Teorie di Campo con Bordo

Relatori:
Prof. Nicola Maggiore
Prof. Nicodemo Magnoli

Correlatore:
Prof. Renzo Collina

Candidato: Stefano Storace

Anno Accademico 2008-2009
Field Theories With Boundary

Supervisors:
Prof. Nicola Maggiore
Prof. Nicodemo Magnoli

Assistant Supervisor:
Prof. Renzo Collina

Candidate: Stefano Storace

Academic Year 2008-2009
Contents

Introduction

1 The scalar field
   1.1 Introduction of a boundary ................................. 7
   1.2 Boundary conditions ..................................... 12

2 Chern-Simons theory ........................................... 17
   2.1 The action and its properties ............................... 24
   2.2 Equations of motion ....................................... 26
   2.3 Introduction of the boundary ............................... 27
   2.4 Boundary conditions ..................................... 30
   2.5 Conserved boundary currents ............................... 31

3 Maxwell-Chern-Simons theory .................................. 35
   3.1 The action and its properties ............................... 41
   3.2 Equations of motion ....................................... 43
   3.3 Introduction of the boundary ............................... 45
   3.4 Boundary conditions ..................................... 53
   3.5 Parameters, boundary conditions
       and residual Ward identity ............................... 57

Conclusions ...................................................... 63

Bibliography .................................................... 66
Introduction

In the last decades, there has been vast interest in the role of the space boundary in Quantum Field Theory [1, 2, 3]. The main reason is that the presence of a boundary changes the properties of a physical system, originating phenomena like the Casimir Effect [4] or the edge states in the Fractional Quantum Hall Effect [5].

In some cases, the boundary even represents the center of the investigation, as in the case of black holes [6], where the horizon of events can be indeed regarded as a boundary. Boundary effects play an important role also in critical phenomena [7].

In particular, the effect of a boundary is extremely interesting in topological field theories. Indeed, topological theories are known to have no local observables but in the case in which the base manifold has a boundary [2, 3]. Our attention will be actually focused on the topological, three dimensional Chern-Simons (CS) model [2, 8, 9].

The introduction of a boundary in CS theory has been investigated in previous works [2, 3], with the result that the local observables arising from the presence of the boundary are two-dimensional conserved chiral currents generating the Kać-Moody algebra [10] of the Wess-Zumino-Witten model [11]. In other words, the CS theory with boundary establishes a strong connection with conformal field theories [12].

An analogous result has been obtained for another topological field theory, namely the topological BF model [13] in three dimensions, for which it has been shown as well the existence of chiral currents living on the boundary and satisfying a Kać-Moody algebra with central extension [14].

The existence of a Kać-Moody algebra with central extension in the CS model with boundary has led to interesting applications due to the connection of the CS theory with several physical systems.

In fact, following the equivalence of the CS theory to the (2+1) dimensional gravity theory with a cosmological constant [15, 16], it has been
stressed that the algebra plays a crucial role in understanding the statistical origin of the entropy of a black hole [6]. It is thus possible to use the algebra to compute the BTZ black hole (negative cosmological constant) entropy [17] and the Kerr-de Sitter space (positive cosmological constant) entropy [18].

In condensed matter physics, the abelian CS model provides an effective low energy theory for the Fractional Quantum Hall Effect [5]. Further, when a boundary is taken into account, the Kač-Moody algebra describes the boundary chiral currents which are indeed observed on the edge of the Hall bar (edge states) [5].

On the other hand, the coupling of the CS model to other theories gives rise to interesting results. For instance, it can be coupled to fermion fields to attach magnetic flux to charge density, thus providing an explicit realization of anyons, i.e. particles living only in systems with two spatial dimensions and satisfying a fractional statistics [19].

Among the others, one of the most striking properties of the CS term is that, when added to the three dimensional Yang-Mills action, it originates a topologically massive gauge theory [20].

Its abelian version, the Maxwell-Chern-Simons (MCS) theory, when coupled with fermions defines a three dimensional modified electrodynamics, in which the “photons” are massive and have a single state of helicity [20]. The Casimir effect for topologically massive electrodynamics could provide, in principle, a way to “measure” the topological mass of the “photons” [21].

Moreover, the addition of a Maxwell term to the effective low energy theory for the Quantum Hall Effect allows the description of the gap between the ground state and the bulk elementary excitations [5].

The Maxwell theory coupled to the CS action on a manifold with boundary has a further application in (2+1) dimensional quantum gravity: the so called Einstein-Maxwell-Chern-Simons theory [22]. In this framework, the boundary can be regarded as the horizon of the black hole solutions, and the gauge field coupled to gravity describes a topologically massive electromagnetic field which provides the black hole with an electric charge, which, coming from the CS term, is a kind of topological. The result is thus called “charged black hole”.

On the other hand, the Maxwell model is not topological, and therefore the question if its addition spoils the chiral current algebra of the CS theory arises naturally: the aim of this thesis is actually to discuss this issue.

To reach this task, we first must face a further problem, i.e. the method. Indeed, the inclusion of a boundary in field theory is a highly non trivial task if one wishes to preserve locality and power counting, the most basic
In 1981 K. Symanzik [1] addressed this question: his key idea was to add to the bulk action a local boundary term which modifies the propagators of the fields in such a way that nothing propagates from one side of the boundary to the other. He called this property “separability” and showed that it requires the realization of a well identified class of boundary conditions that can be implemented by a local bilinear interaction.

These ideas strongly inspired the authors of [23] and [24], who used a closely related approach to compute the chiral current algebra living on the boundary of the three-dimensional CS model.

Indeed, the authors of [23] added to the action local boundary terms compatible with power counting, using a covariant gauge fixing. In [24], on the other hand, a regularization-free procedure was followed: the equations of motion, rather than the action, were modified by appropriate boundary terms, and a non-covariant axial gauge was preferred.

The main reason for the latter choice was that the main advantage of a covariant gauge, i.e., Lorentz invariance, already fails due to the presence of the boundary. On the other hand, the axial choice does not completely fix the gauge [25], and a residual gauge invariance exists, implying the existence of a Ward identity which plays a crucial role since, when restricted to the boundary, it might generate a chiral current algebra, as we shall see.

In both these works, the explicit computation of the propagators of the theory seems to be necessary. However, this step could be quite difficult in other theories, like MCS that we are going to study, and another way of investigation is worthy.

This is precisely the choice that we make. Our approach is actually more similar to that used in [24], since we focus on the effect of the boundary on the equations of motion rather than on the action, and we adopt a noncovariant axial gauge rather than a covariant one.

However, the original part of this work does not consist only in the study of the MCS theory with boundary, but also in the method, which is new, since it allows to avoid the explicit computation of the Green functions of the theory.

The basic idea is that, after modifying the equations of motion by means of boundary terms satisfying general basic requirements, we integrate them in proximity of the boundary and use Symanzik’s idea of separability to determine the boundary conditions on the propagators, that can be expressed as boundary conditions on the fields [1]. These, in turn, have an effect on
the boundary breaking term of the residual Ward identity that generates the Kač-Moody algebra living on the boundary.

This thesis is organized as follows. In Chapter 1 we illustrate our method by applying it to the simple case of a scalar field theory, showing that it leads to the correct known results found in [1, 26]; most of the conventions are there also established.

In Chapter 2 we examine the abelian CS theory with boundary, and we show that our method leads again to the correct results found in [23] and [24] for both the boundary conditions and the residual Ward identity, and we illustrate how the Ward identity generates the Kač-Moody algebra. We then make some physical remarks about this result.

Although the method illustrated in Chapters 1 and 2 cannot be entirely found elsewhere already, the main original results of this work can be found in Chapter 3, where we introduce the Maxwell term in the CS theory. We follow the same scheme of Chapter 2 to investigate whether, also in this case, conserved chiral currents exist which satisfy some kind of algebra, like in the pure CS theory. We stress again that the answer to this question is not at all taken for granted, due to the non-topological character of the bulk theory.

We close this thesis summarizing our results and drawing some conclusions, with some further suggestions of possible applications and extensions.
Chapter 1

The scalar field

In this Chapter we want to illustrate the method and the tools adopted in this work. For this purpose, let us consider the free massive scalar field theory in four spacetime dimensions; the bulk classical action is (we use natural units, \( \hbar = c = 1 \))

\[
S = \int d^4x \left[ \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi + \frac{1}{2} m^2 \varphi^2 \right],
\]

(1.1)

where \( \varphi(x) \) is a scalar field and \( m \) is the mass of the theory. Repeated indices are summed and the metric of the space is chosen to be flat euclidean, the infinitesimal distance being

\[
ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.
\]

(1.2)

The use of natural units gives sense to the notion of canonical mass dimension, because every quantity has a dimension which can be expressed in terms of powers of mass. In natural units, indeed, time is equivalent to the inverse of a mass; the same applies to length, while energy and impulse are equivalent to a mass. For each quantity \( K \) we can thus define its mass dimension \([K]\) as \( K \sim m^{[K]} \). We then have \([d^4x] = -4\) and \([\partial_\mu] = 1\). From now on, when we talk about dimension we will systematically mean canonical mass dimension.

Starting from the action (1.1), we can define the path integral in euclidean metric:

\[
Z[J] = \int D\varphi e^{-S_J},
\]

(1.3)

where \( S_J \) is the modified action

\[
S_J = S - \int d^4x J \varphi,
\]

(1.4)
$J(x)$ being an external source coupled to $\varphi(x)$. The definition (1.3) implies that $S_J$ must be dimensionless; for what we have said above, this means that $[\varphi]=1$ and $[J]=3$.

The path integral (1.3) represents the probability amplitude that the system evolves from vacuum initial state to vacuum final state. If the source is “switched off”, we have

$$Z[J=0] = 1,$$

(1.5)
since in this case the system is free and therefore it cannot change its initial state. Moreover, the $n$-point Green functions of the theory, defined as the mean value of $n$ time-ordered fields on the vacuum state, are generated by $Z$ through

$$\langle T(\varphi(x_1)\ldots\varphi(x_n)) \rangle = \frac{\delta^n Z}{\delta J(x_1)\ldots\delta J(x_n)} \bigg|_{J=0}.\quad (1.6)$$

In particular, for $n=1$ we have

$$\langle \varphi(x) \rangle = \frac{\delta Z}{\delta J(x)} \bigg|_{J=0}. \quad (1.7)$$

The equation of motion for the action $S_J$ is

$$\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = 0, \quad (1.8)$$

which has mass dimension three. From [27] we know that the solution $\varphi_J(x)$ of the quantum equation of motion in presence of the external source $J(x)$ is

$$\varphi_J(x) = \frac{\delta W}{\delta J(x)}, \quad (1.9)$$

where $W$ is the generator of the connected Green functions, defined as

$$W \equiv \ln Z. \quad (1.10)$$
1.1 Introduction of a boundary

We now want to introduce a boundary in the theory, and we choose the planar surface $x_3 = 0$.

The introduction of a boundary in a quantum field theory is not straightforward a task, and many ways have been studied to reach this goal; let us see how we can proceed.

One possibility [4] may consist in translating the features of the boundary directly on the field itself: in such a procedure, particular boundary conditions on the boundary (compatible with the bulk equation of motion) are set on the field and its derivatives, and the mathematical description of the boundary becomes a consequence of these constraints. However, this approach is restrictive because in a certain way it implies a sort of a priori choice of the physics, while we would like to find which are the consequences of the boundary itself on the physics.

A second possibility [1, 26] is offered by the so-called Lagrangian approach, which preserves the idea of describing the entire model through the action. The lagrangian density is thus modified by adding boundary terms. However, this procedure implies the presence of Dirac delta functions to keep unaltered the $d^4x$-integration and this, in turn, gives rise to ill-defined quantities.

A third possibility [14, 24, 28] is offered by acting directly on the equations of motion: the idea is that the presence of a boundary affects the dynamics of the system, so it is reasonable to assume that this effect can be directly seen on the equations of motion describing the dynamics itself.

The choice of the way to follow is just a matter of taste, since in the known cases they give the same results; we shall adopt the last one for the simplicity of the involved calculations and for the possible extensions.

We thus add breaking terms to the original equations of motion, which must reflect the effect of the boundary according to some basic principles, as follows [1]:

**Locality:** A first principle is that the boundary contribution must be local. In other words, we assume that the interactions due to the boundary take place only on its surface, while, away from it, the ordinary field equations hold. Since locality is the fundamental principle on which field theories are based, this requirement is necessary if we want to discuss a field theory with boundary. In other words, locality is a general principle that must always be observed.

**Separability:** The second principle that we will use in the construction
of a quantum field theory with boundary refers to Symanzik’s original idea [1] and is more related to the nature itself of a boundary. Indeed, the boundary does not only break the isotropy of space by dividing it into a left side \((x_3 < 0)\) and a right side \((x_3 > 0)\), but it also separates them completely. In other words, nothing can go across the boundary, and therefore the left side and the right side of space are now two separate worlds. This is why Symanzik called this feature separability. This is a very powerful constraint, as it automatically imposes the vanishing of the \(n\)-point Green functions which involve two fields computed in points belonging to opposite sides of space. In particular, the propagators of the theory are the 2-point Green functions. Therefore, they must satisfy this constraint as well:

\[
x_3y_3 < 0 \Rightarrow \Delta(x - y) = \langle T(\varphi(x)\varphi(y)) \rangle = 0. \tag{1.11}
\]

This property is satisfied by propagators which split according to

\[
\Delta(x - y) = \theta(x)\Delta_+(x - y) + \theta(y)\Delta_-(x - y), \tag{1.12}
\]

where \(\Delta_+(x - y)\) and \(\Delta_-(x - y)\) are respectively the propagators for the right and the left side of spacetime,

\[
\theta_{\pm} \equiv \theta(\pm x_3)\theta(\pm y_3) \tag{1.13}
\]

and \(\theta(x)\) is the step function, defined as usual:

\[
\theta(x) \equiv \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0
\end{cases}. \tag{1.14}
\]

The relation (1.12) is of primary importance for what follows, since it formally expresses the idea that the dynamics of the opposite sides are independent; we refer to it also as “decoupling condition”. The decoupling condition, in turn, induces the decomposition of \(\mathcal{W}\) according to

\[
\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^- \tag{1.15}
\]

where \(\mathcal{W}^+\) and \(\mathcal{W}^-\) are the generators of the connected Green functions for the right side and the left side of spacetime, respectively.

**Linearity:** Finally, we require that the boundary contributions to the equations of motion be linear in the fields because, in general, the symmetries of the classical action, if only linearly broken at the classical level, nonetheless remain exact symmetries of the quantum action [29]. Moreover, we are considering a free field theory and therefore nonlinear terms in the
field must not occur in the equations of motion.

We can now proceed writing down the most general boundary terms that can break the equation of motion (1.8), according to what above stated: the breaking terms must be local, linear in the quantum fields and analytic in the parameters, and they must respect the dimensions ("power counting") and the quantum numbers characterizing the bulk equation of motion. Let us see how these requirements select possible candidates as boundary terms:

1. **Locality** imposes that all possible breaking terms have the form
   \[ \delta^{(n)}(x_3)X(x_1, x_2, x_3), \]  
   where \( \delta^{(n)}(x_3) \) is the \( n \)-order derivative of the Dirac delta function with respect to its argument, and \( X(x_1, x_2, x_3) \) is a local functional. The presence of \( \delta^{(n)}(x_3) \) is necessary because its support is the boundary itself;

2. **Linearity in the quantum fields** imposes that the boundary terms contain the field \( \varphi(x) \) or its derivatives only linearly;

3. **Analyticity in the parameters** requires that inverse powers of any parameter must not occur;

4. The **power counting** of the boundary terms must be the same of (1.8), i.e. three (as we have seen above). For this purpose, it is useful to recall the definition of \( \delta(x) \) as \( \int dx \delta(x) = 1 \), which sets its dimension to one and that of \( \delta^{(n)}(x_3) \) to \( n + 1 \);

5. In this scalar theory there are no conserved quantum numbers.

Moreover, a simple argument of quantum field theory [27] shows that there must not be negative-dimensional parameters, otherwise they could be used to add infinite terms to the action: the theory would then lose its predictive power and would not be renormalizable. This last constraint, together with power counting, limits the possible boundary terms to a finite number. More precisely, the presence of both \( \varphi(x) \) or its derivatives and \( \delta(x_3) \) or its derivatives allows only parameters of dimension one or zero.

At the end, the most general equation of motion satisfying all the above requests is:

\[
\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = c_1^+ \delta(x_3) \varphi_+ + c_1^- \delta(x_3) \varphi_- + c_2^+ \delta(x_3) (\partial_3 \varphi)_+ + c_2^- \delta(x_3) (\partial_3 \varphi)_- + c_3^+ \delta'(x_3) \varphi_+ + c_3^- \delta'(x_3) \varphi_- .
\] (1.17)
where the prime indicates derivative with respect to the argument, and $c_i^\pm$ are constant parameters; notice that $c_i^\pm$ are one-dimensional, while the others are dimensionless. Moreover, we have introduced the boundary fields on the right, respectively on the left, of the boundary:

$$
\varphi_\pm(x_0, x_1, x_2) \equiv \lim_{x_3 \to 0^\pm} \varphi(x_0, x_1, x_2, x_3) \quad (1.18)
$$

$$
(\partial_3 \varphi)_\pm(x_0, x_1, x_2) \equiv \lim_{x_3 \to 0^\pm} \partial_3 \varphi(x_0, x_1, x_2, x_3) \quad . \quad (1.19)
$$

Let us now write the equation of motion in a functional way as

$$
M(x)W = J(x) \quad , \quad (1.20)
$$

where $M(x)$ is a suitable local functional operator. Now, from (1.9) it follows that $M(x)$ can be expressed in terms of functional derivatives with respect to $J(x)$, according to

$$
M(x) = - \left[ \nabla^2 + \partial_0^2 - m^2 \right] \frac{\delta}{\delta J(x)} \quad , \quad (1.21)
$$

which can be easily checked by substitution in (1.20) and using (1.9). On the other hand, if we apply $M(x')$ again to (1.20) we get

$$
M(x')M(x)W = M(x')J(x)
$$

$$
= - \left[ \nabla^2 + \partial_0^2 - m^2 \right] \frac{\delta J(x)}{\delta J(x')}
$$

$$
= - \left[ \nabla^2 + \partial_0^2 - m^2 \right] \delta(x - x')
$$

$$
= - \left[ \nabla^2 + \partial_0^2 - m^2 \right] \delta(x' - x)
$$

$$
= - \left[ \nabla^2 + \partial_0^2 - m^2 \right] \frac{\delta J(x')}{\delta J(x)}
$$

$$
= M(x)J(x')
$$

$$
= M(x)M(x')W \quad . \quad (1.22)
$$

In the fourth passage we have used the symmetry of the delta function in its argument, which implies the same property for its even derivatives as well. In other words, (1.22) states that the operators $M(x)$ and $M(x')$ satisfy the commutative algebra

$$
[M(x'), M(x)] W = 0 \quad . \quad (1.23)
$$

In quantum field theory, the operatorial algebras are fundamental because they are associated to intrinsic properties of the theory; therefore, we ask that the algebraic structure is preserved also in the theory including the boundary.
We thus write the broken equation of motion as well in the functional form (1.20). This time, however, the decoupling condition allows to write the broken equation of motion separately for the two sides of space, in the form

\[ M^\pm(x)W^\pm = J(x) , \]

where \( M^\pm(x) \) are the operators for the two sides of spacetime

\[
M^\pm(x) = M(x) + \delta(x_3)c_1^\pm \frac{\delta}{\delta J(x)} + \delta(x_3)c_2^\pm \partial_3 \frac{\delta}{\delta J(x)} + \delta'(x_3)c_3^\pm \left( \frac{\delta}{\delta J(x)} \right)_{x_3=0} ,
\]

and \( M(x) \) is the operator for the bulk equation of motion defined in (1.21). We thus have an operator for each side of space, and (1.23) splits into the two analogous conditions

\[
\left[ M^\pm(x'), M^\pm(x) \right] W^\pm = 0 \tag{1.26}
\]

that we want to satisfy in order to preserve the algebra of the operators in each side of space. We will refer to these conditions using the name of “compatibility”, as in [24]. Using (1.24) in (1.26), and taking into account (1.25), we get

\[
0 = +c_1^+ \delta(x'_3)\delta(x_3 - x'_3) - c_1^- \delta(x_3)\delta(x'_3 - x_3) + c_2^+ \delta(x'_3)\partial_3 \delta(x_3 - x'_3) - c_2^- \delta(x_3)\partial_3 \delta(x'_3 - x_3) + c_3^+ \delta'(x'_3)\delta(x_3) - c_3^- \delta'(x_3)\delta(x'_3) . \tag{1.27}
\]

Using the relations between distributions [30]

\[
\delta(x)\delta(y - x) = \delta(x)\delta(y) \tag{1.28}
\]

\[
\delta(x)\partial_x \delta(x - y) = -\delta(x)\delta'(y) , \tag{1.29}
\]

we finally get the two constraints

\[
c_2^+ = -c_3^- . \tag{1.30}
\]

Notice that the commutators between operators acting on opposite sides of the boundary vanish, since the decoupling condition imposes

\[
M^\mp(x')M^\pm(x)W^\mp = 0 = M^\mp(x')M^\pm(x)W^\mp . \tag{1.31}
\]

The broken equation of motion thus becomes

\[
\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = \delta(x_3) \left[ c_1^+ \varphi_+ + c_1^- \varphi_- - c_3^+ (\partial_3 \varphi)_+ - c_3^- (\partial_3 \varphi)_- \right] + \delta'(x_3) \left( c_3^+ \varphi_+ + c_3^- \varphi_- \right) . \tag{1.32}
\]

This, in turn, determines boundary conditions on the field near the separation, which are the effect of the boundary on the physics. The next task will thus be to study what this effect consists of.
1.2 Boundary conditions

The result (1.32) shows that the requests for locality, separability and compatibility do not fix the boundary terms uniquely, since there are still four free parameters. In other words, the possible terms that satisfy all the constraints are more than one; indeed, $c^+_1$ and $c^+_3$ identify four distinct families of boundary terms. In general, it is thus possible to make a high variety of choices corresponding to the values of the parameters, and each of them is associated to a different boundary condition for the field.

We can see how a particular choice for the parameters affect the boundary conditions by means of two integration processes:

1. First of all we set the sources to zero. We then integrate (1.32) from one side of the boundary to the other one along a path perpendicular to the boundary:

$$\int_{-\varepsilon}^{\varepsilon} dx_3 \left( \partial_0^2 \varphi + \partial_1^2 \varphi + \partial_2^2 \varphi + \partial_3^2 \varphi - m^2 \varphi \right) =$$

$$= \int_{-\varepsilon}^{\varepsilon} dx_3 \delta(x_3) \left( c^+_1 \varphi_+ + c^-_1 \varphi_- \right)$$

$$- \int_{-\varepsilon}^{\varepsilon} dx_3 \delta(x_3) \left[ c^+_3 (\partial_3 \varphi)_+ + c^-_3 (\partial_3 \varphi)_- \right]$$

$$+ \int_{-\varepsilon}^{\varepsilon} dx_3 \delta'(x_3) \left( c^+_3 \varphi_+ + c^-_3 \varphi_- \right),$$

(1.33)

where $\varepsilon$ is a positive infinitesimal constant. Since we are treating a boundary of infinitesimal thickness, we consider the limit $\varepsilon \to 0$; in this process, we reasonably assume that the field and its derivative is limited everywhere, in particular near the boundary. This yields:

$$(\partial_3 \varphi)_+ - (\partial_3 \varphi)_- = c^+_1 \varphi_+ + c^-_1 \varphi_- - c^+_3 (\partial_3 \varphi)_+ - c^-_3 (\partial_3 \varphi)_-. \quad (1.34)$$

We now rearrange (1.34) as:

$$\left(1 + c^+_3 \right) (\partial_3 \varphi)_+ - c^+_1 \varphi_+ = \left(1 - c^-_3 \right) (\partial_3 \varphi)_- + c^-_1 \varphi_- . \quad (1.35)$$

Now, let us write (1.35) in the functional way using (1.9), derive the result with respect to $J(x')$ and then set the external source to zero. We thus get an equation between propagators:

$$\lim_{x_3 \to 0^+} \left[ (1+c^+_3) \partial_3 \langle T(\varphi(x')\varphi(x)) \rangle - c^+_1 \langle T(\varphi(x')\varphi(x)) \rangle \right] =$$

$$= \lim_{x_3 \to 0^-} \left[ (1-c^-_3) \partial_3 \langle T(\varphi(x')\varphi(x)) \rangle + c^-_1 \langle T(\varphi(x')\varphi(x)) \rangle \right].$$

(1.36)
CHAPTER 1: THE SCALAR FIELD

If $x'$ belongs to the right side of spacetime, the right-hand side vanishes due to the decoupling condition; since it must hold for arbitrary $x'$, we get

$$\left(1 + c_3^+\right) (\partial_3 \varphi)_+ - c_1^+ \varphi_+ = 0 .$$  \hspace{1cm} (1.37)

Similarly, if $x'$ belongs to the left side, the left-hand member vanishes, yielding

$$\left(1 - c_3^-\right) (\partial_3 \varphi)_- + c_1^- \varphi_- = 0 .$$  \hspace{1cm} (1.38)

2. On the other hand, (1.32) involves second-order derivatives; we can thus get additional conditions on the boundary fields by integrating (1.32) twice. After setting the source to zero, let us perform the first integration according to

$$\int_{-\infty}^{x_3} dx' \left( \partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi \right) = \int_{-\infty}^{x_3} dx' \delta(x'_3) \left( c_1^+ \varphi_+ + c_1^- \varphi_- \right)$$

$$- \int_{-\infty}^{x_3} dx' \delta(x'_3) \left[ c_3^+ (\partial_3 \varphi)_+ + c_3^- (\partial_3 \varphi)_- \right]$$

$$+ \int_{-\infty}^{x_3} dx' \delta'(x'_3) \left( c_3^+ \varphi_+ + c_3^- \varphi_- \right) .$$  \hspace{1cm} (1.39)

Always assuming, like before, that the field and its derivative vanish at infinity, this yields

$$\int_{-\infty}^{x_3} dx' \left( \partial_0^2 + \partial_1^2 + \partial_2^2 - m^2 \right) \varphi + \partial_3 \varphi = \theta(x_3) \left( c_1^+ \varphi_+ + c_1^- \varphi_- \right)$$

$$- \theta(x_3) \left[ c_3^+ (\partial_3 \varphi)_+ + c_3^- (\partial_3 \varphi)_- \right]$$

$$+ \delta(x_3) \left( c_3^+ \varphi_+ + c_3^- \varphi_- \right) ,$$  \hspace{1cm} (1.40)

where we have used the relation between distributions [30]

$$\theta'(x) = \delta(x) .$$  \hspace{1cm} (1.41)

We then perform a second integration like we have done in 1.:

$$\int_{-\varepsilon}^{\varepsilon} dx_3 \int_{-\infty}^{x_3} dx' \left( \partial_0^2 + \partial_1^2 + \partial_2^2 - m^2 \right) \varphi + \int_{-\varepsilon}^{\varepsilon} dx_3 \partial_3 \varphi =$$

$$= \int_{-\varepsilon}^{\varepsilon} dx_3 \theta(x_3) \left( c_1^+ \varphi_+ + c_1^- \varphi_- \right)$$

$$- \int_{-\varepsilon}^{\varepsilon} dx_3 \theta(x_3) \left[ c_3^+ (\partial_3 \varphi)_+ + c_3^- (\partial_3 \varphi)_- \right]$$

$$+ \int_{-\varepsilon}^{\varepsilon} dx_3 \delta(x_3) \left( c_3^+ \varphi_+ + c_3^- \varphi_- \right) .$$  \hspace{1cm} (1.42)
which leads to

\[
\int_{-\infty}^{x_3} dx_3 \int_{-\infty}^{x_3} dx_3' \left( \partial^2_{0} + \partial^2_{1} + \partial^2_{2} - m^2 \right) \varphi + \varphi_+ - \varphi_- = \\
= \varepsilon \left( c^+_3 \varphi_+ + c^-_3 \varphi_- \right) - \varepsilon \left[ c^+_3 \left( \partial_3 \varphi \right)_+ + c^-_3 \left( \partial_3 \varphi \right)_- \right] \\
+ c^+_3 \varphi_+ + c^-_3 \varphi_- .
\]  

(1.43)

Taking again the limit \( \varepsilon \to 0 \), in the same hypothesis as in 1., we get

\[
\left( 1 - c^+_3 \right) \varphi_+ = \left( 1 + c^-_3 \right) \varphi_- ,
\]

(1.44)

which splits as well into two distinct equations due to the decoupling condition:

\[
\left( 1 - c^+_3 \right) \varphi_+ = 0 \quad (1.45) \\
\left( 1 + c^-_3 \right) \varphi_- = 0 . \quad (1.46)
\]

We have thus found four conditions, (1.37), (1.38), (1.45) and (1.46), which must be simultaneously satisfied. Summarizing:

\[
\left\{ \begin{array}{l}
\left( 1 + c^+_3 \right) \left( \partial_3 \varphi \right)_+ - c^+_3 \varphi_+ = 0 \\
\left( 1 - c^-_3 \right) \left( \partial_3 \varphi \right)_+ + c^-_3 \varphi_- = 0 \\
\left( 1 - c^+_3 \right) \varphi_+ = 0 \\
\left( 1 + c^-_3 \right) \varphi_- = 0 .
\end{array} \right.
\]  

(1.47)

The system (1.47) proves that each choice for the parameters determines a different set of conditions on the boundary fields, i.e. a different set of boundary conditions and therefore a different physics. There are four types of acceptable choices:

1. \( c^+_3 = \pm 1 \). In this case, (1.47) yields Robin conditions on both sides of the boundary:

\[
\left\{ \begin{array}{l}
\left( \partial_3 \varphi \right)_+ = \frac{c^+_3}{2} \varphi_+ \\
\left( \partial_3 \varphi \right)_- = -\frac{c^-_3}{2} \varphi_- ,
\end{array} \right.
\]  

(1.48)

and the broken equation of motion is

\[
\partial^2_{0} \varphi + \nabla^2 \varphi - m^2 \varphi + J = \delta(x_3) \left( \frac{c^+_3}{2} \varphi_+ + \frac{c^-_3}{2} \varphi_- \right) + \delta'(x_3) \left( \varphi_+ - \varphi_- \right) .
\]  

(1.49)
CHAPTER 1: THE SCALAR FIELD

2. \( c_3^\pm = \mp 1 \). With this choice, (1.47) leads to Dirichlet conditions on both sides of space:

\[
\varphi_+ = \varphi_- = 0 ,
\]

and the broken equation of motion is

\[
\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = \delta(x_3) \left[ (\partial_3 \varphi)_+ - (\partial_3 \varphi)_- \right] .
\]

3. \( c_3^\pm = 1 \). In this situation, from (1.47) we get a Robin condition on the right side and a Dirichlet condition on the left side:

\[
\begin{cases}
(\partial_3 \varphi)_+ = \frac{c_3^+}{2} \varphi_+ \\
\varphi_- = 0 ,
\end{cases}
\]

and for the broken equation of motion we get

\[
\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = \delta(x_3) \left[ \frac{c_3^+}{2} \varphi_+ - (\partial_3 \varphi)_- \right] + \delta'(x_3) \varphi_+ .
\]

4. \( c_3^\pm = -1 \). This is the complementary case of 3., therefore we have the boundary conditions

\[
\begin{cases}
\varphi_+ = 0 \\
(\partial_3 \varphi)_- = -\frac{c_3^+}{2} \varphi_-
\end{cases}
\]

and the broken equation of motion

\[
\partial_0^2 \varphi + \nabla^2 \varphi - m^2 \varphi + J = \delta(x_3) \left[ (\partial_3 \varphi)_+ + \frac{c_3^+}{2} \varphi_- \right] - \delta'(x_3) \varphi_- .
\]

Besides these above, other choices exist, where at least one of the parameters \( c_3^\pm \) is different from \( \pm 1 \). In these cases, (1.47) makes both the field and its derivative vanish on the same side of the boundary:

\[
\varphi_\pm = \partial_3 \varphi_\pm = 0 .
\]

This is too strong a condition, for which only the trivial solution

\[
\varphi(x) = 0 \text{ everywhere}
\]

exists, and therefore such choices for the parameters are forbidden [31].

A comparison should be made with [1] and [26], where the scalar field theory with boundary has been considered to study the Casimir effect.
There are differences in the results, concerning the number of the possible choices for the parameters (two in [1, 26] and three in our case). This is just a consequence of the difference in the approach. If we require that the boundary terms in the equation of motion (1.32) come from a boundary action, in order to make contact with the approach presented in [1] and [26], we land on the additional constraints on the parameters $c_1^+ = c_1^-$ and $c_3^+ = c_3^-$. In this case, the only possible choices are 3. and 4., in complete agreement with [1] and [26].
Chapter 2

Chern-Simons theory

The Chern-Simons (CS) model is a topological gauge field theory defined on a three-dimensional manifold [2, 8, 9]; in general, it is based on a nonabelian gauge group. Topological means that it depends only on global features of the manifold on which it is defined, and no local observables depending on the metric exist [9]. This theory has been extensively studied in the last decades because of its relevance in many topics of physics and mathematics. Here we stress only a few of the points of interest of the CS theory; therefore, the reader will be addressed to references, the list of which is inevitably incomplete.

From the mathematical point of view, the central idea is that it offers a means of computing invariants of knot and link configurations on an arbitrary three-manifold by means of field theory tools [2]. Furthermore, a remarkable property of the CS theory is its finiteness [32], i.e. the vanishing of the beta function of its unique coupling constant. This is a general property of topological field theories [9] and, under some constraints [33], of supersymmetric field theories as well. This feature is extremely important from the physical point of view. In fact, since the early days of quantum field theory, physicists considered the property of finiteness as one of the most appealing features that a theory could possess, according to the belief that the ultimate Theory of Nature should be finite [34].

In this sense, an application is the description of gravity in a (2+1) - spacetime as a first approach to quantum gravity [15]. Indeed, it turns out that in this case the theory can be described in terms of a CS action. Moreover, the same theory for a manifold with boundary can be used to describe black holes in (2+1) dimensions [6, 17].

In condensed matter physics, the abelian CS theory is useful to describe the Fractional Quantum Hall Effect [5]. Furthermore, the introduction of a planar boundary in the theory gives a description of the so called edge
The CS theory can also be coupled to matter fields (fermions for instance), thus giving a description of anyons [19], i.e. particles existing only in a three-dimensional spacetime and obeying a fractional statistics.

From now on, we shall consider the abelian CS action, which is given by

$$S_{mink}^{cs} = \frac{k}{2} \int d^3 \hat{x} \varepsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho ,$$

(2.1)

where $k$ is a constant, $A_\mu (\hat{x})$ is a gauge field and $\varepsilon^{\mu \nu \rho}$ is the Levi-Civita tensor, i.e. the completely antisymmetric symbol defined as usual:

$$\varepsilon^{\mu \nu \rho} = \begin{cases} 
1 & \iff \mu \nu \rho = \text{even permutation of } 012 \\
-1 & \iff \mu \nu \rho = \text{odd permutation of } 012 \\
0 & \iff \text{otherwise}
\end{cases}$$

(2.2)

The spacetime is now three-dimensional with flat minkowskian metric, the infinitesimal interval being

$$ds^2 = - (d\hat{x}^0)^2 + (d\hat{x}^1)^2 + (d\hat{x}^2)^2 .$$

(2.3)

Notice that the constant $k$ could be reabsorbed by a redefinition of the gauge field

$$A_\mu \rightarrow \frac{A_\mu}{\sqrt{k}} ;$$

(2.4)

however, $k$ has a direct connection with physical observables - the filling factor of the Fractional Quantum Hall Effect, for instance. Therefore, the normalization of the gauge field is fixed once for all, and we keep $k$ in the theory.

The action (2.1) is left invariant by the following gauge transformation of the field $A_\mu (\hat{x})$:

$$\delta A_\mu = \partial_\mu \theta ,$$

(2.5)

where $\theta (\hat{x})$ is a local parameter.

The action (2.1) is therefore gauge invariant, and the path integral of the theory requires a gauge choice [35]. A way to see this is to notice that the action (2.1) does not admit a propagator. Indeed, let us write the abelian CS action in momentum space as

$$S_{CS}^{mink} = \frac{ik}{2} \int \frac{d^3 p}{(2\pi)^3} \varepsilon^{\mu \nu \rho} \tilde{A}_\mu (-p) p_\nu \tilde{A}_\rho (p) ,$$

(2.6)

where $\tilde{A}_\mu (p)$ is the Fourier transform of the gauge field $A_\mu (\hat{x})$

$$A_\mu (\hat{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{ip\cdot \hat{x}} \tilde{A}_\mu (p) ,$$

(2.7)
and we have used the relation
\[ \int \frac{d^3 \hat{x}}{(2\pi)^3} e^{ip \cdot \hat{x}} = \delta^3(p). \]  
(2.8)

The propagator is obtained by inverting the operator \( ik\varepsilon^{\mu\nu\rho} p_\nu \) appearing in the momentum space action (2.6), but this is impossible without fixing a gauge, like in any gauge field theory. In this case, for instance, one can note that
\[ [ik\varepsilon^{\mu\nu\rho} p_\nu] p_\mu = 0, \]  
(2.9)
which shows that the kernel of \( ik\varepsilon^{\mu\nu\rho} p_\nu \) is nontrivial. This problem coincides with the requirement of restricting the space on which the path integral
\[ Z = \int D A_\mu e^{iS(A_\mu)} \]  
(2.10)
is computed. This is necessary because the gauge transformation
\[ A_\mu \rightarrow A_\mu + \partial_\mu \theta, \]  
(2.11)
where \( \theta(\hat{x}) \) is a local parameter, renders the path integration redundant, and a representative for each orbit defined by the gauge transformation is necessary. The standard way to do this [35], is to intersect the space of the orbits with an hypersurface
\[ F(A_\mu) = 0, \]  
(2.12)
where \( F(A_\mu) \) is a scalar function of the gauge field. The problem of translating this geometrical construction into a restriction of the path integral has been solved by Faddeev and Popov [36]. The solution is to add the action a gauge fixing term
\[ S_{gf} = \int d^4x \frac{1}{2\xi} F^2(A_\mu), \]  
(2.13)
where \( \xi \) is the gauge parameter, and a term depending on additional fields - the ghosts and antighosts [36] - which do not play any role in abelian gauge field theories, since they are decoupled from the gauge field and hence their contribution to the path integral \( Z \) can be factorized. At this point, the propagator can be computed.

For the CS theory we can see this, for instance, in the case of a covariant gauge
\[ \partial^\mu A_\mu = 0, \]  
(2.14)
corresponding to the gauge fixing term
\[ S_{gf}^{mink} = \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \]  
(2.15)
in the limit $\xi \to 0$ (Landau gauge), as we shall see in a moment. In this case, the total action
\[ S_{mink} = S_{mink}^{CS} + S_{mink}^{gf} \] (2.16)
in momentum space, reads
\[ S_{mink} = \frac{1}{2} \int \! \frac{d^3p}{(2\pi)^3} \tilde{A}_\mu (-p) \left[ ik \varepsilon^{\mu\nu\rho} p_\nu + \frac{p^\mu p^\rho}{\xi} \right] \tilde{A}_\rho (p) , \] (2.17)
and the operator to be inverted is now $ik \varepsilon^{\mu\nu\rho} p_\nu + \frac{p^\mu p^\rho}{\xi}$. We can do this by means of the most general rank-2 tensor in momentum space
\[ \tilde{\Delta}_{\rho\lambda} = A g_{\rho\lambda} + B p_\rho p_\lambda + C \varepsilon_{\rho\lambda\tau} p^\tau , \] (2.18)
where $A (p^2)$, $B (p^2)$ and $C (p^2)$ are scalar functions of $p^2$ and $g_{\rho\lambda}$ is the minkowskian metric tensor
\[ g_{\rho\lambda} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \] (2.19)
The requirement that (2.18) inverts the quadratic part of the action is expressed by the equation
\[ \left( ik \varepsilon^{\mu\nu\rho} p_\nu + \frac{1}{\xi} p^\mu p^\rho \right) \tilde{\Delta}_{\rho\lambda} (p) = \delta^\mu_\lambda , \] (2.20)
which yields the condition
\[ ik A \varepsilon^{\mu\nu\rho} g_{\rho\lambda} p_\nu + ik C p^2 \delta^\mu_\lambda + \left( \frac{A + B p^2}{\xi} - ik C \right) p^\mu p_{\lambda} = \delta^\mu_\lambda . \] (2.21)
This is satisfied if
\[ A = 0 \] (2.22)
\[ B = \frac{\xi}{(p^2)^2} \] (2.23)
\[ C = -\frac{i}{kp^2} . \] (2.24)
The momentum space propagator of the CS theory is thus
\[ \tilde{\Delta}_{\rho\lambda} = \frac{\xi k p_\rho p_\lambda - i p^2 \varepsilon_{\rho\lambda\tau} p^\tau}{k (p^2)^2} . \] (2.25)
The two most popular choices for the gauge parameter $\xi$ are $\xi = 0$, which corresponds to the Landau gauge, and $\xi = 1$, known as the Feynman gauge.

The gauge fixing term (2.13) can be rewritten introducing an additional field: the Lagrange multiplier $b(x)$ [35]. Indeed, the gauge fixing term

$$S_{gf} = \int d^4x \left[ F \left( A_\mu \right) b - \frac{\xi}{2} b^2 \right]$$

(2.26)

reduces to (2.13) once the integration on the $b$-field in the path integral

$$Z = \int DA_\mu Db \ e^{i \left( S_{CS}(A_\mu) + S_{gf}(A_\mu, b) \right)}$$

(2.27)

is performed. Indeed, the equation of motion of the Lagrange multiplier $b(x)$ reads

$$F \left( A_\mu \right) - \xi b = 0 \ .$$

(2.28)

In the limit $\xi \to 0$ and in the case of the covariant choice $F \left( A_\mu \right) \equiv \partial^\mu A_\mu$, the Landau gauge (2.14) is recovered.

As a last comment, we stress that the hypersurface $F \left( A_\mu \right) = 0$, of which we gave the covariant example $\partial^\mu A_\mu = 0$, could be chosen also of non-covariant type:

$$F \left( A_\mu \right) \equiv n^\mu A_\mu = 0 \ ,$$

(2.29)

where $n^\mu$ is a constant vector. In particular, the case $n_\mu n^\mu = -1$ corresponds to the temporal gauges, while $n_\mu n^\mu = 1$ selects the axial gauges. Usually, the disadvantages of such non-covariant choices are more than the advantages; nevertheless, this is actually the choice that we shall adopt. In fact, the presence of a boundary already breaks the Lorentz invariance of the theory, hence the main advantage of the choice (2.14), i.e. the covariance, is already lost. The other important reason which leads us to a non-covariant gauge is the existence of a residual, local, gauge invariance, expressed by a local (and a fortiori integrated) Ward identity [25], which will be crucial for what follows.

From now on, closely following [28], we will work in euclidean metric, defined as in (1.2); therefore, we recall the relations between a minkowskian vector $\hat{Y}^\mu$ and its euclidean counterpart $Y^\mu$:

$$\begin{cases}
\hat{Y}^0 = -iY^0 \\
\hat{Y}^i = Y^i 
\end{cases}$$

(2.30)

while the relation between the measures, to be always intended in the sense of the wedge product, is

$$d^3\hat{x} = -id^3x \ .$$

(2.31)
Therefore, (2.1) becomes

\[ S_{\text{cs}}^{\text{smink}} \rightarrow -\frac{ik}{2} \int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \]  

(2.32)

and from the definition of the path integral

\[ Z \equiv \int \mathcal{D}\Phi \ e^{iS^{\text{smink}}} \equiv \int \mathcal{D}\Phi \ e^{-S^{\text{eucl}}}, \]

(2.33)

we can read off the euclidean CS action

\[ S_{\text{cs}}^{\text{eucl}} \equiv -\frac{k}{2} \int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \]  

(2.34)

Moreover, like in [28], it will be convenient to work in euclidean light-cone coordinates, defined as:

\[
\begin{align*}
    u &= x_2 \\
    z &= \frac{1}{\sqrt{2}}(x_1 - ix_0) \\
    \bar{z} &= \frac{1}{\sqrt{2}}(x_1 + ix_0)
\end{align*}
\]

\[ \Rightarrow \quad \begin{align*}
    \partial_u &= \partial_2 \\
    \partial\bar{z} &= \frac{1}{\sqrt{2}}(\partial_1 + i\partial_0) \\
    \bar{\partial}\bar{z} &= \frac{1}{\sqrt{2}}(\partial_1 - i\partial_0),
\end{align*} \]  

(2.35)

which induce similar definitions in the space of the fields:

\[
\begin{align*}
    A_u &= A_2 \\
    A &= \frac{1}{\sqrt{2}}(A_1 + iA_0) \\
    \bar{A} &= \frac{1}{\sqrt{2}}(A_1 - iA_0)
\end{align*}
\]  

(2.36)

After this change of coordinates, reminding the relation between the measures

\[ d^3x = idudzd\bar{z}, \]  

(2.37)

the CS action reads

\[ S_{\text{cs}} = -k \int dudzd\bar{z} \ (\bar{A}\partial_u A + A_u \partial\bar{A} - A_u \bar{\partial}A). \]  

(2.38)

We need now to add the gauge fixing term (2.26). As anticipated, we adopt an axial gauge fixing. Since we are going to introduce the boundary \( u = 0 \), the natural choice is

\[ A_u = 0, \]  

(2.39)

which corresponds to

\[ F(A_\mu) = A_u. \]  

(2.40)
The gauge fixing term (2.26) thus takes the form

$$S_{gf} = - \int dudzd\bar{z} A_u b .$$  \hfill (2.41)

We stress again that this choice is not covariant, but the introduction itself of a boundary already breaks Lorentz invariance and, on the other hand, the axial gauge implies the existence of a residual local Ward identity [25], which will be the core of our discussion. Therefore, the complete action is

$$S = S_{cs} + S_{gf} .$$  \hfill (2.42)

As in (1.3) for the scalar case, $S$ appears in the generating functional of the Green functions according to

$$Z[J_{\Phi}] = \int \mathcal{D}\Phi \exp \left[ - \left( S - \int dudzd\bar{z} \sum_{\Phi} J_{\Phi} \Phi \right) \right]$$ \hfill (2.43)

where, similarly to the scalar case, $J_{\Phi} = J_u, \bar{J}, J, J_b$ are external sources coupled to the fields $\Phi = A_u, A, \bar{A}, b$ respectively, and now

$$x \equiv (z, \bar{z}, u) .$$  \hfill (2.44)

The relation (1.6) for the $n$-point Green functions is then generalized according to

$$\langle T (\Phi_1(x_1) \ldots \Phi_n(x_n)) \rangle = \frac{\delta^n Z}{\delta J_{\Phi_1}(x_1) \ldots \delta J_{\Phi_n}(x_n)} \bigg|_{J=0} ;$$  \hfill (2.45)

in particular, for $n = 1$ we have

$$\langle \Phi(x) \rangle = \frac{\delta Z}{\delta J_{\Phi}(x)} \bigg|_{J=0} .$$  \hfill (2.46)

\footnote{In three dimensions and in the axial gauge, the gauge parameter turns out to have negative mass dimension; hence we choose $\xi = 0$ in order to preserve power counting and renormalizability.}
2.1 The action and its properties

From (2.43) we get the modified action

\[ S_J \equiv S - \int dudzd\bar{z} \sum_{\Phi} J_\Phi \Phi \, . \]  

(2.47)

Let us see its properties and symmetries in detail.

**Power counting:** the mass dimension of the measure is \([dudzd\bar{z}] = -3\). Therefore, since the action \(S_J\) must be mass dimensionless, the dimensions of the fields are

\[ [A] = [\bar{A}] = [A_u] = 1 = [J_b] \]  

(2.48)

\[ [\bar{J}] = [J] = [J_u] = 2 = [b] \, , \]  

(2.49)

and \(k\) is a dimensionless constant.

**Symmetries and quantum numbers:**

1.) The gauge fixing breaks the three-dimensional Lorentz invariance, but the action \(S\) still remains invariant under the *two-dimensional Lorentz transformations* in the planes \(u = \text{constant}\). This property can be encoded through the association of a quantum number \(h\) (*helicity*) to the quantities \(z\) and \(\bar{z}\)-defined. If we set

\[
\begin{align*}
    h(z) &= -1 \\
    h(\bar{z}) &= +1 \\
    h(u) &= 0 
\end{align*}
\]  

(2.50)

the other assignments are then straightforward and each term appearing in the action \(S_J\) has vanishing helicity, as it can be directly checked from (2.47) by using the summarizing Table 1. This means that \(h\) is a conserved quantum number, and this property expresses the two-dimensional Lorentz invariance.

<table>
<thead>
<tr>
<th></th>
<th>(A_u)</th>
<th>(A)</th>
<th>(\bar{A})</th>
<th>(b)</th>
<th>(J_u)</th>
<th>(J)</th>
<th>(\bar{J})</th>
<th>(J_b)</th>
<th>(\partial_u)</th>
<th>(\partial)</th>
<th>(\bar{\partial})</th>
<th>(u)</th>
<th>(z)</th>
<th>(\bar{z})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dim</strong></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td><strong>hel</strong></td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>+1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2.) The choice of the axial gauge does not completely fix the gauge: a residual local gauge invariance remains [25]. Indeed, the residual gauge transformation

$$
\begin{align*}
\delta A &= \partial \theta \\
\delta \bar{A} &= \bar{\partial} \theta \\
\delta A_u &= 0 \\
\delta b &= 0
\end{align*}
$$

(2.51)

is a symmetry of the action $S$; here $\theta(z, \bar{z})$ is a local gauge parameter that does not depend on the variable $u$. The residual gauge invariance (2.51) is expressed by a local residual Ward identity which, for the case in hand, is

$$
\partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_u b = 0 .
$$

(2.52)

This, once integrated, gives

$$
\int du \left( \partial \bar{J} + \bar{\partial} J \right) = 0 .
$$

(2.53)

3.) The action (2.47) is left invariant by the discrete transformation involving at the same time coordinates and fields:

$$
\begin{align*}
z &\leftrightarrow \bar{z} \\
u &\rightarrow -u \\
A &\leftrightarrow \bar{A} \\
A_u &\rightarrow -A_u \\
b &\rightarrow -b \\
J &\leftrightarrow \bar{J} \\
J_u &\rightarrow -J_u \\
J_b &\rightarrow -J_b
\end{align*}
$$

(2.54)

which we will refer to by using the term “parity”, as in [24] and [28].
2.2 Equations of motion

The equations of motion generated by the action (2.47) are

\[ k \left( \bar{\partial} A_u - \partial_u \bar{\partial} A \right) + \bar{J} = 0 \quad (m = 2, h = -1) \] (2.55)

\[ k \left( \partial_u A - \partial A_u \right) + J = 0 \quad (m = 2, h = 1) \] (2.56)

\[ k \left( \partial \bar{\partial} A - \partial A \bar{\partial} + \frac{1}{k} b \right) + J_u = 0 \quad (m = 2, h = 0) \] (2.57)

\[ A_u + J_b = 0 \quad (m = 1, h = 0) \] (2.58)

where we have explicitly indicated the mass dimension \( m \) and the value of the quantum number of helicity \( h \). The fields solutions of the quantum equations of motion are given by the generalization of (1.9) for the scalar field:

\[ \Phi_J(x) = \frac{\delta W}{\delta J(x)} , \] (2.59)

where the symbols have the same meaning as in the scalar case.

Since the equations of motion are generated by the action \( S_J \), it is clear that each of its symmetries has a consequence on the equations themselves. Let us see this point in more detail.

**Residual Lorentz invariance:** as we have previously said, the invariance of the action under the two-dimensional Lorentz transformations on the planes \( u = \text{constant} \) has been encoded through the introduction of the helicity, that hence must be conserved. Therefore, each equation of motion has a defined value of \( h \), that we have explicitly written aside.

**Residual gauge invariance:** applying \( \partial \) to the first equation of motion (2.55), \( \bar{\partial} \) to the second (2.56), \( \partial_u \) to the third (2.57) and then summing, we get the identity

\[ \partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_v b = 0 , \] (2.60)

which is the local Ward identity (2.52) expressing the residual gauge invariance of the theory. It is worth pointing out that local identities are stronger than global ones. In our case, this additional property is due to the choice of the axial gauge \( A_u = 0 \).

**Parity:** the invariance of the action under the parity transformation (2.54) implies that its effect on the equations of motion is

\[
\begin{align*}
(2.55) & \leftrightarrow (2.56) \\
(2.57) & \leftrightarrow - (2.57) \\
(2.58) & \leftrightarrow - (2.58)
\end{align*}
\] (2.61)
CHAPTER 2: CHERN-SIMONS THEORY

2.3 Introduction of the boundary

We now introduce in the theory the planar boundary

\[ u = 0 \quad (2.62) \]

Following the guidelines discussed in Chapter 1, Symanzik’s decoupling condition [1] implies the decomposition of the generating functional of the connected Green functions

\[ W = W_+ + W_- \quad (2.63) \]

Hence the propagators take the form

\[
\Delta_{\Phi_1\Phi_2}(x_1, x_2) = \langle T(\Phi_1(x_1)\Phi_2(x_2)) \rangle = \theta_+ \Delta_+(x_1, x_2) + \theta_- \Delta_-(x_1, x_2) \quad (2.64)
\]

where

\[ \theta_{\pm} \equiv \theta(\pm u_1)\theta(\pm u_2) \quad (2.65) \]

is equal to 1 if \( u_1 \) and \( u_2 \) are both in the same half-space, and equal to zero otherwise.

For what concerns the equations of motion, they are broken by boundary terms which must respect all the discussed constraints of locality, linearity in the quantum fields, power counting and helicity. The most general broken equations of motion which satisfy all the requirements are

\[
k (\partial A_u - \partial_u \bar{A}) + \bar{J} = \delta(u)(c_1^+ A_\mp + c_1^- \bar{A}_\mp) \quad (2.66)
\]

\[
k (\partial_u A - \partial A_u) + J = \delta(u)(c_2^+ A_\mp + c_2^- \bar{A}_\mp) \quad (2.67)
\]

\[
k (\partial \bar{A} - \partial \bar{A} + \frac{1}{k}) + J_u = \delta(u)(c_3^+ A_u + c_3^- \bar{A}_u) \quad (2.68)
\]

\[ A_u + J_b = 0 \quad (2.69) \]

where the \( c_i^\pm \) are constant parameters, and \( A_{\pm}(Z) \), \( \bar{A}_{\pm}(Z) \) and \( A_{\mp}(Z) \) are, as in the scalar case, the boundary fields on the right, respectively on the left, of the boundary:

\[
A_{\pm}(Z) = \lim_{u \to 0^\pm} A(x) \quad (2.70)
\]

\[
\bar{A}_{\pm}(Z) = \lim_{u \to 0^\pm} \bar{A}(x) \quad (2.71)
\]

\[
A_{\mp}(Z) = \lim_{u \to 0^\pm} A_u(x) \quad , \quad (2.72)
\]

with \( Z \equiv (z, \bar{z}) \). Under parity (2.54), the boundary fields transform according to

\[
A_{\pm} \leftrightarrow \bar{A}_{\mp} \quad (2.73)
\]

\[
A_{\mp} \rightarrow -A_{\mp} \quad (2.74)
\]
Let us impose the parity constraint (2.61) on the broken equations of motion, getting
\[
\delta(-u)(c_1^+ A_- + c_1^- A_+) = \delta(u)(c_2^+ A_+ + c_2^- A_-) \quad (2.75)
\]
\[
\delta(-u)(c_2^+ \bar{A}_- + c_2^- \bar{A}_+) = \delta(u)(c_1^+ \bar{A}_+ + c_1^- \bar{A}_-) \quad (2.76)
\]
\[
-\delta(-u)(c_3^+ A_{u-} + c_3^- A_{u+}) = \delta(u)(c_3^+ A_{u+} + c_3^- A_{u-}) \quad (2.77)
\]
which lead to
\[
c_1^\pm = c_2^\mp \quad (2.78)
\]
\[
c_3^\pm = -c_3^\mp \equiv c_3 \quad (2.79)
\]
The broken equations of motion thus become
\[
k \left( \partial A_{u} - \partial_{u} \bar{A} \right) + \bar{J} = \delta(u)(c_1^+ \bar{A}_+ + c_1^- \bar{A}_-) 
\]
\[
k \left( \partial A - \partial A_{u} \right) + J = \delta(u)(c_1^+ A_+ + c_1^- A_-) \quad (2.81)
\]
\[
k \left( \partial\bar{A} - \partial\bar{A} + \frac{1}{k}b \right) + J_{u} = c_3 \delta(u)(A_{u+} - A_{u-}) \quad (2.82)
\]
\[
A_{u} + J_{b} = 0 \quad (2.83)
\]
We now move our attention to the algebraic constraint that in the previous Chapter we called compatibility (1.26). Using (2.59), we write the bulk equations in a functional way as
\[
M_{\Phi}(x)W = J_{\Phi}(x) \quad (2.84)
\]
where the operators $M_{\Phi}(x)$ are given by
\[
k \left( \partial_{u} \frac{\delta}{\delta J(x)} - \bar{\partial} \frac{\delta}{\delta J_{u}(x)} \right) \equiv \bar{M}(x) \quad (2.85)
\]
\[
k \left( \partial \frac{\delta}{\delta J_{u}(x)} - \partial_{u} \frac{\delta}{\delta J(x)} \right) \equiv M(x) \quad (2.86)
\]
\[
k \left( \bar{\partial} \frac{\delta}{\delta J(x)} - \partial \frac{\delta}{\delta J_{u}(x)} - \frac{1}{k} \frac{\delta}{\delta J_{b}(x)} \right) \equiv M_{u}(x) \quad (2.87)
\]
\[
-\frac{\delta}{\delta J_{u}(x)} \equiv M_{b}(x) \quad (2.88)
\]
We see that this time there are four operators acting on $W$, instead of one, as in the scalar case. However, it can be easily checked that they satisfy a commutative algebra as well:
\[
[M_{\Phi}(y), M_{\Phi'}(x)]W = 0 \quad (2.89)
\]
CHAPTER 2: CHERN-SIMONS THEORY

After the introduction of the boundary, the equations of motion take the functional form

\[ M_\Phi^\pm(x) W_\pm = J_\Phi(x), \] 

(2.90)

where \( M_\Phi^\pm(x) \) are the operators modified by the boundary:

\[ \overline{M}^\pm(x) \equiv M(x) + \delta(u) c_1^\pm \frac{\delta}{\delta J(x)} \] 

(2.91)

\[ M^\pm(x) \equiv M(x) + \delta(u) c_1^\mp \frac{\delta}{\delta J(x)} \] 

(2.92)

\[ M_u^\pm(x) \equiv M_u(x) \pm \delta(u) c_3 \frac{\delta}{\delta J_u(x)} \] 

(2.93)

\[ M_b^\pm(x) \equiv M_b(x), \] 

(2.94)

the operators without superscript being the same defined in (2.85)-(2.88).

For the same reasons illustrated in Chapter 1, we want that the boundary does not break the bulk algebra. We therefore impose the compatibility conditions

\[ \left[ M_\Phi^\pm(x'), M_\Phi^\pm(x) \right] W_\pm = 0. \] 

(2.95)

It can be easily checked that only the commutator

\[ \left[ M^\pm(x'), \overline{M}^\pm(x) \right] W_\pm \] 

(2.96)

is nontrivial. Using (2.91) and (2.92), and requiring that it vanishes, we get the condition

\[ c_1^+ \delta(u') \delta(u - u') - c_1^- \delta(u) \delta(u' - u) = 0, \] 

(2.97)

which leads to

\[ c_1^+ = c_1^- = c, \] 

(2.98)

where we have used the property of the Dirac delta function [30]

\[ \delta(y) \delta(x - y) = \delta(y) \delta(x). \] 

(2.99)

Notice that the parity constraint implies that the preservation of the algebra on one side of the boundary is a consequence of the preservation of the algebra on the opposite side.

At the end, the final form of the broken equations of motion which respect parity and compatibility is

\[ k \left( \partial A_u - \partial_u A \right) + \bar{J} = c \delta(u)(A_+ + A_-) \] 

(2.100)

\[ k \left( \partial A - \partial A_u \right) + J = c \delta(u)(A_+ + A_-) \] 

(2.101)

\[ k \left( \partial A - \partial A + \frac{1}{k} b \right) + J_u = c_3 \delta(u)(A_{u+} - A_{u-}) \] 

(2.102)

\[ A_u + J_b = 0. \] 

(2.103)
### 2.4 Boundary conditions

After setting the sources to zero, let us integrate (2.100) and (2.101) with respect to $u$ from $-\varepsilon$ to $\varepsilon$, analogously to what we did in (1.33) for the scalar case. Taking into account also (2.103), we get

$$
(k - c) \bar{A}_- = (k + c) \bar{A}_+ \tag{2.104}
$$

$$
(k + c) A_- = (k - c) A_+ . \tag{2.105}
$$

Due to the decoupling condition, each side of the above identities must vanish separately:

$$
(k - c) \bar{A}_- = 0 \tag{2.106}
$$

$$
(k + c) \bar{A}_+ = 0 \tag{2.107}
$$

$$
(k + c) A_- = 0 \tag{2.108}
$$

$$
(k - c) A_+ = 0 . \tag{2.109}
$$

The nontrivial solutions are given by

$$
c = k \Rightarrow \bar{A}_+ = 0 = A_- \tag{2.110}
$$

$$
c = -k \Rightarrow \bar{A}_- = 0 = A_+ . \tag{2.111}
$$

This is exactly the same result previously found in [26, 24, 28]: the fields obey Dirichlet boundary conditions on both sides of the dividing plane $u = 0$.

It is clear that the choices (2.110) and (2.111) essentially describe the same physics, since they are related by the parity transformation (2.54). Therefore, in what follows we choose the solution with $c = k$, keeping in mind that a kind of “mirror solution” exists.

The equations of motion thus become

$$
k \left( \partial A_u - \partial_u \bar{A} \right) + \bar{J} = k \delta(u) \bar{A}_- \tag{2.112}
$$

$$
k \left( \partial_u A - \partial A_u \right) + J = k \delta(u) A_+ \tag{2.113}
$$

$$
k \left( \partial A - \partial \bar{A} + \frac{1}{k} b \right) + J_u = c_3 \delta(u)(A_{u+} - A_{u-}) \tag{2.114}
$$

$$
A_u + J_b = 0 . \tag{2.115}
$$

Correspondingly, the local Ward identity acquires a boundary breaking:

$$
\partial \bar{J} + \partial J + \partial_u J_u + \partial_u b = k \delta(u) \left( \partial A_+ + \partial \bar{A}_- \right) + \partial_u [c_3 \delta(u)(A_{u+} - A_{u-})] , \tag{2.116}
$$

which, once integrated, yields the Ward identity

$$
\frac{1}{k} \int du \left( \partial \bar{J} + \partial J \right) = \partial A_+ + \partial \bar{A}_- . \tag{2.117}
$$
2.5 Conserved boundary currents

We now see how the Ward identity (2.117) implies the existence of a Kač-Moody algebra [10] of conserved chiral currents on the boundary.

Using (2.59) we can rewrite the Ward identity in a functional way as

\[ \frac{1}{k} \int du \left[ \partial \tilde{J}(x) + \bar{\partial} J(x) \right] = \bar{\partial} \left. \frac{\delta \mathcal{W}_+}{\delta J(x)} \right|_{u=0^+} + \partial \left. \frac{\delta \mathcal{W}_-}{\delta J(x)} \right|_{u=0^{-}}. \] (2.118)

We then differentiate with respect to \( \bar{\partial} J(x') \), with \( x' \) lying on the right side of space next to the boundary, and then set the sources to zero. We thus get

\[ \frac{1}{k} \int du \partial \delta^3(x - x') = \left. \left( \bar{\partial} \frac{\delta^2 \mathcal{W}_+}{\delta J(x') \delta J(x)} \right) \right|_{J_\Phi=0} \quad + \quad \left. \left( \partial \frac{\delta^2 \mathcal{W}_-}{\delta J(x') \delta J(x)} \right) \right|_{J_\Phi=0}. \] (2.119)

Recalling that \( \mathcal{W}_\pm \) are the generators of the connected Green functions for the \( u > 0 \) and \( u < 0 \) sides of spacetime respectively, the right-hand side of this expression involves propagators, the second of which vanishes due to the decoupling condition. Therefore, we are left with

\[ \frac{1}{k} \partial \delta^3(Z - Z') = \bar{\partial} \left. \langle T(A_+(Z')A_+(Z)) \rangle \right| \cdot \] (2.120)

Keeping in mind the definition of the time-ordering operator \( T \), we have to specify the role of time in light-cone variables. This issue has already been extensively studied in the literature [37]. The outcome is that a possibility is to identify the light-cone variable \( \bar{z} \) with time.

We can now explicitly compute the right-hand side of (2.120):

\[ \bar{\partial} \left. \langle T(A_+(Z')A_+(Z)) \rangle \right| = \]

\[ = \bar{\partial} \left( \theta(z' - \bar{z})A_+(Z')A_+(Z) + \theta(\bar{z} - \bar{z}')A_+(Z)A_+(Z') \right) \]

\[ = -\left( \delta(z' - \bar{z})A_+(Z')A_+(Z) + \theta(\bar{z} - \bar{z}')A_+(Z')\bar{\partial}A_+(Z) \right) \]

\[ + \left( \delta(\bar{z} - \bar{z}')A_+(Z)A_+(Z') + \theta(\bar{z} - \bar{z}')\bar{\partial}A_+(Z)A_+(Z') \right) \]

\[ = \left( \theta(z' - \bar{z})A_+(Z')\bar{\partial}A_+(Z) + \theta(\bar{z} - \bar{z}')\bar{\partial}A_+(Z)A_+(Z') \right) \]

\[ + \delta(\bar{z} - \bar{z}') \left( [A_+(Z), A_+(Z')] \right). \] (2.121)
On the other hand, after setting the sources to zero, the Ward identity (2.117) and the decoupling condition yield the *chirality condition* [23, 24, 28]:

\[
0 = \tilde{\partial} A_+ \Rightarrow A_+ = A_+ (z) \tag{2.122}
\]

\[
0 = \partial \tilde{A}_- \Rightarrow \tilde{A}_- = \tilde{A}_- (\tilde{z}) \tag{2.123}
\]

Substituting (2.122) into (2.121) and then into (2.120) we get

\[
\frac{1}{k} \delta^2(Z-Z') = \frac{1}{k} \delta(\tilde{z}-\tilde{z}') \langle \partial \delta(z-z') \rangle = \delta(\tilde{z}-\tilde{z}') \langle [A_+(z), A_+(z')] \rangle , \tag{2.124}
\]

which finally yields the commutation relation

\[
[A_+(z), A_+(z')] = \frac{1}{k} \delta (z - z') . \tag{2.125}
\]

This is the abelian counterpart of the Kač-Moody algebra [10] of the Wess-Zumino-Witten model [11] generated by the chiral currents found in [3, 23, 24]. Under this respect, the coefficient \( \frac{1}{k} \) can be seen as the central charge of the Kač-Moody algebra. Note that the parity symmetry (2.61) implies the mirror algebra

\[
[A_-(\tilde{z}), A_-(\tilde{z}')] = \frac{1}{k} \tilde{\partial} \delta(\tilde{z} - \tilde{z}') \tag{2.126}
\]

on the opposite side of the boundary.

Moreover, from (2.122) and (2.110), we get

\[
\bar{\partial} A_+ + \partial \bar{A}_+ = 0 , \tag{2.127}
\]

which is the conservation relation for the planar boundary field, written in light-cone coordinates [28]. Indeed, recalling (2.36), we can come back to the euclidean components of the gauge field:

\[
A_+ \equiv \frac{1}{\sqrt{2}} (A_{1+} + iA_{0+})
\]

\[
\tilde{A}_+ \equiv \frac{1}{\sqrt{2}} (A_{1+} - iA_{0+}) . \tag{2.128}
\]

The relation (2.127) then takes the form

\[
\partial_1 A_{1+} + \partial_0 A_{0+} = 0 , \tag{2.129}
\]

which is easily identified with a continuity relation involving a density \( A_{0+} \) and a current \( A_{1+} \).
Furthermore, from (2.110) and (2.122) follows
\[
\partial_1 A_{0+} - i \partial_0 A_{0+} = 0 ,
\]  
which, compared to (2.129), implies the identification of \( A_{0+} \) and \( A_{1+} \)
\[
A_{1+} = i A_{0+} .
\]  
Consequently, the algebra (2.125) can be written in terms of the density \( A_{0+} \):
\[
[A_{0+}(z), A_{0+}(z')] = \frac{1}{2k} \delta(z - z') .
\]  
Summarizing, the commutation relation (2.125), obtained in the framework of a gauge field theory, can be interpreted as an abelian Kač-Moody algebra of conserved chiral currents.

This turns out to coincide with the known result of the Tomonaga-Luttinger theory [38] for a monodimensional liquid of interacting electrons. In particular, we can come back to the minkowskian spacetime where, in the usual model of the edge states of the Fractional Quantum Hall Effect [5], the quantity
\[
\rho^M_+ = -i A_{0+}
\]  
is proportional to the electron density on the edge of the Hall bar, and therefore can be regarded as a physical quantity.

In conclusion, we recovered the general result [2, 3] that a topological field theory acquires local observables only when a boundary is introduced. In the CS case, the observables are conserved chiral currents living on the boundary and satisfying a Kač-Moody algebra, whose central charge is the inverse of the CS coupling constant, as discussed in [23, 24, 28].

In [23], the covariant Landau gauge has been adopted, and the effect of the boundary has been encoded into an additional term in the action.

In [24, 28] a noncovariant axial gauge has been preferred, and the effect of the boundary revealed in a modification of the equations of motion of the CS gauge field.

Our approach has been closely related with that presented in [24, 28], leading us to get the same results, but with the remarkable difference that we have used an algebraic method which has allowed to avoid the explicit computation of the propagators of the theory. This will be extremely useful in cases, like that described in the next Chapter, in which the explicit computation of correlators is not that easy.
Chapter 3
Maxwell-Chern-Simons theory

The (minkowskian) Maxwell theory

\[ S_{M}^{\text{mink}} = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \]  

(3.1)
is the abelian gauge field theory involving a vector field \( A_{\mu}(\hat{x}) \) which describes the photons. In fact, the equations for the gauge field \( A_{\mu}(\hat{x}) \) are the Maxwell equations

\[ \partial_{\nu} F^{\mu\nu} = 0 , \]  

(3.2)
and the vector field is massless and characterized by two spin 1 degrees of freedom, which can be regarded as the helicity states of the photons.

The action (3.1) is invariant under the infinitesimal gauge transformation

\[ \delta A_{\mu} = \partial_{\mu} \theta , \]  

(3.3)
where \( \theta(\hat{x}) \) is a local parameter. To describe massive vector fields, one can add to the Maxwell action (3.1) the mass term

\[ S_{M}^{\text{mass}} = \frac{m^2}{2} \int d^4 x A_{\mu} A^{\mu} . \]  

(3.4)
Indeed, the resulting equations of motion are

\[ \partial_{\nu} \partial^{\nu} A_{\mu} - \partial_{\mu} \partial^{\nu} A_{\nu} + m^2 A_{\mu} = 0 , \]  

(3.5)
whose divergence yield

\[ \partial^{\mu} A_{\mu} = 0 . \]  

(3.6)
Consequently, (3.5) yields the Klein-Gordon equation

\[ \left( \partial_{\nu} \partial^{\nu} + m^2 \right) A_{\mu} = 0 , \]  

(3.7)
which identifies $m$ with the mass of the theory.

However, (3.4) is not invariant under the gauge transformation (3.3); therefore, the simple addition of the term (3.4) to the action (3.1) does not solve the problem of describing gauge invariant massive models.

This issue is highly relevant in the four dimensional Yang-Mills theory, which is the nonabelian extension of the Maxwell theory (3.1) [35], due to its physical consequences in elementary particle physics. The only way out seems to be the introduction of an additional scalar field, so far unobserved, by means of the Higgs mechanism [35].

However, an alternative way to give masses to gauge fields without introducing any additional fields was proposed in [20] in three dimensions. We refer to the Yang-Mills-Chern-Simons theory, where the addition of the CS term to the Yang-Mills model gives to the gauge field a mass, which is therefore called “topological”.

The topological mass does not only provide a solution to the problem of the description of a massive gauge field, but, at the quantum level, provides also an infrared cutoff for the three-dimensional vector gauge theories, thus providing a cure for the infrared problem without affecting the ultraviolet gauge aspects of the theory [39]. Unfortunately, this topological method is peculiar to the three dimensional spacetime only.

In this thesis we study the abelian Maxwell-Chern-Simons (MCS) theory, which turns out to describe a massive spin 1 particle with a single state of helicity, and which can be coupled with fermions to define a modified electrodynamics of fermions interacting with each other and with topologically massive “photons” [20].

On the other hand, adding a Maxwell term to the CS theory breaks its topological character, since the Maxwell model (3.1), contrarily to the CS action (2.1), depends on the metric. In this sense, our aim is actually to study how the addition of this non-topological term affects the results found in the previous Chapter.

Let us now see how the addition of a CS term gives a mass to the gauge field. The MCS action in the minkowskian metric (2.3) can be written as

$$S_{MCS}^{\text{mink}} = \frac{k}{2} \int d^3 \tilde{x} \varepsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho - \frac{\alpha}{4} \int d^3 \tilde{x} F^{\mu \nu} F_{\mu \nu}, \quad (3.8)$$

where the second term represents the Maxwell action, and $F_{\mu \nu}(\tilde{x})$ is the field strength defined in (3.1).

Notice that the theory actually depends only on one parameter, that we identify with $k$, which represents the topological mass, as we are going to see in a moment. In fact, the parameter $\alpha$ in front of the Maxwell term can be reabsorbed by a redefinition of the gauge field $A_\mu(\tilde{x})$. Nevertheless, we
prefer to keep it in order to be able, at a later step, to switch off the Maxwell
term and make contact with the CS case, discussed in the previous Chapter.
Another good reason to keep $\alpha$, is that in the nonabelian extension it is not
possible to reabsorbe it, and therefore in the general case it is a real coupling
constant.

The action (3.8) generates the Euler-Lagrange equations for $A_\mu(\hat{x})$:

$$\alpha (\partial_\rho \partial^\rho A^\sigma - \partial_\sigma \partial^\rho A^\rho) + k \varepsilon^{\sigma \rho \mu} \partial_\rho A_\mu = 0 ,$$

(3.9)

which, written for $F_{\mu\nu}(\hat{x})$, are:

$$\alpha \partial_\rho F^{\rho \sigma} + \frac{k}{2} \varepsilon^{\sigma \rho \mu} F_{\rho \mu} = 0 .$$

(3.10)

We introduce the dual field strenght $^* F^\mu(\hat{x})$

$$^* F^\mu \equiv \frac{1}{2} \varepsilon^{\mu \nu \rho} F_{\nu \rho} \Rightarrow F^{\mu \nu} = \varepsilon^{\mu \nu \rho} F_\rho ,$$

(3.11)

which is a vector in three dimensions, whose divergence vanishes due to (3.10):

$$\partial_\sigma ^* F^\sigma = 0 .$$

(3.12)

The equations of motion (3.10) can then be restated in the equivalent, dual form

$$\alpha \partial_\mu ^* F^\nu - \alpha \partial_\nu ^* F^\mu - k F^\mu_\nu = 0 ,$$

(3.13)

whose divergence, taking into account (3.12) and (3.10), yields

$$\left( \partial_\mu \partial_\mu + k^2 \frac{1}{\alpha^2} \right) ^* F^\nu = 0 .$$

(3.14)

The equation (3.14) represents a Klein-Gordon equation for the dual field

$$^* F^\mu(\hat{x}) ,$$

which clearly shows that $\frac{k}{\alpha}$ is a mass for the theory.

Moreover, as discussed in Chapter 2, fixing the gauge allows to compute
the propagator of the theory. For instance, in the covariant gauge

$$\partial^\mu A_\mu = 0 ,$$

(3.15)

that we used for the computation of the propagator of the CS theory, the
gauge fixing term (2.15) modifies the action in momentum space according to

$$S_{MCS}^{\text{mink}} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \tilde{A}_\mu(-p) \left[ ik \varepsilon^{\mu \nu \rho} p_\rho + \left( \alpha + \frac{1}{\xi} \right) p^\mu p^\rho - \alpha p^2 g^{\mu \rho} \right] \tilde{A}_\rho(p) .$$

(3.16)
The propagator of the theory is the inverse of the operator

\[ ik\varepsilon^{\mu\nu\rho} p_\rho + \left( \alpha + \frac{1}{\xi} \right) p^\mu p^\rho - \alpha p^2 g^{\mu\rho}, \quad (3.17) \]

where \( g^{\mu\rho} \) is the minkowsian metric tensor (2.19). As we did in the previous Chapter, we compute it by writing down the most general rank-2 tensor in momentum space

\[ \tilde{\Delta}_{\mu\lambda} = A g_{\rho\lambda} + B p_\rho p_\lambda + C \varepsilon_{\rho\lambda\tau} p^\tau, \quad (3.18) \]

\( A (p^2), B (p^2) \) and \( C (p^2) \) being scalar functions of \( p^2 \), and impose the condition

\[ \left[ ik\varepsilon^{\mu\nu\rho} p_\rho + \left( \alpha + \frac{1}{\xi} \right) p^\mu p^\rho - \alpha p^2 g^{\mu\rho} \right] \tilde{\Delta}_{\rho\lambda} = \delta_{\mu}^{\lambda}. \quad (3.19) \]

From this we get

\[ - (ikA + \alpha p^2 C) g^{\mu\rho} \varepsilon_{\rho\lambda\tau} p^\tau + (ik p^2 C - \alpha p^2 A) \delta_{\lambda}^{\mu}, \quad (3.20) \]

which yields

\[ \begin{align*}
0 &= ikA + \alpha p^2 C & \quad (3.21) \\
1 &= ik p^2 C - \alpha p^2 A & \quad (3.22) \\
0 &= \frac{A + B p^2}{\xi} + \alpha A - ikC, \quad (3.23) 
\end{align*} \]

and hence

\[ \begin{align*}
A &= \frac{\alpha}{k^2 - \alpha^2 p^2} & \quad (3.24) \\
B &= \frac{\xi}{p^2} \left[ \frac{1}{p^2} - \frac{\alpha}{\xi (k^2 - \alpha^2 p^2)} \right] & \quad (3.25) \\
C &= \frac{-ik}{p^2 (k^2 - \alpha^2 p^2)}. \quad (3.26) 
\end{align*} \]

The MCS momentum space propagator is thus given by

\[ \tilde{\Delta}_{\rho\lambda} = \frac{\alpha p^2 g_{\rho\lambda} - ik\varepsilon_{\rho\lambda\tau} p^\tau - \alpha p_\rho p_\lambda}{p^2 (k^2 - \alpha^2 p^2)} + \frac{\xi p_\rho p_\lambda}{(p^2)^2}, \quad (3.27) \]
which identifies the topological mass through the pole at \( p^2 = \frac{k^2}{\alpha^2} \). As we have pointed out in the previous Chapter, for \( \xi = 0 \) we get the Landau gauge, while the Feynman gauge corresponds to \( \xi = 1 \).

Since our aim is to study the effect of the Maxwell term on the results found in the previous Chapter, we now write down the Maxwell action in the euclidean light-cone coordinates defined in (2.35) and (2.36). As it can be read off from (3.8), the Maxwell action in minkowskian metric (2.3) is

\[
S_{\text{mink}}^M = -\frac{\alpha}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} .
\] (3.28)

Passing to euclidean coordinates (1.2), this becomes

\[
S_{\text{mink}}^M \rightarrow i\frac{\alpha}{4} \int d^3x F_{\mu\nu} F_{\mu\nu} ,
\] (3.29)

where we have used the relations (2.30) and (2.31) between minkowskian vectors and their euclidean counterparts. From the definition of the path integral

\[
Z = \int \mathcal{D}\Phi \ e^{iS_{\text{mink}}} \equiv \int \mathcal{D}\Phi \ e^{-S_{\text{eucl}}}
\] (3.30)

we can read off the euclidean Maxwell term

\[
S_{\text{eucl}}^M = \frac{\alpha}{4} \int d^3x F_{\mu\nu} F_{\mu\nu} ,
\] (3.31)

and recalling the relation between the measures

\[
d^3x = idudzd\bar{z} ,
\] (3.32)

we finally get the Maxwell action in light-cone coordinates:

\[
S_M = -\frac{i\alpha}{2} \int dudzd\bar{z} \left[ \partial \bar{\partial} A + \partial \bar{\partial} \bar{A} - 2\partial \bar{\partial} A \right. \\
\left. + 2\partial_u A \bar{\partial} A_u + 2\partial_u \bar{\partial} A_u - 2\partial_u \bar{\partial} \bar{A}_u - 2\partial A_u \bar{\partial} A_u \right] .
\] (3.33)

Recall from Chapter 2 that the CS theory in euclidean light-cone coordinates is given by

\[
S_{\text{cs}} = -k \int dudzd\bar{z} \ (\bar{A}\partial_u A + A_u \partial \bar{A}) ,
\] (3.34)

and that the gauge fixing term for the axial gauge \( A_u = 0 \) with \( \xi = 0 \) is

\[
S_{gf} = -\int dudzd\bar{z} A_u b .
\] (3.35)
The complete action for the theory that we now consider is therefore

\[ S = S_{CS} + S_M + S_{gf}, \tag{3.36} \]

which defines the generating functional of the Green functions:

\[ Z[J_\Phi] = \int \mathcal{D}\Phi \exp \left[ -\left( S - \int dud\bar{z} d\bar{\bar{z}} \sum_\Phi J_\Phi \Phi \right) \right], \tag{3.37} \]

as in the previous Chapter. The external sources \( J_\Phi \equiv J_u, \bar{J}, J, J_b \) are coupled to the fields \( \Phi \equiv A_u, A, \bar{A}, b \), respectively, and now

\[ x \equiv (z, \bar{z}, u). \tag{3.38} \]

From Chapter 2 we recall also the definition of the \( n \)-point Green functions

\[ \langle T(\Phi_1(x_1) \ldots \Phi_n(x_n)) \rangle = \left. \frac{\delta^n Z}{\delta J_{\Phi_1}(x_1) \ldots \delta J_{\Phi_n}(x_n)} \right|_{J=0}. \tag{3.39} \]
3.1 The action and its properties

From (3.37) we get the modified action

\[ S_J \equiv S - \int dudzd\bar{z} \sum_\Phi J_\Phi \Phi . \]  

(3.40)

As we did in the previous Chapter, let us examine its properties and symmetries:

**Power counting:** while in the CS case the mass dimension of the gauge field is one, the addition of the Maxwell term changes its dimension to \( \frac{1}{2} \) in order to have a dimensionless action. Therefore, the dimensions of the fields, sources and parameters are

\[ [A] = [\bar{A}] = [A_u] = \frac{1}{2} = [J_b] \]  

(3.41)

\[ [\bar{J}] = [J] = [J_u] = \frac{5}{2} = [A_b] \]  

(3.42)

\[ [k] = 1 . \]  

(3.43)

As we said, \( \alpha \) is a massless parameter kept only to be able to switch off the Maxwell term.

**Symmetries and quantum numbers:**

1.) The helicity \( h \), which we introduced in the previous Chapter to encode the two-dimensional Lorentz invariance in the planes \( u = \text{constant} \), is still conserved, as it can be checked by using the summarizing Table 2.

<table>
<thead>
<tr>
<th></th>
<th>( A_u )</th>
<th>( A )</th>
<th>( \bar{A} )</th>
<th>( b )</th>
<th>( J_u )</th>
<th>( J )</th>
<th>( \bar{J} )</th>
<th>( J_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>( 1/2 )</td>
<td>( 1/2 )</td>
<td>( 1/2 )</td>
<td>( 5/2 )</td>
<td>( 5/2 )</td>
<td>( 5/2 )</td>
<td>( 1/2 )</td>
<td></td>
</tr>
<tr>
<td>hel</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( -1 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \partial_u )</th>
<th>( \partial )</th>
<th>( \bar{\partial} )</th>
<th>( u )</th>
<th>( z )</th>
<th>( \bar{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( -1 )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>hel</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( -1 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
2.) The residual gauge transformation
\[
\begin{align*}
\delta A &= \partial \theta \\
\delta \bar{A} &= \bar{\partial} \theta \\
\delta A_u &= 0 \\
\delta b &= 0,
\end{align*}
\]
\(\theta(z, \bar{z})\) being a local gauge parameter that does not depend on the variable \(u\), is still a symmetry of the action. This implies the existence of the local Ward identity
\[
\partial \bar{J} + \bar{\partial} J + \partial_u J_u + \partial_u b = 0,
\]
which can be integrated in \(u\), yielding
\[
\int du \left( \partial \bar{J} + \bar{\partial} J \right) = 0,
\]
as in the CS case.

3.) The action (3.40) is still invariant under the parity transformation (2.54):
\[
\begin{align*}
z &\leftrightarrow \bar{z} \\
u &\rightarrow -u \\
A &\leftrightarrow \bar{A} \\
A_u &\rightarrow -A_u \\
b &\rightarrow -b \\
J &\leftrightarrow \bar{J} \\
J_u &\rightarrow -J_u \\
J_b &\rightarrow -J_b.
\end{align*}
\]
3.2 Equations of motion

The equations of motion generated by the action (3.40) are

\[
\begin{align*}
&k(\partial u A - \partial A u) + i\alpha[\partial(\partial A - \partial A) + \partial u(\partial A u - \partial A)] + J = 0 \quad (3.48) \\
&k(\partial u A - \partial A u) + i\alpha[\partial(\partial A - \partial A) + \partial u(\partial A u - \partial A)] + J = 0 \quad (3.49) \\
&k(\partial A - \partial A) + i\alpha[2\partial \partial A u - \partial u(\partial A + \partial A)] + b + J_u = 0 \quad (3.50) \\
&A_u + J_b = 0 \quad (3.51)
\end{align*}
\]

and the fields solutions of the quantum equations of motion are given by

\[
\Phi_j(x) = \frac{\delta W}{\delta J_b(x)} ,
\]

\(W\) being the generating functional of the connected Green functions

\[
W = \ln Z .
\]

As we have already pointed out in Chapter 2, the symmetries of the action have consequences on the equations of motion. Let us summarize:

**Residual Lorentz invariance:** This symmetry has been encoded by means of the quantum number \(h\). We summarize the value of \(h\) together with the mass dimension \(m\) of each equation of motion:

\[
\begin{align*}
(3.48) & \rightarrow (m = \frac{5}{2}, h = -1) \quad (3.54) \\
(3.49) & \rightarrow (m = \frac{5}{2}, h = 1) \quad (3.55) \\
(3.50) & \rightarrow (m = \frac{5}{2}, h = 0) \quad (3.56) \\
(3.51) & \rightarrow (m = \frac{1}{2}, h = 0) . \quad (3.57)
\end{align*}
\]

**Residual gauge invariance:** Exactly as in the CS theory, applying \(\partial\) to the first equation of motion (3.48), \(\bar{\partial}\) to the second (3.49), \(\partial_u\) to the third (3.50) and then summing, yields the local Ward identity (3.45), which expresses the residual local gauge invariance.

**Parity:** The fact that the parity transformation (3.47) is a symmetry of the action implies that its effect on the equations of motion is analogous to...
what we found for the pure CS equations:

\[
\begin{align*}
(3.48) & \leftrightarrow (3.49) \\
(3.50) & \rightarrow -(3.50) \\
(3.51) & \rightarrow -(3.51).
\end{align*}
\]
3.3 Introduction of the boundary

As we did in the previous Chapter, we now introduce the planar boundary \( u = 0 \). The decoupling condition imposes that the generating functional of the connected Green functions \( W \) can be written as

\[
W = W_+ + W_-
\]

and, consequently, the propagators of the theory take the form

\[
\Delta_{\Phi_1\Phi_2}(x_1, x_2) = \langle T(\Phi_1(x_1)\Phi_2(x_2)) \rangle = \theta_+ \Delta_+(x_1, x_2) + \theta_- \Delta_-(x_1, x_2), \tag{3.60}
\]

where

\[
\theta_\pm \equiv \theta(\pm u_1)\theta(\pm u_2) \tag{3.61}
\]

is equal to 1 if \( u_1 \) and \( u_2 \) are both in the same half-space, and equal to zero otherwise.

The presence of the boundary induces in the equations of motion boundary terms that must respect all the constraints of locality, linearity in the quantum fields, power counting, conserved quantum numbers and analyticity in the parameters. The most general broken equations of motion satisfying all the requirements are

\[
k(\bar{\partial}A_u - \partial_u \bar{A}) + i\alpha [\bar{\partial}(\partial \bar{A} - \partial A) + \partial_u (\partial_u \bar{A} - \partial A_u)] + J = \\
= \delta(u) \left[ \alpha^+_4 \bar{A}_+ + \alpha^-_4 \bar{A}_- + \alpha^+_5 (\partial_u \bar{A})_+ + \alpha^-_5 (\partial_u A)_- \right] \\
+ \delta'(u) \left[ \alpha^+_4 \bar{A}_+ + \alpha^-_4 \bar{A}_- \right] + \delta(u) \left[ \alpha^+_6 \partial A_{u+} + \alpha^-_6 \partial A_{u-} \right] \tag{3.62}
\]

\[
k(\partial_u A - \partial A_u) + i\alpha [\partial(\bar{\partial}A - \bar{\partial}A) + \partial_u (\partial_u A - \partial A_u)] + J = \\
= \delta(u) \left[ \gamma^+_1 \bar{A}_+ + \gamma^-_1 \bar{A}_- + \gamma^+_2 (\partial_u \bar{A})_+ + \gamma^-_2 (\partial_u A)_- \right] \\
+ \delta'(u) \left[ \gamma^+_1 \bar{A}_+ + \gamma^-_1 \bar{A}_- \right] + \delta(u) \left[ \gamma^+_4 \partial A_{u+} + \gamma^-_4 \partial A_{u-} \right] \tag{3.63}
\]

\[
k(\bar{\partial} - \partial \bar{A}) + i\alpha [2\bar{\partial}A_u - \partial_u (\bar{\partial}A + \partial \bar{A})] + b + J_u = \\
= \delta(u) \left[ \alpha^+_4 \bar{A}_+ + \alpha^-_4 \bar{A}_- + \alpha^+_5 \partial A_+ + \alpha^-_5 \partial A_- \right] \\
+ \delta'(u) \left[ \alpha^+_8 A_{u+} + \alpha^-_8 A_{u-} \right] + \delta(u) \left[ \alpha^+_9 (\partial_u A_{u})_+ + \alpha^-_9 (\partial_u A_{u})_- \right] \\
+ \delta'(u) \left( \alpha^+_5 A_{u+} + \alpha^-_5 A_{u-} \right) \tag{3.64}
\]

\[A_u + J_b = 0. \tag{3.65}\]
In the above equations, $\alpha_\pm$ and $\gamma_\pm$ are constant parameters to be partially fixed by imposing the constraints discussed in the previous Chapters. Again, we have used the boundary fields on the left, respectively on the right of the boundary:

$$\bar{A}_\pm(Z) = \lim_{u \to 0^\pm} \bar{A}(x)$$  \hspace{1cm} (3.66)
$$A_\pm(Z) = \lim_{u \to 0^\pm} A(x)$$  \hspace{1cm} (3.67)
$$A_{u\pm}(Z) = \lim_{u \to 0^\pm} A_{u}(x)$$  \hspace{1cm} (3.68)
$$(\partial_u \bar{A})_\pm(Z) = \lim_{u \to 0^\pm} \partial_u \bar{A}(x)$$  \hspace{1cm} (3.69)
$$(\partial_u A)_\pm(Z) = \lim_{u \to 0^\pm} \partial_u A(x)$$  \hspace{1cm} (3.70)
$$(\partial_u A_{u})_\pm(Z) = \lim_{u \to 0^\pm} \partial_u A_{u}(x)$$, \hspace{1cm} (3.71)

where $Z \equiv (z, \bar{z})$. Under the parity symmetry (3.47), these transform according to

$$\begin{cases}
A_\pm & \leftrightarrow \bar{A}_\mp \\
A_{u\pm} & \rightarrow -A_{u\mp} \\
(\partial_u A)_\pm & \leftrightarrow - (\partial_u \bar{A})_\mp \\
(\partial_u A_{u})_\pm & \leftrightarrow (\partial_u A_{u})_\mp .
\end{cases}$$  \hspace{1cm} (3.72)

Notice that the lower dimension of the gauge field, with respect to the CS case, allows the presence of differentiated delta functions $\delta'(u)$ in the right-hand sides of the broken equations of motion.

We now proceed by imposing the constraints that we discussed in the previous Chapters. First of all, we examine the parity constraint, which yields the following conditions:
\[ \delta(-u) \left[ \alpha_1^+ A_- + \alpha_1^- A_+ - \alpha_2^+ (\partial_u A)_- - \alpha_2^- (\partial_u A)_+ \right] + \delta'(-u) \left[ \alpha_3^+ A_- + \alpha_3^- A_+ \right] - \delta(-u) \left( \alpha_6^+ \partial A_{u-} + \alpha_6^- \partial A_{u+} \right) = \]
\[ \delta(u) \left[ \bar{\gamma}_1^+ A_+ + \bar{\gamma}_1^- A_- + \bar{\gamma}_2^+ (\partial_u A)_+ + \bar{\gamma}_2^- (\partial_u A)_- \right] + \delta'(u) \left[ \bar{\gamma}_3^+ A_+ + \bar{\gamma}_3^- A_- \right] + \delta(u) \left( \bar{\gamma}_4^+ \partial A_{u+} + \bar{\gamma}_4^- \partial A_{u-} \right) \] (3.73)

\[ \delta(-u) \left[ \gamma_1^+ \bar{A}_- + \gamma_1^- \bar{A}_+ - \gamma_2^+ (\partial_u \bar{A})_- - \gamma_2^- (\partial_u \bar{A})_+ \right] + \delta'(-u) \left[ \gamma_3^+ \bar{A}_- + \gamma_3^- \bar{A}_- \right] - \delta(-u) \left( \gamma_4^+ \partial \bar{A}_{u-} + \gamma_4^- \partial \bar{A}_{u+} \right) = \]
\[ \delta(u) \left[ \alpha_1^+ \bar{A}_+ + \alpha_1^- \bar{A}_- + \alpha_2^+ (\partial_u \bar{A})_+ + \alpha_2^- (\partial_u \bar{A})_- \right] + \delta'(u) \left[ \alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_- \right] + \delta(u) \left( \alpha_4^+ \partial \bar{A}_{u+} + \alpha_4^- \partial \bar{A}_{u-} \right) \] (3.74)

\[ \delta(-u) \left[ \alpha_4^+ \partial \bar{A}_- + \alpha_4^- \partial \bar{A}_- + \alpha_5^+ \partial A_- + \alpha_5^- \partial A_- \right] - \delta(-u) \left( \alpha_6^+ A_{u-} + \alpha_6^- A_{u+} \right) + \delta'(-u) \left[ \alpha_9^+ (\partial_u A)_- + \alpha_9^- (\partial_u A)_+ \right] - \delta(-u) \left( \alpha_{10}^+ A_{u-} + \alpha_{10}^- A_{u+} \right) = \]
\[ \delta(u) \left[ \alpha_1^+ \partial A_+ + \alpha_1^- \partial A_- + \alpha_2^+ \partial A_+ + \alpha_2^- \partial A_- \right] - \delta(u) \left( \alpha_3^+ \partial A_{u+} + \alpha_3^- \partial A_{u-} \right) - \delta(u) \left( \alpha_4^+ \partial A_{u+} + \alpha_4^- \partial A_{u-} \right) \] (3.75)

Using the properties of the Dirac delta function
\[ \delta(-x) = \delta(x) \] (3.76)
\[ \delta'(-x) = -\delta'(x) \] (3.77)
we get
\[ \alpha_1^\pm = \gamma_1^\mp \] (3.78)
\[ \alpha_2^\pm = -\gamma_2^\mp \] (3.79)
\[ \alpha_3^\pm = -\gamma_3^\mp \] (3.80)
\[ \alpha_6^\pm = -\gamma_4^\mp \] (3.81)
\[ \alpha_4^\pm = -\alpha_3^\mp \] (3.82)
\[ \alpha_8^\pm = \alpha_8 \] (3.83)
\[ \alpha_9^\pm = -\alpha_9 \] (3.84)
\[ \alpha_{10}^\pm = -\alpha_{10} \] (3.85)
Therefore, the broken equations of motion become
\[ k(\bar{\partial}A_u - \partial A_u) + i\alpha[\bar{\partial}(\partial A - \partial A) + \partial_u(\partial A - \bar{\partial}A_u)] + J = \]
\[ = \delta(u) \left[ \alpha_1^+ A_+ + \alpha_1^- A_+ + \alpha_2^+ \left(\partial_u \bar{A}\right)_+ + \alpha_2^- \left(\partial_u A\right)_- \right] \]
\[ + \delta'(u) \left[ \alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_- \right] + \delta(u) \left( \alpha_6^+ \bar{A}_u + \alpha_6^- \partial A_u \right) \]
(3.86)

\[ k(\partial_u A - \partial A_u) + i\alpha[\bar{\partial}(\partial A - \partial \bar{A}) + \partial_u(\partial A - \bar{\partial}A_u)] + J = \]
\[ = \delta(u) \left[ \alpha_4^- \left(\bar{\partial}A_+ - \bar{\partial}A_\right) + \alpha_4^+ \left(\partial \bar{A}_+ - \partial \bar{A}_\right) \right] \]
\[ + \delta'(u) \left[ \alpha_5^+ (A_u + A_u) + \alpha_5^- \left(\partial_u A_u \right) - \left(\partial_u A_u \right) \right] \]
\[ + \delta'(u) \left[ \alpha_7^+ (A_u + A_u) - \left(\partial_u A_u \right) \right] \]
(3.87)

\[ k(\partial \bar{A} - \partial \bar{A}) + i\alpha[2\partial \bar{A}_u - \partial_u(\bar{\partial} A + \partial A)] + b + J_u = \]
\[ = \delta(u) \left[ \alpha_8^- \left(\bar{\partial} \bar{A}_+ - \bar{\partial} \bar{A}_- \right) + \alpha_8^+ \left(\partial \bar{A}_+ - \partial \bar{A}_- \right) \right] \]
\[ + \delta'(u) \left[ \alpha_9^+ (A_u + A_u) - \alpha_9^- \left(\partial_u A_u \right) \right] \]
(3.88)

\[ A_u + J_b = 0 \]  
(3.89)

We now consider the algebraic constraint of “compatibility”. Let us use (3.52) to write the bulk equations of motion in a functional way according as
\[ M_\Phi(x) \mathcal{W} = J_\Phi(x) \]  
(3.90)

where the operators \( M_\Phi(x) \) are given by
\[ M(x) \equiv k \left( \frac{\partial}{\partial J(x)} - \bar{\partial} \frac{\delta}{\delta J_u(x)} \right) - i\alpha \bar{\partial} \left( \partial \frac{\delta}{\delta J(x)} - \bar{\partial} \frac{\delta}{\delta J(x)} \right) \]
(3.91)

\[ M(x) \equiv k \left( \frac{\partial}{\partial J_u(x)} - \partial \frac{\delta}{\delta J(x)} \right) - i\alpha \bar{\partial} \left( \partial \frac{\delta}{\delta J(x)} - \bar{\partial} \frac{\delta}{\delta J(x)} \right) \]
(3.92)

\[ M_u(x) \equiv k \left( \bar{\partial} \frac{\delta}{\delta J(x)} - \partial \frac{\delta}{\delta J(x)} - \frac{1}{k} \frac{\delta}{\delta J_b(x)} \right) \]
\[ - i\alpha \left[ 2\bar{\partial} \frac{\delta}{\delta J_u(x)} - \partial \left( \bar{\partial} \frac{\delta}{\delta J(x)} + \partial \frac{\delta}{\delta J(x)} \right) \right] \]
(3.93)

\[ M_b(x) \equiv - \frac{\delta}{\delta J_u(x)} \]  
(3.94)
CHAPTER 3: MAXWELL-CHERN-SIMONS THEORY

It can be directly checked that these operators satisfy the same commutative algebra found in the pure CS case:

\[ [M_{\Phi}(x'), M_{\Phi}(x)] \mathcal{W} = 0 \, . \] (3.95)

After the introduction of the boundary, the equations of motion take the form

\[ M_\Phi^\pm(x) \mathcal{W}^\pm = J_\Phi(x) \, , \] (3.96)

where \( M_\Phi^\pm(x) \) are the operators modified by the boundary:

\[
M^\pm(x) \equiv M(x) + \delta(u) \left( \alpha_1^\pm \frac{\delta}{\delta J(x)} + \alpha_2^\pm \partial_u \frac{\delta}{\delta J(x)} \right) \\
+ \delta'(u) \alpha_3^\pm \left( \frac{\delta}{\delta J(x)} \bigg|_{u=0} \right) \\
- \delta'(u) \alpha_3^\mp \left( \frac{\delta}{\delta J(x)} \bigg|_{u=0} \right) - \delta(u) \alpha_6^\pm \partial_u \frac{\delta}{\delta J(x)} \] (3.97)

\[
M^\pm_a(x) \equiv M_a(x) + \delta(u) \left( \alpha_1^\pm \partial_u \frac{\delta}{\delta J(x)} + \alpha_2^\pm \partial_u \frac{\delta}{\delta J(x)} \right) \\
+ \delta(u) \left( \alpha_8^+ \frac{\delta}{\delta J_a(x)} + \alpha_9^+ \partial_u \frac{\delta}{\delta J_a(x)} \right) \\
+ \delta'(u) \alpha_{10}^+ \left( \frac{\delta}{\delta J_a(x)} \bigg|_{u=0} \right) \] (3.98)

\[
M^\pm_b(x) \equiv M_b(x) - \delta(u) \left( \alpha_1^\pm \partial_u \frac{\delta}{\delta J(x)} + \alpha_2^\pm \partial_u \frac{\delta}{\delta J(x)} \right) \\
+ \delta(u) \left( \alpha_8^+ \frac{\delta}{\delta J_b(x)} - \alpha_9^+ \partial_u \frac{\delta}{\delta J_b(x)} \right) \\
- \delta'(u) \alpha_{10}^+ \left( \frac{\delta}{\delta J_b(x)} \bigg|_{u=0} \right) \] (3.100)

the operators without superscript being the bulk operators defined in (3.91)-(3.94).
It can be checked that the nontrivial commutators (3.96) are those corresponding to

\[ \Phi = A, \quad \Phi' = \bar{A} \]  
(3.102)
\[ \Phi = A, \quad \Phi' = A_u \]  
(3.103)
\[ \Phi = \bar{A}, \quad \Phi' = A_u \]  
(3.104)
\[ \Phi = A_u, \quad \Phi' = A_u. \]  
(3.105)

These respectively yield the conditions

\[
\begin{align*}
\alpha_1^+ \delta(u')\delta(u - u') + \alpha_2^+ \delta(u')\partial_u \delta(u - u') + \alpha_3^+ \delta'(u')\delta(u) \\
-\alpha_1^- \delta(u)\delta(u' - u) + \alpha_2^- \delta(u)\partial_u \delta(u' - u) + \alpha_3^- \delta'(u)\delta(u') &= 0
\end{align*}
\]  
(3.106)

\[
\begin{align*}
\alpha_6^+ \delta(u')\delta^3(x - x') - \alpha_4^+ \delta(u)\partial \delta^3(x' - x) &= 0 \\
-\alpha_6^- \delta(u')\partial \delta^3(x - x') - \alpha_5^+ \delta(u)\delta \delta^3(x' - x) &= 0
\end{align*}
\]  
(3.107)

\[
\begin{align*}
\alpha_8^+ \delta(u')\delta(u - u') + \alpha_9^+ \delta(u')\partial_u \delta(u - u') + \alpha_{10}^+ \delta'(u')\delta(u) \\
-\alpha_8^- \delta(u)\delta(u' - u) - \alpha_9^+ \delta(u)\partial_u \delta(u' - u) - \alpha_{10}^+ \delta'(u)\delta(u') &= 0
\end{align*}
\]  
(3.108)

which lead to

\[
\begin{align*}
\alpha_1^+ &= \alpha_1^- \equiv \alpha_1 \\
\alpha_2^+ &= \alpha_3^+ \\
\alpha_6^+ &= -\alpha_4^+ \\
\alpha_6^- &= \alpha_5^+ \\
\alpha_9^+ &= -\alpha_{10}^+.
\end{align*}
\]  
(3.110)
CHAPTER 3: MAXWELL-CHERN-SIMONS THEORY

51

The broken equations of motion thus become

\[ k(\bar{\partial}A_u - \partial_u \bar{A}) + i\alpha[\bar{\partial}(\partial \bar{A} - \partial A) + \partial_u(\partial_u \bar{A} - \bar{\partial}A_u)] + \bar{J} = \]

\[ = \delta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- \right] \]

\[ + \delta'(u) \left[ \alpha_3^+ \bar{A}_+ + \alpha_3^- \bar{A}_- \right] - \delta(u) \left( \alpha_4^+ \bar{\partial}A_{u+} - \alpha_4^- \bar{\partial}A_{u-} \right) \]  

(3.115)

\[ k(\partial_u A - \bar{\partial}A_u) + i\alpha[\bar{\partial}(\bar{\partial}A - \partial A) + \partial_u(\partial_u A - \bar{\partial}A_u)] + J = \]

\[ = \delta(u) \left[ \alpha_1 (A_+ + A_-) - \alpha_3^+ (\partial_u A)_+ - \alpha_3^- (\partial_u A)_- \right] \]

\[ - \delta'(u) \left[ \alpha_3^+ A_+ + \alpha_3^- A_- \right] - \delta(u) \left( \alpha_4^+ \partial A_{u+} - \alpha_4^- \partial A_{u-} \right) \]  

(3.116)

\[ k(\bar{\partial}A - \bar{\partial}A) + i\alpha[2\bar{\partial}A_u - \partial_u(\bar{\partial}A + \partial A)] + b + J_u = \]

\[ = \delta(u) \left[ \alpha_4^+ (\bar{\partial}A_+ - \bar{\partial}A_-) + \alpha_5^+ (\bar{\partial}A_+ - \bar{\partial}A_-) \right] \]

\[ + \delta(u) \left[ \alpha_8^+ (A_u + A_u) - \alpha_1^+ [\partial_u A]_+ - \partial_u A_+ \right] \]

\[ + \delta'(u) \alpha_1^+ (A_u + A_u) - \alpha_1^+ [\partial_u A]_+ - \partial_u A_+ \]  

(3.117)

\[ A_u + J_b = 0 . \]  

(3.118)

Correspondingly, the local Ward identity (3.45) acquires a boundary breaking:

\[ \partial \bar{J} + \bar{\partial}J + \partial_u J_u + \partial_u b = \delta(u) \left[ \alpha_1 (\partial \bar{A}_+ + \partial \bar{A}_- + \bar{\partial}A_+ + \bar{\partial}A_-) \right] \]

\[ + \delta(u) \left\{ \alpha_3^- \left[ \partial (\partial_u \bar{A})_+ - \bar{\partial} (\partial_u A)_- \right] + \alpha_3^+ \left[ \partial (\partial_u \bar{A})_- - \bar{\partial} (\partial_u A)_+ \right] \right\} \]

\[ - \delta(u) \left( \alpha_4^+ + \alpha_5^+ \right) \partial A_{u+} - \left( \alpha_4^+ + \alpha_5^+ \right) \partial A_{u-} \]

\[ + \delta'(u) \left[ \alpha_3^+ (\bar{\partial}A_+ - \bar{\partial}A_-) + \alpha_3^- (\bar{\partial}A_+ - \bar{\partial}A_-) \right] \]

\[ + \delta'(u) \left[ \alpha_8^+ (A_u + A_u) - \alpha_1^+ [\partial_u A]_+ - \partial_u A_+ \right] \]

\[ + \delta''(u) \alpha_1^+ (A_u + A_u) - \alpha_1^+ [\partial_u A]_+ - \partial_u A_+ , \]  

(3.119)

which, once integrated and taking into account the axial gauge condition \( A_u = 0 \), yields the residual Ward identity

\[ \int du \left( \partial \bar{J} + \bar{\partial}J \right) = \alpha_1 \left( \partial \bar{A}_+ + \partial \bar{A}_- + \bar{\partial}A_+ + \bar{\partial}A_- \right) + \]

\[ + \alpha_3^- \left[ \partial (\partial_u \bar{A})_+ - \bar{\partial} (\partial_u A)_- \right] + \alpha_3^+ \left[ \partial (\partial_u \bar{A})_- - \bar{\partial} (\partial_u A)_+ \right]. \]  

(3.120)
In Chapter 2 we have seen that, in the CS theory with boundary, it is the residual Ward identity which implies the existence of an algebra of chiral conserved currents on the boundary. Our aim will now be to study the change of the boundary conditions induced by the additional Maxwell term on the fields, and the consequences on the algebra: indeed, it is not obvious that there will be still an algebra of boundary currents.

Therefore, our primary interest will be focused on the residual Ward identity (3.120).
3.4 Boundary conditions

We now study which are the boundary conditions on the fields. For this purpose, we perform the two integrations illustrated in the previous Chapters:

1.) We set the sources to zero and integrate the broken equations of motion between the two sides of the boundary, thus getting the conditions

\[
 k (\bar{A}_- - \bar{A}_+) + i \alpha \int_{-\varepsilon}^{\varepsilon} du \left( \partial \bar{A} - \bar{\partial} A \right) - i \alpha \left[ (\partial_u \bar{A})_- - (\partial_u A)_+ \right] = \alpha_1 (\bar{A}_+ + \bar{A}_-) + \alpha_3^- (\partial_u \bar{A})_+ + \alpha_3^+ (\partial_u \bar{A})_- 
\]  

(3.121)

\[
 k (A_+ - A_-) + i \alpha \int_{-\varepsilon}^{\varepsilon} du \left( \partial A - \partial \bar{A} \right) - i \alpha \left[ (\partial_u A)_+ - (\partial_u \bar{A})_- \right] = \alpha_1 (A_+ + A_-) - \alpha_3^- (\partial_u A)_+ - \alpha_3^+ (\partial_u A)_- 
\]  

(3.122)

\[
 k \int_{-\varepsilon}^{\varepsilon} du \left( \partial \bar{A} - \bar{\partial} A \right) - i \alpha \left( \partial A_+ - \partial A_- + \partial \bar{A}_+ - \partial \bar{A}_- \right) + \int_{-\varepsilon}^{\varepsilon} du b = \alpha_4^+ (\partial A_+ - \partial A_-) + \alpha_5^+ \left( \partial \bar{A}_+ - \partial \bar{A}_- \right) , 
\]  

(3.123)

where we have also used the axial gauge condition \( A_u = 0 \). Notice that (3.122) can be obtained from the first one by means of the parity transformation (3.47). Therefore, we can consider only (3.121) and then apply parity to the resulting conditions to get the analogous ones generated by (3.122). Taking the limit \( \varepsilon \to 0 \) in (3.121) and (3.123) we get the two conditions

\[
 (\alpha_1 + k) \bar{A}_+ + (\alpha_3^- - i \alpha) (\partial_u \bar{A})_+ = 0 
\]  

(3.124)

\[
 (\alpha_4^++i\alpha) \partial \bar{A}_- + (\alpha_5^++i\alpha) \bar{\partial} A_- = 0 
\]  

(3.125)

Let us focus on (3.125) first. As usual, separability splits it into

\[
 (\alpha_4^++i\alpha) \partial A_+ + (\alpha_5^++i\alpha) \partial A_+ = 0 
\]  

(3.126)

\[
 (\alpha_4^++i\alpha) \partial \bar{A}_- + (\alpha_5^++i\alpha) \bar{\partial} A_- = 0 . 
\]  

(3.127)

Since the parameters \( \alpha_4^+ \) and \( \alpha_5^+ \) do not appear in the Ward identity, we can fix them as we wish; we choose

\[
 \alpha_4^+ = \alpha_5^+ \neq -i \alpha , 
\]  

(3.128)
so that (3.126) and (3.127) yield
\[ \bar{\partial}A_+ + \partial \bar{A}_+ = 0 \quad (3.129) \]
\[ \bar{\partial}A_- + \partial \bar{A}_- = 0 \quad (3.130) \]
which, as we have seen in the previous Chapter, can be interpreted as conservation relations for currents living on the right, respectively on the left of the boundary.

On the other hand, the request for separability splits (3.124) into
\[ (\alpha_1 + k) \bar{A}_+ + (\alpha_3^- - i\alpha) \left( \partial_u \bar{A} \right)_+ = 0 \quad (3.131) \]
\[ (\alpha_1 - k) \bar{A}_- + (\alpha_3^+ + i\alpha) \left( \partial_u \bar{A} \right)_- = 0 \quad (3.132) \]
Using the parity transformation (3.47) on these conditions we get the analogous ones generated by (3.122):
\[ (\alpha_1 + k) A_- - (\alpha_3^- - i\alpha) \left( \partial_u A \right)_- = 0 \quad (3.133) \]
\[ (\alpha_1 - k) A_+ - (\alpha_3^+ + i\alpha) \left( \partial_u A \right)_+ = 0 \quad (3.134) \]

2.) As in the scalar case, the equations of motion contain second-order derivatives and therefore we can integrate them twice. Let us set the sources to zero and then integrate (3.115):
\[ \int_{-\infty}^{u} du' \left\{ k(\bar{\partial}A_u - \partial_u \bar{A}) + i\alpha[\bar{\partial}(\bar{\partial}A - \partial A) + \partial_u(\partial_u \bar{A} - \bar{\partial}A_u)] + \bar{J} \right\} = \]
\[ = \int_{-\infty}^{u} du' \delta(u) \left[ \alpha_1 \left( \bar{A}_+ + \bar{A}_- \right) + \alpha_3^- \left( \partial_u \bar{A} \right)_+ + \alpha_3^+ \left( \partial_u \bar{A} \right)_- \right] + \int_{-\infty}^{u} du' \delta'(u) \left( \alpha_3^- \bar{A}_+ + \alpha_3^- \bar{A}_- \right) \quad (3.135) \]
from this we will get also the conditions generated by (3.116) by means of the parity transformation (3.47), like we have done above. From (3.135) we get
\[ -k \bar{A} + i\alpha \int_{-\infty}^{u} du' \bar{\partial}(\bar{\partial}A - \partial A) + i\alpha \partial_u \bar{A} = \]
\[ = \theta(u) \left[ \alpha_1 \left( \bar{A}_+ + \bar{A}_- \right) + \alpha_3^- \left( \partial_u \bar{A} \right)_+ + \alpha_3^+ \left( \partial_u \bar{A} \right)_- \right] + \delta(u) \left( \alpha_3^- \bar{A}_+ + \alpha_3^- \bar{A}_- \right) \quad (3.136) \]
where we have assumed that the fields and their derivatives vanish at infinity, and we have used the relation between distributions $\theta'(x) = \delta(u)$. We then integrate (3.136) between the two sides of the (infinitesimal) boundary:

$$
-\int_{-\varepsilon}^{\varepsilon} du\, k\bar{A} + i\alpha \int_{-\varepsilon}^{\varepsilon} du \int_{-\infty}^{u} du' \bar{\delta}(\partial\bar{A} - \partial A) + i\alpha \int_{-\varepsilon}^{\varepsilon} du \, \partial_u \bar{A} = \\
\int_{-\varepsilon}^{\varepsilon} du \,\theta(u) \left[ \alpha_1 (\bar{A}_+ + \bar{A}_- - i\alpha^+ A_+ + \alpha^- A_-) \right] \\
+ \int_{-\varepsilon}^{\varepsilon} du \, \delta(u) \left( \alpha^+_3 \bar{A}_+ + \alpha^-_3 \bar{A}_- \right).
$$

(3.137)

Taking the limit $\varepsilon \to 0$ we finally get

$$
-i\alpha \bar{A}_- + i\alpha \bar{A}_+ = \alpha^+_3 \bar{A}_+ + \alpha^-_3 \bar{A}_-,
$$

(3.138)

which can be rearranged as

$$
-\left( i\alpha + \alpha^-_3 \right) \bar{A}_- = \left( -i\alpha + \alpha^+_3 \right) \bar{A}_+.
$$

(3.139)

The request for separability again splits this condition into two separate ones for the opposite sides of the boundary:

$$
\left( i\alpha + \alpha^-_3 \right) \bar{A}_- = 0
$$

(3.140)

$$
\left( -i\alpha + \alpha^+_3 \right) \bar{A}_+ = 0.
$$

(3.141)

Applying the parity transformation (3.47) to these conditions we get those generated by (3.116)

$$
\left( i\alpha + \alpha^-_3 \right) A_+ = 0
$$

(3.142)

$$
\left( -i\alpha + \alpha^+_3 \right) A_- = 0.
$$

(3.143)

At the end, we have obtained a set of eight boundary conditions involving both the fields and their first derivatives, which must be simultaneously satisfied.

Summarizing:

$$
(\alpha_1 + k) \bar{A}_+ + \left( \alpha^-_3 - i\alpha \right) \left( \partial_u \bar{A} \right)_+ = 0
$$

(3.144)

$$
(\alpha_1 - k) \bar{A}_- + \left( \alpha^+_3 + i\alpha \right) \left( \partial_u \bar{A} \right)_- = 0
$$

(3.145)

$$
(\alpha_1 + k) A_- - \left( \alpha^-_3 - i\alpha \right) \left( \partial_u A \right)_- = 0
$$

(3.146)

$$
(\alpha_1 - k) A_+ - \left( \alpha^+_3 + i\alpha \right) \left( \partial_u A \right)_+ = 0
$$

(3.147)

$$
\left( i\alpha + \alpha^-_3 \right) \bar{A}_- = 0
$$

(3.148)
\[ (-i\alpha + \alpha^+_3) \bar{A}_+ = 0 \]  
\[ (i\alpha + \alpha^-_3) A_+ = 0 \]  
\[ (-i\alpha + \alpha^+_3) A_+ = 0 , \]  
which must be considered together with the conservation condition (3.128):

\[ \alpha^+_4 = \alpha^+_5 \neq -ia \]  
\[ \bar{\partial} A_+ + \partial \bar{A}_+ = 0 \]  
\[ \bar{\partial} A_- + \partial \bar{A}_- = 0 . \]  

We are now left with the task of solving the above constraints, thus identifying all the possible choices for the parameters and their consequences on the boundary fields and the residual Ward identity.
3.5 Parameters, boundary conditions and residual Ward identity

First of all, we note that the situation is much more complicated with respect to the pure CS model: we now have to solve eight conditions instead of four. Moreover, the number of parameters involved is higher: three instead of one. Therefore, there are many more possible choices for the parameters. However, the request to recover the CS result in the limit $\alpha \to 0$ implies a dependence of the parameter $\alpha_1$ on $\alpha$ of the type

$$\alpha_1 = \pm k (1 + 2f(\alpha)) \, , \, \lim_{\alpha \to 0} f(\alpha) = 0 \quad (3.155)$$

but, as we will show later, the case $f(\alpha) \neq 0$ is incompatible with the conditions (3.153) and (3.154) expressing the conservation condition. Therefore, the choice

$$\alpha_1 = \pm k \quad (3.156)$$

is compulsory. Moreover, the case $\alpha_1 = -k$ can be obtained from $\alpha_1 = k$ by means of the changes

$$\bar{A}_\pm \to \bar{A}_\mp \quad (3.157)$$

$$A_\pm \to A_\mp \quad (3.158)$$

$$(\partial_\nu \bar{A})_\pm \to (\partial_\nu \bar{A})_\mp \quad (3.159)$$

$$(\partial_\nu A)_\pm \to (\partial_\nu A)_\mp \quad (3.160)$$

$$k \to -k \quad (3.161)$$

$$\alpha_3^* \to -\alpha_3^* \quad (3.162)$$

as it can be directly checked by comparison with (3.144)-(3.151).

Therefore, we can focus on the eight cases with $\alpha_1 = k$. Reminding that we are mainly interested in the residual Ward identity (3.120), for each solution we write the corresponding Ward identity:

1. )

$$\begin{cases} 
\alpha_1 &= k \\
\alpha_3^+ &= -i\alpha \\
\alpha_3^- &= +i\alpha 
\end{cases} \Rightarrow A_\pm = \bar{A}_\mp = 0$$

$$\int du \left( \partial \bar{J} + \bar{\partial} J \right) = i\alpha \partial \left[ (\partial_\nu \bar{A})_+ - (\partial_\nu \bar{A})_- \right]$$

$$+ \, i\alpha \bar{\partial} \left[ (\partial_\nu A)_+ - (\partial_\nu A)_- \right] ; \quad (3.163)$$
2. \[
\begin{aligned}
\alpha_1 &= k \\
\alpha_2^+ &= -i\alpha \\
\alpha_3^- &= -i\alpha
\end{aligned}
\Rightarrow \begin{cases} 
\bar{A}_+ = 0 = A_- \\
\left(\partial_u \bar{A}\right)_+ = 0 = \left(\partial_u A\right)_-
\end{cases}
\]
\[
\int \! du \left( \partial \bar{J} + \partial J \right) = k \left( \partial \bar{A}_- + \partial A_+ \right) + i\alpha \left[ \partial \left( \partial_u A \right)_+ - \partial \left( \partial_u \bar{A} \right)_- \right] 
\]
\hspace{1cm} (3.164)

3. \[
\begin{aligned}
\alpha_1 &= k \\
\alpha_2^+ &= +i\alpha \\
\alpha_3^- &= -i\alpha
\end{aligned}
\Rightarrow \begin{cases} 
\left(\partial_u A\right)_+ = \left(\partial_u \bar{A}\right)_- = 0 \\
\bar{k} A_+ - i\alpha \left(\partial_u \bar{A}\right)_+ = 0 \\
k A_- + i\alpha \left(\partial_u A\right)_- = 0
\end{cases}
\]
\[
\int \! du \left( \partial \bar{J} + \partial J \right) = k \left( \partial \bar{A}_- + \partial A_+ \right) 
\]
\hspace{1cm} (3.165)

4. \[
\begin{aligned}
\alpha_1 &= k \\
\alpha_2^+ &= -i\alpha \\
\alpha_3^- &\neq \mp i\alpha
\end{aligned}
\Rightarrow \begin{cases} 
\bar{A}_+ = 0 = A_- \\
\left(\partial_u \bar{A}\right)_+ = 0 = \left(\partial_u A\right)_-
\end{cases}
\]
\[
\int \! du \left( \partial \bar{J} + \partial J \right) = i\alpha \left[ \partial \left( \partial_u A \right)_+ - \partial \left( \partial_u \bar{A} \right)_- \right] 
\]
\hspace{1cm} (3.166)

5. \[
\begin{aligned}
\alpha_1 &= k \\
\alpha_2^+ &\neq -i\alpha \\
\alpha_3^- &= +i\alpha
\end{aligned}
\Rightarrow \begin{cases} 
\bar{A}_+ = 0 = A_- \\
\left(\partial_u A\right)_+ = 0 = \left(\partial_u \bar{A}\right)_-
\end{cases}
\]
\[
\int \! du \left( \partial \bar{J} + \partial J \right) = i\alpha \left[ \partial \left( \partial_u A \right)_+ - \partial \left( \partial_u \bar{A} \right)_- \right] 
\]
\hspace{1cm} (3.167)

6. \[
\begin{aligned}
\alpha_1 &= k \\
\alpha_2^+ &= +i\alpha \\
\alpha_3^- &\neq \mp i\alpha
\end{aligned}
\Rightarrow \begin{cases} 
A_+ = \bar{A}_- = \left(\partial_u A\right)_+ = \left(\partial_u \bar{A}\right)_- = 0 \\
2k \bar{A}_+ + \left(-i\alpha + \alpha_3^\pm\right) \left(\partial_u \bar{A}\right)_+ = 0 \\
2k A_- - \left(-i\alpha + \alpha_3^\pm\right) \left(\partial_u A\right)_- = 0
\end{cases}
\]
\[
\int \! du \left( \partial \bar{J} + \partial J \right) = k \frac{i\alpha + \alpha_3^-}{i\alpha - \alpha_3^\pm} \left( \partial \bar{A}_+ + \partial A_- \right) 
\]
\hspace{1cm} (3.168)
7.
\[
\begin{align*}
\alpha_1 &= k \\
\alpha_3^+ &\neq \mp i\alpha \\
\alpha_3^- &= -i\alpha \\
\Rightarrow \quad \bar{A} &= A_- = 0 \\
\int du (\partial \bar{J} + \bar{\partial} J) &= k (\partial \bar{A}_- + \bar{\partial} A_+) ; \\
\end{align*}
\]

8.
\[
\begin{align*}
\alpha_1 &= k \\
\alpha_3^+ &\neq \mp i\alpha \\
\alpha_3^- &\neq \mp i\alpha \\
\Rightarrow \quad A_\pm &= \bar{A}_\pm = 0 \\
\int du (\partial \bar{J} + \bar{\partial} J) &= 0 .
\end{align*}
\]

However, not all these solutions of the system (3.144)-(3.151) are acceptable. Indeed, we must keep in mind that taking the limit \(\alpha \to 0\), which corresponds to switching off the Maxwell term, we must recover the result found for the CS theory.

This condition eliminates the cases 1.), 4.), 5.) and 8.). Furthermore, the conservation relations (3.153) and (3.154), together with the boundary conditions, imply that the Ward identity for the cases 6.) and 7.) becomes
\[
\int du (\partial \bar{J} + \bar{\partial} J) = 0 ,
\]
while for the case 2.) it takes the form
\[
\int du (\partial \bar{J} + \bar{\partial} J) = +i\alpha \left[ \bar{\partial} (\partial A)_+ - \partial (\partial_\alpha \bar{A})_+ \right] .
\]
These, in turn, do not have the correct limit \(\alpha \to 0\) either, and therefore they are forbidden as well.

At the end, we are left only with the case 3.), which corresponds to Neumann and Robin boundary conditions on the opposite sides of the boundary:

\[
\begin{align*}
\alpha_1 &= k \\
\alpha_3^+ &= +i\alpha \\
\alpha_3^- &= -i\alpha \\
\alpha_4^+ &= \alpha_5^+ \neq -ik \\
(\partial A)_+ = (\partial_\alpha \bar{A})_- &= 0 \\
k\bar{A}_+ - i\alpha (\partial_\alpha \bar{A})^+ &= 0 \\
kA_- + i\alpha (\partial_\alpha A)_- &= 0 \\
\bar{\partial} A_+ + \bar{\partial} \bar{A}_+ &= 0 \\
\bar{\partial} A_- + \bar{\partial} A_- &= 0 \\
\int du (\partial \bar{J} + \bar{\partial} J) &= k (\partial \bar{A}_- + \bar{\partial} A_+) .
\end{align*}
\]
The Ward identity (3.175) is the same as in the CS case, and it implies both the chirality

\[ \bar{\partial}A_+ = 0 \]  
\[ \partial \bar{A}_- = 0 \]  

and the boundary Kač-Moody algebras

\[ [A_+(z), A_+(z')] = \frac{1}{k} \partial \delta(z - z') \]  
\[ [\bar{A}_-(\bar{z}), \bar{A}_-(\bar{z}')] = \frac{1}{k} \bar{\partial} \delta(\bar{z} - \bar{z'}) \]

involving the conserved chiral currents \( A_+(z) \) and \( \bar{A}_-(\bar{z}) \).

This is a remarkable result, since we have found not only that the boundary terms are uniquely fixed, but also that there is still a Kač-Moody algebra, which is the same as in the CS case. In other words, the Maxwell term does not affect the boundary physics of the light-cone currents.

This result is in agreement with the outcomes of [40] and [41], where different approaches have been adopted, and a disk instead of a plane has been considered.

The claim of [40] and [41] is that the boundary is characterized by Kač-Moody algebras of conserved charges with the same structure of the CS model, thus concluding that the algebra is a consequence of the CS term rather than of the nature of the entire model. We refer to our conclusions for a more precise comparison.

We conclude this Chapter by illustrating the reason why the parameter \( \alpha_1 \) must be set to the value \( k \). As we previously pointed out, the request that the limit \( \alpha \to 0 \) lead to the CS result implies (3.155). In such a case, it can be checked that the only acceptable solution of the system (3.144)-(3.151) is

\[
\begin{align*}
\alpha_1 &= k (1 + 2f(\alpha)) \\
\alpha_3^+ &= +i\alpha \\
\alpha_3^- &= -i\alpha \\
\alpha_4^+ &= \alpha_5^+ \neq -i\alpha
\end{align*}
\]

\[ \Rightarrow \left\{ \begin{array}{l}
k (1 + f(\alpha)) \bar{A}_+ - i\alpha (\partial_u \bar{A})_+ = 0 \\
k f(\alpha) \bar{A}_- + i\alpha (\partial_u \bar{A})_- = 0 \\
k (1 + f(\alpha)) A_+ + i\alpha (\partial_u A)_+ = 0 \\
k f(\alpha) A_- - i\alpha (\partial_u A)_- = 0
\end{array} \right. ,
\]

which corresponds to the broken residual Ward identity

\[
\int du \left( \partial \bar{J} + \bar{\partial} J \right) = k f(\alpha) \left( \partial \bar{A}_+ + \bar{\partial} A_- \right) + k (1 + f(\alpha)) \left( \partial \bar{A}_- + \bar{\partial} A_+ \right).
\]

(3.180)
Equation (3.180), with the external sources set to zero, together with separability, implies the relations

\[ kf(\alpha) \partial \bar{A}_+ + k (1 + f(\alpha)) \bar{\partial} A_+ = 0 \]  
\[ kf(\alpha) \partial \bar{A}_- + k (1 + f(\alpha)) \bar{\partial} A_- = 0 , \]

which are incompatible with the existence of conserved quantities as expressed by (3.153) and (3.154) unless \( f(\alpha) = 0 \). Notice that this argument shows also that the request that the boundary currents are conserved automatically implies their chirality.
Conclusions

In this thesis we addressed the problem of how to treat the effect of a boundary in quantum field theory, following the pioneering work of Symanzik [1]. In particular, we studied the cases of a scalar field theory (Chapter 1) and of the topological Chern-Simons (CS) theory (Chapter 2), but the main original results have been achieved in Chapter 3, where the three dimensional Maxwell-Chern-Simons (MCS) theory with planar boundary has been considered.

We adopted a method similar to the one introduced in [24], which avoids the problem of regularizing the boundary action, which is necessarily $\delta$-dependent, but we modified it in a significant manner. In fact, one of the main step towards the realization of the Symanzik’s constraint of separability, which implements the introduction of a boundary in quantum field theory, is the direct computation of the propagators of the theory, taking into account, of course, the boundary interactions. While this is quite feasible in the cases of the scalar field theory and in the CS theory, in the MCS case it is a more difficult task.

Amongst the original results presented in this thesis is the fact that we have been able to reach the same goals, as in the cited simpler cases, without a direct computation of the propagators of the theory.

Entering more in the details of what we have done, we stress that the addition of the Maxwell term to the CS action spoils its topological character, and thus it is not at all granted that the same results can be obtained.

Indeed, we found that, under certain conditions,

1. chiral conserved currents living on the boundary exist,

2. which satisfy, like in the CS case, a Kač-Moody algebra, whose central charge coincides with the inverse of the CS coupling constant.

In particular, we have found that there are many possible boundary terms compatible with Symanzik’s constraint of separability, each associated with different boundary conditions on the fields. However, the requirement to
recover the CS case in the limit of vanishing Maxwell term, uniquely sets the boundary parameters to values which imply *Neumann and Robin boundary conditions* on the fields on opposite sides of the boundary. In this situation, the same residual Ward identity of the CS case holds, thus implying the results 1. and 2. above mentioned.

This result leads us to conclude that the boundary physics is independent of the Maxwell term which breaks the topological character of the CS model, even though the Maxwell boundary contributions to the equations of motion are nontrivial. This is a remarkable result, since it suggests that the existence of a boundary Kac-Moody algebra of conserved chiral currents depends only on the presence of a CS term in the theory, rather than on the particular theory itself.

We point out that our main results hold also for the nonabelian model, namely the Yang-Mills-Chern-Simons action. Indeed, the conservation of boundary currents, their chirality and the central charge of their Kac-Moody algebra are entirely determined by the quadratic part of the action only, and therefore are shared with the nonabelian extension, which effects the vertices of the theory.

Our result, concerning the existence of a Kac-Moody boundary algebra with the same structure of the CS model, somehow agrees with Deser's recent statement that, for spin one and in three spacetime dimensions, "everything is CS" [42]. In Ref. [42], in fact, he shows how a Yang-Mills-Chern-Simons (YMCS) action can be rewritten in the form of a pure CS action in terms of a new field.

But, even in the earlier work [43], three dimensional Yang-Mills gauge theories in the presence of the Chern-Simons action were shown to be generated by the pure topological Chern-Simons term through nonlinear covariant redefinitions of the gauge field. And indeed these claims are supported by our results, which do not depend on the presence of a bulk Maxwell term.

Another point of contact is with the outcome of [22] for the (2+1) dimensional black hole coupled with electrodynamics, where it is shown that the black hole coupled to the pure CS theory does not change configuration when a Maxwell term is "turned on".

In the framework of the Fractional Quantum Hall Effect, our result justifies the fact that an additional Maxwell term to the pure CS effective low energy model describes the gaps between the elementary bulk excitations, without affecting the properties of the edge states [5].

Finally, we are in agreement also with [40] and [41], where the nonabelian YMCS theory defined on a manifold with boundary is considered in the Weyl gauge. In Ref. [40], a (2+1) dimensional disk is considered, and the Dirac's procedure is followed. The boundary Kac-Moody algebra is then obtained
as the projection of the Dirac’s brackets on the boundary of the disk.

In [41] the discussion is instead carried out in terms of the Gauss law generators, and the algebra in terms of the fields is just a consequence. Nevertheless, they both find that also in the YMCS theory there is a boundary algebra, which is the same as that of the CS model.

However, our approach is not only simpler, but also stays in a more general framework. In fact, in both [40] and [41] the boundary conditions are chosen \textit{a priori}, provided that they satisfy some general requirements, while we find all possible boundary conditions compatible with the simple and fundamental requests of locality and separability, without requiring any other constraint.

Moreover, in our description the boundary is the most general one compatible with very general principles - like locality, power counting and helicity conservation - and with the algebraic structure of the theory (see end of Chapter 1).

For these reasons, our method is suitable also for a different kind of investigation. Indeed, the approach in [40] and [41] is good to describe just a portion of space: there is not a \textit{beyond the boundary}. This is not a problem for the description of systems like electrodynamics on a disk [44] or a Hall bar in the framework of the Quantum Hall Effect [5], in which one is interested in the dynamics of the internal system only.

However, the description of what there is beyond the boundary is fundamental in the physics of \textit{defects} [45], and our approach is actually suitable for this task. In the study of semiconductors [46], for instance, it is important to be able to describe local interactions between the particles and the imperfections of the material, approximated by $\delta$-type interactions.

On the other hand, topological defects are gathering more and more importance also in the framework of astroparticle physics, where they are proposed as an explanation for the formation of cosmic structures and for the generation of extremely high-energy cosmic rays [47, 48].

For these reasons, even considering an impenetrable boundary, we have kept both the left and the right side of spacetime. Indeed, we have fixed the parameters of our general description in such a way to decouple the opposite sides of spacetime completely, and therefore in principle we are able to examine just one of the two sides, if we wish. In a certain way, this is actually what we have done by imposing the conservation of the bulk discrete symmetry (3.47) which we called parity.

In other words, according to Symanzik’s approach, the boundary is \textit{defined} by the decoupling condition, which prevents correlations between points belonging to opposite sides of the boundary. A defect could be treated by using the same approach described in this thesis, by simply relaxing the
decoupling condition. In this way, the resulting modified theory describes interactions between a bulk and a \( \delta \)-type insertion in the action, \textit{i.e.} a defect, as it is done in [50].

We conclude our discussion by indicating other possible further developments.

In fact, the three dimensional CS theory is not the only possible topological field theory. The other important Schwarz type topological field theory [8] is represented by the BF models [13], which, contrarily to the CS theory, which is intrinsically three dimensional, exist in an arbitrary number of dimensions.

In three dimensions, the BF model with boundary has been studied in [14], where a richer algebraic structure than the CS case has been found, always of the Kač-Moody type.

A natural possible extension could therefore be the study of the effect of a three dimensional boundary in the four dimensional BF theory. The investigation of the possible boundary algebra in that case is a challenge. Analogously, to our knowledge, the Yang-Mills-BF theory has not been studied yet either in three or in four spacetime dimension, with or without boundary.
Bibliography


   X.-G. Wen, Quantum Field Theory of Many Body Systems: from the
   Origin of Sound to an Origin of Light and Electrons (Oxford Graduate
   Texts) (Oxford University Press, USA, 2007).

   (1992);
   M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D

   Surface, in Phase Transitions and Critical Phenomena Vol. 8, (edited


[34] J. Schwinger, Phys. Rev. 82, 914 (1951).


