U(1)$, whose Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + (\mathcal{D}_\mu H_A)^\dagger \mathcal{D}^\mu H_A$$

$$- \frac{e^2}{2} \left| H_A^t t^0 H_A - \frac{v^2}{\sqrt{2N}} \right|^2 - \frac{g^2}{2} |H_A^t t^a H_A|^2 .$$

(10.1.1)

The matter content of the model consists of $N$ flavors of Higgs scalar fields in the fundamental representation, with a common $U(1)$ charge, written as a color-flavor mixed $N \times N$ matrix $H$. This model is a direct generalization of (3.1.2), which has $G' = SU(N)$. The flavor symmetry of the model is $SU(N)_F$. It is important, as always, to remember that the model can be embedded in an $\mathcal{N} = 2$ supersymmetric gauge theory, which explains the
particular form of the potential, ensuring at the same time its stability against radiative corrections.

An important consequence of this is that we have several useful descriptions of the moduli space of vacua, which will be important in the upcoming discussion [161]:

$$\mathcal{M}_G = \{ H \mid \text{D-term conditions} \} / G$$

(10.1.2a)

$$= \{ H \} // G^c$$

(10.1.2b)

$$= \{ I_{G}^{ij}, \text{holomorphic } G\text{-invariants} \} / \{ \text{algebraic relations} \}$$

(10.1.2c)

Furthermore, $N = 2$ supersymmetry implies the existence of BPS saturated vortex solutions which are supported by the non-trivial first homotopy group $\pi_1(\mathcal{G} = G' \times U(1)) = \mathbb{Z}$. The standard Bogomol'nyi completion for static, $x^3$-independent, configurations

$$T = \int d^2x \left[ \frac{1}{2e^2} \left| F_{12}^0 - e^2 \left( H_A^1 t^0 H_A - \frac{v^2}{\sqrt{2N}} \right) \right|^2 + 4 |D_z H|^2 + \frac{1}{2g^2} \left| F_{12}^a - g^2 H_A^1 t^a H_A - \frac{v^2}{\sqrt{2N}} F_{12}^0 \right|^2 \right] \geq - \frac{v^2}{\sqrt{2N}} \int d^2x F_{12}^0,$$

(10.1.3)

gives the BPS equations for vortices.

We will focus our attention on the classical Lie groups $SU(N)$, $SO(2M)$ and $USp(2M)$. For $G' = SO(2M)$, $USp(2M)$ their group elements are embedded into $SU(N)$ ($N = 2M$) by constraints of the form, $U^T J U = J$, where $J$ is the rank-2 invariant tensor

$$J = \begin{pmatrix} 0_M & 1_M \\ \epsilon_1 M & 0_M \end{pmatrix},$$

(10.1.4)

where $\epsilon = +1$ for $SO(2M)$, while $\epsilon = -1$ for $USp(2M)$. Specializing (10.1.3) to these cases we get the BPS vortex equations

$$D_z H = 0,$$

(10.1.5a)

$$\frac{\sqrt{2N}}{e^2} F_{12}^0 - (\text{Tr}(HH^\dagger) - v^2) = 0,$$

(10.1.5b)

$$\frac{4}{g^2} F_{12}^a t^a - (HH^\dagger - J^\dagger (HH^\dagger)^T J) = 0,$$

(10.1.5c)

where the complex coordinate $z \equiv x^1 + ix^2$ has been introduced. Eq. (10.1.5c) reads for $G' = SU(N)$ instead:

$$\frac{2}{g^2} F_{12}^a t^a - \left[ HH^\dagger - \frac{1}{N} \text{Tr}(HH^\dagger) \right] = 0.$$
10.2 The Strong Coupling Limit

Another class of theories we are interested in, is the set of non-linear sigma models which can be obtained from the theories (10.1.1) by taking the strong coupling limit: \( e, g \to \infty \). In a fully Higgsed vacuum, all gauge fields are massive and can be integrated out, and the limit is reliable. The resulting theory is an \( \mathcal{N} = 2 \) NL\( \sigma \)M in 3+1 dimensions whose target space is the vacuum manifold of the gauge theory:

\[
L_{NL\sigma M} = g_{i\bar{j}} \phi_i \partial \mu \phi_i \bar{\phi}_j, \quad g_{i\bar{j}} = \partial \phi_i \partial \phi_j K(\phi_i, \bar{\phi}_j), \quad \mathcal{M}_{NL\sigma M} = \mathcal{M}_G. \tag{10.2.1}
\]

The fields \( \phi_i \) are a conveniently chosen set of independent coordinates on the target space while \( K \) is the Kähler potential of the gauge theory.

This class of models also admits BPS, \( x^3 \)-independent, stringy soliton called lumps. This kind of solitons are supported by a non-trivial homotopy group \( \pi_2(\mathcal{M}_{NL\sigma M}) \). It is indeed very easy to find the equations for lumps without going into the explicit computation of the Kähler potential. In fact, taking the strong coupling limit in (3.1.2) without explicitly integrating out the gauge fields, we obtain a description of the models (10.2.1) in terms of the fields of the original gauge theory \( (H, W_\mu) \). Lump equations are simply given by the strong coupling limit of the vortex equations (10.1.5).

\[
D_{\bar{z}} H = 0, \tag{10.2.2a}
\]
\[
\text{Tr}(H H^\dagger) - v^2 = 0, \tag{10.2.2b}
\]
\[
H H^\dagger - J^\dagger (H H^\dagger)^T J = 0. \tag{10.2.2c}
\]

Notice that Eq. (10.2.2b) and Eq. (10.2.2c) are nothing but the D-term vacuum equations, whose solutions define \( \mathcal{M}_{NL\sigma M} \).

A general property of the models (10.2.1) is the classical scale invariance. Lumps develop a moduli space of degenerate solutions with arbitrary size and consequently there appear singular solutions with zero size, called small-lump singularities. Another way to understand these singularities is to notice that the NL\( \sigma \)Ms are well defined only on a fully Higgsed vacuum, while the theories (3.1.2) have a moduli space of vacua that usually includes many branches with different physical properties: Higgs and Coulomb branches, mixed Higgs-Coulomb phases in which the gauge symmetry is only partially broken. Lump solutions can be understood as maps from the \( \mathbb{C} \)-plane orthogonal to the \( x^3 \)-axis, into the target space \( \mathcal{M}_{NL\sigma M} \) and we expect singularities when this map hit a point which corresponds to a phase of the original gauge theory with an unbroken gauge symmetry. In fact, in this points, the appearance of massless particles gives rise to singularities in the Kähler potential.
of the gauge theory, and thus on the target manifold of the $\text{NL} \sigma \text{M}$. For example, small-lump singularities are usually related to conical singularities in the target space.

Before giving the recipes for the construction of lumps, we will comment, in the next section, on the correspondence between vortices in the gauge theories (3.1.2) and lumps in the related $\text{NL} \sigma \text{M}$ (10.2.1). A consequence of this correspondence is that we get, as promised, the moduli space of vortices.

10.3 Vortex-Lump Correspondence

The deep connection between vortices and lumps was first noticed by Hindmarsh [74]. In the extended Abelian Higgs model “semilocal” vortices [73, 163] appear, which are generalizations of the usual Abrikosov-Nielsen-Olesen (ANO) vortices: they possess zero modes related to an arbitrary size parameter. Semilocal vortices “interpolate” between ANO vortices and $\mathbb{C}P^n$ lumps, in the sense that zero-size semilocal vortices are ANO (“local”) vortices, while semilocal vortices with increasing size become identical to lumps.

We consider this relationship from another, but equivalent, point of view [6]: taking the strong coupling limit $e, g \to \infty$ we map the gauge theory (3.1.2) into the sigma model (10.2.1); at the same time, we map any vortex appearing in the first theory into a lump of the latter:

$$G' \times U(1) \text{ Gauge theory } \longrightarrow \text{ NL} \sigma \text{M on } \mathcal{M}_G$$
$$\text{Vortices } \longrightarrow \text{ Lumps.} \quad (10.3.1)$$

This correspondence can also be better understood if one consider more quantitative relations.

**BPS correspondence.** Vortices and lumps are BPS saturated objects. Their masses do not depend on the gauge couplings, being proportional to the central charge of the supersymmetry algebra: $T_{\text{vorf}} = M_{\text{lump}}$.

**Topological correspondence.** In a generic case in which the vacuum manifold is simply connected, the following relations hold: $\pi_2(\mathcal{M}_G) = \pi_2(\mathcal{M}_{\text{vacman}}/G) = \pi_1(G)$, which define a correspondence between the topological charges of vortices and lumps.

It is important, however, to notice that some vortex configurations will be mapped into singular lumps. For example, ANO vortices, with a typical size

$^1$ $\mathcal{M}_{\text{NL} \sigma \text{M}}$ can contain also different kind of singularities, like curvature singularities. They give rise to different kinds of singular lumps [162].

$^2$ This property of the vacuum manifold is the necessary condition for the existence of semilocal vortices [163].
\[ M_{\text{vor}} = M_{\text{lum}} \oplus \{ \text{singular lumps} \} \] (10.3.2)

### 10.4 A Generalized Rational Map Construction for Lumps

After the general discussion of the previous Section, we are now ready to explicitly construct the moduli space of lumps arising from (10.2.1).

We choose the fully Higgsed, color-flavor locked vacuum: \( H_{\text{vev}} = \frac{v}{\sqrt{N}} 1_N \). The \( G' \times U(1) \times SU(N)_F \) invariance of the theory is broken to the global color-flavor diagonal \( G'_{C+F} \). The first step toward the solutions of Eq. 10.2.2 is to switch from the description (10.1.2a) for the moduli space of vacua to that of (10.1.2b). We can thus forget the D-term vacuum equations (10.2.2b) and (10.2.2c), at the prize of lifting the gauge symmetry \( G \) to its complexification \( G_C \). Eq. 10.2.2a has now a very intuitive meaning. It can be thought as a “covariant holomorphicity” condition. This means that \( H \) is holomorphic, up to a complexified gauge transformation:

\[
H = S^{-1} H_0(z) = S_c^{-1} S'^{-1} H_0(z),
\] (10.4.1)

where \( S \in G_C \), while \( H_0(z) \) is a matrix whose elements are holomorphic in \( z \). We denote \( H_0(z) \) the moduli matrix, as it encodes all moduli parameters\(^3\). Eq. 10.4.1 does not fix completely \( G_C \). \( H_0(z) \) is in fact only defined up to holomorphic gauge transformations \( V(z) \):

\[
H_0 \sim V(z) H_0 = V_c V'(z) H_0, \quad V' \in G^C, \quad V_c \in \mathbb{C}^*. \tag{10.4.2}
\]

The next step is to consider the holomorphic invariants \( I_G^{(i,j)}(H) \) made of \( H \), which are invariant under \( G^C \), with \( (i, j) \) labeling them. This is equivalent to changing our description of \( M_G \) from Eq. 10.1.2b to Eq. 10.1.2c. The key point here is that, thanks to Eq. 10.4.1, these invariant are also holomorphic functions of the coordinate \( z \):

\[
I_G^{(i,j)}(H) = I_G^{(i,j)}(S^{-1} H_0(z)) = I_G^{(i,j)}(H_0)(z). \tag{10.4.3}
\]

\(^3\)To reconstruct the solution in terms of the original Higgs fields \( H \), one has still to solve Eq. 10.2.2 as algebraic equations for \( S \).
We have thus identified a lump as a holomorphic map from the complex plane $\mathbb{C}$ to the set of holomorphic invariants $I_G^{(i,j)}$. The last step is to impose the following boundary conditions to the holomorphic invariants:

$$ I_G^{(i,j)}(H)|_{|z| \to \infty} = I_{\text{vev}}^{(i,j)}. \quad (10.4.4) $$

It is convenient to translate the above conditions in terms of the holomorphic invariants of $G'$, $I_G^{(i,j)}(H_0)$:

$$ I_G^{(i,j)}(H_0) \equiv \frac{I_G^i(H_0)}{I_G^j(H_0)}, \quad (10.4.5) $$

where we must take ratios of invariants with the same $U(1)$ charge: $n_i = n_j$.

Eq. 10.4.5 defines the generalized rational map construction for lumps.

It is easy to check that Eq. 10.4.4 and Eq. 10.4.5 together imply the following:

$$ I_G^i(H_0)|_{|z| \to \infty} = I_{\text{vev}}^i z^{\nu n_i}. \quad (10.4.6) $$

where $\nu$ is an integer common to all invariants. As $I_G^i(H_0(z))$ are holomorphic, the above condition means that $I_G^i(H_0(z))$ are polynomials in $z$. Furthermore, $\nu n_i$ must be a positive integer for all $i$:

$$ \nu n_i \in \mathbb{Z}_+ \quad \Rightarrow \quad \nu = k/n_0, \quad k \in \mathbb{Z}_+, \quad (10.4.7) $$

where (GCD = the greatest common divisor)

$$ n_0 \equiv \text{GCD}\{n_i \mid I^i_{\text{vev}} \neq 0\}. \quad (10.4.8) $$

Note that a $U(1)$ gauge transformation $e^{2\pi i/n_0}$ leaves $I_G^i(H)$ invariant:

$$ I_G^i(H') = e^{2\pi i n_i/n_0} I_G^i(H) = I_G^i(H) : (10.4.9) $$

the phase rotation $e^{2\pi i/n_0} \in \mathbb{Z}_{n_0}$ changes no physics, and the true gauge group is thus $G = U(1) \times G'/\mathbb{Z}_{n_0}$, where $\mathbb{Z}_{n_0}$ is the center of $G'$. A simple homotopy argument tells us that $1/n_0$ is the $U(1)$ winding for the minimal,

$^4$All the points at the infinity of $z$ are identified, so that the map is in fact defined on $\mathbb{C}P^1$.

$^5$The $U(1)$ charge $n_i$ is simply defined as: $I_G^i(\alpha H_0) = \alpha^{n_i} I_G^i(H_0)$, for any complex number $\alpha$. 168
\[ k = 1, \text{ topologically non-trivial configuration. Finally, for a given } k \text{ the following important relation holds} \]

\[ I_{G'}^\prime(H_0) = I_{vev}^i z^{kn_i/n_0} + \mathcal{O}(z^{kn_i/n_0-1}) , \quad (10.4.10) \]

which implies nontrivial constraints on \( H_0(z) \). The set of all inequivalent \( H_0(z) \) satisfying Eq. (10.4.10) identify a lump solution, and thus defines the full moduli space.

### 10.5 Moduli Space of Vortices

The lump construction explained in the last Section is only partially complete, because one has to identify all singular lump solutions, and this is not easy in general. This problem disappears when we lift the construction for lump to that for vortices in the related gauge theory, according to the discussion of Sec. [10.3](#). As explained there, singular lumps are regularized into vortices as we turn on finite gauge couplings, and Eq. (10.4.10) now give the full and complete description of the moduli space of non-Abelian vortices.

To reconstruct the original Higgs fields \( H = S^{-1}H_0 \) we now have to solve the BPS differential equations Eq. (10.1.5b) and Eq. (10.1.5c) (or Eq. (10.1.6)) in terms of \( S \), while from Eq. (10.1.5a) we find the gauge fields:

\[ W_1 + iW_2 = -2iS^{-1}\tilde{\partial}S. \quad (10.5.1) \]

The tension of the BPS vortices can be written as

\[ T = -\frac{v^2}{\sqrt{2N}} \int d^2x F_{12}^0 = 2v^2 \int d^2x \tilde{\partial}\tilde{\partial} \log(S e^S_e). \quad (10.5.2) \]

The asymptotic behavior \( S_e \sim |z|^\nu \) then determines the tension

\[ T = 2\pi v^2 \nu , \quad (10.5.3) \]

thus \( \nu \) has to be identified with the \( U(1) \) winding number of the vortex configuration.

Let us now apply the general discussion above to concrete examples. For \( G' = SU(N) \), with \( N \) flavors, there only exists one invariant

\[ I_{SU} = \det(H) , \quad (10.5.4) \]

with charge \( N \). Thus the minimal winding is equal to \( 1/N \) and the condition for \( k \) vortices is given by:

\[ A_{N-1} : \det H_0(z) = z^k + \mathcal{O}(z^{k-1}), \quad \nu = k/N . \quad (10.5.5) \]
For $G' = SO(2M), USp(2M)$, there are $N(N \pm 1)/2$ invariants
\[(I_{SO,USp})_{r,s} = (H^T J H)^{r,s}, \quad 1 \leq r \leq s \leq N,
\] in addition to (10.5.4). The constraints are:
\[C_M, D_M : H_0^T (z) J H_0(z) = z^k J + O(z^{k-1}), \quad \nu = k/2.
\] Thus, vortices in the $SO(2M)$ and $USp(2M)$ theories are quantized in half integers of the $U(1)$ winding [164].

Explicitly, the minimal vortices in $SU(N)$ and $SO(2M)$ or $USp(2M)$ theories are given respectively by the moduli matrices
\[H_0 = \begin{pmatrix} z - a & 0 \\ b & 1_{N-1} \end{pmatrix}, \quad \begin{pmatrix} z 1_M - A & C_{S/A} \\ B_{A/S} & 1_M \end{pmatrix},
\] together with all permutations of lines and columns. The moduli parameters are all complex. For $SU(N)$, $a$ is just a number; $b$ is a column vector. For $SO(2M)$ or $USp(2M)$, the matrix $C_{S/A}$ for instance is symmetric or antisymmetric, respectively. And vice versa for $B$. A detailed derivation of the matrices written above, together with the construction of moduli matrices for $SO(2M + 1)$ as well as those for $k = 2$ vortices in $SU, SO, USp$ theories, will be given explicitly in the next Chapter (see Refs. [162, 165] for more details).

The complex dimension of the moduli space, given by the number of complex parameters in the moduli matrices (10.5.8), is
\[
\text{dim}_C (\mathcal{M}_{G',k}) = \frac{k N^2}{n_0}.
\] This result was firstly obtained in Ref. [7] for $SU(N)$; The dimension of the moduli space, inferred from the moduli matrices, agrees with the one given in Eq. (10.5.9) and is the same for any choice of the gauge group $G'$, where $N$ is the dimension of the fundamental representation of $G'$ for which $\mathbb{Z}_{n_0}$ is the center.

Except for the $SU(N)$ case, our model has a non-trivial Higgs branch (flat directions). The color-flavor locked vacuum $H_{\text{vev}} \propto 1_N$ is just one of the possible (albeit the most symmetric) choices for the vacuum; our discussion can readily be generalized to a generic vacuum on the Higgs branch. This fact, however, implies that our non-Abelian vortices have semi-local moduli even for $N_f = N$. In contrast to the Abelian or $SU(N)$ cases, moreover, they exhibit new, interesting phenomena such as “fractional” vortices [165].

The generalization to exceptional groups can be done if the invariant tensors of each group are known. Indeed, this is the case, and Tab. [10.1] lists them all.
Table 10.1: The dimension of the fundamental representation ($R$), the rank of the other invariants \([166]\) and the minimal tension $\nu = 1/n_0$ i.e. the center $\mathbb{Z}_{n_0}$ of $G'$. The determinant of the $R \times R$ matrix gives one invariant with charge, $\text{dim } R$.

### 10.6 Local (ANO-like) Vortices

For various considerations, we are interested in knowing which of the moduli parameters describe the local vortices, the ANO-type vortices with exponential tails. The moduli space of fundamental local vortices, for example, is completely generated by the global symmetries of the vacuum (see below) and it can eventually survive non-BPS deformations that preserve these symmetries. Furthermore, we found in Section 5.4 that local vortices correspond to the subset of solutions with the maximum number of normalizable moduli.

Local vortices correspond, as we mentioned, to small lump singularities in the lump construction. These configurations are obtained when the rational map construction is degenerate, e.g., when all the invariants $I_i$ share a common zero:

$$I_i^{G'} = (z - z_0)^{n_i/n_0} I_{G'}^n.$$  \hfill (10.6.1)

The rational map in Eq. (10.4.5), does not feel any of these common zeros, which can be interpreted as insertions of singular spikes of energy, or local vortices. Configurations with only local vortices are obtained imposing a complete degeneration of the rational map:

$$I_i^{G'}(H_{0,\text{local}}) = \left[ \prod_{\ell=1}^{k} (z - z_{0\ell}) \right]^{n_i/n_0} I_{\text{vev}}^i.$$ \hfill (10.6.2)

For $G' = SO(2M), USp(2M)$ with $I_{SO,USp}$ of Eq. (10.5.6) we find that the condition for vortices to be of local type is

$$H_{0,\text{local}}^T(z)JH_{0,\text{local}}(z) = \prod_{\ell=1}^{k} (z - z_{0\ell}) J.$$ \hfill (10.6.3)
Let us now discuss a few concrete examples. The general solution for the minimal vortex \((10.5.8)\) for \(G' = \{SU(N), SO(2M), USp(2M)\}\) is reduced to a local vortex if we restrict it to be of the form:

\[
H_{0,\text{local}} = \begin{pmatrix}
    z - a & 0 \\
    b & 1_{N-1}
\end{pmatrix}, \begin{pmatrix}
    (z - a)1_M & 0 \\
    B_{A/S} & 1_M
\end{pmatrix}.
\] (10.6.4)

The vortex position is given by \(a\), while \(b\) for \(SU(N)\) and \(B_{A/S}\) for \(SO(2M)\) or \(USp(2M)\) encode the Nambu-Goldstone modes associated with the breaking of the color-flavor symmetry by the vortex \(G'_{C+F} \to H_{G'}\). The moduli spaces are direct products of a complex number and the Hermitian symmetric spaces

\[
\mathcal{M}^{\text{local}}_{G',k=1} \simeq C \times G'_{C+F}/H_{G'} .
\] (10.6.5)

\(H_{SU(N)} = U(N-1)\) while \(H_{SO(2M),USp(2M)} = U(M)\). The \(SU(N)\) and \(SO(2M)\) cases have already been studied in Refs. [7, 16, 164]. In the next Chapter we will going to discuss in more details all these cases. Furthermore, we will generalize these results to \(USp\) and \(SO(2M + 1)\) theories.
Chapter 11

Orthogonal and Symplectic Vortices
Chapter 12

Semi-classical hints of GNOW
Duality from vortex side
Conclusions and Outlook

The developments in our knowledge about non-Abelian vortices, like those I described in this Thesis are a new, big, step in our capacity to handle solitons in supersymmetric gauge theories. In particular, the potential extension of the moduli matrix formalism to all gauge theories supporting BPS solitons, will give us the possibility to significantly improve our understanding about the physics of solitons.

Now we have the technologies to start a broad and systematic study of solitons, for which the following points are only possible examples of future line of research.

Constructing new types of solitons

It will be interesting to explicitly apply our construction to models different from those studied until now. The first step has been already partially addressed, constructing non-Abelian vortices in theories with $SO(N) \times U(1)$ or $USp(2M) \times U(1)$ gauge groups. Other interesting possibilities would be to study how the properties of solitons change with the choice of the matter content and in particular with respect to the choice of the representations of the fields. Exceptional groups can also be considered. Groups like $F_4$ and $G_2$ have attracted much attention, especially in lattice simulations. They have a trivial center, avoiding the possibility that confinement is given by condensation of center vortices (instead of being due to the condensation of monopoles).

The generalized moduli matrix construction should also be applied, in the near future, to explore different kinds of solitons in this new set of theories. We think of domain-walls and monopoles, together with composite configuration of solitons, like domain-wall-vortex junctions, monopole as confined kinks on vortices and so on. These configurations have already been studied in $U(N)$ gauge theories with fields in the fundamental representation. They can now be constructed in a more general set-up, giving rise to the possibility of making a real complete and systematic survey of BPS solitons in supersymmetric gauge theories.
This program could be useful in the “reverse engineering” of string theory from field theory. As is well-known, string theory has been used to construct, with a geometrical approach, field theories and their solitons. These constructions have been very useful in the understanding of many properties of gauge theories. More recently, it has become clear that the inverse process is also possible. For example, domain-walls have been used to “reproduce” the physics on D-branes. The biggest difficulty in this project is to show the actual localization of a non-Abelian gauge field on the domain-wall. From a general point of view, a wider understanding of solitons is crucial to fully appreciate the relations between string theory and gauge theories.

**Non-Abelian vortices and 4-d gauge dynamics**

There are many interesting developments which could be realized along this line of research. Until now, the matching between the vortex theory and the bulk theory has been demonstrated only in the case of $U(N)$ gauge theories. The first step forward would be the study of the quantum physics of the vortex theory in the generic case $G' \times U(1)$. This would give us a new set of 2d/4-d correspondences.

A more recent progress is regarding to the study of the same issue in theories with $\mathcal{N} = 1$ supersymmetry. The quantum physics of the vortex (now dubbed heterotic) has been shown to capture the physics of the bulk theory. In particular, the quantum deformations of the moduli space of vacua are reproduced from the vortex side, together with supersymmetry breaking and a successive restoration mechanism, as soon as a meson field expectation value is turned on. It is very interesting to consider the possibility that the vortex theory could also mimic the physics of Seiberg duality. To push forward this program, it is crucial to understand how the meson fields of the Seiberg-dual theory will deform the vortex theory. Semi-local vortices will also play a crucial role in this context. These considerations may indicate the existence of dual (vortex) theories. Indeed, there are hints that the moduli space of vortices transforms in an interesting fashion under the transformation $N_f \to N_f - N_c$.

**Non-Abelian monopoles from the vortex side**

Non-Abelian vortices may also help to push forward the study of non-Abelian monopoles, using similar strategies to those I have used in my past research. The main step forward will be the generalization of previous works on the hierarchical system $G \to G' \times U(1) \to 1$, where $G' \times U(1)$ is a subgroup of a generic simple group $G$. We have now the technology to construct
non-Abelian vortices arising from the low-energy breaking \( G' \times U(1) \to 1 \). These vortices will confine a very general class of non-Abelian monopoles which arise from the \( G \to G' \times U(1) \) breaking. We believe that the moduli space of the confining vortex captures the important properties of that of the confined monopole, including the quantum ones. This strategy seems to be consistent in the \( G = U(N) \) case. From this point of view, the vortex should provide a regularization of the non-normalizable modes of the monopoles (these modes are responsible for the well-known difficulties in quantizing them). The proposed generalization is mostly important in the context of verifying GNOW duality. While the \( U(N) \) group is self-dual, other gauge groups have really different duals, for example: \( SO \leftrightarrow USp \). This is crucial to really have strong hint of the appearance of new symmetries in theories supporting non-Abelian monopoles.

**Further research lines**

It is tempting to speculate that non-Abelian vortices may play a role in the physics of a newly proposed mechanism of dynamical supersymmetry breaking \([167]\). These new results have been important for a renewed hope to construct supersymmetric models with a true phenomenological relevance. This new framework is intriguing. First it gives a solution to the long-standing problem of supersymmetry breaking. Moreover, it puts together the phenomenology of supersymmetry with the (strong coupling) physics of Seiberg duality. Seiberg duality may be the link between dynamical supersymmetry breaking and the vortex physics. In fact, a study of vortices in metastable vacua has already started in some depth.

Another interesting and recently developed field is the gauge/gravity correspondence (ADS/CFT). The correspondence gives us the possibility to study the strong coupling regime, and confinement, from a different perspective. Gravity duals of vortices in confining, \( \mathcal{N} = 1^* \), theories have already been studied, and a Casimir scaling law for the tension has been checked. It would be interesting to construct gravity duals of non-Abelian vortices in theories like those we have previously considered.

Non-Abelian vortices may also be relevant in future experiments of condensed matter. They appear in many low-energy phenomena, like superconductivity and superfluidity. Their importance is also recognized in the phenomenology of the fractional quantum Hall effect. In the last case, vortices appear as solitons in a (2+1)d Chern-Simons effective theory. Non-abelian vortices can be detected in systems with non-Abelian statistic, which can be realized, for example, in multi-layered materials. One interesting question is the formation of lattices. In a type II superconductor, for example,
vortices form the hexagonal Abrikosov lattice. Recently found 1.5 superconductors admit vortices which have interaction similar to those we found in the non-Abelian case, with static inter-vortex forces, which change sign with respect to the relative distance. Our study about interactions is a first step to understand the shape and the structure of lattices of non-Abelian vortices.
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