The particle interpretation of $N = 1$ supersymmetric spin foams

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Love.

The journey is long
and the path is pathless
and one has to be alone.
There is no map and no one to guide.

But there is no alternative
One cannot escape it,
one cannot evade it.
One has to go on the journey.

The goal seems impossible
but the urge to go on it is intrinsic.
The need is deep in the soul.

Really, you are the urge, you are the need
and consciousness cannot be otherwise
because of this challenge
and because of this adventure.

So do not waste time – begin.
Do not calculate – begin.
Do not hesitate – begin.
Do not look back – begin

And always remember old Lao Tzu’s words:
A tree that takes both arms to encircle
grows from a tiny rootlet.
A many-storied pagoda
is built by placing one brick upon another brick.
A journey of three thousand miles
is begun by a single step.

OSHO - A Cup of Tea
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Chapter 1

Introduction

The theory of General Relativity formulated by Einstein in 1915, and the theory of Quantum Mechanics formulated during the first half of the twentieth century are certainly two of the greatest achievements of physics of the past century. Both describe in a comprehensive way the physical phenomena that fall under their domains, respectively cosmology and atomic and subatomic particles physics, and both do so to an astonishing degree of accuracy. Yet they offer us strikingly different pictures of physical reality. Indeed, the description of ‘reality’ given by the two theories seems to be quite in contradiction.

Quantum Mechanics and its relativistic counterpart, Quantum Field Theory, are formulated using dynamical fields on a fixed background, Minkowski spacetime, and they have an intrinsic probabilistic nature. On the other hand the main and most interesting features of General Relativity are its completely background independent formulation and the interpretation of gravity as a geometric property of spacetime. Moreover, it is also a classical theory meaning it does not have any probabilistic/quantum behavior.

Therefore these two theories turn out to be logically incompatible when applied to systems where they are both non negligible such as particle-physics processes at energy scales of the order of the Planck Energy $E_P = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{28}\text{eV}$.

However, keeping in mind that the energy reachable by present experiments, as in the new accelerator LHC, will be around $14 \times 10^{12}\text{eV}$, we could in principle maintain this double attitude.

Nevertheless this picture is highly unsatisfactory from a conceptual point of view. Indeed the successful quantum formulation and unification of three of the four fundamental interactions (Electromagnetic, Strong and Weak, excluding Gravity) led to the search for a Theory of Everything, of which the two pictures we are currently using can be considered approximations in their respective domains. The attempts to find such an ultimate and unified theory have been ongoing, leading to several possible ways of addressing the problem,
one of which is String Theory. Other approaches try instead to achieve perhaps a more modest goal and just formulate a quantum theory of gravity. Examples of such approaches are Loop Quantum Gravity [52, 57], Causal Dynamical Triangulations [3], Quantum Regge Calculus [50] and Non-Commutative Geometries [14]. Among these headlines lies our topic of interest: the so-called Spin Foam models.

The approaches developed to address quantum gravity can be separated in two main currents: perturbative and non-perturbative quantum gravity methods. The difference between the two is quite substantial.

In the standard perturbative approach, one attempts to describe the gravitational interaction using the same technique applied to quantum field theory, i.e. defining a non-dynamical (and non-physical) background on which the dynamical fields evolve and interact. The degrees of freedom of the gravitation field are therefore split in terms of a fixed background geometry $\eta_{\mu\nu}$ and dynamical metric fluctuations $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (1.1)$$

However this separation of the degrees of freedom is utterly arbitrary, since we can choose a different background term $\tilde{\eta}_{\mu\nu}$ and fluctuations $\tilde{h}_{\mu\nu}$. The consequence is that in general the light-cone structure of the two backgrounds is different, providing a different notion of causality. Furthermore the theory turns out to be non-renormalizable; this is thought to be an effect of having neglected some microscopic degrees of freedom by fixing a background (i.e., by imposing a fixed background, we are losing some degrees of freedom that can become explicit only at the Planck scale).

On the other hand, the non-perturbative approaches want to preserve the main lesson of general relativity, that is, that the spacetime geometry is fully dynamical and therefore the notion of absolute space on top of which fields interact and evolve does not make any sense any longer.

Among all the possible non-perturbative models used to quantize gravity, we shall now focus on a particular approach called Loop Quantum Gravity. This is explicitly formulated as a background independent theory therefore the standard perturbative quantization method used in quantum field theory (that yields Feynman diagrams) cannot be employed. Instead the canonical Dirac program is used. The very new idea, besides the stress on the background independence, is the introduction of the Ashtekar-Barbero connection variables that allows one to use a formulation of Einstein’s equations in which parallel transport, rather than the metric, plays the main role. For more detailed information about LQG, we refer the interested reader to [4], [5], [43], [51] and references therein.

The steps by which LQG carries out the quantization process are, firstly, to recast the Einstein-Hilbert action in terms of a tetrad and connection field:
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the Einstein-Cartan action. One then adds a topological term to the action. This term is known as the Holst modification and comes multiplied by an arbitrary coefficient known as the Immirzi parameter [28–30]. The classical theory is insensitive to this new term, but it gains remarkable importance in the loop quantization procedure, since different values of the Immirzi parameter label unitarily inequivalent quantum theories. Now that one has this Einstein-Cartan-Holst action, one performs a 3+1 decomposition and uses up some of the gauge-freedom of theory to fix the normal to the spatial hypersurface. Thus, the dynamical variables at any given time are a triad (the spatial part of the tetrad since the time component is fixed using the gauge freedom) and the connection mentioned above. Note that the Ashtekar-Barbero connection is explicitly dependent on the Immirzi parameter. Furthermore, the Hamiltonian of the theory is a function of these variables and certain Lagrange multipliers imposing constraints. As is indicative of generally covariant systems, the Hamiltonian comprises simply of constraints, there is no true dynamics in the usual sense. It is an expression of the arbitrariness in the choice of spacetime coordinates. The constraints fall into three sets: the Gauge constraints imposing local internal gauge invariance; the Vector constraint imposing spatial diffeomorphism invariance; and finally the Scalar constraint imposing time-diffeomorphisms. This final constraint is often termed the Hamiltonian constraint, since it evolves the spatial hypersurface (in co-ordinate time).

Secondly, the implementation of canonical Dirac program requires the definition of the kinematical Hilbert space. Here, the important factor is that connection plays the role of the configuration variable, while the triad is its canonical momentum. Thus, the state space is made up of functionals of the Ashtekar-Barbero connection and having imposed quantum versions of the Gauge and Vector constraints, one arrives at the Kinematical Hilbert space. A basis for this state space is given by the so-called Spin networks, see Fig. 1.2. They have support on graphs in the spatial hypersurface. Each edge is labelled by the parallel transport along that edge, with respect to the local internal gauge group, in a given representation (in this case SU(2) elements and representations, since some of the gauge symmetry was used to fixed the time component of the tetrad). The vertices are labelled by SU(2)-invariant tensors which intertwine these representations. Spin networks were first introduced by Penrose in order to give a purely combinatorial description of spacetime, see [42]. In the context of loop quantum gravity, Rovelli and Smolin, discovered that they are better suited to describe space rather than spacetime [54]. Thus, they describe the quantum geometry of a spatial hyper-surface at a fixed time.
The dynamical evolution of the states enters the theory in the form of the Hamiltonian or Scalar constraint that generates coordinate time evolution. Unfortunately the solutions to the Hamiltonian constraint are still not completely understood and so neither is the dynamics of the theory, see [43].

The idea is to solve this problem was to consider the quantum Hamiltonian constraint as a Hamiltonian and attempt to treat it as one often does in quantum field theory, that is, by providing loop quantum gravity with an appropriate path integral formalism. Following the lessons of quantum field theory, we may try to calculate transitions amplitudes as a sum over paths interpolating between two states. Since the states of quantum geometry have support on a graphs, a generalized Feynman diagram-like structure emerges, denoted a spin foam. A path interpolating between two spin network states is a two dimensional cellular complex built from vertices, edges and polygonal faces. A generic slice of a spin foam is a spin network, see Fig. 1.3.
To reiterate, the goal we would like to achieve using a sum over such paths or spin foams, is to calculate the transition amplitude between states by summing over spin foams going from one spin network to another.

Starting from the existing loop quantum structure, this program turned out to be harder than expected. Recasting the operator $e^{i\hat{H}t}$ ($\hat{H}$ is the quantum Hamiltonian constraint and $t$ the co-ordinate time) as a sum over spin foams is possible but the real aim of the program has been only partially achieved. It is difficult to make sense of the sum. Although the spin foam amplitude stalled along this avenue in four dimensional loop quantum gravity, it has been brought to completion in the three dimensional setting [38]. Furthermore, it gave tantalizing clues as to the possible state sum structure. The Feynman diagrams are 2d cellular complexes labelled by representations and intertwiners of an appropriate gauge group. In fact, the spin foam framework has already arisen in the context of a different theory: $BF$ theory and its quantization.

As we just said, spin foam models also arise in a completely independent context. Indeed, many of the known topological quantum field theories, like Ponzano-Regge model of gravity in three dimensions [46], the Ooguri model [40] in four dimensions, which was later generalized by Crane, Kauffman and Yetter [16], can all be considered as spin foam models.

It has been shown that [24] the state sum models mentioned above correspond to some well-known topological field theories such as $BF$ in three and four dimensions. What is interesting from this point of view is that gravity may be seen to arise from four dimensional $BF$ theory after one imposes some constraints, known as the simplicity constraints. The constrained $BF$ theory action is known at the Plebanski action. Simply speaking, the spin foam quantization program in this context, begins by replacing the continuum manifold with a simplicial complex (triangulation); the spin foam is then a subset of the topologically dual cellular complex to this triangulation. After this, one formulates
a discrete version of the Plebanski action [18, 48, 49]. Since it is $BF$ theory with constraints, one uses the spin foam state sum for $BF$ theory and attempts to impose quantum versions of the simplicity constraints. This constraints form a highly non-trivial set and there are several proposals as to their quantum imposition, leading to the Barrett-Crane model [8], the EPRL model [21] and the FK model [26]. An interesting feature of the latter two is that one can include a Holst modification once more, thus including an Immirzi parameter and leading to greater agreement with the loop quantum gravity theory. Ultimately, one can view spin foam models as emerging from a lattice gauge theory approach for these models, using a random lattice. In comparison with loop quantum gravity, these cellular complex is labelled by representations and intertwiners, although the weights differ. Most strikingly, we have only used one Feynman diagram (or spin foam) coming from this covariant avenue. It is expected that in the context of four dimensional gravity, this is insufficient to capture all the degrees of freedom, since discretization breaks diffeomorphism invariance. Then, to recapture these degrees of freedom, one should sum over spin foams once again.

So far we have considered the quantization of pure gravity. However, in order to observe how the intrinsic fabric of space-time changes at the quantum level we need a way to probe it. For that reason, it is crucial to define physically relevant observables.

Amid several strategies, coupling gravity to particles is the simplest and most effective way to understand the physics and the dynamics of the quantum space-time and to construct of above-mentioned observables. One probes the quantum spacetime by examining the (hopefully) detectable effects that quantum gravity might have on the matter theories, for example, modifying dispersion relations in quantum field theory.

So far, the attempts made to couple matter within a spin foam model are of two kinds, [20, 22, 27, 34, 35]. In three dimensions, where the pure gravity theory is topological, the most tractable and indeed most successful approach, directly embedded the Feynman diagram structure of the quantum field theory into the spin foam structure. This allowed one to maintain the topological nature of the quantum state sum except in the neighbourhood of the particle trajectory. Thus, it was possible to re-sum the gravitational degrees of freedom so as to arrive at an effective field theory for the matter sector. This field theory turned out to be a non-commutative field theory, thus with modified dispersion relations. The other method discretizes the field directly on the spin foam, more in the vein of lattice field theory. Since in the continuum theory, matter fields generically have a non-trivial energy-momentum tensor, they affect the state sum globally and thus it loses its topological nature completely. Calculations are cumbersome to compute and exact results are scarce.

Alongside the several models proposed already, lies the supergravity ap-
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proach, where we couple matter degrees of freedom to gravity via supersymmetrising the gravity theory. We concentrate on this approach here, and our choice stems from various problems we want to address.

As a non-perturbative approach to quantum gravity, spin foam models remain completely separate from the other main strand addressing the same issue, the perturbative approach of string theory. Therefore, it would be a very important achievement to connect the two theories, that would also allow us to compare the major results of such completely different approaches.

However, since one of the main requirements of string theory is supersymmetry, we need to modify the spin foam model substituting gauge groups with gauge supergroups. This task has been carried out and achieved in the 3d case (but it can easily be generalized to n dimensions) in [34].

In particular, one important goal would be the development of the formalism in order to study extended supersymmetry. This is crucial to examine the properties of BPS states in a non-perturbative setting, which could also provide a way to compare results pertaining to the black hole entropy, obtained in both string theory and loop quantum gravity, see [35].

Besides the connection with string theory, supersymmetry is also one of the most suitable techniques to include matter degrees of freedom into a pure gravity spin foam model. So far, various proposals have been made both for fermions [15, 36] and for gauge fields [37, 41], but supersymmetry remains by far the sleeker method. The reason for this is the usual one. In spin foam models, supersymmetry is included by modifying the gauge group. Thus, one keeps the essential structural properties of the original spin foam. In three dimensions for example, the classical theory is still topological but the Lagrangian automatically contains the fermionic degrees of freedom. The super field formulation allows one to hide them inside the gauge field. At the quantum level, one get a topological state sum, once again. Furthermore, supergravity theories also have the characteristic of taming the infinities of their non-supersymmetric counterparts and this is also true in the spin foam context.

In the three dimensional model we discuss here, the generalization from fields to superfields can be thought as occurring at the level of gauge group, which becomes a supergroup. Hence, we can obtain three dimensional supergravity by proceeding in analogy with ordinary gravity in three dimensions; in this case we re-express the Lagrangian, employing a superframe and superconnection, [1, 2], (instead of the frame and connection of ordinary gravity) and the gauge group becomes UOSP(1|2), in the $N = 1$ ($p = 1$ $q = 1$) version of supergravity. Even if in this thesis work we explicitly consider only supergravity in three dimensions, this approach is in principle extendible to higher dimensions, meaning that the Lagrangian of supergravity in any dimension can be written as that of super $BF$ theory plus some constraints [25], [32]. Thus, from the spin foam point of view, the most suitable way to quantize supergravity would be by quantizing the super
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$BF$ theory first and then implementing a quantum version of the constraints. Having said that, this might be the starting point for the development of a non-perturbative approach to M-theory via 11–dimensional quantum supergravity.

However, with the introduction of a gauge supergroup we cannot make a straightforward use of the procedure used so far. Indeed, by substituting the algebra with a graded algebra and a super-algebra the theory is not in principle unitary any longer, this fact bringing about problems with the use of representation. In order to overcome this issue, we shall employ the formalism of circuit diagrams, a diagrammatic language to represent naturally the representation theory (intertwiners etc.) of groups, supergroups or quantum groups using notions from category theory. As a consequence of the generality of such a formalism, the final result can be extended to any dimension and any appropriate gauge group, justifying what we said above.

In this thesis work, we shall show how this formalism is applied to calculate the partition function for supersymmetric $BF$ theory and, based on the property of the representation theory of UOSP$(1|2)$, we shall identify the fermions in the spin foams. In general, the supersymmetric partition function can be understood as a sum over gravity+fermions configurations, which are defined through the identification of fermionic Feynman diagrams embedded in the spin foam. Nonetheless, in the supersymmetric context, the fermionic degrees of freedom are mixed together with the gravity degrees of freedom in such a way that we can view the model both as a gravity+fermions system or as a manifestly supersymmetric system, containing a sum over pure supergravity configurations, which highlights the topological nature of the theory.

The further step we shall take is to investigate the validity of the results obtained. In order to do that we ‘jump’ to the other side of the theory by suitably preparing the usual SU(2) Ponzano-Regge model so that we can insert fermionic observables into the amplitude. We shall see that those reproduce the supersymmetric amplitude exactly. This is our main result.

We conclude this thesis work by exploring the geometric properties of the fermionic observables and their effects on spacetime.
Chapter 2

Spin Foams Models

To place spin foams in a historical setting, they arose first within the framework of loop quantum gravity, where the spin foam structure was observed to occur in an attempt to impose the Hamiltonian constraint, that is, in a dynamical context. Just as Feynman diagrams arise as the trajectories of point particles, spin foams arise as the trajectories of spin network states (the states of quantum 3-geometry). Since spin network states already have a graph structure, these trajectories are higher-dimensional analogues of Feynman diagrams (in fact, they are 2d cellular complexes). This cellular structure had already been observed, in various lattice approaches, namely in (quantum) Regge calculus and in the geometric quantization of the simplest 4d discrete geometry (the 4-simplex) by Barrett and Crane. There the spin foam is the topological dual of the triangulated manifold. Spin foam models, however, were soon seen to arise independently as a generic tool to quantise certain topological field theories and their (non-topological) variants. Ultimately, they can be considered as an attempt to formulate a rigorous state sum formulation for quantum gravity.

Rather than proceeding from the Einstein-Hilbert action, as used in Regge calculus, or the Holst action, as used in loop quantum gravity, modern spin foam models originate from the Plebanski action. This is a reformulation of general relativity in terms of a bi-vector field and a connection. The action contains two terms, one of which is topological and the other which constrains the theory, introducing local degrees of freedom. In essence, in moving to these variables we have moved away from a geometric theory and the constraints serve to bring us back to the geometric sector. The topological theory that we use is BF theory. For example, in three dimensions, the constraints are trivial and general relativity coincides with BF theory, as we shall show in Section 2.2.1. Thus, we arrive at the well-known fact that 3d gravity is topological. In dimension \( n > 3 \), the Hilbert-Palatini action arises as a sector of the theory by requiring the bi-vector field to satisfy the so-called ‘simplicity’ constraints, as we shall show for the 4d case in Section 2.2.2.

BF theory is considered as an advantageous theory to use as a genesis, since
spin foams are very well adapted to its quantization; a natural well-defined measure on the space of field configurations exists for the path integral. This is in marked contrast to the situation in quantum Regge calculus [46], where arguments as to the naturalness of various measures still continue [46]. The price one has to pay for this initial boon is that one must impose the simplicity constraints on the state sum, which is a highly non-trivial exercise. In essence, the various spin foam models differ due to their imposition of these constraints.

Moreover, BF theory is a gauge theory. There is a gauge invariance which must be gauge fixed. For pure discrete BF this has been solved, (via a consistent gauge-fixing). One may also even alter the gauge symmetry by introducing a cosmological constant. Such theories have been linked to topological quantum state sums such as the Turaev-Viro model [58] and its higher dimensional generalizations [24].

In this chapter, we shall give the essential properties of spin foam models. To this effect, in Section 2.1 we shall describe classical BF theory and its basic properties, and use this as a platform to investigate how it is related to classical general relativity in three and four dimensions in Section 2.2.

Following that, we shall embark on a procedure of quantization in Section 2.3. Our first stop, Section 2.3.1, is to catalogue some pertinent aspects of path integral quantization in general. Afterwards, we illustrate in detail how to triangulate the manifold, which shall bring us into the realm of discrete BF theory in Section 2.3.5. Our purpose is to use this discrete theory to give a rigorous definition of the quantum theory, more precisely, the path integral. In Section 2.3.6, we shall give as general a description of spin foam quantization as is feasible but shall specialize to three and four dimensions for clarity.

2.1 BF theory - the classical theory

BF theory is a gauge theory dependent on two fields, traditionally denoted $B$ and $W$. Generically, it can be defined over an $n$-dimensional, oriented, smooth manifold $M$, which in our context plays the role of the space-time manifold. Moreover, at this point, we shall also describe the theory utilising an unspecified Lie (super-)group $G$, although, some particular choices are of greater interest than others. We shall denote the algebra related to $G$ as $\mathfrak{g}$. The $B$ field is a $\mathfrak{g}$-valued $(n-2)$-form, while $W$ is a $G$-connection, that is, a $\mathfrak{g}$-valued 1-form. It arises in the theory via its curvature $F[W] := dW + W \wedge W$, a $\mathfrak{g}$-valued 2-form.\footnote{In the language of fibre-bundles: the basic dynamical fields are a connection $A$ on $P$, a principle $G$-bundle over $M$, and a field $B$ that takes values on the vector bundle associated to $P$. We then choose a local trivialization in order to write the fields in the form occurring in the main text.}
CHAPTER 2. SPIN FOAMS MODELS

Action and equations of motion

The $BF$ theory action is particularly simple:

$$S_{BF,M} = \int_M \text{tr}(B \wedge F[W]),$$  \hspace{1cm} (2.1)

where $\text{tr}$ is the trace on the fundamental representation of $\mathfrak{g}$. In fact, we can see immediately that the $B$ field arises as a Lagrange multiplier. Varying with respect to $B$ and $W$ yields the equations of motion:

$$F[W] = 0, \quad d_W B = 0.$$  \hspace{1cm} (2.2)

The first equation states the connection is flat while the second requires $B$ to be covariantly constant with respect to this flat connection. As we shall see, equations (2.2) place severe restrictions on $B$ and $W$, so much so that the theory is a topological field theory. Before demonstrating this fact, however, we shall study the symmetries of the action.

Symmetries: gauge and translation invariance

As one might imagine in a gauge theory, there is a gauge invariance. The $B$, $W$ and $F$ fields transform as:

$$B \to B' = h^{-1}Bh,$$
$$W \to W' = h^{-1}Wh + h^{-1}dh,$$
$$F \to F' = h^{-1}Fh,$$  \hspace{1cm} (2.3)

where $h \in G$. As expected, $B$ and $F$ transform in the adjoint representation, while $A$ transforms as a connection. The action is clearly invariant under such transformations.

Another (so-called translation) symmetry manifests itself in the case of $BF$ theory:

$$B \to B' = B + d_W \eta,$$
$$W \to W' = W,$$
$$F \to F' = F,$$  \hspace{1cm} (2.4)

where $\eta$ is a $\mathfrak{g}$-valued $(n - 3)$-form. In order to verify that $S_{BF}$ is invariant under this transformation, one must integrate by parts and employ the Bianchi identity $d_W F = 0$.

Manifolds with boundary

If $\mathcal{M}$ possesses a boundary $\partial\mathcal{M}$, one needs to be more careful. The pullback of $B$ and $W$ to the boundary are canonically conjugate variables. Generically, one fixes half the boundary variables and varies the action under these conditions. If one chooses to fix the boundary connection variables $W$, then the action (2.1) above is consistent with subsequent variations. On the other hand, if one wishes
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to fix the $B$ variables on the boundary one must add the following boundary term to the action:

$$S_{BF, \partial M} := - \int_{\partial M} \text{tr}(B \wedge W),$$

else one does not get $dW B = 0$ as an equation of motion. In this work, however, we shall not concern ourselves with this subtlety explicitly, since we shall interest ourselves solely in the partition function.

Returning to the solution space

We are now ready to say something more about the classical solutions. First of all, for flat connections, one can always find locally a transformation $h$ so that $W' = 0$. In other words, all connections are locally gauge equivalent (to the zero connection). Furthermore, given a flat connection, and a $B$ field satisfying the equation $dW B = 0$, one can find locally an $(n-3)$-form $\xi$ such that $B = dW \xi$. This a consequence of the fact that all closed forms are exact. Thus, given the translation symmetry, all solutions are locally equivalent. The only subtlety in these arguments rises from the fact that there may be topological obstructions to making this argument globally. To recapitulate, the theory has no local degrees of freedom, only topological ones.

2.2 BF theory in three and four dimensions

In the light of our programme, we focus on examples of $BF$ theory in three and four dimensions and their relation with a corresponding theory of gravity. While in 3d there exists a $BF$ theory which translates immediately into a theory of gravity, in 4d, one must constrain the $B$ field in order to arrive at a gravitational theory.

2.2.1 General relativity in 3d as an SO(1, 2) BF theory

In three space-time dimensions, there exists a $BF$ theory that coincides with pure general relativity. We consider a 3-manifold $M$ and choose the gauge group to be $G = \text{SO}(1, 2)$. Since general relativity is a metric theory, the main part of the process is to define the metric and the Ricci scalar.

To start with, $B$ is an $\mathfrak{so}(1, 2)$-valued 1-form and so we choose to reinterpret it as the co-triad or co-frame field, while $W$ is identified with the gravitational spin connection. If this is indeed viable, then the equations of motion (2.2) state that the connection is flat and that the co-triad is compatible with the spin connection, that is, there is zero torsion. Our aim is now to identify this $\text{SO}(1, 2)$ $BF$ theory with 3d gravity in its first order formalism. With this in mind, there is an obvious definition for the metric tensor in terms of $B$, namely:

$$g_{\mu \nu} = \eta_{IJ} B^{I}_{\mu} B^{J}_{\nu},$$

(2.6)
where $\eta_{IJ}$ is the Minkowski tensor and $B = B_\mu^I \sigma_I \, dx^\mu$.\footnote{The Minkowski metric in 3d: \[ \eta_{IJ} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (2.7)} Furthermore, the space-time connection and Riemann tensor may be written in terms of $W$ and $F$ as:
\[ \Gamma^\lambda_{\mu\nu} := B^\lambda_{\mu I} B'^\nu_{J} W_{\mu I} - B^\nu_{J} \partial_\mu B^\lambda_I, \]
\[ R_{\mu\nu\rho\sigma} \Gamma^\nu_{\rho} = B_{\rho I} B_{\sigma J} F_{\mu}^{I} W_{\rho}^{J}, \]
where $W_{\mu I} := \epsilon^{IJK} W_{\mu}^{K}$ and $F_{\mu}^{I} := \epsilon^{IJK} F_{\mu}^{K}$, while $W = W_{\mu}^{I} \sigma_I \, dx^\mu$ and $F = \frac{1}{2} F_{\mu}^{I} \sigma_I \, dx^\mu \wedge dx^\nu$. $\sigma_I$ are the generators of $\mathfrak{so}(1, 2)$ in the fundamental representation.\footnote{The generators of $\mathfrak{so}(1, 2)$ in the fundamental representation are multiples of the Pauli matrices:}

Coming back to the action and utilising the trace on $\mathfrak{so}(1, 2)$, we can recast the gravity $BF$ theory action as:
\[ S_{BF} = \int d^3x \epsilon^{IJK} B_I \, F_{\mu}^{J} K, \]
\[ = \int d^3x \epsilon^{IJK} B_I \, B_{\mu}^{J} B_{\nu}^{K} R_{\mu\nu\rho\sigma} \Gamma^\nu_{\rho}, \]
\[ = \int d^3x \sqrt{-g} R_{\Gamma}. \]

where we used the fact that in 3d: $\epsilon^{\lambda\mu\nu} \epsilon_{IJK} B^I = |B| B^{[I} \, B_{J]}$, and $|B| = \sqrt{-g}$. Thus we arrive at the Hilbert-Palatini action as promised.

This demonstrates that $BF$ theory with gauge group $SO(1, 2)$ is an alternative formulation of Lorentzian general relativity. Furthermore, in the case of 3d gravity, one sees that the $BF$ equations of motion (2.2) mean that the connection is flat and torsion free. Something to notice is that even if the metric $g_{\mu\nu}$ is degenerate, the $BF$ field equations still hold. Thus, $BF$ theory provides an extension of the vacuum Einstein equations to the case of degenerate metrics.

**Some considerations about gauge groups**

Let us consider our choice of gauge group for the moment. If we choose the gauge group to be $SO(3)$ and proceed along similar lines to those given above, the difference is that we arrive at Riemannian general relativity; the internal tensor: $\eta_{\mu\nu} = \text{diag}(1, 1, 1)$ is the Euclidean tensor and so forth. In the quantum regime, we shall focus our attention on this case since it is technically simpler to deal with than Lorentzian gravity. (Our quantization depends heavily on the use of group representations and the compactness of $SO(3)$ simplifies matters greatly.)
Other gauge groups of interest are their respective double covers: Spin(1, 2) or Spin(3) ∼ SU(2). The BF theory obtained using one of these groups is still equivalent to Lorentzian (resp. Riemannian) general relativity. The differences become manifest once one couples fermionic fields; at the quantum level the use of SU(2) as the gauge group becomes essential.

2.2.2 Plebanski formulation of general relativity in four dimensions as a constrained BF theory

Before we move on the quantization of the theory, let us at least give some details as to how this formalism deals with the more realistic theory of gravity in four dimensions. We select a 4-manifold $M$ and the gauge group $SO(1, 3)$ for BF theory. Understandable, the relation with general relativity is not as straightforward as in the 3d case, since 4d gravity is not topological. To obtain general relativity, one must constrain the $B$ field. Remember that in pure BF theory, the $B$ field acts as a Lagrange multiplier imposing the flatness of the connection; hence one is in fact ‘constraining the constrainer’.

The constrained BF theory leading to 4d gravity is known as the Plebanski model [45], whose action takes the form:

$$S_{Pleb}[B, W, \phi] = \int_M \left( B^{IJ} \wedge F_{IJ}(W) - \frac{1}{2} \phi_{IJKL} B^{IJ} \wedge B^{KL} \right),$$

(2.10)

where $B$ is an $so(4)$-valued 2-form, $W$ is an $so(4)$-valued 1-form connection, $F[W]$ is once more its curvature and finally the Lagrange multipliers $\phi$, which in terms of internal components form a symmetric traceless matrix $\phi_{[IJ][KL]}$. One sees immediately that the Plebanski action (2.10) is a BF action with constraints imposed by varying $\phi$.

As usual, varying the action (2.10) with respect each variable, we obtain the equations of motion (see [44]) but to show its relationship with general relativity, we need only solve for the constraint imposed by varying $\phi$:

$$\delta S \over \delta \phi \rightarrow B^{IJ} \wedge B^{KL} = e_{e} e^{IJKL} \quad \text{with} \quad e := \frac{1}{4} \epsilon_{IJKL} B^{IJ} \wedge B^{KL}.$$  (2.11)

This equation has four sectors of solutions, that is, there exists a real tetrad field $e^{I} = e_{\mu}^{I} \, dx^{\mu}$ such that one of the following equalities is satisfied:

$$I^{+} : \quad B^{IJ} = +e^{I} \wedge e^{J},$$
$$I^{-} : \quad B^{IJ} = -e^{I} \wedge e^{J},$$
$$II^{+} : \quad B^{IJ} = +\frac{1}{2} e^{IJ}_{KL} e^{K} \wedge e^{L},$$
$$II^{-} : \quad B^{IJ} = -\frac{1}{2} e^{IJ}_{KL} e^{K} \wedge e^{L}.$$  (2.12)

The constraint (2.11) is known as the simplicity constraint, since it imposes that the bi-vector $B^{IJ}$ is simple, that is, it is composed as the wedge product of two vectors (or their dual).
Substituting the solution $II^\mp$ into (2.10) we obtain the Einstein-Cartan action:

$$S_{II^\mp}[e,W] = \pm \int_M \left( \epsilon_{IJKL} e^I \wedge e^J \wedge F_{KL}(W) \right).$$

(2.13)

Up to a proportionality factor, the case $II^+$ is the Einstein-Cartan action for general relativity in four dimensions. The sector $II^-$ is still gravity but with the opposite sign.

For completeness sake, the solutions $I^\pm$ yields instead a topological theory:

$$S_{I^\pm}[e,W] = \pm \int_M \left( e^I \wedge e^J \wedge F_{IJ}(W) \right).$$

(2.14)

The importance of the Plebanski action stems from the central role it plays in the definition of the spin foam model in four dimensions. Indeed, the various spin foam models on the market [9, 21, 26] are based around this action.

2.3 Spin foams as path integral for pure gravity

Before moving onto quantization of our random lattice theory, let us first mention some basic properties of the path integral approach in general.

2.3.1 Generic path integrals - some brief points

Path integrals were introduced by Feynman utilising a cardinal idea of quantum mechanics: the superposition principle: the transition amplitude between two states of the system could be equivalently re-expressed as the sum over all possible paths joining the two configurations. These paths are in turn weighted by the exponential of $i$ times the action $S$ relevant for the theory in question.

For example, in scalar field theory, a textbook manipulation gives one:

$$\langle \phi_b(x) | e^{-iHT} | \phi_a(x) \rangle = \int \mathcal{D} \phi(x) \exp \left[ i \int_0^T d^4x L \right].$$

(2.15)

In this formalism, one may also neatly extract some properties of interacting field theories, $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$, by considering a weak interaction and using a perturbative expansion in Feynman diagrams. Generally, one adds a source field $J(x)$ and utilizes a generating functional:

$$Z_{\text{free}}[J] = \int \mathcal{D} \phi(x) \exp \left[ i \int d^4x (\mathcal{L}_{\text{free}}[\phi(x)] + J(x)\phi(x)) \right].$$

(2.16)

Then, the transition amplitude takes the form:

$$\langle \phi_b(x) | e^{-iHT} | \phi_a(x) \rangle = \exp \left[ i \int d^4x \mathcal{L}_{\text{int}} \left[ \frac{\delta}{i \delta J(x)} \right] \right] Z_{\text{free}}[J] \bigg|_{J=0}$$

(2.17)

Moreover, the path integral approach is particularly useful when dealing with gauge theories, but there are a number of subtleties pertaining to the proper
treatment of gauge symmetries. In such theories, gauge symmetries relate physically indistinguishable configurations of the theory. Thus, in the path integral, we wish to take gauge equivalence classes of paths as the fundamental entities in our sum. Were one to assemble a naïve sum over paths, one would encounter infinities arising from the fact that one encounters the same configuration over and over again. One can pick a single member of each equivalence class by introducing a gauge-fixing term in the action. To counterbalance this, one must also include the so-called Fadeev-Popov determinant:

\[
Z[0] = \Delta_{FP} \int \mathcal{D}\phi(x) \exp \left[ i \int d^4x (\mathcal{L}[\phi(x)] + \mathcal{L}_{gf}[\phi(x)]) \right],
\]

where \( \mathcal{L}_{gf} \) is the gauge-fixing term in the action and \( \Delta_{FP} \) denotes the Fadeev-Popov determinant. Non-trivial determinants lead to ghost terms in the action.

As we mentioned earlier, spin foams arose from even more formal yet similar considerations. Naturally, one would love to set up a neat correspondence as in (2.15) and to some extent, the gravitational case can be recast in such a manner. On the left hand side, the operator \( H \) is the quantum Hamiltonian constraint of loop quantum gravity, while the states \( |\phi_a\rangle \) are spin network states of quantum 3-geometry. Following roughly the same argument that is put forward in the scalar field scenario, one can manipulate this entity into a sum over ‘Feynman graphs’, where the graph is a 2d cellular complex interpolating between the two spin networks.

The re-summation of these paths into a field theory path integral was not achieved, but all the same, it provides tantalising evidence, that the quantum gravitational amplitude may be restated in such a form.

Indeed, one may take the following view of the coming spin foam quantization. One discretizes the manifold, and analyses a quantum theory with support on this structure. A single simplicial manifold is in fact related to a single Feynman graph in the above sum (although with a different amplitude).

In 3d, where the theory is topological, one Feynman diagram is sufficient to capture the full quantum theory. This has been supported by a canonical analysis [38]. In 4d, one expects that a single Feynman diagram is insufficient and one should once again consequently sum over all spin foam diagrams interpolating between the two spin networks in question.

### 2.3.2 Path integrals for BF theory

At this point, let us still keep a modicum of generality. Our free field theory is the BF theory outlined earlier. Other terms that might occur in the action are treated as interactions, for example the terms in four dimensions imposing the
simplicity of the bi-vector field:

\[ S = S_{\text{free}} + S_{\text{int}}, \]

\[ S_{\text{free}} = \int_{\mathcal{M}} \text{tr}(B \wedge F[W]), \]

\[ S_{\text{int}} = \int \mathcal{L}_{\text{int}}[B, W]. \]  

(2.19)

We shall deal explicitly only with the free-field theory in this work: \( \mathcal{L}_{\text{int}} = 0 \), since our aim is to study the properties of the 3d theory. Fortunately, in this context we may also dispense with adding a source term (which is dealt with comprehensively in [24]).

Our purpose in section is to make sense of the following statement:

\[ Z_{\mathcal{M}} = \int DW \, DB \, e^{iS[B,W]} \quad "=\quad \int DW \, \delta(\mathcal{F}[W]). \]  

(2.20)

With this in mind, we embark on a process of discretization.

### 2.3.3 Discretization of the manifold

As promised, our first step is to discretize the manifold \( \mathcal{M} \). We shall attempt to keep the arguments general but shall give specific three-dimensional examples where it clarifies matters. Our aim is to formulate a discrete version of BF theory, first classically and then quantum mechanically. With this in mind, we replace the manifold \( \mathcal{M} \) with an \( n \)-dimensional simplicial complex \( \Delta \) of the same topology. We should say here that the fundamental building blocks of \( \Delta \), the \( n \)-simplices and the sub-simplices, are flat but that this still leaves the possibility that non-trivial curvature arises once one glues these objects. This is not the only discrete structure, in which we shall be interested. We outline them below:

\( \Delta \): This is the simplicial complex that replaces \( \mathcal{M} \). It is composed of \( n \)-simplices glued along their \((n-1)\)-dimensional boundaries. The boundary of a given \( n \)-simplex is composed of \((n-1)\)-simplices, which are glued along their \((n-2)\)-dimensional boundaries. One may proceed inductively down to the 0-simplices.

For example, a 3d simplicial complex is made up of a collection of tetrahedra. The tetrahedra are glued along their triangles. The triangles of a given tetrahedron are glued along their edges and these edges meet at vertices:
Figure 2.1: One tetrahedron in a 3d simplicial complex. The tetrahedron and its triangles, edges and vertices are labelled respectively by \( t, f, e \) and \( v \).

\( \Delta^* \): This is the topological dual to the simplicial complex \( \Delta \). One may construct it in the following fashion. One places a 0-cell or dual vertex at the baricentre of each \( n \)-simplex. If two \( n \)-simplices are glued along a common \((n - 1)\)-simplex, then one joins the dual vertices by a 1-cell or dual edge. Generically, a number of \((n - 1)\)-simplices share a common \((n - 2)\)-simplex. One identifies this loop of dual edges as the boundary of a dual face. As before, one may proceed inductively, with each \((n - k)\)-simplex in \( \Delta \) being identified with a \( k \)-cell in \( \Delta^* \).

In 3d, one places a dual vertex at the baricentre of each tetrahedron. If two tetrahedra share a common triangle, then one joins them by a dual edge. In a simplicial complex, many triangles may share an edge. Their dual edges form a loop in \( \Delta^* \), which we identify as the boundary of a dual face. Moreover, many edges in \( \Delta \) may share a vertex. Their dual faces form the boundary of a dual 3-cell:

Figure 2.2: A dual 3-cell in \( \Delta^* \). The dual 3-cells, faces, edges and vertices are respectively labelled by \( t^*, f^*, e^* \) and \( v^* \).

There is a one-to-one correspondence between the \((n - k)\)-simplices and the \( k \)-cells. In the 3d case, we have: \( v \sim t^* \), \( e \sim f^* \), \( f \sim e^* \) and \( t \sim v^* \).

\( \Delta^*_2 \): This is a subset of the dual complex \( \Delta^* \) known as its 2-skeleton. It contains the dual vertices, edges and faces only: \( \Delta^*_2 = \{ v^*, e^*, f^* \} \subset \Delta^* \).
$\tilde{\Delta}^*_2$: We need to introduce a refinement of the dual 2-skeleton [47], which we shall refer to as the wedge graph. Ultimate, we wish to identify precisely the part of the $\Delta^*_2$ which is contained with the shell of a $n$-simplex.

Going directly to the 3d scenario. To this effect, one introduces new dual vertices $v^*_f$ at the midpoints of the dual edges. This splits the dual edges into two which are now labelled by $e^*_t$. There are also new dual vertices $v^*_e$ at the centre of dual faces $f^*$. One joins $v^*_f$ and $v^*_e$ by new dual edges labelled $e^*_{e,f}$. The edges $e^*_t$ and $e^*_{e,f}$ form the boundaries of the wedges. The wedges $w^*$ are that part of the dual cellular complex contained within one tetrahedron. We draw the new elements in Fig. 2.3:

![Diagram](image)

Figure 2.3: Depiction of wedges $w^*$ in a dual face $f^*$.

Perhaps one can get a better overview of the structure if we draw exactly that part of the $\tilde{\Delta}^*_2$ contained within the shell of a tetrahedron.
While the wedge formulation is somewhat an overkill for pure BF theory, it is necessary to deal with interaction terms and, moreover, to deal with the matter observables we wish to extract from the supersymmetric theory.

We now move to formulating the a discrete version of BF theory.

### 2.3.4 Discretization of the fields

We want to define the discrete counterparts of the fields $B$, $W$ and $F$. We shall specialize to the case where the gauge group $G = SU(2)$. The fields are defined as integrals over appropriate objects. The $B$ field, being a $(n-2)$-form, is integrated over the $(n-2)$-simplices. The connection 1-form $W$ is integrated over dual 1-cells. In the three dimensional case, the fields become:

$$
\begin{align*}
B & \rightarrow B_{w^*} = \int_{e^* \sim w^*} B \in su(2), \\
W & \rightarrow g_{e^*} = \mathcal{P} \exp \left[ \int_{e^*} W \right] \in SU(2), \\
\mathcal{F}[W] & \rightarrow G_{w^*} = \prod_{e^* \subset \partial w^*} g_{e^*}^{(e^*, f^*)} \in SU(2),
\end{align*}
$$

(2.21)

where $\epsilon_{(e^*, f^*)} = \pm 1$ depends on the relative orientation of $e^*$ and $f^*$; the $\mathcal{P}$ denotes the path-ordered exponential and in this context $e^*$ is shorthand for $e^*_c, f$ and $e^*_t$. One may also notice that we have assigned a discrete version of the $B$ field to each wedge $w^*$ of the dual face $f^* \sim e$, rather than just one for the whole dual face, as might have been expected and that furthermore, we have replaced the $su(2)$-valued curvature by the holonomy around a wedge, which is...
an element of the group. We shall explain these choices in due course, but first there are two points that must be mentioned here:

**Orientation:** Briefly, since we chose $\mathcal{M}$ (and subsequently $\Delta$) to be orientable, we can assign a consistent choice of orientation to the sub-simplices of $\Delta$. Moreover, the dual cells of $\Delta^*$ inherit this orientation. For example, in 3d, one can use the familiar right-hand rule to get the orientation for $f^*$ from $e$, etc.

**Frames of reference:** A number of properties which we shall describe shortly stem from the fact that we attach a frame of reference to each tetrahedron, triangle and edge of the simplicial complex. Then, there are transformation matrices which map between these frames of reference. These are exactly the parallel transport matrices constructed from the curvature. $g_{\gamma^*}$ parallel transports objects from the reference frame attached to the tetrahedron to that attached to the triangle, while $g_{e^* f^*}$ parallel transports from the triangle to the edge:

![Figure 2.5: Here we show a single wedge and parallel transport matrices from the tetrahedral frame of reference to that of the face and then onto the edge.]

Now we are in a position to describe some properties of the discrete fields.

**$B$ field:** As we mentioned above $B \rightarrow B_w^*$, one for each wedge $w^* \in \Delta^*$. Technically, consider an edge $e \in \Delta$. The $B$ field is integrated along this edge but in the reference frame of a given tetrahedron. Thus, we arrive at one discrete $B$ variable per wedge. However, all the variables $B_{w^*}$ that correspond to an edge $e$ are related by an SU(2) transformation to $B_e$, the $B$ field integrated in the edge reference frame:

$$B_{w^*} = (g_{\gamma^* e^*} g_{e^*}) B_e (g_{\gamma^* e^*} g_{e^*})^{-1}. \quad (2.22)$$

Since the are related in this manner, they have the same length (when seen as vectors in $\mathbb{R}^3$), that is, $|B_{w^*_1}| = |B_{w^*_2}|$ for $w^*_1, w^*_2 \subset \sim e$.

We shall be able to trace this coincidence of lengths through our quantization. Furthermore, once we move to the supersymmetric theory, we shall
see that this an interesting point of discussion. In the case of our supersymmetric theory, the analogue of \( B_{w^*} \) must have the same ‘supersymmetric length’, as seen from a different reference frames, but its geometric length can change.

**W field:** For the most part, we have already described the discrete analogue of the connection. Its most natural counterpart is the parallel transport matrix which maps between reference frames.

**F field:** We now switch our attention to the other field that appears in the \( BF \) action, the curvature \( F[W] \). In the continuum, one would calculate the curvature by performing the parallel transport of a vector along a closed loop, which is ultimately the holonomy of the connection \( W \). In the discrete setting, the holonomy is given by the trace of the product of parallel transport matrices around that loop in question and thus we have something which is intimately related to the curvature, see [60]. Indeed for small wedges \( w^* \):

\[
G_{w^*} \approx \mathbb{I} + \epsilon^2 F_{w^*}.
\]

(2.23)

where \( \epsilon^2 \) is the area of the wedge and \( F_{w^*} \) is the curvature field \( F \) evaluated as some point within the wedge.

As we have already mentioned, \( G_{w^*} \) comprises of the \( g_{e^*_f} \) and \( g_{e^*_t} \) elements occurring around the boundary of the wedge. The order is determined by the orientation of the dual face and the relative orientation of the \( e^* \) and \( f^* \) may cause the inverse element to occur. There is still an ambiguity in where one choses the base point for the holonomy. One can fix this ambiguity requiring that the discrete action is gauge-invariant, as it is in our model.

As a final comment on the discrete variables, it might seem that by introducing the wedge variables, the number of independent variables of the theory has increased. In reality the physical content of the theory is the same since all the \( B_{w^*} \) occurring in a dual face are gauge equivalent.

**Some comments on distributional fields**

A posteriori, one can see that our process of discretization is equivalent to considering only field configurations, \( B \) and \( W \), which are distributional with support on the relevant sub-simplices of \( \Delta \) and sub-cells of \( \Delta^* \).

In other words, proceeding in the opposite direction, in order to discretize the fields \( B, W \) and \( F[W] \), we could have replaced them with distributional configurations. In this light, one might question if this is a correct method. We respond here in the affirmative.

Firstly, if we investigate the form of the quantum state in the canonical theory, namely loop quantum gravity, we see that they are of the form:

\[
\Psi \left( h_{e^*_1}(W), \cdots, h_{e^*_n}(W) \right).
\]

(2.24)
where $e^*$ are edges of the boundary spin network graph. In other words, their dependence on the connection $W$ is through holonomies $h_{e^*}$ along some graphs in space, whose edges are labeled by $e_i^*$, see [43]. (Note, these are states of a continuum theory; no discretization is assumed here.) At the classical level, the conjugate momentum to the connection $W$ is the $B$ field; therefore at the quantum level this will become the functional operator $\delta / \delta W(x)$, the variational derivative with respect to $W$. Thus, when the operator $\hat{B}$ acts on the state $\Psi$, it will give zero for all points $x$ except those points lying on the edges $e_i$, thus $B$ is distributional with support on the edges $e_i$ of the corresponding graph. Moreover, let us consider the $B$ field for a moment. $\Psi$ is a state at a given time, hence $B$ at each time $t$ has support along graph edges. This suggests that the ‘history’ of the $B$ field configurations can be depicted as a collection of two-dimensional surfaces in spacetime, along which $B$ is distributional. (The surfaces of support of $B$ are always two dimensional, independently of the dimension of the spacetime manifold.)

Secondly, another argument emanates from the fact that similar distributional solutions of the equations of motion arise naturally. It has been shown [7], that for this kind of solution, the field $B$ is zero except in a neighborhood of a two-dimensional surface in spacetime. The connection field $W$ away from the surface must be either arbitrary or flat, depending on the Lagrangian. For more details see [24].

2.3.5 The classical discrete theory

Action and equations of motion

We are ready to write down the action. From the definition of the $B_{w^*}$ variables, we can say that the first step is:

$$ S = \sum_{w^*} \int_{w^*} \text{tr}(B_{w^*} F(W)). $$

(2.25)

The next step is to use the approximation (2.23):

$$ S \approx \sum_{w^*} \text{tr}(B_{w^*} G_{w^*}). $$

(2.26)

We take as our discrete action:

$$ S_{BF,\Delta} = \sum_{w^*} \text{tr}(B_{w^*} G_{w^*}). $$

(2.27)

Let us say few words about the validity of the discrete action $S_{BF,\Delta}$ in its role as an approximation to the continuum version. The approximation is good for $G_{w^*}$ close to identity element of the Lie group, a case that is certainly valid for $BF$ theory, where we expect only connections close to the flat one to be relevant in quantum theory. (In the cases when this is not the case anymore, for example $BF$ theory with a cosmological term, one would expect the approximation to
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become better and better as the triangulation of the manifold becomes finer and thus, the dual faces become smaller. Moreover, we expect finer triangulations to have a higher relevance in quantum gravity. Therefore, the approximation is justified.)

The equations of motion come from varying $B^{w^*}$, $g_{e^*_f}^{w^*}$ and $g_{e^*_t}^{w^*}$:

$$G^{w^*} = 0,$$
$$g_{e^*_t}^{w^*}B^{w^*}G^{w^*}g_{e^*_t}^{-1} - g_{e^*_t}^{w^*}B^{w^*}G^{w^*}g_{e^*_t}^{-1} = 0, \quad (2.28)$$
$$B^{w^*}G^{w^*} + B^{w^*}G^{w^*} + B^{w^*}G^{w^*} = 0$$

In the second equation, $w^*$ and $\tilde{w}^*$ are the two wedges sharing the dual edge $e^*_f$ being varied, while $e^*_t$ and $\tilde{e}^*_t$ are the dual edges joining the tetrahedra $t$ and $\tilde{t}$ to their common triangle. Moreover, in the third equation, $w^*$, $\tilde{w}^*$ and $\hat{w}^*$ are the three wedges shared by the dual edge $e^*_t$ being varied. It is easy to see using the first equation of motion we can simplify the latter ones to:

$$g_{e^*_t}^{w^*}B^{w^*}g_{e^*_t}^{-1} - g_{e^*_t}^{w^*}B^{w^*}g_{e^*_t}^{-1} = 0,$$
$$B^{w^*} + B^{\tilde{w}^*} + B^{\hat{w}^*} = 0 \quad (2.29)$$

The first states that for any two wedges in a face, their $B$ variables are related by a gauge transformation. Remember that to arrive at the second relation, we varied with respect to $g_{e^*_t}^{w^*}$. The three wedges incident at $e^*_t$ are dual to the three edges of a triangle. Thus, the second relation ensures the closure of the triangle. This is the discrete counterpart of the torsion-free condition in (2.2).

Note also that we have dispensed with an explicit imposition of the length constraint, which imposes that $|B^{w^*}_{e^*_t}| = |B^{w^*}_{e^*_t}|$ for $w^*_{e^*_t}, w^*_{e^*_t} \subset f^*$. We are really taking the $B^{w^*}$ as completely independent variables. Note, however, that this condition now occurs as the second equation of motion. So classically, the theories coincide. Of course, in the quantum regime, dispensing with this constraint, leads to a different quantization. For $BF$ theory, however, we shall see explicitly that the path integral sums only over paths which satisfy this constraint. Furthermore, for $BF$ theories with interaction terms, there are arguments that the saddle point analysis leads to semi-classical theories once again satisfying this constraint [11].

Symmetries: gauge and translation invariance

The gauge invariance survives into the discrete regime:

$$B^{w^*} \rightarrow B'^{w^*} = h_{v^*}^{w^*}B^{w^*}h_{v^*}^{-1},$$
$$g_{e^*_t}^{w^*} \rightarrow g'^{e^*_t}_t = h_{v^*}^{-1}g_{e^*_t}^{w^*}h_{v^*},$$
$$g_{e^*_f}^{w^*} \rightarrow g'^{e^*_f}_f = h_{v^*}^{-1}g_{e^*_f}^{w^*}h_{v^*},$$
$$G^{w^*} \rightarrow G'^{w^*} = h_{v^*}^{-1}G^{w^*}h_{v^*}.$$  \quad (2.30)
We note that the gauge parameters all reside at the various dual vertices $v^*$, $v_f^*$ and $v_e^*$:

The translation symmetry is more difficult to quantify but has been explicitly shown when one takes the (equivalent) action:

$$S'_{BF,\Delta} = \sum_{w^*} \text{tr}(B_{w^*} Z_{w^*}), \quad \text{where} \quad Z_{w^*} = \log(G_{w^*}). \quad (2.31)$$

In this case, the discrete form of the translation symmetry and as in the continuum setting, the discrete form of the Bianchi identity is required to show that the symmetry holds. We refer the reader to [27] for a comprehensive exposition. In the case where one takes the action $S_{BF,\Delta}$, the symmetry takes a yet more complicated form. The important thing for us is that the symmetry is parameterized by Lie algebra elements $\Phi_v$ attached to the vertices $v$ of the triangulation:

$$B_{w^*} \rightarrow B_{w^*} + f_{w^*}([\Phi_v], \{G_{w^*}\}), \quad (2.32)$$

where $f_{w^*}$ is an $\mathfrak{su}(2)$-valued function. Its arguments are the $\mathfrak{su}(2)$ elements $\Phi_v$ attached to the two vertices of $e \sim w^*$ and the holonomies around the wedges which lie on the boundary of the same dual 3-cells as $w^*$.

### 2.3.6 Quantum discrete BF theory

#### State sum measure

The definition of the discretized fields allows us to specify a natural measure of the functional integral. Namely, $B_{w^*} \in \mathfrak{su}(2)$ and $g_{w^*} \in \text{SU}(2)$, so we can use the Lebesgue measure on $\mathbb{R}^3$ and the normalized Haar measure on $\text{SU}(2)$.\footnote{The Haar measure for the group $\text{SU}(2)$ is:}

$$\int DWDDB \rightarrow \int \prod_{\gamma_i^*} dg_{\gamma_i^*} \prod_{\gamma_{i,f}^*} dg_{\gamma_{i,f}^*} \prod_{w^*} dB_{w^*}. \quad (2.34)$$

Moreover, this measure invariant under the symmetries of the action.

\footnote{The normalization implies $\int_{\text{SU}(2)} dg = 1$}
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Partition function

Using the results we compiled so far, we can write the discrete partition function as:

\[ Z_{\Delta, SU(2)} = \int \prod_{e} dg_{e} \prod_{e,f} dg_{e,f} \prod_{w} dB_{w*} e^{iS_{\Delta}[B_{w*}, g_{e*}, g_{e,f}]}, \]  

(2.35)

where the action \( S_{\Delta} \) is:

\[ S_{\Delta}[B_{w*}, g_{e*}, g_{e,f}] = \sum_{w* \in \Delta^*} \text{tr} (B_{w*} G_{w*}). \]  

(2.36)

On the road to the state sum

In order to calculate the amplitude, the first step is to perform the integration over the \( B_{w*} \) variables. These variables are elements of \( su(2) \) which is isomorphic to \( \mathbb{R}^3 \) as a vector space. Thus it can be written as:

\[ B_{w*} = i\vec{B}_{w*} \cdot \vec{\sigma}, \quad \text{where} \quad \vec{B}_{w*} = \frac{1}{2i} \text{tr}(\vec{\sigma} B_{w*}), \]  

(2.37)

and \( \sigma \) refers to the Pauli matrices. The holonomy \( G_{w*} \in SU(2) \) may also be written as:

\[ G_{w*} = \exp \left( i\frac{\theta_{w*}}{2} \vec{n}_{w*} \cdot \vec{\sigma} \right) = \cos \theta_{w*} \mathbb{1} + i \sin \theta_{w*} \vec{n}_{w*} \cdot \vec{\sigma} \]  

(2.38)

where \( \vec{n}_{w*} \) is a unit vector.\(^5\) Therefore, the action takes the form:

\[ S_{\Delta} = \sum_{w*} i\vec{B}_{w*} \cdot \vec{n}_{w*} \sin \theta_{w*}. \]  

(2.39)

We may now directly perform the \( B_{w*} \) integral using the standard Lebesgue rule:\(^6\)

\[ Z_{\Delta, SU(2)} = \int \prod_{e} dg_{e} \prod_{e,f} dg_{e,f} \prod_{w*} \delta^{(3)}(\vec{n}_{w*} \sin \theta_{w*}). \]  

(2.41)

Using a standard relationship for \( \delta \)-functions,\(^7\) the \( \delta^{(3)}(\vec{n}_{w*} \sin \theta_{w*}) \) can be manipulated into \( \delta(G_{w*}) = \delta \left( \exp \left[i\frac{\theta_{w*}}{2} \vec{n}_{w*} \cdot \vec{\sigma} \right] \right). \) In fact, as shown in Appendix A, it is in fact related to a linear combination of \( \delta(G_{w*}) \) and \( \delta(-G_{w*}) \). This unwanted second peak may be removed upon the insertion of the observable:

\(^5\)\( \vec{n}_{w*} := (\sin \psi_{w*} \cos \phi_{w*}, \sin \psi_{w*} \sin \phi_{w*}, \cos \psi_{w*}) \)

\(^6\)Explicitly, the integral for each wedge is:

\[ \int dB_{w*} e^{(\vec{B}_{w*} \cdot \vec{n}_{w*} \sin \theta_{w*})} = \delta^{(3)}(\vec{n}_{w*} \sin \theta_{w*}) \]  

(2.40)

\(^7\)The relationship in question is : \( \delta(f(x)) = \sum_{x=x_i} \frac{\delta(x - x_i)}{|f'(x_i)|} \) where \( x_i \) are the zeros of the function \( f(x) \).
\( \frac{1}{2} (1 + \cos \theta_w^*) \) into the partition function and we retroactively do so. In other words:

\[
\int dB_{w^*} \exp \left[ \frac{1}{2i} \text{tr}(B_{w^*} G_{w^*}) \right] (1 + \cos \theta_{w^*}) = 8\pi \delta(G_{w^*}),
\]

(2.42)

We shall see that in the supersymmetric case that the presence of the fermionic degrees of freedom will automatically ‘cure’ this discrepancy occurring in the \( SU(2) \) case.

With this in mind our partition function becomes:

\[
Z_{\Delta, SU(2)} = \int \prod e^* t dg e^* t \prod e^* e, f dg e^* e, f \prod w^* \delta(G_{w^*})
\]

(2.43)

We see from (2.43) that in the quantum case, we sum over flat configurations \( G_{w^*} = I \) (due to the \( \delta \)-function amplitude). Thus, when we integrate out the \( g_{e^*, f} \) elements, we shall implicitly impose that we keep configurations that satisfy the first relation in (2.29). A posteriori, we see that we did not need to impose that \( |B_{w^1}| = |B_{w^2}| \) explicitly in the partition function from the beginning. It comes about automatically.

**A diversionary note on gauge-fixing**

Now that we have some feel for the amplitude, let us describe the gauge-fixing procedure. The symmetries of the theory are parameterized by group elements \( h_{v^*} \in SU(2) \) and algebra elements \( \Phi_v \in \text{su}(2) \). The gauge-fixing procedure is familiar from lattice field theory. One needs the concept of a maximal tree in a graph. A maximal tree \( T \) of a graph \( \Delta \) is a sequence of edges in \( \Delta \) which touch all of its vertices but contains no loops of edges. For the gauge symmetry one picks a maximal tree \( T^* \) in \( \Delta_2^* \) and uses the \( h_{v^*} \) to fix certain group elements:

\[
g_{e^*} = I \quad \text{if} \quad e^* \in T^*.
\]

(2.44)

Likewise, for the translation symmetry, one picks a maximal tree \( T \) in \( \Delta \) and one uses the elements \( \Phi_v \) to fix:

\[
B_{w^*} = 0 \quad \text{for only one} \quad w^* \sim e \in T.
\]

(2.45)

The \( SU(2) \) gauge symmetry is a compact symmetry and would not have caused infinities in the amplitude (although, this is not case for the more realistic Lorentzian scenario). The translation symmetry, however, is a non-compact symmetry and would certainly lead to infinities. We can see this immediately from the form of the amplitude (2.43). We note that we have a \( \delta \)-function for each wedge, forcing the holonomy around the wedge to be the identity. But the discrete Bianchi identity (which is intimately related to the satisfaction of the translation symmetry) means that the product of the holonomies for wedges on the boundary of a dual 3-cell is the identity:
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Figure 2.6: A dual 3-cell \( t^* \) with its boundary. The discrete Bianchi identity reads: \( \prod_{w^* \in \partial t^*} G_{w^*} = I \). Note that there is a subtlety in constructing this identity since one must pick a dual vertex \( v^* \in \partial t^* \) and transport each wedge holonomy to this base point before multiplying.

This means that given a dual 3-cell whose boundary contains \( m \) wedges, then the quantum amplitude imposes that all \( m \) are flat. But after \( (m - 1) \) wedges are imposed to be flat, the final one is automatically flat using the discrete Bianchi identity. This means that one acquires a factor of \( \delta(I) \). Our gauge-fixing kills this contribution. Explicitly, the gauge-fixing observable that one includes in the path integral (2.35) is:

\[
O_{ gf, \Delta, T, T^*}[g_{e^*}, B_{w^*}] := \prod_{e^* \in T^*} \delta(g_{e^*}) \prod_{\text{one } w^* \sim e \in T} \delta(B_{w^*}) \tag{2.46}
\]

The next task to construct the Fadeev-Popov determinant \( \Delta_{FP, \Delta, T, T^*} \). This has been carried out explicitly in [27] and was found to equal the unit element: \( \Delta_{FP, \Delta, T, T^*} = 1 \). We shall not derive this explicitly here. In fact, the whole gauge-fixing procedure is independent of the maximal trees chosen. Furthermore, we shall leave the gauge-fixing implicit in the following analysis, but it may be carried all the way through to the end.

Back on the road again

It is now time to integrate over the group elements \( g_{e^*} \) and \( g_{t^*} \). We use the Peter-Weyl decomposition for a function of group elements:

\[
f(g) = \sum_{j=0, j=\frac{1}{2}} i^{j} f_{m}^{n} i^{D_{m}^{n}}(g), \quad \text{where} \quad i^{j} f_{m}^{n} := \int_{SU(2)} dg f(g) i^{j} D_{m}^{n}(g), \tag{2.47}
\]

where \( i^{j} D_{m}^{n}(g) \) is the irreducible representation of \( SU(2) \), labelled by the half-integer \( j \) and in the basis of labeled by magnetic states: \( |j, m\rangle \). This is essentially the Fourier transform on the group \( SU(2) \); taking the momentum space to be the group, then the position space is the collection of representation vectors spaces. Since \( SU(2) \) is compact, this position space is discrete. For future reference,
there is a difference between upper and lower indices on tensors. The raising and lowering operators for SU(2) tensors are:
\[
\begin{align*}
    j^m_{\lambda\mu} &= (-1)^{k-m} \delta^{m+n,0}, \\
    j_m^{\lambda\mu} &= (-1)^{k-n} \delta^{m+n,0}.
\end{align*}
\]
This convention allows one to neatly keep track of minus signs. This implies the following relation for the \(\delta\)-function:
\[
\delta(g) = \sum_{j=0, j=\frac{N}{2}} \dim_j \chi_j(g).
\]
where \(\dim_j = (2j + 1)\) is the dimension of the irreducible representation and \(\chi_j(g) = \sum_m j^m_{\lambda\mu}(g)\) is the character of the \(j\) representation. Thus, our amplitude now takes the form:
\[
Z_{\Delta, SU(2)} = \int \prod e^*_t \prod e^*_{e,f} \prod w^* \sum j_w^* \dim_j j_w^* \chi_j(G_w^*).
\]
The integration over the group elements attached to \(e^*_e\) is the same for any dimension of the discrete manifold \(\Delta\), since this edge is always shared by just two wedges \(w^*\) in a dual face \(f^* \in \Delta^*_2\). The remaining integration over the edges \(e^*_f\) depends instead on the dimension. To perform the integration explicitly, one must split apart these character functions using the formula that the representation of a product of elements is the same as the product of the representation of each element:
\[
\chi_j(G_w^*) = \prod_{e^*_e \subset \partial w^*} g_{e^*_e}^{j^*_w(e^*_e^*, f^*)} = \text{tr} \left[ \prod_{e^*_e \subset \partial w^*} j_w^* D \left( g_{e^*_e}^{j^*_w(e^*_e^*, f^*)} \right) \right]
\]
The wedges inherit their orientation from the face \(f^*\), in which the reside. Thus, the group \(g_{e^*_e,f}^*\) occurs with opposite orientation in the two \(G_w^*\), which contain it:

Figure 2.7: The edge \(e^*_e,f\) occurs with both orientations.
Thus we find integrals of the following form in the partition function:

\[ \int dg_{e^*_f} \ j_{w^*} D_{m,n}^o (g_{e^*_f}) \ j_{\tilde{w}^*} \tilde{D}_{\tilde{m},\tilde{n}} (g_{e^*_f}) = \]

\[ = \int dg_{e^*_f} \ j_{w^*} D_{m,n}^o (g_{e^*_f}) \ j_{\tilde{w}^*} \tilde{D}_{\tilde{m},\tilde{n}} (g_{e^*_f}) = \frac{1}{\dim_j} \delta_{j_{w^*} j_{\tilde{w}^*}} \delta_{m_{\tilde{n}}} \delta_{n_{\tilde{m}}}. \]  

(2.52)

where \( \tilde{D} \) denotes the complex conjugate matrix. To arrive at the second equation, we have used that the group is unitary, that is, \( jD_{m,n} (g^{-1}) = jD_{m,n} (g^1) = (jD^\dagger)^{m,n} (g) = jD^\dagger_{n,m} (g) \), where \( D^\dagger \) denotes the Hermitian conjugate matrix.

Note that the integration enforces that \( j_{w^*} = j_{\tilde{w}^*} \), which is the counterpart in representations of \( |B_{w^*}| = |B_{\tilde{w}^*}| \).

In the amplitude, we replace these pairs of matrices by the Kronecker deltas, as prescribed by (2.52). The resulting amplitude is rather simple:

\[ Z_{\Delta, SU(2)} = \int \prod_{e^*_t} dg_{e^*_f} \prod_{f^*} \sum_{j_{f^*}} \dim_{j_{f^*}} j_{f^*} \chi (G_{f^*}) \]  

(2.53)

where \( G_{f^*} = \prod_{w^* \subset f^*} G_{w^*} \). (Note that in such a product the elements \( g_{e^*_f} \) do not occur explicitly since they occur side-by-side with opposite orientations.)

In fact, we have entered into a slight amount of gratuitous detail in this case, since \( g_{e^*_f} \) occur in just two \( \delta \)-functions, we could have utilized the fact that they are indeed \( \delta \)-functions to arrive immediately at this result. But it served to introduce the general scheme of integration that occurs in more complicated scenarios later.

We move onto the edges \( e^*_t \). One should notice that \( g_{e^*_t} \) occurs in as many representations as there are wedges \( w^* \) sharing \( e^*_t \). Grouping together these representations referring to the same edge, the partition function is:

\[ Z_{\Delta, SU(2)} = \int \prod_{e^*_t} dg_{e^*_f} \left( \prod_{f^*} \sum_{j_{f^*}} \dim_{j_{f^*}} \prod_{e^*_t} j_{f^*} \right) \left( \prod_{e^*_f} \prod_{f^* \supset e^*_f} j_{f^*} \ D \left( (g_{e^*_f})^{e_{t^*}(f^*)} \right) \right) \]  

(2.54)

where \( \epsilon_{i^*_t(f^*)} \) keeps track of the relative orientation of \( e^*_t \) and \( f^* \). The number of elements in the product depends directly on the dimension of the manifold. Namely, for an \( n \)-dimensional manifold, there are \( n \) elements in each such product.

Let us consider the specific examples of three and four dimensions.

**Three dimension gravity**

In three dimensions, one can see that these dual faces share each dual edge:
Figure 2.8: Here we see the three wedges containing $e^*_t$.

In this case the following equation for the matrix elements holds

$$
\int dg_1^* j_1^D^{m_1} n_1 (g_1^*) j_2^D^{m_2} n_2 (g_2^*) j_3^D^{m_3} n_3 (g_3^*)
= (-1)^{j_1 + j_2 + j_3} C^{m_3 m_2 m_1}_{j_3 j_2 j_1} C^{j_1 j_2 j_3}_{n_1 n_2 n_3},
$$

where the factor $C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ is a $\{3j\}$-symbol for SU(2) and $j_i$ stands for $j^*_f$ etc.\(^8\)

Now we must substitute this into the amplitude to get:

$$
\mathcal{Z}_{\Delta, \text{SU}(2)} = \left[ \prod_{f^*} \sum_{j^{f^*}_f} \text{dim}_{j^{f^*}_f} \right] \prod_{e_t^*} (-1)^{j_1 + j_2 + j_3} C^{m_3 m_2 m_1}_{j_3 j_2 j_1} C^{j_1 j_2 j_3}_{n_1 n_2 n_3} e^*_t.
$$

It is time once again to repartition the amplitude. If we consider a dual edge $e^*_t$, it joins the centre of the tetrahedron $t$ to the centre of the face $f \sim e$. Thus, we split the amplitude:

$$
(-1)^{j_1 + j_2 + j_3} C^{m_3 m_2 m_1}_{j_3 j_2 j_1} C^{j_1 j_2 j_3}_{n_1 n_2 n_3}.
$$

Each triangle $f$ gets contributions from two $e^*_t$, so its amplitude is in the end:

$$
A_{e^*} = A_f = (-1)^{(j_1 + j_2 + j_3)^*},
$$

while each tetrahedron $t$ gets contributions from four $e^*_t$, so its amplitude is in the end:

$$
A_{e^*} = A_t = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} v^*.
$$

where:

$$
\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} := C^{j_1 j_2 j_3}_{m_1 m_2 m_3} C^{j_4 j_5 j_6}_{m_4 m_5 m_6} C^{m_4 m_2 m_6}_{j_4 j_2 j_6} C^{m_1 m_5 m_6}_{j_1 j_5 j_6}.
$$

\(^8\)It is related to the Clebsch-Gordan coefficient by:

$$
C^{j_1 j_2 j_3}_{m_1 m_2 m_3} = \frac{(-1)^{j_1 - j_2 + m_3}}{\sqrt{2j_3 + 1}} \left( \langle j_1, m_1 \rangle \otimes \langle j_2, m_2 \rangle \right) \langle j_3, m_3 \rangle.
$$
where one may also refer to [59]. The partition function becomes:

$$Z_{\Delta, SU(2)} = \sum_{j^*, f^*} \prod_{e^*} (\text{dim}_{j^*}) \prod_{v^*} (-1)^{(j_1 + j_2 + j_3)} \prod_{e^*} \left\{ \frac{1}{j_4 \ j_5 \ j_6} \right\},$$

(2.62)

are the $\{6j\}$-symbols for SU(2). This is the standard amplitude for the Ponzano-Regge model. As one can see, each of the elements of the 2-skeleton $\Delta^*_2$ is assigned an amplitude.

**Four dimension case**

In order to show how the final result for the partition function changes as one changes the dimension of the manifold, we shall briefly illustrate how one calculates the SPIN(4) BF theory partition function on a 4d simplicial manifold (SPIN(4) is the double cover of SO(4)). We shall use the property that $\text{spin}(4) \equiv \text{su}(2) \oplus \text{su}(2)$, that is, the algebra (and the group) can be split into self-dual and anti-self-dual sectors.

Although the manifold $M$ is replaced by a 4d simplicial complex $\Delta$, one is still really interested in the wedge graph $\tilde{\Delta}^*_2$, which has a similar form to that in the 3d. Indeed, the partition function once again takes the form:

$$Z_{\Delta, \text{SPIN(4)}} = \prod_{e^*_t} dg_{e^*_t} \prod_{e^*_t, f^*_j} dg_{e^*_t, f^*_j} \prod_{w^*_v} \delta(G_{w^*_v})$$

(2.63)

where $g_{e^*_t}$ and $g_{e^*_t, f^*_j}$ are elements of SPIN(4). Once again, each $g_{e^*_t, f^*_j}$ is shared by two wedges, but the big difference is that each $g_{e^*_t}$ is shared by four wedges. Remember that in four dimensions, the wedges are in fact dual to triangles $f \in \Delta$ (not edges, as in 3d) and four triangles form the boundary of a tetrahedron $t \in \Delta$ (just like three edges form the boundary of a triangle in the 3d scenario). Thus, one must be able to deal with integrals of the form:

$$\int_{\text{SPIN(4)}} dg_{e^*_t} J_1 D^{N_1} g_{e^*_t} J_2 D^{N_2} g_{e^*_t} J_3 D^{N_3} g_{e^*_t} J_4 D^{N_4} g_{e^*_t}.$$

(2.64)

But first we shall use the fact that SPIN(4) splits into two chiral sectors to rewrite the partition function as:

$$Z_{\Delta, \text{SPIN(4)}} = Z_{\Delta, \text{SU(2)}^+, \text{SU(2)}^-}.$$ 

(2.65)

Then, we can bring across trivially a lot of the tools described in detail in the previous section. We can perform the integral over $g_{e^*_t, f^*_j}$ to get factors of $\delta(G_{f^*_j})$. We can subsequently decompose these $\delta$-functions in terms of representations of $\text{SU(2)}^\pm$. Then, we perform the integrals over $g_{e^*_t}$, where we need
the result:

\[
\int_{SU(2)} dg_1 \, D_{m_1 n_1} (g_1) \, D_{m_2 n_2} (g_1) \, D_{m_3 n_3} (g_1) \, D_{m_4 n_4} (g_1)
\]

\[
= \sum_i \dim_i \, i C_{j_1 \ldots j_4}^{m_1 m_2 m_3 m_4}
\]

(2.66)

where \(i C_{j_1 \ldots j_4}^{m_1 m_2 m_3 m_4} \equiv C_{j_1 \ldots j_4}^{m_1 m_2 m_3 m_4} \). Now, we regroup these intertwiners. It turns out that their are five associated to each 4-simplex \(\tau(\sim v^*)\) in \(\Delta\) (just as there were four associated to each tetrahedra in 3d), normalisation factors once again label \(e^* \in \Delta^*\), and factors of dimension dual 2-cells \(f^*\). Thus, one arrives at the amplitude:

\[
Z_{\Delta, SPIN(4)} = Z_{\Delta, SU(2)_+} Z_{\Delta, SU(2)_-}
\]

\[
= \sum_{j_1 \ldots j_4} \prod_{f^*} \dim_{j_1} \prod_{e^*} \dim_{i_{e^*}} \prod_{v^*} \{15j\}_{v^*},
\]

\[
\{15j\} \equiv i a C_{m_1 m_2 m_3 m_4}^{j_1 j_2 j_3 j_4} i b C_{m_5 m_6 m_7}^{j_5 j_6 j_7} i c C_{m_8 m_9 m_10}^{j_8 j_9 j_{10}}
\]

\[
\times i d C_{m_{11} m_{12} m_{13} m_{14}}^{j_1 j_2 j_3 j_4}
\]

(2.67)

The amplitude (2.67) is for \(BF\) theory. The main aim of the spin foam approach in four dimensions is to impose a quantum version of the simplicity constraints (written down in Section 2.2.2) on this state sum so that one obtains a gravitational state sum. It is not straightforward to implement these constraints in the simplicial (let alone quantum) setting, and the various proposals to do so lead to different models such as the Barrett-Crane (BC) model, the Engle-Livine-Pereira-Rovelli (ELPR) model or the Freidel-Krasnov (FK) model, see for example [8], [10], [26], [53].
Chapter 3

The group $\text{UOSP}(1|2)$

In order to introduce the supersymmetric Ponzano-Regge model in Chapter 4, it is perhaps clearer that we give some mathematical preliminaries and introduce the new gauge group that we shall use in our theory. For this reason, we devote this chapter to the introduction of some of the main characteristics of the group with the Grassmann structure $\text{OSP}(1|2)$ and its compact analogue $\text{UOSP}(1|2)$.

In the first section, we shall recall some features of Grassmann algebras. This is indispensable, since one does not parameterize supersymmetric gauge algebras and groups with complex numbers but rather with elements of these Grassmann algebras. Furthermore, they are an essential tool for the introduction of fermionic degrees of freedom into a path integral quantization. Following this, we are ready to introduce the algebra $\mathfrak{uosp}(1|2)$, where we outline its defining properties and those of its representation theory. Interestingly, we shall find that the basis of a generic $\mathfrak{uosp}(1|2)$ module comprises of two $\mathfrak{su}(2)$ modules with the odd generators mapping between them. This substructure is very important for our subsequent analysis of the supersymmetric spin foam model.

In Section 3.3 we shall focus on the corresponding group $\text{UOSP}(1|2)$. After briefly analyzing the form of the group elements, we shall see that the representation matrices are endowed with an SU(2) substructure, a property that will become useful for recasting the super $\{3j\}$-symbols and $\{6j\}$-symbols, illustrated in the last two sections, in terms of SU(2) $\{3j\}$-symbols and $\{6j\}$-symbols.

For the sake of brevity, the portrait we give here merely reports the mathematical concepts that we need to develop and examine the supersymmetric model. For a more comprehensive description, we refer the reader to [13, 55] and references therein.
3.1 Grassmann algebra

A Grassmann algebra \( G \), with \( N \) generators \( \xi_1, \ldots, \xi_N \) satisfies \( \xi_i \xi_j + \xi_j \xi_i = 0 \). (N may be both finite and infinite.) The element:

\[
\alpha = \sum_{m \geq 0} \sum_{i_1 \ldots i_m} \alpha_{i_1 \ldots i_m} \xi_{i_1} \ldots \xi_{i_m},
\]

is called even if only the coefficients with even \( m \) are non-zero and odd if only the coefficients with odd \( m \) are non-zero. The sets of even and odd elements are denoted \( G_0 \) and \( G_1 \) respectively, and \( G = G_0 \oplus G_1 \).

The parity function \( \lambda(\alpha) \) is defined on \( G \) as:

\[
\lambda(\alpha) = \begin{cases} 
0 & \text{if } \alpha \in G_0, \\
1 & \text{if } \alpha \in G_1.
\end{cases}
\]

We define a complex conjugation operation, \( \Box \), on \( G \) with the following properties:

\[
(\alpha \beta) \Box = \alpha \Box \beta \Box, \quad (c \alpha) \Box = \bar{c} \alpha \Box, \quad (\alpha \Box) \Box = (-1)^{\lambda(\alpha)} \alpha,
\]

where \( \alpha, \beta \in G \) and \( c \in \mathbb{C} \) and \( \bar{c} \) denotes standard complex conjugation on \( \mathbb{C} \). There are a number of ways to define such an operation on a Grassmann algebra, so one must pick one and adhere to it.

We can define an integration theory for functions \( f : G \to G \). First of all, analytic functions on \( \mathbb{C} \) have a natural extension to superanalytic functions on \( G \):

\[
f(\alpha) = \sum_{n \geq 0} \frac{f^{(n)}(\alpha_*)}{n!} (\alpha - \alpha_*)^n,
\]

where \( \alpha_* \) is known as the body of \( \alpha \) in DeWitt’s terminology [19]. It is the \( m = 0 \) term in (3.1) (while the remainder \( \alpha - \alpha_* \) is its soul). We can define a measure to integrate functions on \( G \). For even elements, the body plays a special role:

\[
\alpha = \alpha_* + \sum_{m \geq 2} \sum_{i_1 \ldots i_m} \alpha_{i_1 \ldots i_m} \xi_{i_1} \ldots \xi_{i_m},
\]

Then, the measure is:

\[
\int d\alpha := \int d\alpha_*,
\]

where one also replaces \( \alpha \) by \( \alpha_* \) in the integrand. Clearly, odd elements have no body. Thus, we need a different definition of the measure. The most general superanalytic function of an odd element is: \( f(\alpha) = (c_1 + c_2 \alpha) \). For odd elements, the measure is defined as:

\[
\int d\alpha (c_1 + c_2 \alpha) = c_2,
\]

which means that

\[
\int d\alpha \alpha f(\alpha) = f(0),
\]

where \( \delta(\alpha_*) \) is the standard distributional one on \( \mathbb{C} \). Hence, we can define a delta function on \( G \). For the functions on \( G_0 \):

\[
\int d\alpha f(\alpha) \delta(\alpha) := \int d\alpha_* f(\alpha_*) \delta(\alpha_*) = f(0),
\]

and for functions on \( G_1 \), we can see that \( \delta(\alpha) := \alpha \), as can be seen in (3.7).
CHAPTER 3. THE GROUP UOSP(1|2)

3.2 Super Lie algebra \( \mathfrak{osp}(1|2) \)

The algebra \( \mathfrak{osp}(1|2) \) is a super Lie algebra [13, 55]. There is a parity function defined on \( \mathfrak{osp}(1|2) \) which divides its elements into *even* and *odd* subsets:

\[
\lambda(X) = \begin{cases} 
0 & \text{if } X \in \mathfrak{osp}(1|2)_0 \\
1 & \text{if } X \in \mathfrak{osp}(1|2)_1.
\end{cases}
\]  

(3.9)

The set \( \mathfrak{osp}(1|2)_0 \sim \mathfrak{su}(2) \) contains three generators \( J_1, J_2, J_3 \), while the set \( \mathfrak{osp}(1|2)_1 \) contains two generators \( Q_{\pm} \).

We define a bracket on this algebra by:

\[
[X_1, X_2] = (-1)^{\lambda(X_1)\lambda(X_2)+1}[X_2, X_1]
\]  

(3.10)

and which satisfies a super Jacobi identity,

\[
(-1)^{\lambda(X_1)\lambda(X_2)}[X_1, [X_2, X_3]] + (-1)^{\lambda(X_2)\lambda(X_3)}[X_2, [X_3, X_1]] \\
+ (-1)^{\lambda(X_3)\lambda(X_1)}[X_3, [X_1, X_2]] = 0.
\]  

(3.11)

All together, the generators satisfy the algebra:

\[
[J_3, J_\pm] = \pm iJ_\pm, \quad [J_+, J_-] = 2iJ_3, \\
[J_3, Q_\pm] = \pm \frac{i}{2}Q_\pm, \quad [J_\pm, Q_\pm] = 0, \quad [J_\pm, Q_\mp] = iQ_\pm, \\
[Q_\pm, Q_\mp] = \mp \frac{i}{2}J_3
\]  

(3.12)

We define a *supertranspose* operation \( \dagger \) on \( \mathfrak{osp}(1|2; \mathbb{C}) \), which has the properties:

\[
(c_1 X_1 + c_2 X_2)\dagger = c_1 X_1\dagger + c_2 X_2\dagger, \\
[X_1, X_2]\dagger = (-1)^{\lambda(X_1)\lambda(X_2)}[X_2, X_1]\dagger, \\
(X\dagger)\dagger = (-1)^{\lambda(X)}X\dagger,
\]  

(3.13a–c)

where \( X_1 \in \mathfrak{osp}(1|2) \) and \( c_i \in \mathbb{C} \). There are two such operations for the generators of \( \mathfrak{osp}(1|2) \):

\[
J_1\dagger = -J_1, \quad Q_1\dagger = (-1)^{\epsilon}Q_-, \quad Q_2\dagger = (-1)^{\epsilon+1}Q_+, 
\]  

(3.14)

for \( \epsilon = 0, 1 \).

Finally, on

\[
\mathfrak{osp}(1|2; \mathcal{G}) = \mathfrak{osp}(1|2; \mathcal{G}_0)_0 \oplus \mathfrak{osp}(1|2; \mathcal{G}_1)_1
\]  

(3.15)

we define a *grade adjoint* operation \( \square \) by

\[
(\alpha_1 X_1 + \alpha_2 X_2)\square = \alpha_1 \square X_1 + \alpha_2 \square X_2.
\]  

(3.16)

For our purposes, we confine to a subalgebra \( \mathcal{G} \subset \mathcal{G} \), such that every element of \( \mathfrak{uosp}(1|2) := \mathfrak{osp}(1|2; \mathcal{G}) \) satisfies \( X\square = -X \). Therefore, depending on the choice of grade adjoint operation, the elements are of the form:

\[
X = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3 + (-1)^{\epsilon}\alpha \square Q_+ + \alpha \square Q_-, 
\]  

(3.17)

where \( \alpha_i \square = \alpha_i \) and \( (\alpha \square)^\square = -\alpha \).
Representations

Primarily, the representations of \( \mathfrak{uosp}(1|2) \) are labelled by a half-integer \( j \) and a parity \( \lambda \in \{0, 1\} \). One has a certain freedom as to the inner product one chooses for a given representation, which is parametrised by two more numbers \( \rho, \tau \in \{0, 1\} \). We shall denote such a representation by: \( R^{j, \lambda, \rho, \tau} \). These representations can be decomposed over the even subalgebra: \( \mathfrak{uosp}(1|2)_0 \cong \mathfrak{su}(2) \).

Each representation \( R^{j, \lambda, \rho, \tau} \) of \( \mathfrak{uosp}(1|2) \) comprises of the direct sum of two representations of \( \mathfrak{su}(2) \).

\[
R^{j, \lambda, \rho, \tau} = V^j \oplus V^{j+1, \lambda, \rho, \tau}.
\]

A generic basis element is \( |j; k, m\rangle \), where \( k \in \{j, j - \frac{1}{2}\} \), \( m \in \{-k, -k + 1, \ldots, k\} \) and we have suppressed the labels \( \lambda, \rho, \tau \) for simplicity. The action of the operators on the representation \( R^{j, \lambda, \rho, \tau} \) is:

\[
J_3 |j; k, m\rangle = i m |j; k, m\rangle,
\]

\[
J_\pm |j; j, m\rangle = i \sqrt{(j + m)(j + m + 1)} |j; j, m \pm 1\rangle,
\]

\[
J_\pm |j; j - \frac{1}{2}, m\rangle = i \sqrt{(j - \frac{1}{2} + m)(j + \frac{1}{2} + m)} |j; j - \frac{1}{2}, m \pm 1\rangle,
\]

\[
Q_\pm |j; j, m\rangle = \mp \frac{1}{2} \sqrt{j + m} |j; j, m \pm \frac{1}{2}\rangle,
\]

\[
Q_\pm |j; j - \frac{1}{2}, m\rangle = - \frac{1}{2} \sqrt{j + \frac{1}{2} + m} |j; j, m \pm \frac{1}{2}\rangle,
\]

where \( J_\pm := J_1 \pm i J_2 \) and \( |j; k, m\rangle \) has parity \( \lambda + 2(j - k) \).

![Structure of the \( R^j \) representation of \( \mathfrak{osp}(1|2) \).](image)

As can be deduced by fig. 3.1, the \( \mathfrak{su}(2) \) generators \( J_{\pm, 3} \) act on each level while the supersymmetric generators \( Q_{\pm} \) allow one to go from one “level” to the other.

The inner product, \( \Phi^{(\rho, \tau)}(\cdot, \cdot) \), on such a representation is defined by:

\[
\Phi^{(\rho, \tau)}(|j; k, m\rangle, |j; k', m'\rangle) := (-1)^{2(j-k)+\rho+\tau} \delta^{km} \delta^{m'm} = (-1)^{\rho} \delta^{kk'} \delta^{mm'},
\]

(3.20)
where we define \( \varphi := 2(j - k)\rho + \tau \) for later convenience. One can show that there is a consistency relation among \( \epsilon, \lambda \) and \( \rho \):

\[
\epsilon + \lambda + \rho + 1 \equiv 0 \pmod{2}. \tag{3.21}
\]

Therefore, once two of these parameters are chosen, the final one is fixed.

From a spin-statistics viewpoint, we would like to endow the integer representations with even parity and the half-odd-integer representations with odd parity, that is, \( \lambda \equiv 2j \pmod{2} \). Furthermore, in the near future, we shall wish to define a measure on the supergroup that can be used to integrate all representation functions. Thus, we must have just one definition of grade adjoint; we shall choose \( \epsilon = 0 \). We conclude that \( \rho \equiv 2j + 1 \pmod{2} \). Ultimately, we are free with our choice of the overall sign \( \tau \), but we must include both choices.

The reason for this will appear shortly when we consider tensor products of representations. In fact, these choices mean that for integer representations, one does not acquire a positive definite inner product on the representation space:

\[
\langle j, \tau; k, m | j, \tau; k', m' \rangle = (-1)^\varphi \delta_{kk'} \delta_{mm'} \quad \text{for } j \in \mathbb{N},
\]

\[
\quad = \begin{cases} 
(-1)^{2(j-k) + \tau} \delta_{kk'} \delta_{mm'} & \text{for } j \in \mathbb{N}, \\
(-1)^\tau \delta_{kk'} \delta_{mm'} & \text{for } j \in \mathbb{N} + \frac{1}{2} 
\end{cases} \tag{3.22}
\]

Furthermore, the tensor product of two representations of \( uosp(1|2) \) satisfies a rule analogous to that of \( su(2) \), except that the sum over \( j \) goes in half-integer steps:

\[
R^{j_1, \tau_1} \otimes R^{j_2, \tau_2} = \bigoplus_{|j_1 - j_2| \leq j_3 \leq j_1 + j_2} R^{j_3(\tau_1, \tau_2)}. \tag{3.23}
\]

Using the properties of the inner product on the representation space, we find that:

\[
\tau_3(\tau_1, \tau_2) = \tau_1 + \tau_2 + 2(j_1 + j_2 + j_3)\lambda_3 + \lambda_1\lambda_2. \tag{3.24}
\]

Thus, we see our initial requirement that we include both values of \( \tau \) is justified; we cannot restrict to one particular choice of \( \tau \) since we will obtain both under tensor composition. We also choose that in the matrix realization:

\[
{j, \tau}T^{(km)}_{(ln)}(X) := \langle j, \tau; k, m | X | j, \tau; l, n \rangle 
= \left(\begin{array}{c|c}
\text{uosp}(1|2)_0 - \text{uosp}(1|2)_0 & \text{uosp}(1|2)_0 - \text{uosp}(1|2)_1 \\
\text{uosp}(1|2)_1 - \text{uosp}(1|2)_0 & \text{uosp}(1|2)_1 - \text{uosp}(1|2)_1 \\
\end{array}\right). \tag{3.25}
\]

The supertrace of a matrix operator \( {j, \tau}M^{(km)}_{(ln)} \) (in the representation \( R^{j, \tau} \)) is defined as:

\[
\text{Str}({j, \tau}M) = \sum_{k,m} (-1)^{\lambda + 2(j-k)} {j, \tau}M^{(km)}_{(km)} = \sum_{k,m} (-1)^{2k} {j, \tau}M^{(km)}_{(km)}. \tag{3.26}
\]
This means that the supertrace of the identity operator is: \( \text{Str}(\mathbf{1}) = (-1)^{2j}. \)

With these choices, the matrix elements of the generators in the fundamental representation \( R^{1,0} = V^0 \oplus V^1 \) are:

\[
J_1 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
Q_+ = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_- = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

(3.27)

The supertrace in the fundamental representation is:

\[
\text{Str}(J_i J_j) = \frac{1}{2} \delta_{ij}, \quad \text{Str}(J_i Q_A) = 0, \quad \text{Str}(Q_A Q_B) = \frac{1}{2} \epsilon_{AB}.
\]

(3.28)

The measure over the algebra is:

\[
\text{d}B = \text{d}b_1 \text{d}b_2 \text{d}b_3 \text{d}b^\square \text{d}b,
\]

(3.29)

where \( B = b_i J_i + b^\square Q_+ + b Q_- \).

### 3.3 Super group UOSP(1|2)

Elements of UOSP(1|2) have the form:

\[
g = u \xi, \quad \text{where} \quad u = e^{\theta \bar{n} J} \quad \text{and} \quad \xi = e^{\eta Q_+ + \eta Q_-},
\]

(3.30)

and \( \bar{n} = (\sin \psi \cos \phi, \sin \psi \sin \phi, \cos \psi) \). More explicitly:

\[
u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta + \frac{i}{2} \sin \theta \cos \psi & \frac{i}{2} \sin \theta \sin \psi e^{-i\phi} \\ 0 & \frac{i}{2} \sin \theta \sin \psi e^{i\phi} & \cos \theta - \frac{i}{2} \sin \theta \cos \psi \end{pmatrix}
\]

(3.31)

and

\[
\xi = \begin{pmatrix} 1 + \frac{1}{2} \eta \square \eta & \frac{i}{2} \eta \square \eta & -\frac{1}{2} \eta \square \eta \\ -\frac{1}{2} \eta \square \eta & 1 - \frac{1}{2} \eta \square \eta & 0 \\ -\frac{1}{2} \eta \square \eta & 0 & 1 - \frac{1}{2} \eta \square \eta \end{pmatrix}
\]

(3.32)

The spinor indices follow the north-west convention so that \( \phi^A = \epsilon^{AB} \phi_B \) and \( \phi_A = \phi^B \epsilon_{BA} \). The metric on the spinor space is the anti-symmetric tensor \( \epsilon_{AB} \) with \( \epsilon_{++} = \epsilon_{+-} = 1 \). This implies \( \epsilon^{AB} \epsilon_{BC} = -\delta^A_C \).
Thus, we arrive at:

\[
g = \begin{pmatrix}
1 + \frac{1}{4} \eta^\Box \eta & \frac{1}{2} \eta & -\frac{1}{2} \eta^\Box \\
g_{21} & (1 - \frac{1}{8} \eta^\Box \eta) u_{22} & (1 - \frac{1}{8} \eta^\Box \eta) u_{23} \\
g_{31} & (1 - \frac{1}{8} \eta^\Box \eta) u_{32} & (1 - \frac{1}{8} \eta^\Box \eta) u_{33}
\end{pmatrix}
\]

(3.33)

with

\[
g_{21} = -\frac{1}{2} \eta^\Box u_{22} - \frac{1}{2} \eta u_{23} \quad \text{and} \quad g_{31} = -\frac{1}{2} \eta^\Box u_{32} - \frac{1}{2} \eta u_{33}.
\]

(3.34)

Interestingly, elements of group UOSP(1|2) satisfy the relations \( g^\dagger g = gg^\dagger = \mathbb{I} \) and \( g^\dagger \zeta g = \zeta \), where \( \zeta := \text{diag}(1_{1\times1}, 1_{2\times2}) \). From the second relation, we can see the origin of the description orthosymplectic. Note here that the grade adjoint and supertranspose are respectively:

\[
g^\dagger = \begin{pmatrix}
1 + \frac{1}{4} \eta^\Box \eta & g_{31} & -g_{21} \\
\frac{1}{2} \eta & (1 - \frac{1}{8} \eta^\Box \eta) u_{33} & -(1 - \frac{1}{8} \eta^\Box \eta) u_{23} \\
\frac{1}{2} \eta & -(1 - \frac{1}{8} \eta^\Box \eta) u_{32} & (1 - \frac{1}{8} \eta^\Box \eta) u_{33}
\end{pmatrix}
\]

(3.35)

and

\[
g^{\dagger} = \begin{pmatrix}
1 + \frac{1}{4} \eta^\Box \eta & -g_{21} & -g_{31} \\
\frac{1}{2} \eta & (1 - \frac{1}{8} \eta^\Box \eta) u_{23} & (1 - \frac{1}{8} \eta^\Box \eta) u_{23} \\
-\frac{1}{2} \eta^\Box & (1 - \frac{1}{8} \eta^\Box \eta) u_{23} & (1 - \frac{1}{8} \eta^\Box \eta) u_{22}
\end{pmatrix}.
\]

(3.36)

**Representations**

The representation matrices of the group elements are denoted by

\[
\begin{align*}
\tau^j, T^{(km)}(n) & = \tau^j, T^{(km)}(n) (\Omega, \eta^\Box, \eta), \\
\end{align*}
\]

where \( \Omega = \{ \psi, \theta, \phi \} \), and have elements:

\[
\begin{align*}
\tau^j T^{(jm)}(j, n) & = (-1)^{\tau} \left( 1 - \frac{1}{4} j \eta^\Box \eta \right)^j D^m_n(\Omega), \\
\tau^j T^{(jm)}(j, -\frac{1}{2} n) & = (-1)^{\tau} \left[ -\frac{1}{2} \sqrt{j + n + \frac{1}{2}} \eta^\Box j D^m_{n + \frac{1}{2}}(\Omega) \\
& \quad -\frac{1}{2} \sqrt{j - n + \frac{1}{2}} \eta^\Box j D^m_{n - \frac{1}{2}}(\Omega) \right], \\
\tau^j T^{(jm)}(j, \frac{1}{2} n) & = (-1)^{\prime + \tau} \left[ -\frac{1}{2} \sqrt{j + n + \frac{1}{2}} \eta^\Box (j + \frac{1}{2}) D^m_{n + \frac{1}{2}}(\Omega) \\
& \quad +\frac{1}{2} \sqrt{j + n + \frac{1}{2}} \eta^\Box (j + \frac{1}{2}) D^m_{n - \frac{1}{2}}(\Omega) \right].
\end{align*}
\]

(3.37a, 3.38a, 3.38b, 3.38c)
where $\rho = 2j + 1$ and $D^j(\Omega)$ is the $j$th representation of the SU(2) group element pertaining to $\Omega$. Matrix multiplication, complex conjugation and the grade adjoint operation satisfy the following relations:

$$j, \tau T^{(km)}(l_n)(g) = j, \tau T^{(km)}(l_n)(g_1g_2) = j, \tau T^{(km)}(k'm')(g_1) j, \tau T^{(k'm')}(l_n)(g_2)$$

(3.39a)

$$j, \tau T^{(km)}(l_n)(g) \Box = j, \tau T^{(km)}(l_n)(g) \Box = j, \tau T^{(km)}(l_n)(g)$$

(3.39b)

$$j, \tau T^{(km)}(l_n)(g)^\dagger = j, \tau T^{(km)}(l_n)(g)^\dagger = j, \tau T^{(ln)}(l_n)(g)$$

(3.39c)

Indices are raised and lowered using a metric and its inverse:

$$J_r^{(km)}(l_n) = (-1)^{2(j-k)(2j+1)+k-m} \delta^{kl} \delta^{kn} ,$$

(3.40)

$$J_r^{(km)}(l_n) = (-1)^{2(j-l)(2l+1)+l-n} \delta^{kl} \delta^{kn} .$$

The representation functions are orthogonal:

$$\int_{UOSP(1/2)} dg \ j_1, \tau T^{(k_1,m_1)}(l_1,n_1)(g) j_2, \tau T^{(k_2,m_2)}(l_2,n_2)(g) \delta^{j_1j_2} \delta^{\tau_1,\tau_2} \delta^{\omega(k_1,m_1)} \delta^{\omega(k_2,m_2)} = \delta^{j_1j_2} \delta^{\tau_1,\tau_2} \delta^{\omega(k_1,m_1)} \delta^{\omega(k_2,m_2)} .$$

(3.41)

### 3.4 Super \{3j\}-symbols

We can derive the Clebsch-Gordan coefficients quite easily from relations given above [17]. The UOSP(1/2) coefficients are defined as:

$$|j_3(j_1,j_2), \tau_3(\tau_1,\tau_2); k_3, m_3\rangle = \tilde{I}^{j_3}_{j_1,j_2}(k_3,m_3)
\times [ |j_1,\tau_1; k_1, m_1\rangle \otimes |j_2,\tau_2; k_2, m_2\rangle ] .$$

(3.42)

where the coefficients are:

$$\tilde{I}^{j_3}_{j_1,j_2}(k_3,m_3) = B_{k_1k_2k_3}^{j_1j_2j_3} C_{m_1m_2m_3}^{k_1k_2k_3} ,$$

(3.43)

where

$$C_{m_1m_2m_3}^{k_1k_2k_3} := \langle k_1, m_1 | \otimes \langle k_2, m_2 | |k_3, m_3\rangle$$

(3.44)

defines SU(2) Clebsch-Gordan coefficients and $B_{k_1k_2k_3}^{j_1j_2j_3}$ are some factors we will see below.

Before the explicit calculus of the interwiners, we consider the implication of half integers values the $j_3$ may assume. First, we have that either case holds $j_1 + j_2 + j_3 \in \mathbb{N}$ or $j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2}$. We consider the isospin decomposition of the representation $R^j = V^{j,\lambda,\rho,\tau} \oplus V^{j,\lambda,\rho,\tau-\frac{1}{2}}$ therefore, from the properties of the SU(2) intertwiner $I^{j_1,j_2,j_3} : V^{j_1} \otimes V^{j_2} \rightarrow V^{j_3} \oplus V^{j_3-\frac{1}{2}}$, the image is $V^{j_3}$ or $V^{j_3-\frac{1}{2}}$ depending on whether $j_1 + j_2 + j_3$ is an integer or not.

Thus, there are two different situations:
1. \( j_1 + j_2 + j_3 \in \mathbb{N} \):
   The only non-vanishing components are \( V^{j_1} \otimes V^{j_2} \rightarrow V^{j_3} \), which is the usual \( \mathfrak{su}(2) \) intertwiner and its supersymmetric counterparts \( V^{j_1+\frac{1}{2}} \otimes V^{j_2} \rightarrow V^{j_3+\frac{1}{2}} \) and \( V^{j_1} \otimes V^{j_2+\frac{1}{2}} \rightarrow V^{j_3+\frac{3}{2}} \).

2. \( j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2} \):
   The only non-vanishing components are \( V^{j_1+\frac{1}{2}} \otimes V^{j_2+\frac{1}{2}} \rightarrow V^{j_3+\frac{1}{2}} \), and its supersymmetric counterparts \( V^{j_1+\frac{1}{2}} \otimes V^{j_2} \rightarrow V^{j_3+\frac{1}{2}} \), \( V^{j_1+\frac{1}{2}} \otimes V^{j_2+\frac{1}{2}} \rightarrow V^{j_3+\frac{3}{2}} \) and \( V^{j_1} \otimes V^{j_2+\frac{1}{2}} \rightarrow V^{j_3} \).

In fig. 3.2 the reader can see a graphical representation of these two cases:

![Diagram]

Figure 3.2: Intertwiner \( \tilde{I}^{j_1,j_2,j_3} : R^{j_1} \otimes R^{j_2} \rightarrow R^{j_3} \) invariant under \sp(1|2): the decomposition on isospins \( V^{j_1} \) (solid lines) and \( V^{j_1-1/2} \) (dotted lines) in the two cases \( j_1 + j_2 + j_3 \in \mathbb{N} \) and \( j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2} \).

The values of factors \( \tilde{B}_{k_1,k_2,k_3}^{j_1,j_2,j_3} \) can be separated as well, depending on \( j_1 + j_2 + j_3 \in \mathbb{N} \) or \( j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2} \). When the sum is integral, we have

\[
\tilde{B}_{j_1,j_2,j_3}^{j_1,j_2,j_3+j_3} = \sqrt{\frac{j_1 + j_2 + j_3 + 1}{2j_3 + 1}},
\]

\[
\tilde{B}_{j_1+\frac{1}{2},j_2,j_3}^{j_1,j_2,j_3} = (-1)^{j_1} \sqrt{\frac{-j_1 + j_2 + j_3}{2j_3}},
\]

\[
\tilde{B}_{j_1,j_2+\frac{1}{2},j_3}^{j_1,j_2,j_3} = \sqrt{\frac{j_1 - j_2 + j_3}{2j_3}},
\]

\[
\tilde{B}_{j_1,j_2,j_3-\frac{1}{2}}^{j_1+1,j_2,j_3} = (-1)^{j_1+1} \sqrt{\frac{j_1 + j_2 - j_3}{2j_3 + 1}}.
\]
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while if the sum is half-integral we have instead

\[
\tilde{B}^{j_1,j_2,j_3}_{j_1-j_2-j_3-\frac{1}{2}} = (-1)^{\lambda_1+1} \sqrt{\frac{j_1 - j_2 + j_3 + \frac{1}{2}}{2j_3 + 1}}.
\]

This means that:

\[
\begin{align*}
\left[ \langle j_1, \tau_1; k_1, m_1 \rangle \otimes \langle j_2, \tau_2; k_2, m_2 \rangle \right] \langle j_3(\lambda_1, \lambda_2, \lambda_3)\rangle_{(k_3 m_3)} & = (-1)^{(\lambda_1+2(j_1-k_1))+(\lambda_2+2(j_2-k_2))+\varphi_1+\varphi_2} I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)} \\
& = (-1)^{\lambda_1+2(j_1-k_1)} I^{\tilde{j_1},\tilde{j_2},\tilde{j_3}}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)},
\end{align*}
\]

These objects do not have simple transformation properties under permutation. On the other hand, the \(\{3j\}_{UOSP(1|2)}\)-symbols do (by definition):

\[
I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)} := B^{j_1,j_2,j_3}_{k_1 k_2 k_3} C^{k_1 k_2 k_3}_{m_1 m_2 m_3},
\]

where

\[
B^{j_1,j_2,j_3}_{k_1 k_2 k_3} := (-1)^{(\lambda_3+1)(2(j_2-k_2)+2(j_1+j_2+j_3))} \sqrt{2k_3 + 1} \tilde{B}^{j_1,j_2,j_3}_{k_1 k_2 k_3},
\]

and

\[
C^{k_1 k_2 k_3}_{m_1 m_2 m_3} := (-1)^{k_1-k_2-m_3} \sqrt{2k_3 + 1} C^{k_1 k_2 k_3}_{m_1 m_2 m_3},
\]

are the \(\{3j\}_{SU(2)}\)-symbols. Under permutation, they satisfy:

\[
I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)} = (|\sigma|) \sum_{k_4}^\lambda (2(j_a-k_a)(2j_a+1)+k_a)
\]

\[
\times I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)},
\]

where \(|\sigma| = \pm 1\) is the signature of the permutation and is similar to the SU(2) case. Reversing the magnetic indices, we find:

\[
I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)} = (-1)^{k_1+k_2+k_3} I^{j_1,j_2,j_3}_{(k_1 m_1)(k_2 m_2)(k_3 m_3)}.
\]

Also, we can raise and lower indices using the invariant metric on the representation space \(R^{\tilde{j}\tilde{k}}\) provided by:

\[
\left[ \langle j_1, \tau_1; k_1, m_1 \rangle \otimes \langle j_2, \tau_2; k_2, m_2 \rangle \right] |0,0;0,0\rangle = (-1)^{2(j_1-k_1)(2j_1+1)+k_1-m_1}
\]

\[
\times \delta^{j_1,j_2} \delta_{k_1 k_2} \delta_{m_1,-m_2}.
\]
CHAPTER 3. THE GROUP UOSP

This has been mentioned already in (3.40). Additionally, they satisfy a pseudo-orthogonality relation:

\[
\sum_{k_1,k_2} (-1)^{\lambda_1+2(j_1-k_1)}(\lambda_2+2(j_2-k_2)) + \varphi_1 + \varphi_2 \, \Theta \, \frac{j_1}{(k_1,m_1)} \frac{j_2}{(k_2,m_2)} \frac{j_3}{(k_3,m_3)} = (-1)^{\varphi_1, \varphi_2} \delta_{k_1,k_2} \delta_{m_1,m_2},
\]

where \( \varphi_3(\varphi_1, \varphi_2) = 2(j_3 - k_3)\rho_3 + \tau_3(\tau_1, \tau_2) \). This implies:

\[
\sum_{k_1,k_2,k_3} (-1)^{\Theta} \, \frac{j_1}{(k_1,m_1)} \frac{j_2}{(k_2,m_2)} \frac{j_3}{(k_3,m_3)} \frac{j_4}{(k_4,m_4)} = 1,
\]

where:

\[
\Theta := (\lambda_1+2(j_1-k_1))(\lambda_2+2(j_2-k_2)) + 2(j_3-k_3) + \varphi_1 + \varphi_2 + \varphi_3(\varphi_1, \varphi_2).
\]

As expected, \( \Theta \) is invariant under permutation.\(^2\)

3.5 Super \( \{6j\}\)-symbols

The supersymmetric \( 6j \)-symbol is defined as the matrix relating two ways of coupling three representations:

\[
R^{j_1,\tau_1} \otimes R^{j_2,\tau_2} \otimes R^{j_3,\tau_3} = \begin{cases} 
\sum_{j_5} (-1)^{(j_5+1)j_2+2j_4+2(\lambda_5+1)} \delta_{j_5,j_2} \delta_{j_5,j_4} R^{j_5(j_1,j_2,j_4),\tau_5(\tau_1,\tau_2,\tau_4)} & \text{if } j_5 = |j_1-j_2| \text{ or } j_5 = |j_1+j_2| \\
\sum_{j_5} (-1)^{(j_5+1)j_2+2j_4+2(\lambda_5+1)} \delta_{j_5,j_2} \delta_{j_5,j_4} R^{j_5(j_1,j_2,j_4),\tau_5(\tau_1,\tau_2,\tau_4)} & \text{if } j_5 = |j_1-j_2| \text{ or } j_5 = |j_2-j_4| 
\end{cases}
\]

The states are related by:

\[
|j_5(j_1,j_6(j_2,j_4),\tau_5(\tau_1,\tau_2,\tau_4)); k_5, m_5) = \sum_{j_3} (-1)^{(j_1+1)j_2+2j_4+2(\lambda_3+1)} \left[ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right] \times |j_5(j_3(j_1,j_2,j_4),\tau_5(\tau_1,\tau_2,\tau_4)); k_5, m_5),
\]

\(\Theta = \sum_{a=1}^3 (\lambda_a + 2j_a - k_a)(\lambda_{a+1} + 2(j_{a+1} - k_{a+1})) + \lambda_a \lambda_{a+1} + 2(j_a - k_a)(\lambda_a + 1),\)

\(\Theta = \sum_{a=1}^3 4k_a k_{a+1} + \lambda_a \lambda_{a+1} + 2(j_a - k_a)(2j_a + 1)\)
where:

\[ I = 2(j_1 + j_2 + j_4 + j_5), \]  

(3.60)

and

\[
\begin{align*}
I_{1,2} &= 2(j_1 + j_2 + j_3), & I_{1,6} &= 2(j_1 + j_6 + j_5), \\
I_{2,4} &= 2(j_2 + j_4 + j_6), & I_{3,4} &= 2(j_3 + j_4 + j_5),
\end{align*}
\]  

(3.61-3.62)

and \( \lfloor \frac{I}{2} \rfloor \) is the integer part of \( \frac{I}{2} \). This means:

\[
\begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} := \sum_{k_i, m_i \leq i \leq 6} \left( -1 \right)^{\sum_{a=1}^{6} (k_a - m_a) + 2(j_a - k_a)(\lambda_a + 1)} \times \prod_{i=1}^{6} \binom{k_{a_i} - m_{a_i}}{k_{a_i} - k_{a_i}} B_{k_{1i} k_{2i} k_{3i}} B_{k_{5i} k_{6i} k_{1i}}
\]  

(3.63)

The SU(2) \{6j\}-symbol is:

\[
\begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix} := (-1)^{\sum_{a=1}^{6} (k_a - m_a)} \prod_{a=1}^{6} \binom{k_{a_i} - m_{a_i}}{k_{a_i} - k_{a_i}} C_{m_{1i} m_{2i} m_{3i}}^{k_{1i} k_{2i} k_{3i}} C_{m_{5i} m_{6i} - m_{1i}}^{k_{5i} k_{6i} k_{1i}} \times C_{m_{4i} m_{4i} - m_{2i}}^{k_{4i} k_{4i} k_{2i}} C_{m_{4i} m_{5i} - m_{3i}}^{k_{4i} k_{5i} k_{3i}}
\]  

(3.64)

One finds that the supersymmetric \( \{6j\}\)-symbol has the same symmetry properties as its SU(2) counterpart.
Chapter 4

$N = 1$ supergravity and the super Ponzano-Regge model

In this chapter, we shall perform in detail the generalization of 3d $BF$ theory to $N = 1$ supersymmetric case. In the Ponzano-Regge model developed in Chapter 2, we took SU(2) $BF$ theory as our foundation and calculated the partition function. For the minimal supersymmetric extension of 3d gravity, one considers instead UOSP(1|2) $BF$ theory, that is, one utilizes $\mathfrak{uosp}(1|2)$-valued gauge fields.

We analyse the classical theory once more and calculate the partition function $Z_{\Delta, \text{UOSP}(1|2)}$ in an analogous manner since we have already catalogued the properties of the UOSP(1|2) representation theory in Chapter 3.1.

We shall see that the $Z_{\Delta, \text{UOSP}(1|2)}$ has the same structure of the $Z_{\Delta, \text{SU}(2)}$, i.e. a product of weights associated to the sub-simplices of the simplicial complex.

Once again, this super-$BF$ theory generalizes to any dimension and one can even describe supergravity theories if one implements the appropriate constraints [31].

As a brief note, this type of theory was first conceived in the context of a non-zero cosmological constant where it is equivalent to a super Chern-Simons theory devised by Achúcarro and Townsend [1, 2].

4.1 UOSP(1|2) $BF$ theory

Classical theory, equations of motion and symmetries

We update our analysis in Chapter 2 to 3d supergravity with zero cosmological constant. The action has the form of a (super) $BF$ theory:

$$S[\mathcal{B}, \mathcal{A}] = \int_{\mathcal{M}} \text{Str} (\mathcal{B} \wedge \mathcal{F}[\mathcal{A}]),$$

(4.1)
CHAPTER 4. N = 1 SUPERGRAVITY AND THE SUPER P-R MODEL

where in this case we have superfields: \( B \) is the supertriad, \( A \) is the superconnection, while \( F[A] = dA + A \wedge A \) is the supercurvature. As in the original model, the fields take values on the super Lie algebra \( \mathfrak{osp}(1|2) \), \( B \) and \( A \) are 1-forms while \( F[A] \) is a 2-form. \( \text{Str} \) is supertrace over the algebra as described in equations (3.28).

The variation of the action \( S[B, A] \) with respect \( B \) and \( A \) yields the equations of motion, that in the supersymmetric form are:

\[
F[A] = 0 \quad \text{and} \quad dA B = 0. \tag{4.2}
\]

Thus, at the supersymmetric level, the classical solutions are field configurations \((B, A)\) such that superconnection \( A \) is both superflat and supertorsion-free, that is, while supertriad \( B \) is compatible with \( A \).

In Section 3.2, we saw that the generators of \( \mathfrak{osp}(1|2) \) are \( J_1, J_2, J_3 \), for the set \( \mathfrak{osp}(1|2)_0 \) and \( Q_\pm \) for the set \( \mathfrak{osp}(1|2)_1 \). In order to understand how the fermionic degrees of freedom enter the theory we write supergravity fields in terms of these generators:

\[
B = E^i J_i + \phi^A Q_A, \quad A = W^i J_i + \psi^A Q_A, \tag{4.3}
\]

\( E \) (formerly denoted \( B \)) and \( W \) are the triad and connection, while \( \phi \) and \( \psi \) represent the fermion field. \( A \in \{\pm\} \) and \( i \in \{1, 2, 3\} \).

The action \( S[B, A] \) may be rewritten in terms of these variables as:

\[
S_{N=1}[E, W, \phi, \psi] = \int_M \text{Str} \left( E \wedge (F[W] + \psi \wedge \psi) + \phi \wedge dW \psi \right). \tag{4.4}
\]

From equations (4.3) and (4.4), we can see that this action describes a fermion field propagating on a manifold \( M \) endowed with a dynamical geometry.

We may also examine, the effect of this split (4.2) at the level of the equations of motion:

\[
F[W]^i + \frac{i}{2} (\sigma^i)^{AB} \psi^A \wedge \psi^B = 0, \quad \tag{4.5}
\]

\[
d\psi^A + \frac{i}{2} (\sigma_i)^{AB} W^i \wedge \psi^B = 0, \quad \tag{4.6}
\]

\[
dW E^i + \frac{i}{2} (\sigma^i)^{AB} \psi^A \wedge \phi^B = 0, \quad \tag{4.7}
\]

\[
d\phi^A + \frac{i}{2} (\sigma_i)^{AB} W^i \wedge \phi^B + \frac{i}{2} (\sigma_i)^{AB} E^i \wedge \psi^B = 0, \quad \tag{4.8}
\]

where \( (\sigma_i)^{AB} \) are the Pauli matrices. Going in descending order, we see that the curvature of the \( \mathfrak{su}(2) \) connection is non-vanishing, it picks up a contribution from the matter sector (4.5). Furthermore, the fermion field, acting as a source for the curvature, is covariantly constant with respect to the \( \mathfrak{su}(2) \) connection (4.6). Equation (4.7) states that \( \mathfrak{su}(2) \) connection is not torsion free, while (4.7) and (4.8) together, show that any change in the triad is compensated by a change in the fermionic fields. This complementary viewpoint shall come into play later in Chapter 6.
Gauge invariance

As one might expect, there is a supergauge invariance:

\[
\begin{align*}
B & \rightarrow B' = h^{-1} Bh, \\
A & \rightarrow A' = h^{-1} Ah + h^{-1} dh, \\
F & \rightarrow F' = h^{-1} Fh,
\end{align*}
\]

(4.9)

where \( h \in \text{OSP}(1\mid2) \) and a the supertranslation symmetry manifests itself as:

\[
\begin{align*}
B & \rightarrow B' = B + dA \eta, \\
A & \rightarrow A' = A, \\
F & \rightarrow F' = F,
\end{align*}
\]

(4.10)

where \( \eta \) is a \( \text{OSP}(1\mid2) \)-valued 0-form and requires that one employs the super-Bianchi identity \( dA F = 0 \).

One may once again relate this to a metric super-gravity theory, although we shall not enter the details, but the procedure is similar to that in the non-supersymmetric case.

Discretization of the manifold and fields

Again, the aim of the game is to rigorously define the path integral:

\[
Z_M = \int DA DB e^{iS_{N=1}[B,A]} \quad "=\quad \int DA \delta(F[A]).
\]

(4.11)

Firstly, we introduce the simplicial complex \( \Delta \) which replaces \( M \); \( \Delta^* \) the dual spin foam graph; \( \Delta^*_2 \) the dual 2-skeleton and \( \Delta^*_w \) its related wedge graph.

The classical fields \( B \) and \( A \) are again replaced by configurations that are distributional with support on subsimplices of \( \Delta \) and its topological dual \( \Delta^* \). We integrate these fields over the appropriate subsimplices. The definition of the integrated fields is:

\[
\begin{align*}
B & \rightarrow B_{w^*} = \int_{e \sim w^*} B \in \text{uosp}(1\mid2), \\
A & \rightarrow g_{e^*} = Pf_{e^*} A \in \text{UOSP}(1\mid2), \\
F[A] & \rightarrow G_{w^*} = \prod_{e^* \subset \partial w^*} \epsilon_{e^*} g_{e^*} \in \text{UOSP}(1\mid2),
\end{align*}
\]

(4.12)

where \( \epsilon_{(e^*, w^*)} = \pm 1 \) depending the relative orientation of \( e^* \) and \( w^* \).

All the considerations we made about the validity of such a substitution can be generalized to the supersymmetric case. For example the field \( B \) is replaced by the element \( B_{w^*} \) of \( \text{uosp}(1\mid2) \), defined on the the wedge \( w^* \). Just as SU(2) case the \( B \) variables attached to the wedges dual to a given edge \( e \in \Delta \) are all supergauge equivalent. Thus, they satisfy \( |B_{w^*_1}| = |B_{w^*_2}| \) if \( w^*_1, w^*_2 \sim e \). But this is the supersymmetric modulus \( |B_{w^*}|^2 := -\text{Str}(B_{w^*} B_{w^*}) \). The geometric
lengths need not equal: \(|E_w| \neq |E_w^*|\), which we shall see arise time and again in our analysis of the model.

The curvature \(\mathcal{F}[\mathcal{A}]\) is substituted by the holonomy of the discrete counterpart of the superconnection \(\mathcal{A}\) along the borders of the wedges \(w^*\); accordingly, it becomes a product of elements \(g^{w^*}_{\epsilon^*,f^*} \in \text{UOSP}(1|2)\).

**The state sum measure**

An important factor in rigorously defining the path integral is the measure; the discrete equivalent is:

\[
\mathcal{D}\mathcal{A}(x) \to \prod_{\epsilon_i^*} dg_{\epsilon_i^*} \prod_{\epsilon_i^*,f^*} dg_{\epsilon_i^*,f^*} \tag{4.13}
\]

where \(dg_{\epsilon_i^*}\) and \(dg_{\epsilon_i^*,f^*}\) are copies of the measure on the group \(\text{UOSP}(1|2)\) and take the generic form:

\[
dg = \frac{1}{\pi^2} (1 - \frac{1}{4}(\eta^\square \eta)) \sin^2 \theta \sin \psi \, d\theta \, d\psi \, d\phi \, d\eta^\square \, d\eta \tag{4.14}
\]

where \(\eta^\square, \eta\) are odd Grassmann variables while the rest are even and parameterise the SU(2) sub-group; we recognize in (4.14) the measure of the subgroup SU(2). We refer to Chapter 3.3 for more details. For the supertria field \(B\) we obtain the discrete measure:

\[
\mathcal{D}B(x) \to \prod_{w^*} dB_{w^*} \tag{4.15}
\]

where \(dB_{w^*}\) is the measure on a copy of \(\text{uosp}(1|2)\):

\[
dB = db_1 \, db_2 \, db_3 \, db^\square \, db \tag{4.16}
\]

the variables \(b_i\) with \(i = 1, 2, 3\) belong to the even part of the group while the variables \(b^\square, b\) are the odd Grassmann variables.

The discretization of the superfields that we have carried out so far corresponds exactly to the discretization of the fields we presented in Chapter 2; we therefore feel authorized to use the same approximation we made in (2.27) to obtain the \(BF\) action on the simplicial manifold \(\Delta\):

\[
S_{\Delta, \text{UOSP}(1|2)}[B_{w^*}, g_{\epsilon_i^*}, g_{\epsilon_i^*,f^*}] = \sum_{w^* \in \Delta^*} \text{Str}(B_{w^*} \cdot G_{w^*}) \tag{4.17}
\]

### 4.2 The Super Ponzano-Regge model

**Quantization**

Thus, the path integral for a discrete manifold takes the form:

\[
Z_{\Delta, \text{UOSP}(1|2)} = \int \prod_{\epsilon_i^*} dg_{\epsilon_i^*} \prod_{\epsilon_i^*,f^*} dg_{\epsilon_i^*,f^*} \prod_{w^*} dB_{w^*} \, e^{iS_{\Delta}[B_{w^*}, g_{\epsilon_i^*}, g_{\epsilon_i^*,f^*}]} \tag{4.18}
\]
To perform the integration over $dB^w$ explicitly, it is necessary to evaluate the supertrace:

$$\text{Str}(BG) = b_i p_i + b □ p □ + b p$$  (4.19)

where

$$\vec{p} = -\frac{1}{2} \left( 1 - \frac{1}{8} \eta □ \eta \right) \begin{pmatrix} i (u_{32} + u_{23}) \\ u_{32} - u_{23} \\ i (u_{22} - u_{33}) \end{pmatrix}$$  (4.20)

and the definitions of $u_{ij}$ are given in Section 3.3. Utilizing the Grassmann measures above:

$$\int dB^w e^i \sum_{w^*} (b^w, p^w, + b □ p □, + b p) \delta^3(\vec{p}^w) = (p^w, p^{w^*}) \delta^3(\vec{p}^{w^*})$$  (4.21)

Therefore, with some mathematical manipulation, the partition function (4.18) becomes:

$$\mathcal{Z}_{\Delta, UOSP(1|2)} = \int \prod_{c_i} dg_{c_i} \prod_{c_i^*} dg_{c_i^*} \prod_{w^*} (p^w, p^{w^*}) \delta^3(\vec{p}^{w^*})$$

$$= \int \prod_{c_i} dg_{c_i} \prod_{c_i^*} dg_{c_i^*} \prod_{w^*} \left[ \delta(p^w) \delta(p^{w^*}) \delta^3(\vec{p}^{w^*}) \right]$$

$$= \int \prod_{c_i} dg_{c_i} \prod_{c_i^*} dg_{c_i^*} \prod_{w^*} \left[ -\frac{1}{8} \eta □ \eta \theta_{w^*} (\cos \theta_{w^*} + 1) \right] \times \delta^3 (\sin \theta_{w^*}, \vec{n}_{w^*})$$  (4.22)

where $\sin \theta \vec{n}$ is the vector parameterizing the SU(2) subgroup.

As we can see above (and calculated explicitly in Appendix A.2), we find that the above integrand is peaked on $G_{w^*} = I$ only. This contrasts with the SU(2) case (see Section 2.3.6, equation (2.42)), where we noted that one must insert an appropriate $G_{w^*}$-dependent observable to kill a second peak for which the SU(2) holonomy is equal to $-I$ rather than $+I$. This observable is essentially the prefactor occurring here in (4.22). The fact that there is only one peak in the full supersymmetric $\delta$-function occurring here is a marked improvement on the non-supersymmetric theory.

Here, in our supersymmetric framework, it is the presence of matter itself that takes care of this issue. Intuitively, the equation of motion 4.6 imposes that the fermionic field $\psi$ has a trivial parallel transport, and thus distinguishes between $+I$ and $-I$ holonomies since we are dealing with spinors. Maybe such a feature can be generalised beyond the supersymmetric theory and we could conjecture that the SU(2) holonomy is necessarily peaked on $+I$ whenever fermions are present in the theory.
Performing once again a change of variables (see Appendix A.2), one arrives at the familiar form of the amplitude:

\[ Z_{\Delta, \text{UOSP}(1|2)} = \int \prod_{e}dg_{e}^{\ast} \prod_{c_{i}^{\ast}, f}dg_{c_{i}^{\ast}, f} \prod_{w^{\ast}} \delta(G_{w^{\ast}}). \]  

(4.23)

Moreover, once we integrate with respect to the \( g_{e}^{\ast} \) variables, we in essence glue the wedges within a face together and the amplitude becomes:

\[ Z_{\Delta, \text{UOSP}(1|2)} = \int \prod_{e}dg_{e}^{\ast} \prod_{f^{\ast}} \delta(G_{f^{\ast}}). \]  

(4.24)

This passage is completely analogous to the previous case we illustrated in Section 2.3.6. Let us highlight a supersymmetric subtlety once more. The wedges glued and thus their UOSP(1|2) representations equal: \( j_{1} = j_{2} \) for the two adjacent wedges \( w_{1}^{\ast} \) and \( w_{2}^{\ast} \) in the dual face. But this does not mean that the labels of the SU(2) modules which label subsets of their respective bases must equal. This is a reflection at the quantum level, in the representation theory, of the classical feature we mentioned earlier in Section 4.1. To state it once more, while the superlength of the supertriads attached to the wedges must equal: \( |B_{w_{1}^{\ast}}| = |B_{w_{2}^{\ast}}| \) if \( w_{1}^{\ast}, w_{2}^{\ast} \sim e \), the length of their associated triads need not: \( |E_{w_{1}^{\ast}}| \neq |E_{w_{2}^{\ast}}| \).

Moving on to the integration of the group elements \( g_{c_{i}^{\ast}}^{\ast} \), we know that for UOSP(1|2) we may decompose the \( \delta \)-function as:

\[ \delta(G) = \sum_{j \in \frac{1}{2}N} \text{Sdim}_{j} S\chi^{j}(G), \]  

(4.25)

where \( \text{Sdim}_{j} = (-1)^{2j} \) is the superdimension, and \( S\chi^{j}(G) \) is the supercharacter, that is, the supertrace of the representation matrix \( jT^{(k,m)}_{(l,n)}(G) \) (see section 3.2 for details).

### 4.2.1 Supersymmetric partition function

Having performed this Peter-Weyl decomposition on each \( \delta \)-function, we are left to manipulate the representation functions and integrate with respect to the group variables. This is necessarily an arduous task in the supergroup case since permuting matrix elements may introduce factors of \((-1)\) if the matrix element are both elements of the odd sector of the Grassmann algebra. However, proceeding intuitively in a similar way to the its non-supersymmetric relative, we obtain the following equation for the amplitude:\footnote{\( \lfloor x \rfloor \) is the floor function.}

\[ Z_{\Delta, \text{UOSP}(1|2)} = \sum_{(j)} (-1)^{2j_{e}} \prod_{f} (-1)^{\lfloor j_{e_{1}} + j_{e_{2}} + j_{e_{3}} \rfloor / 2} \prod_{t} \left[ \begin{array}{cccc} j_{e_{1}} & j_{e_{2}} & j_{e_{3}} \\ j_{e_{4}} & j_{e_{5}} & j_{e_{6}} \end{array} \right]_{t}. \]  

(4.26)
This has a similar form to (2.62) and can therefore be considered as a super Ponzano-Regge model. Note, if we consider the amplitudes of the sub-simplices of $\Delta$ individually, the edges, triangles and tetrahedra carry the weights: $(-1)^{2j_e}$ (the superdimension); $(-1)^{\lfloor j_{e_1} + j_{e_2} + j_{e_3} \rfloor}$ (the normalisation of the supersymmetric $\{3j\}$-symbol); and the supersymmetric $\{6j\}$-symbol, respectively.

In order to calculate amplitude (4.26), there also exists an efficient method to manage the bookkeeping of these factors of $(-1)$. It allows us to calculate the amplitude rigorously and is beneficial for the analysis of the model. It is a graphical calculus prevalent in some approaches to braided monoidal categories and we shall present it chapter 5.
Chapter 5

Diagrammatical calculus

We have seen in Chapters 2 and 4 that a spin foam amplitude generically factorises upon integration of the group variables and amplitudes may be assigned to sub-cells. We shall show here that we can perform this factorisation at a graphical level and evaluate the amplitude by considering simpler diagrams. Thus, we shall introduce some basic elements of a diagrammatical calculus in order to graphically evaluate the amplitude (4.26) and by extension, any quantity involving tensor products of group representations.

In the first section, we shall retain a certain level of generality and define the essential elements of our graphical calculus. We shall not present detailed checks of their consistency and the interested reader can find more details in [34, 39] and references therein.

After having prepared all the necessary tools, we shall be able to calculate some more complicated diagrams, which will arise later once we attempt to derive the various state sum amplitudes (4.26) and (2.62). The SU(2) case shall provide a consistency check for the graphical method.

5.1 Basic elements of diagrammatical calculus

First of all, let introduce some basic elements of the diagrammatic calculus for general vector spaces V, W. We denote their duals by V*, W*, respectively. A map f : V → W and its dual map f* : W* → V* are denoted by:

\[
\begin{array}{c}
V \xrightarrow{f} W \\
W \xleftarrow{f^*} V^*
\end{array}
\]

A diagram is always read from top to bottom. We note the direction of the
arrows on the diagram, downward arrows map to and from the vector spaces, while upward arrows pass to and from the duals. Composition of maps follows as one would expect. The identity maps \( \text{id}_V : V \to V \) and \( \text{id}_{V^*} : V^* \to V^* \) are drawn above also. Finally, we mention the crossing map \( \Psi_{V,W} : V \otimes W \to W \otimes V \).

In our case, we must define evaluation and coevaluation maps along with their duals:

\[
\begin{align*}
\text{coev}_V : \mathbb{C} &\to V \otimes V^*; \ 1 \mapsto \sum_m e_m \otimes f^m, \\
ev_V : V^* \otimes V &\to \mathbb{C}; \ f^m \otimes e_n \mapsto f^m(e_n) = \delta^m_n,
\end{align*}
\]

\[
\begin{align*}
= & \ \\
\text{coev}_V^*: V \otimes V^* &\to \mathbb{C}; \ \text{coev}_V^* := ev_v \circ \Psi_{V,V^*},
\end{align*}
\]

\[
\begin{align*}
= & \ \\
\text{ev}_V^*: V^* \otimes V &\to \mathbb{C}; \ \text{ev}_V^* := \Psi_{V^*,V} \circ \text{coev}_V,
\end{align*}
\]

where \( e_m \) is a basis for \( V \) with \( f^m \) its dual basis.

We are interested in orthogonal and symplectic vector spaces (since these play a role in both SU(2) and OSP(1,2)), that is, vector spaces endowed with either an orthogonal (symmetric, non-degenerate) metric \( \sigma_r \) or a symplectic (anti-symmetric, non-degenerate) metric \( \sigma_s \):

\[
\begin{align*}
\sigma_r : V \otimes V &\to \mathbb{C}; \\
\sigma_s : V \otimes V &\to \mathbb{C};
\end{align*}
\]

where the diagrams demonstrate the (anti-)symmetry. The existence of a non-degenerate metric allows us to define maps between the vector spaces and their duals, namely, the raising and lowering operators. The raising operator is given
by:
\[ \sharp : V \to V^* ; \quad v \mapsto \begin{cases} o^r(v, \cdot) , \\ s^r(v, \cdot) , \end{cases} \quad (5.1) \]

while the lowering operator \( \flat \) is defined such that \( \flat \circ \sharp = \mathbb{I}_V \) and \( \sharp \circ \flat = \mathbb{I}_{V^*} \). Thus from a map \( f : V \to W \), we can form another map \( f^\flat : V^* \to W^* ; f^\flat := f \circ \flat \).

Moreover, once such a metric has been defined, there is a natural explicit realisation of the crossing map:
\[ \Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v, \quad \text{where} \quad |v| = \begin{cases} 0 & \text{for } V \text{ orthogonal} , \\ 1 & \text{for } V \text{ symplectic} . \end{cases} \quad (5.2) \]

and similarly for \( w \in W \). Thus, we can now give a more explicit definition of the dual evaluation and coevaluation maps:

\[ \begin{align*}
\text{coev}_V^\flat(e_m \otimes f^n) & = (-1)^{|e_m||f^n|} f^n(e_m) = (-1)^{|e_m||f^n|} \delta^n_m , \\
\text{ev}_V^\flat(1) & = \sum_m (-1)^{|e_m||f^m|} f^m \otimes e_m .
\end{align*} \quad (5.3) \]

5.2 The cases of SU(2) and UOSP(1|2)

Now, let us specialise to the representation spaces of SU(2) and UOSP(1|2). It has been shown in [12] that in order to obtain a topological state-sum in the SU(2) case (the Ponzano-Regge model), one must choose the carrier spaces \( V^k, k \in \mathbb{N} \), to be orthogonal and the \( V^k, k \in \mathbb{N} + \frac{1}{2} \), to be symplectic. The representation spaces for UOSP(1|2) fit nicely in with this choice, since the \( R^j \) are endowed with an orthosymplectic metric \( osr \), that is, it is orthogonal on \( V^k \subset R^j, k \in \mathbb{N} \), and symplectic on \( V^k \subset R^j, k \in \mathbb{N} + \frac{1}{2} \). The metric and its inverse for the vector spaces of SU(2) in their standard bases are:

\[ \begin{align*}
k^{mn}_{r} & = (-1)^{k-m} \delta^{m+n,0} , \\
k^{rm}_{m} & = (-1)^{k-n} \delta^{m+n,0} . \quad (5.4) \end{align*} \]

Note that the orthogonal and symplectic nature of the metric is taken care of implicitly in the definition and we can drop the subscripts. In the course of this work, we consider the irreducible representations of SU(2) within the larger reducible representations \( R^j \), see 3.3, and we also may change the inner product on \( V^k \) by an overall sign:

\[ \begin{align*}
j^{mn}_{r} & = (-1)^{2(j-k)(2j+1)+k-m} \delta^{m+n,0} , \\
j^{rm}_{m} & = (-1)^{2(j-k)(2j+1)+k-n} \delta^{m+n,0} . \quad (5.5) \end{align*} \]

where we shall refer to the diagrammatic evaluations SU(2) with this choice of inner product as SÜ(2) evaluations. We have already stated the metric for UOSP(1|2) in (3.40).
5.2.1 Simple loop

We are ready to evaluate the simple loop diagram in the three different contexts: for SU(2) with the standard irreducible representations, for SU(2) with the altered inner product and for UOSP(1|2).

\[ \text{Simple loop diagram:} \quad \begin{array}{c}
\bigcirc \\
\end{array} \quad \Rightarrow \quad \text{coev}_V^\ast \circ \text{coev}_V. \]

We shall compute the SU(2) case explicitly:

\[ \text{coev}_{V^k}^\ast \circ \text{coev}_{V^k} : 1 \mapsto k e_m \otimes k f^m \]
\[ \mapsto \sum_m (-1)^k |e_m| |f^m| k f^m(k e_m) \]
\[ = (-1)^{2k} (2k + 1). \]

\[ (5.6) \]

Completing an identical calculation yields:

\[ \text{SU}(2) : \quad (-1)^{2k} (2k + 1) \]
\[ \text{UOSP}(1|2) : \quad (-1)^{2j}. \]

\[ (5.7) \]

Once we start coupling matter, see 6.2, we start seeing the appearance of loops on SU(2) with projector maps:

\[ k e_{\pm k} k f_{\pm k} : v \mapsto k e_{\pm k} k f_{\pm k}(v) = k e_{\pm k} v_{\pm k}. \]

Moreover, it is a projector so it does not matter how many times it occurs in a loop, the result is the same.

5.2.2 Theta

This amplitude labels the triangles of a tetrahedron, as we shall see in 6.1.1. This involves the definition of a new map:

\[ C_{k_1 k_2 k_3} : V^{k_1} \otimes V^{k_2} \otimes V^{k_3} \rightarrow \mathbb{C}, \]

\[ (5.8) \]

with components\( C_{m_1 m_2 m_3}^{k_1 k_2 k_3} \). Dualising and applying the lowering operator, one arrives at a map:

\[ \tilde{C}_{k_2 k_3 k_1} : \mathbb{C} \rightarrow V^{k_1} \otimes V^{k_2} \otimes V^{k_3}, \]

\[ (5.9) \]
which, in our cases, we know has components $\tilde{C}_{k_3 k_2 k_1}^{m_3 m_2 m_1} = C_{k_3 k_2 k_1}^{m_3 m_2 m_1}$. Thus, the amplitude for the theta diagram is:

$$\begin{align*}
\text{SU}(2) : & \quad C_{m_1 m_2 m_3}^{k_1 k_2 k_3} C_{k_3 k_2 k_1}^{m_3 m_2 m_1} = (-1)^{k_1 + k_2 + k_3}, \\
\text{SU}(2) : & \quad C_{m_1 m_2 m_3}^{k_1 k_2 k_3} C_{k_3 k_2 k_1}^{m_3 m_2 m_1} = (-1)^{\sum_{a=1}^{3}(2j_a - k_a + k_a)}, \\
\text{USP}(1|2) : & \quad f_{j_1}^{j_2 j_3} f_{j_2}^{j_3 j_1} f_{j_3}^{j_1 j_2} = (-1)^{|j_1 + j_2 + j_3|}.
\end{align*}$$

(5.10)

When one embeds the fermion diagrams, some edge amplitudes are no longer labelled by a loop but by a certain type of theta diagram. The following is such an example:

$$\begin{align*}
(+)\begin{array}{c}
\bigcirc \bigcirc \\
\bigcirc
\end{array} &= (-1)^{2j+1}(2j+1) C_j^j \frac{1}{2} \left(\frac{j}{2} + \frac{j}{2} \right) C_{\frac{j}{2} - \frac{j}{2}}^{\frac{j}{2} - \frac{j}{2}} \frac{1}{2} \frac{j}{2} = (-1)^{2j+1} \\
\text{since:} \quad C_j^j \frac{1}{2} \left(\frac{j}{2} + \frac{j}{2} \right) = (-1)^{2j+1} \frac{1}{2j+1}. 
\end{align*}$$

(5.11)

So we see that the clasp is just to counteract the factor of $\frac{1}{2j+1}$ in the denominator of the $\{3j\}$-symbol, see Eq.(3.50). There is also further types of diagram contributing to the bosonic sector:

$$\begin{align*}
(+)\begin{array}{c}
\bigcirc \bigcirc \\
\bigcirc
\end{array} &= (-1)^{2(j-k)(2j+1)} (-1)^{2k}(2k + 1).
\end{align*}$$

5.2.3 Tetrahedron

The tetrahedral diagram turns out to be rather simple contraction of four intertwiners:

\[^1\text{Remember that the fermion lines are missing due to the retracing identity.}\]
Following the trend set so far, when we couple fermionic Feynman diagrams, we shall find that some triangle amplitudes are in the form of a tetrahedral graph. For example:

occurs when we pass from representations in the upper module on the right to the lower on the left. Moreover:

There are factors of the square root of various dimensions, but there are factors of dimension multiplying each wedge at the beginning of the calculation (before integration) and there is exactly the correct factor left over to deal with this denominator.

5.2.4 Performing integrals graphically

The ultimate power of this formalism is that one can perform integrals over tensor product of group elements using the rule:

where the dots denote lines not explicitly drawn and:
denotes:

\[\tilde{\text{SU}}(2) : \int_{\text{SU}(2)} dg^k D^{m_1}_{n_1}(g) \cdots k^r D^{m_r}_{n_r}(g),\]

\[\text{OSP}(1|2) : \int_{\text{OSP}(2)} dg^j T^{(k_1 m_1)}_{(l_1 n_1)}(g) \cdots j^r T^{(k_r m_r)}_{(l_r n_r)}(g)\]

and

\[\tilde{\text{SU}}(2) : C^{k_2 \ldots k_r}_{m_1 \ldots m_r},\]

\[\text{OSP}(1|2) : I^{j_1 \ldots j_r}_{(k_1 m_1) \ldots (k_r m_r)},\]

where \(C\) and \(I\) is an intertwiner on the representation space of \(\text{SU}(2)\) and \(\text{UOSP}(1|2)\), respectively.

### 5.3 Graphical evaluation of the SU(2) and UOSP(1|2) amplitude

In Sections 2.3.6 and 4.2.1, we wrote the partition functions for the (super)symmetric Ponzano-Regge as:

\[Z_{\Delta, \text{SU}(2)} = \sum_{j_{r+1}} \prod_e (\text{dim}_{j_{r+1}}) \prod_f (-1)^{(j_1 \cdot j_2 + j_3)} \prod_t \left\{ \frac{j_1}{j_2} \frac{j_2}{j_3} \frac{j_3}{j_4} \frac{j_4}{j_5} \frac{j_5}{j_6} \right\} , \quad (5.13)\]

and:

\[Z_{\Delta, \text{UOSP}(1|2)} = \sum_{j_{r+1}} \prod_e (-1)^{2j_e} \prod_f (-1)^{(j_{e_1} + j_{e_2} + j_{e_3})} \prod_t \left[ \frac{j_{e_1}}{j_{e_2}} \frac{j_{e_2}}{j_{e_3}} \frac{j_{e_3}}{j_{e_4}} \frac{j_{e_4}}{j_{e_5}} \frac{j_{e_5}}{j_{e_6}} \right] . \quad (5.14)\]

respectively. Our aim here is to show that we can also represent the initial form of the amplitude graphically:

\[Z_{\Delta, G} = \int \prod_{e_i^1} dg_{e_i^1} \prod_{e_{i,f}} dg_{e_{i,f}} \prod_{w^*} \delta(G_{w^*}) \] (5.15)
and then proceed using just graphical manipulations to the final form of the amplitude (5.15); $G$ is the gauge group, SU(2) or UOSP(1|2) depending on the case.

In order to do so, one must go from the spin foam graph $\tilde{\Delta}^*_2$ to its corresponding circuit diagram, which we shall define presently.

Firstly, let us decompose the $\delta$-function on $G$ as in (2.49) to get:

$$Z_{\Delta, G} = \int \prod_{e^*_f} dg_{e^*_f} \prod_{e^*_f} dg_{e^*_f} \prod_{w^*} \sum_{j_{w^*} = 0, j_{w^*} \in \frac{1}{2}} \dim j_{w^*} \chi(G_{w^*}) \quad (5.16)$$

where the dimension and the character of the representations are appropriately defined for the specific group. Our main tool is to draw the dimension as a closed loop labelled by $j_{w^*}$ and the character as another closed loop labelled by $j_{w^*}$ and $G_{w^*}$. Thus, to every wedge of the wedge graph $\tilde{\Delta}^*_2$, we associate such loops. We must remember, however, that the element $G_{w^*}$ is actually a function of the fundamental variables $g_{e^*_f}$ and $g_{e^*_f}$. Furthermore, several wedges share each of these group elements, as we have illustrated in Chapter 2. In this event, we bind these loops together to form ‘wires’. In the end, the circuit diagram related to a wedge graph $\tilde{\Delta}^*_2$ is a collection of wires, the strands of which get rerouted amongst this collection.

Let us illustrate explicitly some parts of our wedge graph and their corresponding part in the circuit diagram. We start with the examination of a triangle $f$ of $\Delta$ and the wedges of $\tilde{\Delta}^*_2$ in its neighbourhood. Three dual faces $f^*$ intersect this triangle $f$ and thus there are six wedges, two per dual face. This sub-part of the wedge graph becomes a sub-part of the circuit diagram constituted of six loops, see Fig. 5.1. In this figure, we have also labelled explicitly a two-strand wire by $g_{e^*_{f^*}, f}$.

![Figure 5.1: The wedges of $\tilde{\Delta}^*_2$ in the neighbourhood of the triangle $f \in \Delta$ and their corresponding circuit diagram. We have labelled explicitly a two-strand wire by $g_{e^*_{f^*}, f}$.](image1)

One can also see some three-strand wires also and we have labelled one explicitly in Fig. 5.2. Here, we have drawn the 2-skeleton structure contained within a tetrahedron $t \in \Delta$. The six wedges become a set of six loops in this circuit diagram arranged in the following configuration:
We can see that there are four three-strand wires, one corresponding to each triangle. We should also mention that to avoid confusion we have only drawn the loops associated to the characters and left the loops associated to the dimension factors implicit. This shall be our general strategy but these factors are naturally essential to arrive at the correct form of the amplitude.

This finishes the construction of the circuit diagram for the amplitude (5.16). Then one can perform the integrals using the relations defined in Section 5.2.4. We start with the familiar integration over the $g_{e,f}$ variables. Our integration takes the graphical form given in Fig. 5.3 up to a normalization factor:

As we can see from Section 5.2.4, the missing normalization factor in this case in a factor of the dimension (diagrammatically, a loop which cancels one of the dimension factors taken implicitly already).

We move to the dual edge $e^*_t$ and the three wedges sharing it. The integral on this three-strand wire is graphically given in Fig. 5.4, again up to a normalization factor:
CHAPTER 5. DIAGRAMMATICAL CALCULUS

Figure 5.4: The graphical equivalent of performing the integration over $g_{e_f}$ (up to normalisation).

The normalisation factor missing here is a basic theta-diagram described earlier in the chapter. Then, we can use this information to evaluate the manipulate more complicated chunks of the circuit diagram, for example, that part associated to a tetrahedron drawn in Fig. 5.5:

Figure 5.5: The graphical manipulation of a typical part of the circuit diagram.

We see that each for tetrahedron $t \sim v^*$, there is an associated tetrahedral graph whose edges are labelled by the representations inherited from the edges of the tetrahedron $t \in \Delta$. We can also see from Fig. 5.5 that this part of the diagram also contributes a three-strand intertwiner to each triangle $f \sim e^*$. When one glues two tetrahedra $t \in \Delta$, however, one sees in the circuit diagram picture that one glues two such intertwiners to form a theta-diagram.

Finally, taking into account all the normalization factors (which we have left implicit to so far) the amplitude (5.16) becomes:

$$Z_{\Delta, G} = \sum_{\{j\}} \prod_{f^*} \prod_{e^*} \prod_{v^*}^{-1}$$

Note that in the amplitude $Z_{\Delta, G}$, the negative exponent in the dual edge amplitude comes from including the the normalization factors.

If we substitute the mathematical definitions into this diagrammatic formula, which we gave above for specific gauge groups, we obtain equations (5.13) and (5.14).
Chapter 6

Analysis of the super Ponzano-Regge model

In chapter 4, we derived the UOSP(1|2) spin foam model directly from a discrete path integral. Ultimately, the amplitude takes the form given in (4.26):

\[ Z_{\Delta, \text{UOSP}(1|2)} = \sum_{\{j\}} \prod_{e} (-1)^{2j_e} \prod_{f} (-1)^{|j_{e_1} + j_{e_2} + j_{e_3}|} \prod_{t} \left[ \begin{array}{ccc} j_{e_1} & j_{e_2} & j_{e_3} \\ j_{e_4} & j_{e_5} & j_{e_6} \end{array} \right] . \]  

(6.1)

Considering the amplitudes of the sub-simplices of \( \Delta \) individually, the edges, triangles and tetrahedra carry the weights: \((-1)^{2j_e}\) (the superdimension); \((-1)^{|j_{e_1} + j_{e_2} + j_{e_3}|}\) (the normalisation of the supersymmetric \(3j\)-symbol); and the supersymmetric \(6j\)-symbol, respectively.

Moreover, given a fixed triangulation \( \Delta \), the sum ultimately includes all admissible configurations of irreducible UOSP(1|2) representations attached to the edges of the triangulation. These configurations are labelled \( \{j_e\} \) with values \( j_e \in \mathbb{N}_2 \). In Section 3.4 we determined that, in order to be an admissible configuration, the representations must satisfy triangle inequalities but the familiar closure condition is relaxed. That is to say, if \( e_1, e_2, e_3 \in \partial f \), then:

\[ |j_a - j_b| \leq j_c \leq j_a + j_b, \quad \text{where } a, b, c \text{ are a permutation of } e_1, e_2, e_3. \]

but

\[ j_{e_1} + j_{e_2} + j_{e_3} \in \mathbb{N}, \quad \text{or} \quad j_{e_1} + j_{e_2} + j_{e_3} \in \mathbb{N} + \frac{1}{2}. \]  

(6.2)

The spin foam amplitude is a function of these representations, as is conventional.

From the classical standpoint, we derived this quantum amplitude beginning with an action displaying supersymmetric gauge invariance. But since the gauge group UOSP(1|2) is built upon the familiar SU(2) Lie group, there is a
nice SU(2) structure nested inside this overarching supersymmetric one. As we uncovered in Chapter 4, upon making this SU(2) structure explicit, one sees that this theory is one describing gravity coupled to Grassmann-valued spin-$\frac{1}{2}$ fields. Now that we have the quantum amplitude in a well-defined discrete setting, we expect that within the supersymmetric partition function lie amplitudes pertaining to gravity coupled to these spin-$\frac{1}{2}$ fermionic fields.

### 6.1 Analysis of the quantum amplitude

Perhaps some intuition for what might happen can be gained by examining a generic parallel transport matrix in a given representation $j_e$, namely

$$j_e T^{(k_e m_e)}(l_e, n_e)(g_{e^*}).$$

(6.3)

Since each tetrahedron contains its own frame of reference, this matrix describes the change in certain properties connected with the edge $e \in \Delta$ as one moves from one tetrahedron to the next along $e^* \in \Delta^*$. One such property is the length of the edge, $e$, as seen in each tetrahedron, which is given by the SU(2) sub-module (labelled by $k_e, l_e$) pertaining each tetrahedron. To spell it out,

$$j_e T^{(k_e m_e)}(l_e, n_e)(g_{e^*}) : V_{l_e}^{k_e} \rightarrow V_{l_e^*}^{k_e^*},$$

where $V_{l_e}^{k_e}$ are SU(2) representations, so that the length of the edge as viewed from the initial tetrahedron is $l_e + \frac{1}{2}$, while from the final tetrahedron it seems that the edge has length $k_e + \frac{1}{2}$. At a hand-waving level, the fact that different observers (here tetrahedra) see different lengths for the same edge comes from a non-vanishing torsion in the theory. Since both $k_e, l_e$ can take the values $j_e, j_e - \frac{1}{2}$ freely, see Section 3.2, the length of the edge may change from reference frame to reference frame.

Note, however, that the matrix elements fall into two classes. On the one hand there are the cases where the edge length does not change:

$$j_e T^{(j_e m_e)}(j_e, n_e)(g_{e^*}) \quad \text{and} \quad j_e T^{(j_e - \frac{1}{2} m_e)}(j_e - \frac{1}{2}, n_e)(g_{e^*}).$$

(6.4)

For example, let us examine:

$$j_e T^{(j_e m_e)}(j_e, n_e)(g_{e^*}) = (1 - \frac{1}{4} j_e \eta_e^\square \eta_e^*) j_e^{D m_e n_e} (\Omega_{e^*}),$$

(6.5)

where $\Omega = \{ \psi, \theta, \phi \}$. Each element is even in the Grassmann algebra. We propose that the $O((\eta_e^\square \eta_{e^*})^0)$ term corresponds to no-fermion propagation, while the $O((\eta_e^\square \eta_{e^*})^1)$ term corresponds to the propagation of both a fermion and an anti-fermion, which together yield a bosonic contribution.

On the other hand, there are the cases where the edge length does change:

$$j_e T^{(j_e m_e)}(j_e - \frac{1}{2}, n_e)(g_{e^*}) \quad \text{and} \quad j_e T^{(j_e - \frac{1}{2} m_e)}(j_e, n_e)(g_{e^*}).$$

(6.6)

\(^{1}\text{In pure gravity, the length an edge of the spin foam is given by } \frac{\dim k_e}{2} = k_e + \frac{1}{2}\)
6.1.1 Analysis of the UOSP

We shall see this prescription become more precise.

Expanding the UOSP (1) the embedded SU(2) substructure explicit at the level of representations, we identify the integer representation with even parity (bosons) and the half-integer with odd parity (fermions); we also sum over

\[ j_e T(j_e \cdot m_e) (g_{e'}), \]

where the appropriate definitions are given in section 3.4. Let us insist that although we sum over one label \( j_e \) per edge, we are also summing over one label \( k_{e,t} \) per edge \( e \) and per tetrahedron \( t \) to which the edge belongs.

In addition, we notice that, we the new definition (6.8) of the supersymmetric \{6j\} symbols in terms of SU(2) \{6j\} symbols, the amplitude associate to triangle \( f \) are now:
For future reference, let us scrutinise this amplitude.

There are a number of forms this can take depending on the values of \( k_e \) and \( k'_e \). To save space, let us denote the element \( k = j \) by \( \uparrow \) and \( k = j - \frac{1}{2} \) by \( \downarrow \). Thus,

\[
A^f_j (j_{e_1}, j_{e_2}, j_{e_3}; j_{e_1} - \frac{1}{2}, j_{e_2} - \frac{1}{2}, j_{e_3}) = A^f_j (\uparrow, \uparrow, \downarrow; \downarrow, \downarrow, \downarrow) .
\]

We note also that the amplitudes are symmetric with respect to the interchange of \( \{k_1, k_2, k_3\} \) and \( \{k'_1, k'_2, k'_3\} \). The possible configurations are (up to flipping entirely \( \{k_1, k_2, k_3\} \) with \( \{k'_1, k'_2, k'_3\} \)), for integral values \( j_{e_1} + j_{e_2} + j_{e_3} \in \mathbb{N} \):

\[
\begin{align*}
A^f_j (\uparrow, \uparrow, \uparrow; \uparrow, \uparrow, \downarrow) &= (-1)^j(j_{e_1} + j_{e_2} + j_{e_3} + 1) \\
A^f_j (\uparrow, \uparrow, \downarrow; \uparrow, \downarrow, \downarrow) &= (-1)^j(-j_{e_1} + j_{e_2} + j_{e_3}) \\
A^f_j (\downarrow, \downarrow, \uparrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(j_{e_1} - j_{e_2} + j_{e_3}) \\
A^f_j (\downarrow, \downarrow, \downarrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(j_{e_1} + j_{e_2} - j_{e_3}) \\
A^f_j (\uparrow, \uparrow, \downarrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_1} + 1) \\
&\quad \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + 1)(j_{e_1} + j_{e_2} + j_{e_3})} \\
A^f_j (\uparrow, \uparrow, \uparrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_2} + 1) \\
&\quad \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + 1)(j_{e_1} + j_{e_2} + j_{e_3})} \\
A^f_j (\uparrow, \uparrow, \downarrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_3} + 1) \\
&\quad \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + 1)(j_{e_1} + j_{e_2} + j_{e_3})} \\
A^f_j (\downarrow, \downarrow, \uparrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_1} + 1)(2j_{e_2} + 1) \\
&\quad \times \sqrt{(-j_{e_1} + j_{e_2} + j_{e_3})(j_{e_1} - j_{e_2} + j_{e_3})} \\
A^f_j (\downarrow, \downarrow, \downarrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_2} + 1)(2j_{e_3} + 1) \\
&\quad \times \sqrt{(-j_{e_1} + j_{e_2} + j_{e_3})(j_{e_1} + j_{e_2} - j_{e_3})} \\
A^f_j (\downarrow, \downarrow, \uparrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_3} + 1)(2j_{e_1} + 1) \\
&\quad \times \sqrt{(-j_{e_1} + j_{e_2} + j_{e_3})(j_{e_1} + j_{e_2} - j_{e_3})} \\
A^f_j (\downarrow, \uparrow, \downarrow; \downarrow, \downarrow, \downarrow) &= (-1)^j(2j_{e_2} + 1)(2j_{e_3} + 1) \\
&\quad \times \sqrt{(j_{e_1} - j_{e_2} + j_{e_3})(j_{e_1} + j_{e_2} - j_{e_3})}
\end{align*}
\]
and for half-integrals values \( j_{e_1} + j_{e_2} + j_{e_3} \in \mathbb{N} + \frac{1}{2} \):

\[
A^{(j)}_{J} (\downarrow, \downarrow; \uparrow, \downarrow) = (-1)^{J+\frac{1}{2}}(j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})
\]

\[
A^{(j)}_{J} (\downarrow, \uparrow; \downarrow, \downarrow) = (-1)^{J+\frac{1}{2}}(-j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})
\]

\[
A^{(j)}_{J} (\uparrow, \uparrow; \downarrow, \downarrow) = (-1)^{J+\frac{1}{2}}(j_{e_1} - j_{e_2} + j_{e_3} + \frac{1}{2})
\]

\[
A^{(j)}_{J} (\uparrow, \uparrow; \downarrow, \uparrow) = (-1)^{J+\frac{1}{2}}(j_{e_1} + j_{e_2} - j_{e_3} + \frac{1}{2})
\]

\[
A^{(j)}_{J} (\downarrow, \downarrow; \downarrow, \downarrow) = (-1)^{J+\frac{1}{2}+(2j_{e_1}+1)} \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})(-j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})}
\]

\[
A^{(j)}_{J} (\downarrow, \downarrow; \downarrow, \downarrow) = (-1)^{J+\frac{1}{2}} \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})(j_{e_1} - j_{e_2} + j_{e_3} + \frac{1}{2})}
\]

\[
A^{(j)}_{J} (\downarrow, \uparrow; \downarrow, \uparrow) = (-1)^{J+\frac{1}{2}+(2j_{e_1}+1)+(2j_{e_2}+1)} \times \sqrt{(j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})(-j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})}
\]

\[
A^{(j)}_{J} (\downarrow, \uparrow; \downarrow, \uparrow) = (-1)^{J+\frac{1}{2}+(2j_{e_1}+1)+(2j_{e_2}+1)} \times \sqrt{(-j_{e_1} + j_{e_2} + j_{e_3} + \frac{1}{2})(j_{e_1} + j_{e_2} - j_{e_3} + \frac{1}{2})}
\]

\[
A^{(j)}_{J} (\downarrow, \uparrow; \downarrow, \uparrow) = (-1)^{J+\frac{1}{2}+(2j_{e_3}+1)+(2j_{e_1}+1)} \times \sqrt{(j_{e_1} - j_{e_2} + j_{e_3} + \frac{1}{2})(j_{e_1} + j_{e_2} - j_{e_3} + \frac{1}{2})}
\]

(6.12)

where \( J = j_{e_1} + j_{e_2} + j_{e_3} \). There are 32 configurations in total (20 are shown here and the other 12 are obtained by utilizing the symmetry given above). These configurations split into two subsets depending on whether \( J \in \mathbb{N} \) or \( J \in \mathbb{N} + \frac{1}{2} \). Thus, 16 are admissible at any instance. Following on from what we mentioned just a little earlier, if the first three arrows do not differ from the second three, then the amplitude contributes to the even or ‘bosonic’ sector of the theory, while if there are two flips, then the amplitude contributes to the fermionic sector. Remember that there can not be one or three flips due to the parity condition for the existence of SU(2) intertwiners. Thus, the top four listed in each subset above are ‘bosonic’ amplitudes while the rest are ‘fermionic’ amplitudes.
Bosonic Amplitude

Let us examine a bosonic amplitude:

$$A_f^{(j)}(\uparrow, \downarrow; \uparrow, \downarrow; \uparrow, \downarrow; \uparrow, \downarrow) = (-1)^{j_3(j_1 + j_2 + j_3 + 1)}.$$  

(6.13)

The amplitude comes about from the coupling of the three quantities:

$$j_e T^{(j_1 m_{j_1})_{(j_1 n_{j_1})}}(g_{e'})\, j_e T^{(j_2 m_{j_2})_{(j_2 n_{j_2})}}(g_{e'})\, j_e T^{(j_3 m_{j_3})_{(j_3 n_{j_3})}}(g_{e'}) .$$

Therefore, as we can see from (6.5):

$$j_e T^{(j_e m_{j_e})_{(j_e n_{j_e})}}(g_{e'}) = (1 - \frac{1}{4} j_e \eta_{e'}^\square \eta_{e'}) j_e D_{m_{n_e}}(\Omega_{e'}),$$  

(6.14)

in the product of these three factors, the coefficient of $(\eta_{e'}^\square \eta_{e'})$ is $(j_1 + j_2 + j_3 + 1)$ (remembering that there is a $(\eta_{e'}^\square \eta_{e'})$ term in the measure). In fact, we interpret that this coefficient gets contributions from four different sources. There may be no-fermion propagation or fermion - anti-fermion propagation on the edge $e_1$ or the edge $e_2$ or the edge $e_3$ (which contribute the $1, j_1, j_2, j_3$ pieces, respectively).

Fermionic Amplitude

Now for a fermionic amplitude:

$$A_f^{(j)}(\uparrow, \downarrow; \uparrow, \downarrow; \uparrow, \downarrow; \uparrow, \downarrow) = (-1)^{j_e - \frac{1}{2} + (2j_1 + 1)} \times \sqrt{(j_1 + j_2 + j_3 + 1)(-j_1 + j_2 + j_3)}.$$  

(6.15)

Once again we see that this arises from coupling:

$$j_e T^{(j_1 m_{j_1})_{(j_1 n_{j_1})}}(g_{e'})\, j_e T^{(j_2 m_{j_2})_{(j_2 n_{j_2})}}(g_{e'})\, j_e T^{(j_3 m_{j_3})_{(j_3 n_{j_3})}}(g_{e'}) .$$

(6.16)

It is not possible to see how the coefficient of $\eta_{e'}^\square \eta_{e'}$ comes about directly, but for our interpretational purpose here, the coefficient gets a contribution from a fermion or anti-fermion propagation on both edges $e_2$ and $e_3$ while there is no-fermion propagation on edge $e_1$.

To conclude this section, while all the tetrahedra sharing an edge are colored with the same UOSP(1|2) representation $j_e$, they need not necessarily share the same SU(2) label $k_e$. This is to be expected since the SU(2)-modules lie within larger UOSP(1|2)-modules. Heuristically, we can divide the triangle amplitudes into two classes, differentiated by the condition $k_{e_e} = k^e_{e_e}$ for each edge of the triangle (keeping in mind that the triangle belongs to two tetrahedra). Those which satisfy this condition are ‘bosonic’, while those triangles for which this condition is not satisfied are ‘fermionic’. We shall now make this interpretation precise, by Fourier transforming to the space of functions on SU(2). It is rather difficult to do this succinctly, but we shall circumvent some of the clumsiness by jumping to the other side of the computation and working back. Indeed, it is more instructive to do so.
6.1.2 SU(2) spin foam amplitudes

The SU(2) Ponzano-Regge state sum amplitudes have the same fundamental building blocks as their UOSP(1|2) counterparts, 4.2.

\[ Z_{\Delta, SU(2)} = \sum_{\{k\}} \prod_{e} (-1)^{2k_e} (2k_e + 1) \prod_{f} (-1)^{(k_{e_1} + k_{e_2} + k_{e_3})} \times \prod_{t} \left\{ \frac{k_{e_1}}{k_{e_4}}, \frac{k_{e_2}}{k_{e_5}}, \frac{k_{e_3}}{k_{e_6}} \right\}. \]  

(6.17)

However, to make a connection with supersymmetric amplitudes, we must first rewrite this amplitude more appropriately to the supersymmetric context. As pointed out in [34], one has some freedom in the properties of the representations occurring in the decomposition of functions over the group. For a start, one may endow the representations with a \( \mathbb{Z}_2 \)-grading, so that vectors in \( V^{k_e}, k_e \in \mathbb{N} + \frac{1}{2} \) are odd and vectors in \( V^{k_e}, k_e \in \mathbb{N} \) are even. Moreover, one can choose the inner product on an irreducible representation to be either positive definite or negative definite. None of these possibilities affects the decomposition of the \( \delta \)-function.

Let us denote the usual characters by \( \chi \) and the grades ones by \( \chi^\pm \) where \( \pm \) labels the choice of inner product, then:

\[ \dim_k \chi^k(g) = \dim_{k, \pm} \chi^{k, \pm}(g) \]  

(6.18)

where \( \dim_{k, \pm} := \chi^{k, \pm}(I) \). So, we may write:

\[
\begin{align*}
\delta(g) &= \sum_k \dim_k \chi^k(g) = \sum_k \dim_{k, \pm} \chi^{k, \pm}(g) \\
&= \frac{1}{2} \sum_{j \in \mathbb{N} + \frac{1}{2}} \left( \dim_{j, +} \chi^{j, +}(g) + \dim_{j, -} \chi^{j, -}(g) \right) \\
&\quad + \frac{1}{2} \sum_{j \in \mathbb{N}} \left( \dim_{j, +} \chi^{j, +}(g) + \dim_{j, -} \chi^{j, -}(g) \right). 
\end{align*}
\]

(6.19)

Thus, instead of viewing the decomposition as a sum over irreducible representations \( V^k \), one can view it as decomposed over the representations \( R^j = V^j \oplus V^{j - \frac{1}{2}} \), where the representations are graded, and the inner product on \( V^j \subset R^j \) is positive for all \( j \), while that on \( V^{j - \frac{1}{2}} \subset R^j \) is positive for \( j \in \mathbb{N} + \frac{1}{2} \) and negative for \( j \in \mathbb{N} \). This choice is compatible the tensor product operation, and mimics the structure in the supersymmetric theory.

Thus, the SU(2) Ponzano-Regge spin foam amplitude may be recast as an amplitude depending on \( R^j \). Upon integration of the group elements in (6.1.2), we obtain the same diagrams as before, but the evaluation of the loop, theta, and tetrahedral diagrams depends on the grading and inner product, and we
find that we can write the amplitude as:

\[
Z_{\Delta, SU(2)} = \sum_{\{j\}} \sum_{\{k\}} \prod_{e} \left(-1\right)^{2j_e} (2k_e + 1) \prod_{f} \left(-1\right)^{\sum_{a=1}^{a=6} \left(2(j_a - k_a) + (2j_a + 1) + k_a\right)} \times \prod_{t} \left(-1\right)^{\sum_{a=1}^{a=6} \left(2(j_a - k_a) + (2j_a + 1)\right)} \left\{ \begin{array}{ccc}
  k_{e_1} & k_{e_2} & k_{e_3} \\
  k_{e_4} & k_{e_5} & k_{e_6}
\end{array} \right\}_t.
\] (6.20)

We must stress, however, that although the amplitude looks different, it is merely a repartitioning of the original state-sum. We evaluate the diagrams explicitly in Chapter 5.

Let us insist also on the fact that both the \(j_e\)'s and \(k_e\)'s depend only on the chosen edge: it is the same \(k_e\) all around the corresponding plaquette and it does not change from tetrahedron to tetrahedron. The fluctuations of \(k_e\) around the plaquette that occur in the supersymmetric theory will come in when we insert fermions in the model, as explained below.

### 6.2 Coupling matter: massless spinning fields

Of course, looking at the SU(2) theory from the lattice gauge theory perspective, a group element cannot map between different irreducible representations. In other words, the edge length cannot change as we move from tetrahedron to tetrahedron. Thus, the non-trivial matter is to allow for a change in edge length and this is where we expect the fermionic degrees of freedom to come into play. We wish to insert fermionic observables into (6.20) so as to get contributions to \(Z_{\Delta, UOSP(1|2)}\). This section will be concerned with the construction of these observables.

Noticing that the odd generators of UOSP(1|2) carry a spin-\(\frac{1}{2}\) representation of SU(2), we follow that argument to its natural conclusion and trace a spin-\(\frac{1}{2}\) representation through the spin foam. Furthermore, remembering the points expounded earlier, we expect that it should be embedded at the gluing point of two tetrahedra, and that if there is no change in edge length then there are either zero or two fermionic lines, while if the edge length changes then there is a single fermionic line embedded there. Indeed, this is how things turn out in the end.

To begin, we need to have a wedge formulation in terms of holonomies and representations \(R^J\). One might expect that this is a trivial manipulation of (6.20), but to maintain the \(R^J\) structure, requires us to navigate certain subtleties. Consider two adjacent wedges within a face, \(w^1\) and \(w^2\). Each has an assigned reducible representation \(R^{jw_1}\) and \(R^{jw_2}\). Whether a fermion observable is inserted or not, we wish wedges to couple only if \(jw_1 = jw_2\). This is naturally the case for irreducible representations of SU(2), but not for reducible ones. To see this, one needs only notice that \(V^{jw^*}\) is contained in \(R^{jw_1^* + \frac{1}{2}}\) and \(R^{jw_2^* + \frac{1}{2}}\). Therefore, \(R^{jw_1^*}\) and \(R^{jw_2^*}\) will couple for \(jw^*_1 = jw^*_2 = jw^*\). We can
cure this ambiguity by inserting a projector into the holonomy matrix attached to each wedge. It projects onto the highest weight state for $V^j \subset R^j$ and onto the lowest weight state for $V_{j-\frac{1}{2}} \subset R^j$. This projector is illustrated by:

$$\pm, k_w^* = (-1)^{2j_w^* - k_w^*}(2j_w^* + 1)|k_w^*, \pm k_w^*\rangle$$

and it is inserted at the point $v_e^*$ in Fig.2.3.

In effect, when one integrates over the $g_{e,f}^*$ variables, one gets a factor of $\delta_{j_w^* j_{w-\frac{1}{2}}} = 0$ from the projectors. Only the $j_w^* = j_{w-\frac{1}{2}}$ term survives. Now, we insert the fermionic observables $O_F$:

$$Z_{\Delta, SU(2), O_F} = \int \prod_{e^*_i} dg_{e^*_i} \prod_{e^*_{i,f}} dg_{e^*_{i,f}} \prod_w \sum_{\{j\}} A_{w^*}(g_{w^*}, j_{w^*}) O_F(\{g\}, \{j\}). \quad (6.21)$$

So let us proceed to the definition of these observables. Diagrammatically, we shall denote a segment of the fermionic observable by a dashed line. Furthermore, we shall need to introduce the projector onto the spin-up and spin-down states:

$$\pm = \langle \frac{1}{2}, \pm \frac{1}{2} | \frac{1}{2}, \pm \frac{1}{2} \rangle$$

One would expect this projection to occur as part of the propagator for spin-$\frac{1}{2}$ fermions [27, 61]. These projectors are inserted into the diagram once again at the points $v_e^*$ and are joined by parallel transport matrices, which closely follows the procedure for the insertion of matter observables in [27]. This charts the progress of the particle in the spin foam formulation. We shall examine more clearly the geometric space-time interpretation of this fermionic path shortly. The dashed line runs along between the wedges, since we want to allow for a change in edge length by $\frac{1}{2}$ as we move between tetrahedra. Furthermore, the fermionic observable knows about the gravity sector. We must insert an operator to extract various factors of edge-length: $\frac{\text{dim}}{2}$. We shall denote this graphically by a clasp joining the fermion projector and the gravity projector:

We give some motivation as to why one would expect such factors of edge length in the observable. Although the matter theory is massless, if one analyzes the classical field theory, one sees that the matter sector has a non-trivial energy-
momentum tensor. In fact, the Hamiltonian for the system is:

\[ H = \frac{1}{2} e_i^a e_j^b \epsilon^a b \left( F[W]_{ab} + \frac{i}{2} (\sigma^k)^{AB} \psi^A \psi^B \right) \quad \text{where} \quad e^i = \frac{1}{2} \epsilon_{ijk} e^a b e^b. \]

(6.22)

We note at this point that the \( e \)-dependent prefactor has dimensions of length. So from this argument, it comes as no surprise that the presence of matter should mean the insertion of a factor of length (that is, a factor linear in \( k \)) multiplying the holonomy.

Of course, there many possible combinations of insertion, but if we break them down into segments, then there are essentially three basic building blocks. We are ready to draw these:

![Figure 6.1: Sample of the fermionic Feynman diagram insertions.](image)

We begin with the no-fermion case in the top left. In going to the wedge formulation of the amplitude we actually already started to process of modifying the amplitudes. The gravity projectors remove the factor of dimension in \( A_e \) since they project just onto the highest/lowest weight state and therefore kill the sum. Furthermore, the factors of \((-1)^{2(j-k)(2j+1)}\) occurring in the projectors kill the same factors occurring in the triangle amplitudes \( A_f \). Notice that the gravity projectors are all at the center of the face. This is to ensure that the final amplitude for the edges \( e \) is correct. The amplitudes for the various
sub-simplices are now:

\[ A_e = (-1)^{2k_e}, \quad A_f = (-1)^{(k_{e_1} + k_{e_2} + k_{e_3})}, \]

\[ A_t = \left[ (-1)^{\sum_{a=1}^n 2(j_{a_1} - k_{a_1})/(2j_{a_1} + 1)} \begin{pmatrix} k_{e_1} & k_{e_2} & k_{e_3} \\ k_{e_4} & k_{e_5} & k_{e_6} \end{pmatrix} \right]_t. \] (6.23)

Then there is the fermionic loop insertion, which also contributes to the bosonic sector of the theory, and which is illustrated in the top right. The fermionic line traces a loop which is inserted between two wedges in one face. Thus, the insertion does not map between SU(2) modules. In fact, using the standard retracing identity, we can remove the fermionic line altogether, but the non-trivial part is the clasp which extracts a factor of \((-1)^{2(j-1)+1}(2k+1)\) for \(V^k \subset R^j\). Thus, the only difference between the no-fermion amplitude and the amplitude containing this fermion loop is the amplitude for one triangle:

\[ A_f = (-1)^{(k_{e_1} + k_{e_2} + k_{e_3})}(-1)^{2(j_{e_1} - k_{e_1}) + 1}(2k_{e_1} + 1), \] (6.24)

where the loop was inserted around edge \(e_1\). Let us take a specific example, say \(k_{e_a} = j_{e_a}\) for all \(a\), and let us sum up the four contributions: the no-insertion and the loop insertions on \(e_1, e_2, e_3\). The resulting amplitude is exactly

\[ -A_f^{\{j\}}(\uparrow, \uparrow; \uparrow, \downarrow). \]

For the triangle given above, ultimately, the amplitude arising from summing over the no-insertion and the three possible loop insertions will lead to all possible bosonic amplitudes depending on whether the edges are in the upper or lower modules.

Let us move onto the fermionic contributions, which rely on non-trivial propagation of the fermion along edges of the simplicial complex. There is essentially one type of diagram, variations of which give the other 23 possibilities occurring in (6.11) and (6.12). We displayed the insertion in the center bottom of Fig.6.1. We shall reproduce the following amplitude: \(A_f^{\{j\}}(\uparrow, \uparrow; \uparrow, \downarrow, \downarrow)\). This triangle amplitude is made up from several sub-diagrams:

\[ = (-1)^{j_1 + j_2 + j_3 + (2j_1 + 1)} \times \sqrt{(j_1 + j_2 + j_3 + 1)} \times \sqrt{(-j_1 + j_2 + j_3)} \] (6.25)

which we note is just minus the amplitudes for which we were hoping. We stress that the diagrams in this equation all contribute to the triangle amplitude.
$A_f$. We get a tetrahedral diagram, because we have coupled an extra spin-$\frac{1}{2}$ between the wedges. The square root of the loops come from factors of dimension which we saw, in the gravity case, arise upon decomposition of the $\delta$-functions. Therefore, we have successfully reproduced the triangle amplitudes, which is what was our goal.

Thankfully, this allows for a thorough description of all the supersymmetric amplitudes. All we need is to take the pure gravity amplitudes and insert all the possible fermionic observables consistent with the above rules and as such we arrive at the supersymmetric amplitude.

6.2.1 Geometrical properties of the fermionic observables

As promised, we conclude with a description of the geometric properties of the fermionic observables. Consider two adjacent wedges. Their intersection is an edge $e_{e,f}^*$, joining the center of a triangle $f \subset \Delta$ with the midpoint of an edge $e \subset \Delta$. The path of the fermion in the spin foam, the dashed line, contains only such edges (see Fig.6.2 for details). The obvious spacetime picture is that the particle propagates along the edges $e$ occurring in $e_{e,f}^*$. This is a perfectly self-consistent propagation and gives a nice geometric viewpoint to the amplitude.

![Figure 6.2: From spin foam to spacetime picture.](image)

At the end of the day, putting aside the trivial fermionic loop insertion of the top-right type in Fig.6.2, a fermionic Feynman diagram is a graph embedded in the triangulation. More precisely, we want sets of closed loops of edges, such that two consecutive edges on a loop belong to a same triangle. This means that we can equivalently think of the fermionic loop as a closed sequence of triangles i.e. a loop in the spinfoam complex. Then considering a given edge $e$ propagating a fermion, and looking at the plaquette around it, the triangle that contains two fermionic edges will trigger a $\frac{1}{2}$-shift of $k_e$. This will happen twice around the plaquette, once by a $+\frac{1}{2}$-shift, once by a $-\frac{1}{2}$-shift. These two triangles correspond to the two triangles of the Feynman diagram sharing the same edge. Now, we can have an arbitrary number of fermionic loops in the Feynman diagram and actually they can share the same edges, since there is no fermionic interaction term here, so that the only interactions are between
the fermionic field(s) and the gravitational degrees of freedom. Thus, in general there can be several pairs of $\pm \frac{1}{2}$-shifts as one goes around a plaquette, each corresponding to a separate Feynman diagram. The last step is to sum over all possible Feynman diagrams in order to reconstitute the full supersymmetric spinfoam amplitude.

This picture is finally slightly different from the one initially envisioned in [34].
Chapter 7

Conclusions

Starting from the topological spinfoam model for $N = 1$ supergravity in 3d gravity, we have analyzed in detail the structure of these spinfoam amplitudes. We have first shown how to derive these spinfoam amplitudes from a discretised $BF$ action on a triangulation by extending the standard bosonic construction of a discretised action for the SU(2) Ponzano-Regge model [27] to include for fermionic degrees of freedom in the connection and triad. In particular, this showed how including fermions can resolve the standard ambiguity that the usual discretised action leads to $SU(2)/\mathbb{Z}_2 \sim SO(3)$ and not exactly to $SU(2)$.

Then we explicitly decomposed all supersymmetric amplitudes into a superposition of the standard SU(2) amplitudes. This is done by decomposing UOSP(1|2) representations into irreducible representations of SU(2). The most striking result is that although a single spin $j_e$ is associated to each edge of the triangulation, the actual length of that edge is a priori different seen from the viewpoint of each tetrahedron to which it belongs: it can be either $k_e = j_e$ or $k_e = j_e - \frac{1}{2}$ depending whether a fermion is traveling through this tetrahedron or not. Pushing this decomposition into SU(2) amplitudes as far as possible, we finally showed that the supersymmetric amplitude can be seen as the coupling of fermionic Feynman diagrams to the gravitational background. Let us emphasise that the geometry is not static but when a fermionic line is inserted, it creates length shifts as mentioned previously.

If we were to go further in the understanding of these $N = 1$ supersymmetric spinfoam models, we could analyze the asymptotics of the susy $\{6j\}$-symbol and see how the Regge action and the fermionic fields appears in the large spin limit [6]. We should also compare our approach to the standard insertion of particles with spin in the Ponzano-Regge model [27] (the actual difference is that our framework takes into account explicitly the feedback of the fermionic fields on the gravitational fields) and to the more recent gravity+fermions models developed in [20, 23].

Finally, the most interesting application to our formalism would be to study the insertion of actual physical non-topological fermionic fields. Starting in
3d, in the present work, we have tracked from the initial continuum action down to the final discretised spinfoam amplitude how the explicit fermionic Feynman diagrams get inserted in the spinfoam amplitude. These fermionic observables come with precise weights (see e.g. eqn. (6.24)-(6.25)). These weights are fine-tuned so as to ensure that the full model ‘gravity+fermions’ is topological. That shows that these spinfoam amplitudes provide the correct quantisation for our supersymmetric theory. As soon as we modify these weights, we would get non-topological amplitudes and it would be interesting to see how we could modify them in order to insert more physical fermionic fields. Then, we hope to apply the same procedure to the four-dimensional case by first deriving the spinfoam quantisation of supersymmetric BF theory and studying how the fermions are coupled to the spinfoam background, and then seeing how this structure is maintained or deformed when we introduce the (simplicity) constraints on the $B$ field in order to go from the topological BF theory down back to proper gravity. Another interesting outlook is to push our analysis to $N = 2$ supersymmetric BF theory, already in three space-time dimensions, following the footsteps of [35]. Indeed, such a theory already include a spin-1 gauge field, and we could study in more detail how the full supersymmetric amplitudes decomposes into Feynman diagrams for the fermions and spin-1 field inserted in the gravitational spinfoam structure. Then we would see how it is possible to deform this structure in such a way that the spin-1 field represents standard gauge fields. This road would provide an alternative way to coupling (Yang-Mills) gauge fields to spinfoam models, which we could then compare to the other approaches developed in this direction [56].
Appendix A

Delta Function Identity for $SU(2)$ and OSP(1|2)

A.1 The SU(2) case

In this section we prove the following formula

$$\int d^3xe^{i\text{tr}(Xg)} (1 \pm \epsilon(g)) = 8\pi \delta(\pm g),$$  \hspace{1cm} (A.1)

where $X = x^i\sigma_i$ is in the Lie algebra, $g = \exp(i\theta n^i\sigma_i)$ is a SU(2) group element with $\theta \in [0, \pi]$ and $n^1n_i = 1$; $\sigma^i$ are the Pauli matrices, $\text{tr}\sigma_i\sigma_j = \delta_{ij}$, $\epsilon(g) = \text{sign}(\cos \theta)$, and $\delta(g)$ is the delta function on the group with respect to the normalized Haar measure. First we evaluate

$$\int d^3xe^{i\text{tr}(Xg)} = (2\pi)^3\delta(3)(\sin \theta \vec{n}).$$  \hspace{1cm} (A.2)

We can use the familiar identity $\delta(3)(\vec{X}) = \frac{1}{4\pi|\vec{X}|}\delta(|\vec{X}|)$ to write this evaluation as

$$\frac{2\pi^2}{(\sin \theta)^2} \delta(|\sin \theta|) = \frac{2\pi^2}{(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi n) \frac{1}{|\cos \theta|},$$  \hspace{1cm} (A.3)

$$= \frac{2\pi^2}{(\sin \theta)^2} \sum_{n \in \mathbb{Z}} (\delta(\theta - \pi 2n) + \delta(\theta - \pi(2n + 1))).$$  \hspace{1cm} (A.4)

The normalized Haar measure on the group is given by

$$dg = \frac{2}{\pi}d\theta(\sin \theta)^2 d^2n,$$  \hspace{1cm} (A.5)

where $d^2n$ is the normalized measure on $S^2$, therefore we can write

$$\delta(g) = \frac{\pi}{2(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi 2n),$$  \hspace{1cm} (A.6)

$$\delta(-g) = \frac{\pi}{2(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi(2n + 1)).$$  \hspace{1cm} (A.7)
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Together with (A.3) this proves that

\[ \int d^3x e^{i \text{tr}(Xg)} = 4\pi (\delta(g) + \delta(-g)). \]  

(A.8)

A similar computation shows that

\[ \int d^3x e^{i \text{tr}(Xg)} \epsilon(g) = 4\pi (\delta(g) - \delta(-g)) \]  

(A.9)

which proves eq(A.1).

A.2 The UOSP(1|2) delta function

In order to show the form and the properties of the \( \delta \)-function calculated for OSP(1|2) elements, we proceed by showing the following:

\[ \int_{\text{UOSP}(1|2)} dg \, \delta(g) f(g) = f(I). \]  

(A.10)

The element \( g \) can be written as \( g = u \xi \), where \( u = e^{\theta \vec{n} \cdot \vec{J}} \) and \( \xi = e^{\eta \vec{Q} + \eta \vec{Q} -} \); see Section 3.3 for definitions.

The normalized Haar measure is:

\[ dg = \frac{1}{\pi^2} \sin^2 \theta \sin \psi \, d\theta \, d\psi \, d\phi \, d\eta \, d\eta \]  

(A.11)

and \( du \) is the measure over the SU(2) subgroup. Equation (A.10) becomes:

\[ \int d\eta \, d\eta \int du \, \delta(g) f(u \xi) = \frac{1}{\pi^2} \int (1 - \frac{1}{4} \eta \vec{\eta}) \sin \theta \sin \psi \, d\theta \, d\psi \, d\phi \, d\eta \, d\eta \delta(\eta) \delta(\eta) \int du \delta^{(3)}(\sin \theta \vec{n}) f(u \xi) \]  

(A.12)

Performing the integration over the even sector of UOSP(1|2) we obtain the same result for the SU(2) \( \delta \) function, i.e. we obtain a second peak.

\[ \int du \delta^{(3)}(\sin \theta \vec{n}) f(u \xi) = \frac{2}{\pi^2} \int_0^{2\pi} d\theta \, \int_0^\pi d\psi \, \int_0^{2\pi} d\phi \, \sin \psi \sin^2 \left( \frac{\theta}{2} \right) \delta^{(3)}(\sin \theta \vec{n}) f \left[ e^{\theta \vec{n} \cdot \vec{J}} (e^{\eta \vec{Q} + \eta \vec{Q} -}) \right] = \frac{2}{\pi^2} \int_0^{2\pi} d\theta \, \int_0^\pi d\psi \, \int_0^{2\pi} d\phi \, \sin \psi \left[ f \left[ e^{\theta \vec{n} \cdot \vec{J}} (e^{\eta \vec{Q} + \eta \vec{Q} -}) \right] + f \left[ e^{i \pi \vec{n} \cdot \vec{J}} (e^{\eta \vec{Q} + \eta \vec{Q} -}) \right] \right] = 4\pi \left[ f \left[ I_{\text{SU}(2)} \cdot (e^{\eta \vec{Q} + \eta \vec{Q} -}) \right] + f \left[ -I_{\text{SU}(2)} \cdot (e^{\eta \vec{Q} + \eta \vec{Q} -}) \right] \right] \]  

(A.13)
APPENDIX A. DELTA FUNCTION IDENTITY FOR SU(2) & OSP(1|2)

Fortuitously, it can be shown that the integration over the odd sector

\[
\frac{1}{4} \int d\eta d\eta \delta(\eta) \delta(\eta) 4\pi \left[ \frac{9}{2} f(\eta^{\mu} Q_{+} + \eta Q_{-}) + \frac{1}{4} f(-\eta^{\mu} Q_{+} + \eta Q_{-}) \right]
\]

contains a factor \((\cos \theta + 1)\) which kills the second peak. Such a factor was introduced by hand for the SU(2) Ponzano-Regge model in [33] in order to kill this same second peak.
Appendix B

The integration of three representation functions

We mentioned in the main text that there was a non-trivial step in passing from a product of representation functions to intertwiners on the space of representations. One examines the representation functions occurring in the integral:

\[
A_{\omega}^\gamma = \int_{\text{UOSP}(1|2)} dg_{\omega}^\gamma T_{(k_1 m_1),(l_1 n_1)}^{j_1, \tau_1} (g_{\omega}^\gamma) T_{(k_2 m_2),(l_2 n_2)}^{j_2, \tau_2} (g_{\omega}^\gamma) T_{(k_3 m_3),(l_3 n_3)}^{j_3, \tau_3} (g_{\omega}^\gamma),
\]

for each choice of \(k_i, l_i\) as given in (3.38). Integrating with respect to \(\eta, \eta^\square\) and \(\Omega\), one should arrive at:

\[
A_{\omega}^\gamma = I_{(k_1 m_1)(k_2 m_2)(k_3 m_3)}^{j_1} I_{(l_1 n_1)(l_2 n_2)(l_3 n_3)}^{j_2} I_{(l_1 n_1)(l_2 n_2)(l_3 n_3)}^{j_3},
\]

as stated in the text. The subtlety becomes clearer when one realises that on the right hand side of (3.38), there is no concept of change of SU(2) module. Fortunately, there exist relations between the SU(2) \(\{\bar{3}_j\}\)-symbols, which provide the missing link between (B.2) from (B.1):

\[
\left[ (j_1 + j_2 + j_3 + \frac{1}{2}) (j_1 - j_2 + j_3 + \frac{3}{2}) \right] \frac{1}{2} C_{n_1 n_2 n_3}^{j_1 - j_2 j_3}
= - \left[ (j_1 + n_1 + \frac{1}{2}) (j_2 + n_2) \right] \frac{1}{2} C_{n_1 + \frac{1}{2} n_2 - \frac{1}{2} n_3}^{j_1 j_2 - \frac{1}{2} j_3}
- \left[ (j_1 - n_1 + \frac{1}{2}) (j_2 - n_2) \right] \frac{1}{2} C_{n_1 - \frac{1}{2} n_2 + \frac{1}{2} n_3}^{j_1 j_2 - \frac{1}{2} j_3},
\]

\[
\left[ (j_1 + j_2 - j_3) (j_1 + j_2 + j_3 + 1) \right] \frac{1}{2} C_{n_1 n_2 n_3}^{j_1 j_2 - j_3}
= \left[ (j_1 + n_1 + \frac{1}{2}) (j_2 + n_2 + \frac{1}{2}) \right] \frac{1}{2} C_{n_1 + \frac{1}{2} n_2 + \frac{1}{2} n_3}^{j_1 j_2 j_3}
- \left[ (j_1 - n_1 + \frac{1}{2}) (j_2 + n_2 + \frac{1}{2}) \right] \frac{1}{2} C_{n_1 + \frac{1}{2} n_2 + \frac{1}{2} n_3}^{j_1 j_2 j_3}.
\]

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\[
[(j_1 + j_2 - j_3) (j_1 + j_2 + j_3 + 1)]^{\frac{1}{2}} C_{n_1 n_2 n_3}^{j_1 j_2 j_3}
= - [(j_1 - n_1) (j_2 + n_2)]^{\frac{1}{2}} C_{n_1 + \frac{1}{2} n_2 - \frac{1}{2} n_3}^{j_1 - \frac{1}{2} j_2 - \frac{1}{2} j_3}
+ [(j_1 + n_1) (j_2 - n_2)]^{\frac{1}{2}} C_{n_1 - \frac{1}{2} n_2 + \frac{1}{2} n_3}^{j_1 - \frac{1}{2} j_2 - \frac{1}{2} j_3},
\] (B.3c)
Bibliography


