Boundary Conformal Field Theory for superconducting devices

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Anno Accademico 2009/2010
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Introduction

Conformal field theory (CFT) with boundaries finds applications both to open string theory and to various quantum impurity problems in condensed matter physics. These generally describe gapless bulk excitations interacting with some localized degrees of freedom. In these problems the gapless bulk excitations can be represented by a conformal field theory in (1+1) spacetime dimensions, often simply free bosons. It is generally found that the boundary dynamics renormalize, at low energies, to a conformally invariant boundary condition.

Boundary field theories appear to be relevant in several different contexts. In condensed matter physics, they are mostly generalizations of quantum impurity models, which may be described using the Tomonaga-Luttinger liquid (TLL)-paradigm[1, 2]; for instance, boundary interactions appear in the analysis of the Kondo problem[3, 4, 5], in the study of a onedimensional conductor in presence of an impurity [6], and in the derivation of the tunneling between edge states of a Hall bar[7, 8, 9]. The TLL paradigm shows that many interactions are simply diagonalizable in the basis of appropriate collective bosonic modes, and that nondiagonalizable interactions usually correspond to exactly solvable Hamiltonians, such as sine-Gordon models[10, 11, 12, 13]. Recently, boundary field theories have been investigated in the context of string theories. For instance, in studying tachyon instabilities[14, 15], one must face the fact that the space of interacting string theory[16] may be mapped onto the space of boundary perturbations of conformal theories[17, 18], and that the renormalization group flow determined by boundary perturbations may be identified with tachyon condensation[19, 20, 21]. Affleck and Ludwig[22] showed that the boundary entropy decreases along the renormalization group trajectories, triggered by the boundary interaction.

The boundary field theory approach developed in Refs. [23, 24, 25] not only allows for an accurate determination of the phases accessible to a superconducting device, but also for a field-theoretical treatment of the phase slips (instantons), describing quantum
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tunneling between degenerate ground-states.

Existence of finite fixed points (FFP) in condensed matter is a rare instance realized, so far, only in quantum systems with pertinent impurities. Remarkable examples of systems exhibiting attractive FFPs are provided by the two-channel single-impurity [26] and two-impurity [27] overscreened Kondo models, as well as by Y-shaped junctions of quantum wires [28, 29, 30]. At variance, Y-shaped junctions of one-dimensional atomic condensates [31] exhibit a repulsive FFP, signaling the existence of a new transition point between the stable weakly and the strongly coupled phases. Boundary conformal field theories are a natural setting to investigate stable phases and phase transitions of quantum impurity systems, once the quantum impurity is traded [32] for a boundary interaction, involving only a subset of the bulk degrees of freedom: the boundary interaction is then renormalized by the bulk degrees of freedom, and the infrared (IR) behavior is determined by the stable fixed point(s) in the phase diagram.

The original contribution provided in this work is the proposal of a device of Josephson junction chains coupled with a central zone, that can be seen as an impurity. The central zone is chosen through a tetraedral geometry and, thanks to its symmetry, it shows a degenerate ground state resistant to external noises when it is crossed by appropriate magnetic fluxes. This device is analyzed using techniques of Boundary Conformal Field Theory to get from it the phase diagram.

The first part (Chapter 1 and 2) is an introduction of the fundamental concepts of conformal theories in the bulk and in the boundary, showing the relevant objects for these theories: primary fields, Operator Product Expansion, scale dimensions. In the Chapter 3 we describe some superconductor devices and the way they can be studied using the Renormalization Group. Then the Chapter 4 deals with a new device of Josephson junction network. We will draw one Luttinger liquid description of device and we will obtain the effective boundary Hamiltonian that describe the network. Within a range of value of parameters that describe the device, the infrared behavior of the network will be driven by a finite coupling fixed point of the phase diagram, at which a two-level quantum system emerges, that -as we will demonstrate- is robust against quantum fluctuations.
Chapter 1

Conformal Field Theory

Second order phase transitions occur in a large number of (statistical mechanical) systems. At second order phase transitions, the first derivatives of like magnetization, entropy, volume, etc. are continuous. However, higher derivatives exhibit discontinuity. Examples of such systems are gas-liquid phase transition or paramagnetic-ferromagnetic phase transition. At such phase transitions one observes large scale correlations. In the gas-liquid system, this (i.e., the increase in correlation length) leads to the phenomenon of critical opalescence, regions the size of microns which is comparable to the wavelength of visible light are seen to fluctuate coherently. At such phase transitions, the most important features are captured by a small number of parameters (fields). These are called the order parameter(s). In the gas-liquid system, the order parameter is the liquid density; in the ferromagnet the order parameter is the spin density. Let \( \phi(x) \) be a generic order parameter and \( \delta \phi \approx \phi(x) - \langle \phi(x) \rangle \) be the fluctuation of the order parameter about its mean value. Away from the critical point, one has

\[
\langle \delta \phi(x) \delta \phi(0) \rangle \sim e^{-\frac{x}{\xi}} \tag{1.1}
\]

where \( \xi \) is the correlation length. The above is no longer true at the critical point. The correlation length diverges, i.e. \( \xi \sim L \), where \( L \) is the length of the system. The two point correlation function is governed by a power law behaviour

\[
\langle \delta \phi(x) \delta \phi(0) \rangle \sim |x|^{-(D-2+\eta)} \tag{1.2}
\]

The divergence of \( \xi \) at the critical point implies that the system is scale invariant. In other words, physical quantities do not depend on the length scale chosen. Evidence, for this is the wide range of systems (and temperature) where the gas-liquid phase
transition has a similar behaviour. The relevant physical quantities are given by the so-called “scaling fields”. Examples of scaling fields are the “spin-density $\sigma$” and the “energy density $\epsilon$” in the 2D Ising Model. Scaling transformations are given by the following

$$x^\mu \to \lambda x^\mu .$$

Under the above transformation, the scaling fields transform as

$$\phi(x) \to \lambda^\Delta \phi(x) ,$$

where $\Delta$ is referred to as the scaling dimension of the field $\phi$. Polyakov conjectured that systems (with assumptions such as isotropy and possibly locality of interactions) exhibiting scale invariance in 2D possess a symmetry larger than simple scaling. This symmetry group is called the conformal group.

1.1 The conformal group

We denote by $g_{\mu\nu}$, the metric tensor in a space-time of dimension $d$. By definition, a conformal transformation of the coordinates is an invertible mapping $x \to x'$, which leaves the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) .$$

In other words, a conformal transformation is locally equivalent to a (pseudo) rotation and a dilation. The set of conformal transformations manifestly forms a group, and it obviously has the Poincaré group as a subgroup, since the latter corresponds to the special case $\Omega(x) \equiv 1$. The epithet onformal derives from the property that the transformation does not affect the angle between two arbitrary curves crossing each other at some point, despite a local dilation: the conformal group preserves angles (see Fig.1.1).

If we consider an infinitesimal change of coordinates $x'^\mu = x^\mu + \epsilon^\mu$, the corresponding change in the metric tensor $g_{\mu\nu}$ is (to first order in $\epsilon$)

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$= \left( \delta^\alpha_\mu - \partial_\mu \epsilon^\alpha \right) \left( \delta^\beta_\nu - \partial_\nu \epsilon^\beta \right) g_{\alpha\beta}$$

$$= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) .$$

(1.6)
1.1 The conformal group

The requirement that the transformation be conformal implies that

\[ \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \]  

(1.7)

The factor \( f(x) \) can be determined by taking the trace on both sides:

\[ f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho, \]  

(1.8)

where we used the property of the metric tensor \( g_{\mu\nu} g^{\mu\nu} = d \). By applying an extra partial derivative on (1.7), permuting indices and taking a linear combination, we obtain

\[ 2 \partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\nu\rho} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f \]  

(1.9)

Hereafter, for simplicity, we assume that the conformal transformation in an infinitesimal deformation of the standard Cartesian metric \( g_{\mu\nu} = \eta_{\mu\nu} \) (for the Minkowsky metric the treatment is identical, except for the explicit form of \( \eta_{\mu\nu} \)). Upon contracting with \( \eta_{\mu\nu} \), the last equation becomes

\[ 2 \partial^2 \epsilon_\mu = (2 - d) \partial_\mu f. \]  

(1.10)

Applying \( \partial_\nu \) on this expression and \( \partial^2 \) on Eq. (1.7), we find

\[ (2 - d) \partial_\mu \partial_\nu f = \eta_{\mu\nu} \partial^2 f \]  

(1.11)

Figure 1.1: A conformal transformation.
and, contracting with $\eta_{\mu\nu}$, we end up with

$$(d - 1) \partial^2 f = 0. \quad (1.12)$$

From this equation and Eq.(1.7) we can derive the explicit form of conformal transformation in $d$ dimensions.

Now we can distinguish three different cases: $d = 1$, $d = 2$ and $d \geq 3$. In the first case, Eq.(1.12) does not impose any constraint on the function $f$, i.e. any smooth transformation is conformal in one dimension. This statement is trivial, in fact the notion of angle does not exist. The second case, $d = 2$, will be studied in the next sections. Hereafter we concentrate on the case $d \geq 3$.

The equations (1.7) and (1.12) imply that

$$\partial_{\mu} \partial_{\nu} f = 0,$$

i.e. the function $f$ can be at most linear in the coordinates:

$$f(x) = A + B_\mu x^\mu \quad (1.13)$$

where $A$ and $B_\mu$ are some constants. If we substitute this expression into Eq.(1.9) we can see that $\partial_{\mu} \partial_{\nu} \epsilon_\rho$ is constant, so we therefore write the general dependence of $\epsilon$ on $x$:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \quad (1.14)$$

with $a_\mu$, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ constants and $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Since constraint Eq. (1.7)-(1.9) hold for all $x$, we may treat each power of the coordinate separately:

1. $\epsilon_\mu = a_\mu$ are the ordinary translations independent of $x$,
2. $\epsilon_\mu = \omega_{\mu\nu} x^\nu$ are rotations,
3. $\epsilon_\mu = \lambda x_\mu$ are scale transformations,
4. $\epsilon_\mu = b_\mu x^2 - 2 x_\mu b \cdot x$ are the so-called special conformal transformations.

Integrating to finite conformal transformations, we find, as expected, the Poincaré group ($\Omega = 1$)

$$x \rightarrow x' = x + a \quad (1.15)$$

$$x \rightarrow x' = \Lambda x \quad (\Lambda_\mu^\nu \in SO(p,q)) \quad (1.16)$$

Adjoined to it, we have dilatations ($\Omega = \lambda^{-2}$)

$$x \rightarrow x' = \lambda x \quad (1.17)$$
and finally the special conformal transformations (SCT)

\[ x \rightarrow x' = \frac{x^b x^2}{1 + 2b \cdot x + b^2 x^2} \quad (\Omega(x) = (1 + 2b \cdot x + b^2 x^2)^2). \quad (1.18) \]

Note that under Eq. (1.18) we have \( x'^2 = x^2/(1 + 2b \cdot x + b^2 x^2) \), so that points on the surface \( 2b \cdot x + b^2 x^2 = 0 \) have their distance to the origin preserved, whereas points on the exterior of this surface are sent to the interior and vice-versa. Under the transformation (1.18) we have \( x'^\mu / x'^2 = x^\mu / x^2 + b^\mu \), i.e. it correspond to an inversion plus a translation.

1.2 The conformal group in two dimensions

Conformal invariance takes a new meaning in two dimensions; the case \( d = 2 \) requires special attention. Indeed, there exists in two dimensions an infinite variety of coordinate transformations that, although not everywhere well-defined, are locally conformal: they are holomorphic mappings from the complex plane (or part of it) onto itself. Among this infinite set of mappings one must distinguish the 6-parameter \( \text{global} \) conformal group, made of one-to-one mappings of the complex plane into itself. However, a local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined. It is local conformal invariance that enables exact solutions of two-dimensional conformal field theories.

1.2.1 Conformal mappings

We consider the coordinates \((z^0, z^1)\) on the plane. Under a change of coordinate system \( z^\mu \rightarrow w^\mu(x) \) the contravariant metric tensor transforms as

\[ g^{\mu\nu} \rightarrow \left( \frac{\partial w^\mu}{\partial z^\alpha} \right) \left( \frac{\partial w^\nu}{\partial z^\beta} \right) g^{\alpha\beta}. \quad (1.19) \]

The condition (1.5) that defines a conformal transformation is \( g'_{\mu\nu}(w) \propto g_{\mu\nu}(z) \) or, explicitly,

\[ \left( \frac{\partial w^0}{\partial z^0} \right)^2 + \left( \frac{\partial w^1}{\partial z^0} \right)^2 = \left( \frac{\partial w^1}{\partial z^1} \right)^2 + \left( \frac{\partial w^0}{\partial z^1} \right)^2 \quad (1.20) \]

\[ \frac{\partial w^0}{\partial z^0} \frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1} \frac{\partial w^1}{\partial z^1} = 0 \quad (1.21) \]
These conditions are equivalent either to
\[
\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \quad (1.22)
\]
or to
\[
\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \quad (1.23)
\]
In Eq. (1.22) we recognize the Cauchy-Riemann equations for holomorphic functions, whereas Eq. (1.23) defines antiholomorphic functions.

This motivates the use of complex coordinates \( z \) and \( \bar{z} \), with the following translation rules:
\[
\begin{align*}
z &= z^0 + iz^1, & z^0 &= \frac{1}{2}(z + \bar{z}) , \\
\bar{z} &= z^0 - iz^1, & z^1 &= \frac{1}{2i}(z - \bar{z}) , \\
\partial_z &= \frac{1}{2}(\partial_0 - i\partial_1), & \partial_0 &= \partial_z + \partial_{\bar{z}} , \\
\partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1), & \partial_1 &= i(\partial_z - \partial_{\bar{z}}) .
\end{align*}
\]
We shall sometimes write \( \partial = \partial_z \) and \( \bar{\partial} = \partial_{\bar{z}} \) when there is no ambiguity about the differentiation variable. In terms of the coordinates \( z \) and \( \bar{z} \), the metric tensor is
\[
g_{\mu\nu} = \begin{pmatrix}
0 & 1/2 \\
1/2 & 0
\end{pmatrix} , \quad g^{\mu\nu} = \begin{pmatrix}
0 & 2 \\
2 & 0
\end{pmatrix} \quad (1.25)
\]
where the index \( \mu \) takes the values \( z \) and \( \bar{z} \), in that order. This metric tensor allows us to transform a covariant holomorphic index into a controvariant holomorphic index and vice versa. The antisymmetric tensor \( \epsilon_{\mu\nu} \) in holomorphic form is
\[
\epsilon_{\mu\nu} = \begin{pmatrix}
0 & \frac{1}{2}i \\
-\frac{1}{2}i & 0
\end{pmatrix} , \quad \epsilon^{\mu\nu} = \begin{pmatrix}
0 & -2i \\
2i & 0
\end{pmatrix} . \quad (1.26)
\]
In this language, the holomorphic Cauchy-Riemann equations become simply
\[
\partial_z w(z, \bar{z}) = 0 , \quad (1.27)
\]
whose solution is any holomorphic mapping (no \( \bar{z} \) dependence):
\[
z \rightarrow w(z) . \quad (1.28)
\]
1.2 The conformal group in two dimensions

It is a well-known result that any analytic mapping of the complex plane onto itself is conformal (i.e., preserves angles). This is made plainly obvious by considering the differential

$$dw = \left( \frac{dw}{dz} \right) dz.$$  \hspace{1cm} (1.29)

The derivative $dw/dz$ contains a dilation factor $|dw/dz|$, along with a phase $\arg(dw/dz)$, which embodies a rotation. The conformal “group” in two dimensions is therefore the set of all analytic maps, wherein the group multiplication is the composition of maps. This set is infinite-dimensional, since an infinite number of parameters (the coefficients of a Laurent series) is needed to specify all functions analytic in some neighborhood. It is precisely this infinite dimensionality that allows so much to be known about conformally invariant field theories in two dimensions.

1.2.2 Conformal generators

As is typical in physics, the local properties are more immediately useful than the global properties, and the local conformal group (the set of all, not necessarily invertible, holomorphic mappings) is of great importance. We now find the algebra of its generators. Any holomorphic infinitesimal transformation may be expressed as

$$z' = z + \epsilon(z), \quad \epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}$$ \hspace{1cm} (1.30)

where, by hypothesis, the infinitesimal mapping admits a Laurent expansion around $z = 0$. The effect of such a mapping (and of its antiholomorphic counterpart) on a spinless and dimensionless field $\phi(z, \bar{z})$ living on the plane is

$$\phi'(z', \bar{z}') = \phi(z, \bar{z})$$

$$= \phi(z', \bar{z}') - \epsilon(z') \partial' \phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}') \bar{\partial'} \phi(z', \bar{z}') \hspace{1cm} (1.31)$$

or

$$\delta \phi = -\epsilon \partial \phi - \bar{\epsilon}(\bar{z}) \bar{\partial} \phi$$

$$= \sum_n \left[ c_n \ell_n \phi(z, \bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z, \bar{z}) \right] \hspace{1cm} (1.32)$$
where we have introduced the generators
\[ \ell_n = -z^{n+1} \partial_z \]
\[ \bar{\ell}_n = -\bar{z}^{n+1} \partial_{\bar{z}} . \] (1.33)

These generators obey the following commutation relations:
\[ [\ell_n, \ell_m] = (n - m) \ell_{n+m} \]
\[ [\bar{\ell}_n, \bar{\ell}_m] = (n - m) \bar{\ell}_{n+m} \]
\[ [\ell_n, \bar{\ell}_m] = 0 . \] (1.34)

Thus the conformal algebra is direct sum of two isomorphic algebras, each with very simple commutation relations. The algebra (1.34) is sometimes called the Witt algebra.

Each of these two infinite-dimensional algebras contains a finite subalgebra generated by \( \ell_{-1}, \ell_0 \) and \( \ell_1 \). This is the subalgebra associated with the global conformal group. Indeed, from the definition (1.33) it is manifest that \( \ell_{-1} = -\partial_z \) generates translations on the complex plane, than \( \ell_0 = z \partial_z \) generates scale transformations and rotations, and that \( \ell_1 = z^2 \partial_z \) generates special conformal transformations. The generators that preserve the real surface \( z_0, z_1 \in \mathbb{R} \) are the linear combinations
\[ \ell_n + \bar{\ell}_n \quad \text{and} \quad i(\ell_n - \bar{\ell}_n) . \] (1.35)

In particular, \( \ell_0 + \bar{\ell}_0 \) generates dilations on the real surface, and \( i(\ell_0 - \bar{\ell}_0) \) generates rotations.

### 1.2.3 Primary fields

In two dimension the definition of quasi-primary fields applied also to fields with spin. Indeed, given a field with scaling dimension \( \Delta \) and planar spin \( s \), we define the holomorphic conformal dimension \( h \) and its antiholomorphic counterpart \( \bar{h} \) as
\[ h = \frac{1}{2}(\Delta + s) \quad , \quad \bar{h} = \frac{1}{2}(\Delta - s) . \] (1.36)

Under a conformal map \( z \rightarrow w(z), \bar{z} \rightarrow \bar{w}(\bar{z}) \), a quasi-primary field transforms as
\[ \phi'(w, \bar{w}) = \left( \frac{dw}{dz} \right)^{-h} \left( \frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}) . \] (1.37)
1.2 The conformal group in two dimensions

The above shows that a quasi-primary field of conformal dimensions \((h, \bar{h})\) transforms like the component of a covariant tensor of rank \(h + \bar{h}\) having \(h\) \(z\)-type indices and \(\bar{h}\) \(\bar{z}\)-type indices. If the map \(z \to w\) is close to the identity, that is if \(w = z + \epsilon(z)\) and \(\bar{w} = \bar{z} + \bar{\epsilon}(z)\) with \(\epsilon\) and \(\bar{\epsilon}\) small, the variation of quasi-primary is

\[
\delta \epsilon, \delta \bar{\epsilon} \equiv \psi'(z, \bar{z}) - \psi(z, \bar{z}) = - (h \phi \partial_z \epsilon + \bar{h} \phi \partial_{\bar{z}} \bar{\epsilon}) - (\bar{h} \phi \partial_z \bar{\epsilon} + \epsilon \partial_{\bar{z}} \psi) .
\]  

(1.38)

In fact, a field whose variation under any local conformal transformation in two dimensions is given by (1.37) is called primary. All primary fields are also quasi-primary, but the reverse is not true. A field may transform according to (1.37) under an element of the global conformal group \(SL(2, \mathbb{C})\), but for those conformal transformation only. A field which is not primary is generally called secondary. For instance, the derivative of a primary field of conformal dimension \(h \neq 0\) is secondary.

1.2.4 Correlation functions

Expressed in terms of holomorphic and antiholomorphic coordinates, the correlation functions of \(n\) primary fields \(\phi_i\) with conformal dimension \(h_i\) and \(\bar{h}_i\) are

\[
\langle \phi_1(w_1, \bar{w}_1) \ldots \phi_n(w_n, \bar{w}_n) \rangle = \prod_{i=1}^{n} \left( \frac{dw}{dz} \right)^{-h_i} \left( \frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}_i} \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle .
\]  

(1.39)

This relation fixes the form of two- and three-point functions. Let us express them in terms of complex coordinates, taking spin into account when imposing rotation invariance. The distance \(x_{ij}\) is equal to \((z_{ij} \bar{z}_{ij})^{1/2}\) and the two-point function is

\[
\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \text{ if } \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} .
\]  

(1.40)

The two-point function vanishes if the conformal dimension of the two fields are different. The additional condition on the conformal dimension comes from rotation invariance: the sum of the spin within a correlator should be zero. The three-point function is

\[
\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3) \rangle = C_{123} \frac{1}{z_{12}^{h_1 + h_2 - h_3} z_{23}^{h_2 + h_3 - h_1} z_{13}^{h_3 + h_1 - h_2}} \times \\
\times \frac{1}{z_{12}^{h_1 + h_2 - h_3} z_{23}^{h_2 + h_3 - h_1} z_{13}^{h_3 + h_1 - h_2}} .
\]  

(1.41)
Again, the sum of the spins of the holomorphic part cancels that of the antiholomorphic part, thus ensuring rotation invariance. As before, global conformal invariance does not fix the precise form of the four-point function and beyond, because of the existence of anharmonic ratios is reduced, since the four points of the ratio are forced to lie in the same plane, which leads to an additional linear relation between them. Indeed, we have

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad 1 - \eta = \frac{z_{14}z_{23}}{z_{13}z_{24}} \quad \frac{\eta}{1 - \eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}}$$

(1.42)

The four-point function may then depend on $\eta$ and $\bar{\eta}$ in an arbitrary way, provided the result is real. The general expression for the four-point correlation function is

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{h_{i}/3 - h_{i} - h_{j}} z_{ij}^{\bar{h}_{i}/3 - h_{i} - \bar{h}_{j}}$$

(1.43)

where $h = \sum_{i=1}^{4} h_{i}$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_{i}$.

### 1.3 Free fields and Operator Product Expansion

It is typical of correlation functions to have singularities when the positions of two or more fields coincide. This reflects the infinite fluctuations of a quantum field taken at a precise position. To be more precise, the average

$$\phi_{av.} \equiv \frac{1}{V} \int_{V} d^{2}x \phi(x)$$

(1.44)

of a quantum field within a volume $V$ has a variance $s\langle \phi_{av.}\phi_{av.} \rangle$ which diverges as $V \to 0$. The operator product expansion (OPE) is the representation of a product of operators (at positions $z$ and $w$, respectively) by a sum of terms, each being a single operator, well-defined as $z \to w$, multiplied by a $c$-number function of $z - w$ possibly diverging as $z \to w$, and which embodies the infinite fluctuations as the two position tend toward each other.

The OPE of the energy-momentum tensor $T(z, \bar{z})$ with primary fields is written simply by removing the brackets $\langle \ldots \rangle$, it being understood that the OPE is meaningful only within correlation functions. For a single primary field $\phi$ of conformal dimension
1.3 Free fields and Operator Product Expansion

$h$ and $\bar{h}$, we have

\[ T(z)\phi(w, \bar{w}) \sim \frac{h}{(z-w)^2} \phi(w, \bar{w}) + \frac{1}{z-w} \partial_w \phi(w, \bar{w}) \]

\[ \bar{T}(\bar{z})\phi(w, \bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \phi(w, \bar{w}) \]  

(1.45)

Whenever appearing in OPEs, the symbol $\sim$ will mean equality modulo expressions regular as $w \to z$. Of course, the OPE contains also an infinite number of regular terms which, for the energy-momentum tensor, cannot be obtained from the conformal Ward identity. In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

\[ A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \]  

(1.46)

where the composite fields $\{AB\}_n(w)$ are nonsingular at $w = z$. For instance, $\{T\phi\}_1 = \partial_w \phi(w)$.

We stress that, so far, the quantities appearing in Eq. (1.45) are not operators but simply fields occurring within correlation functions. We shall now specific examples, in order to familiarize ourselves with basic techniques and with simple systems.

1.3.1 The free boson

From the point of view of the canonical or path integral formalism, the simplest conformal field theory is that of a free massless boson $\varphi$, with the following action:

\[ S = \frac{g}{2} \int d^2 x \partial_\mu \varphi \partial^\mu \varphi \]  

(1.47)

where $g$ is some normalization parameter. The two-point function, or propagator,is:

\[ \langle \varphi(x)\varphi(y) \rangle = -\frac{1}{4\pi g} \ln(x-y)^2 + \text{const.} \]  

(1.48)

In terms of complex coordinates, this is

\[ \langle \varphi(z, \bar{z})\varphi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} [\ln(z-w) + \ln(\bar{z}+\bar{w})] + \text{const.} \]  

(1.49)
The holomorphic and antiholomorphic components can be separated by taking the
derivatives $\partial_z \varphi$ and $\partial_{\bar{z}} \varphi$:

$$
\langle \partial_z \varphi(z, \bar{z}) \partial_{\bar{w}} \varphi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(z-w)^2}
$$

$$
\langle \partial_{\bar{z}} \varphi(z, \bar{z}) \partial_{\bar{w}} \varphi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(\bar{z}-\bar{w})^2} .
$$  (1.50)

In the following we shall concentrate on the holomorphic field $\partial \varphi \equiv \partial_z \varphi$. It is now clear
that the OPE of this field with itself is

$$
\partial \varphi(z) \partial \varphi(w) \sim \frac{1}{4\pi g} \frac{1}{(z-w)^2} .
$$  (1.51)

This OPE reflects the bosonic character of the field: exchanging the two factors does not affect the correlator.

The energy-momentum tensor associated with the free massless boson is

$$
T_{\mu\nu} = g \left( \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \varphi \partial^\rho \varphi \right) .
$$  (1.52)

Its quantum version in complex coordinates is

$$
T(z) = -2\pi g \, : \partial \varphi \partial \varphi : .
$$  (1.53)

Like all composite fields, the energy-momentum tensor has to be normal ordered, in
order to ensure the vanishing of its vacuum expectation value. More explicitly, the
exact meaning of the above expression is

$$
T(z) = -2\pi g \lim_{w \to z} (\partial \varphi(z) \partial \varphi(w) - \langle \partial \varphi(z) \partial \varphi(w) \rangle)
$$  (1.54)

The OPE of $T(z)$ with $\partial \varphi$ may be calculated from Wick’s theorem:

$$
T(z) \partial \varphi(w) = -2\pi g \, : \partial \varphi(z) \partial \varphi(z) : \partial \varphi(w)
$$

$$
\sim -4\pi g \, : \partial \varphi(z) \partial^2 \varphi(z) : \partial \varphi(w)
$$

$$
\sim \frac{\partial \varphi(z)^2}{(z-w)} .
$$  (1.55)

By expanding $\partial \varphi(z)$ around $w$, we arrive at the OPE

$$
T(z) \partial \varphi(w) \sim \frac{\partial \varphi(w)}{(z-w)^2} + \frac{\partial^2 \varphi(w)}{(z-w)} .
$$  (1.56)
This shows that $\partial \phi(z)$ is a primary field with conformal dimension $h = 1$. This was expected, since $\phi$ has no spin and no scaling dimension; hence its derivative has scaling dimension 1.

Wick’s theorem also allows us to calculate the OPE of the energy-momentum tensor with itself:

$$T(z)T(w) = 4\pi^2 g^2 : \partial \phi(z) \partial \phi(z) :: \partial \phi(w) \partial \phi(w) :$$

$$\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial \phi(z) \partial \phi(w) :}{(z-w)^2}$$

$$\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} .$$

(1.57)

In the second equation the first term is the result of two double contractions, whereas the second term comes from four single contractions. We immediately see that the energy-momentum tensor is not strictly a primary field, because of the anomalous term $\propto (z-w)^{-4}$, which does not appear in Eq.(1.45).

### 1.3.2 The free fermion

In two dimension, the Euclidean action of free Majorana fermion is

$$S = \frac{g}{2} \int d^2 x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi$$

(1.58)

where the Dirac matrices $\gamma^\mu$ satisfy the so-called Dirac algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} .$$

(1.59)

If $\eta^{\mu\nu} = \text{diag}(1,1)$, a representation thereof is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(1.60)

and therefore

$$\gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1) = 2 \begin{pmatrix} \partial z & 0 \\ 0 & \partial z \end{pmatrix} .$$

(1.61)

Writing the two-component spinor $\Psi$ as $(\psi, \bar{\psi})$, the action becomes

$$S = g \int d^2 x \left( \bar{\psi} \partial \bar{\psi} + \psi \partial \psi \right) .$$

(1.62)
The classical equation of motion are $\bar{\partial}\bar{\psi} = 0$ and $\partial\psi = 0$, whose solutions are any holomorphic function $\psi(z)$ and any antiholomorphic function $\bar{\psi}(\bar{z})$.

Our first task is to calculate the propagator $\langle \Psi_i(x)\Psi_j(y) \rangle$ ($i, j = 1, 2$). This is done by expressing the action as

$$S = \frac{1}{2} \int d^2x d^2y A_{ij}(x, y)\Psi_i(x)\Psi_j(y) \tag{1.63}$$

where we have defined the kernel

$$A_{ij}(x, y) = g\delta(x - y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu . \tag{1.64}$$

From previous knowledge of Gaussian integrals of Grassmann variables, the two-point function is then $K_{ij}(x, y) = (A^{-1})_{ij}(x, y)$, or

$$g\delta(x - y)(\gamma^0\gamma^\mu)_{ik}\frac{\partial}{\partial x^\mu}K_{kj}(x, y) = \delta(x - y)\delta_{ij} . \tag{1.65}$$

In terms of complex coordinates, this becomes

$$2g \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} \langle \psi(z, \bar{z})\psi(w, \bar{w}) \rangle & \langle \bar{\psi}(z, \bar{z})\bar{\psi}(w, \bar{w}) \rangle \\ \langle \bar{\psi}(z, \bar{z})\psi(w, \bar{w}) \rangle & \langle \bar{\psi}(z, \bar{z})\bar{\psi}(w, \bar{w}) \rangle \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} . \tag{1.66}$$

where we translated $x \rightarrow (z, \bar{z})$ and $y \rightarrow (w, \bar{w})$. The solution of the above matrix equation is easily read off:

$$\langle \psi(z, \bar{z})\psi(w, \bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{z - w} \tag{1.67}$$
$$\langle \bar{\psi}(z, \bar{z})\bar{\psi}(w, \bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{\bar{z} - \bar{w}}$$
$$\langle \psi(z, \bar{z})\bar{\psi}(w, \bar{w}) \rangle = 0$$

These, after differentiation, imply

$$\langle \partial_z \psi(z, \bar{z})\psi(w, \bar{w}) \rangle = -\frac{1}{2\pi g} \frac{1}{(z - w)^2} \tag{1.68}$$
$$\langle \partial_z \psi(z, \bar{z})\partial_w\psi(w, \bar{w}) \rangle = -\frac{1}{\pi g} \frac{1}{(z - w)^3}$$

and so on. The OPE of the fermion with itself (holomorphic components) is then

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g} \frac{1}{z - w} . \tag{1.69}$$
Again, this OPE reflects the anticommuting character of the field: exchanging the two factors \( \psi(z) \) and \( \psi(w) \) produces a sign that is mirrored in the two-point function.

Second, we wish to calculate the OPE of the energy-momentum tensor with \( \psi \) and with itself. The canonical energy-momentum tensor for the above action is given by

\[
\begin{align*}
T_{\bar{z}z} &= 2g \bar{\psi} \partial \psi \\
T_{zz} &= 2g \bar{\psi} \partial \bar{\psi} \\
T_{z\bar{z}} &= -2g \psi \partial \bar{\psi}
\end{align*}
\]

We see that the energy-momentum tensor is not identically symmetric, since \( T_{z\bar{z}} \neq 0 \). However, \( T_{z\bar{z}} \) vanishes if we use the classical equation of motion. The standard holomorphic component is then

\[
T(z) = -2\pi T_{zz} = -\frac{1}{2}\pi T_{\bar{z}z} = -\pi g :\psi(z)\partial \psi(z) :
\]

where, as before, we have used the normal-ordered product:

\[
: \psi(z) \partial \psi(z) := \lim_{w \to z} (\psi(z) \partial \psi(w) - \langle \psi(z) \partial \psi(w) \rangle) \, .
\]

Again, the OPE between \( T \) and the fermion \( \psi \) is calculated using Wick’s theorem:

\[
T(z)\psi(w) = -\pi g :\psi(z)\partial \psi(z) : \psi(w) \sim \frac{1}{2} \frac{\partial \psi(z)}{z-w} + \frac{1}{2} \frac{\psi(z)}{(z-w)^2}
\]

\[
\sim \frac{\frac{1}{2} \psi(w)}{(z-w)^2} + \frac{\partial \psi(w)}{z-w} \, .
\]

In contracting \( \psi(z) \) with \( \psi(w) \) we have carried \( \psi(w) \) over \( \partial \psi(z) \), thus introducing a \((-\) sign by Pauli’s principle. We see from this OPE that the fermion has a conformal dimension \( h = \frac{1}{2} \).

The OPE of \( T(z) \) with itself is calculated in the same way, with, however, a greater number of contractions:

\[
T(z)T(w) = \pi^2 g^2 :\psi(z)\partial \psi(z) :: \psi(w)\partial \psi(w) : \sim \frac{1}{4} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \, .
\]
This OPE has the same form as Eq.(1.57) except for a numerical difference in the anomalous term.

1.3.3 The central charge

The specific models treated in the last section leads us naturally to the following general OPE of the energy-momentum tensor:

\[ T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \]  (1.75)

where the constant \( c \) depends on the specific model under study: it is equal to 1 for free boson and \( \frac{1}{2} \) for the free fermion. This model-dependent constant is called the central charge. Except for this anomalous term, the OPE (1.75) simply means that \( T \) is a quasi-primary field with conformal dimension \( h = 2 \).

The central charge may not be determined from symmetry considerations: its value is determined by the short-distance behavior of the theory. For free fields it is determined by applying Wick’s theorem on the normal-ordered energy-momentum tensor. When two decoupled systems (e.g., two free fields) are put together, the energy-momentum tensor of the total system is simply the sum of the energy-momentum tensors associated with each part, and the associated central charge is simply the sum of the central charges of the parts. Thus, the central charge is somehow an extensive measure of the number of degrees of freedom of the system.

1.4 The operator formalism

1.4.1 Radial quantization

The operator formalism distinguishes a time direction from a space direction. This is natural in Minkowski space-time, but somewhat arbitrary in Euclidian space-time. In the context of statistical mechanics, choosing time and space directions amounts to selecting a direction in the lattice (e.g., rows) that we call “space”, and defining a space of states spanned by all the possible spin configurations along that direction. The time direction is then orthogonal to space, and the transfer matrix makes the link between state spaces at different times. In the continuum limit the lattice spacing disappears and we are free to choose the space direction in more exotic ways, for instance along
concentric circles centered at the origin. This choice of space and time leads to the so-called radial quantization of two-dimensional conformal field theories. In order to make this choice more natural from a Minkowski space point of view (in particular in the context of string theory), we may initially define our theory on an infinite space-time cylinder, with time $t$ going from $-\infty$ to $\infty$ along the “flat” direction of the cylinder, and space being compactified with a coordinate $x$ going from $0$ to $L$, the points $(0, t)$ and $(L, t)$ being identified. If we continue to Euclidean space, the cylinder is described by a single complex coordinate $\xi = t + ix$ (or equivalently $\xi = t - ix$). We then “explode” the cylinder onto the complex plane (or rather, the Riemann sphere) via the mapping

$$z = e^{2\pi \xi / L}. \quad (1.76)$$

The remote past ($t \to -\infty$) is situated at the origin $z = 0$, whereas the remote future ($t \to -\infty$) lies on the point at infinity on the Riemann sphere.

We must also assume the existence of a vacuum state $|0\rangle$ upon which a Hilbert space is constructed by application of creation operators (or their likes). In free-field theories, the vacuum may be defined as the state annihilated by the positive frequency part of the field. For an interacting field $\phi$, we assume that the Hilbert space is the same as for a free field, except that the actual energy eigenstates are different. We suppose then that the interaction is attenuated as $t \to \pm \infty$ and that the asymptotic field

$$\phi_{\text{in}} \propto \lim_{t \to -\infty} \phi(x, t) \quad (1.77)$$

is free. Within radial quantization, this asymptotic field reduces to a single operator, which, upon acting on $|0\rangle$, creates a single asymptotic “in” state:

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle. \quad (1.78)$$

**The Hermitian product**

On this Hilbert space we must also define a bilinear product, which we do indirectly by defining an asymptotic “out” state, together with the action of Hermitian conjugation on conformal fields. In Minkowsky space, Hermitian conjugation does not affect the space-time coordinates. Things are different in Euclidean space, since the Euclidean time $\tau = it$ must be reversed ($\tau \to -\tau$) upon Hermitian conjugation if $t$ is to be left unchanged. In radial quantization this corresponds to the mapping $z \to 1/z^*$. This
(almost) justifies the following definition of Hermitian conjugation on the real surface $\bar{z} = z^*$.

$$[\phi(z, \bar{z})] = \bar{z}^{-2h} z^{-2h} \phi(1/\bar{z}, 1/z) \quad (1.79)$$

where by assumption $\phi$ is a quasi-primary field of dimensions $h$ and $\bar{h}$. The prefactors on the r.h.s. may be justified by demanding that the asymptotic "out" state

$$\langle \phi_{\text{out}} | = | \phi_{\text{in}} \rangle \quad (1.80)$$

have a well-defined inner product with $| \phi_{\text{in}} \rangle$. Following the definition (1.79) of the Hermitian conjugation, this inner product is

$$\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle = \lim_{z, \bar{z}, w, \bar{w} \to 0} \langle 0 | \phi(z, \bar{z}) \phi(w, \bar{w}) | 0 \rangle$$

$$= \lim_{z, \bar{z}, w, \bar{w} \to 0} \bar{z}^{-2h} z^{-2h} \langle 0 | \phi(1/\bar{z}, 1/z) \phi(w, \bar{w}) | 0 \rangle$$

$$= \lim_{\xi, \bar{\xi} \to \infty} \bar{\xi}^{2h} \xi^{2h} \langle 0 | \phi(\bar{\xi}, \xi) \phi(0, 0) | 0 \rangle \quad (1.81)$$

Notice that the passage from a vacuum expectation value to a correlator in the last equation is correct since the operators are already time-ordered within radial quantization: the first one is associated with $t \to +\infty$ and the second one to $t \to -\infty$.

**Mode expansion**

A conformal field $\phi(z, \bar{z})$ of dimension $(h, \bar{h})$ may be mode expansion as follows:

$$\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}$$

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{m+h-1} \phi(z, \bar{z}) \quad (1.82)$$

A straightforward Hermitian conjugation on the real surface yields

$$\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}^\dagger \quad (1.83)$$

while the definition (1.79) gives instead

$$\phi(z, \bar{z}) = \bar{z}^{-2h} z^{-2h} \phi(1/\bar{z}, 1/z)$$

$$= z^{-2h} \bar{z}^{-2h} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \phi_{m,n} z^{m+h} \bar{z}^{n+\bar{h}}$$

$$= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \phi_{-m,-n} z^{-m-h} \bar{z}^{-n-\bar{h}} \quad (1.84)$$
These two expressions for the Hermitian conjugate of the mode expansion are compatible provided

$$\phi_{m,n}^\dagger = \phi_{-m,-n}.$$  

(1.85)

This is the usual expression for the Hermitian conjugate of modes, and justifies the extra powers of $h$ and $\bar{h}$ occurring in Eq.(1.82). If the “in” and “out” states are to be well-defined, the vacuum must obviously satisfy the condition

$$\phi_{m,n}|0\rangle = 0 \quad (m > -h, n > -\bar{h}).$$  

(1.86)

In the following, we shall lighten the notation by dropping the dependence of the field upon the antiholomorphic coordinate. Thus the mode expansion (1.82) will take the following simplified form:

$$\phi(z) = \sum_{m \in \mathbb{Z}} z^{-m-h} \phi_m$$

$$\phi_m = \frac{1}{2\pi i} \oint dz z^{m+h-1} \phi(z).$$  

(1.87)

### 1.4.2 Radial ordering and OPE

Within radial quantization, the time ordering that appears in the definition of correlation functions becomes a radial ordering, explicitly defined by

$$\mathcal{R} \Phi_1(z) \Phi_2(w) = \begin{cases} 
\Phi_1(z) \Phi_2(w) & \text{if } |z| > |w| \\
\Phi_2(w) \Phi_1(z) & \text{if } |z| < |w| 
\end{cases}.$$  

(1.88)

If the two fields are fermions, a minus sign is added in front of the second expression. Since all field operators within correlation functions must be radially ordered, so must be the l.h.s. of an OPE if it is to have an operator meaning. In particular, the OPEs written previously have an operator meaning only if $|z| > |w|$. We shall not write the radial ordering symbol $\mathcal{R}$ every time, but radial ordering will be implicit.

We now relate OPEs to commutation relations. Let $a(z)$ and $b(z)$ be two holomorphic fields, and consider the integral

$$\oint_w dz a(z)b(w)$$  

(1.89)

wherein the integration contour circles counterclockwise around $w$. This expression has an operator meaning within correlation functions as long as it is radially ordered.
Accordingly, we split the integration contour into two fixed-time circles (see Fig. 1.2) going in opposite directions. Our integral is now seen to be a commutator:

\[
\oint_w da(z)b(w) = \oint_{C_1} da(z)b(w) - \oint_{C_2} db(w)a(z)
\]

\[
= [A, b(w)]
\] (1.90)

where the operator \( A \) is the integral over the space at fixed time (i.e., a contour integral) of the field \( a(z) \):

\[
A = \oint a(z)dz
\] (1.91)

and where \( C_1 \) and \( C_2 \) are fixed-time contours (circles centered around the origin) of radii respectively equal to \(|w|+\varepsilon\) and \(|w|−\varepsilon\), \( \varepsilon \) being infinitesimal. We must allow an arbitrary number of different fields to lie beside \( b(w) \) and \( a(z) \) within a generic correlator; the decomposition into two contours is valid as long as \( b(w) \) is only other field having a singular OPE with \( a(z) \), which lies between the two circles \( C_1 \) and \( C_2 \); this is the reason for taking the limit \( \varepsilon \to 0 \). The commutator obtained is then, in some sense, an equal time commutator. We note that if \( a \) and \( b \) are fermions, the commutator is replaced by an anticommutator. In practice, the integral (1.89) is evaluated by substituting the OPE of \( a(z) \) with \( b(w) \), of which only the term in \( 1/(z-w) \) contributes, by the theorem of residues.

The commutator \([A, B]\) of two operators, each the integral of a holomorphic field,
1.4 The operator formalism

is obtained by integrating Eq.(1.90) over \( w \):

\[
\left[ A, B \right] = \oint_{w} d\omega \oint_{w} d\omega a(\omega)b(w)
\]

(1.92)

where the integral over \( z \) is taken around \( w \), and the integral over \( w \) around the origin, and where

\[
A = \oint a(z)dz \\
B = \oint b(z)dz.
\]

(1.93)

Formulas (1.90) and (1.92) relate OPEs to commutation relations, and allow us to translate into operator language the dynamicla or symmetry statements contained in the OPE.

1.4.3 The Virasoro algebra

Conformal generators

We let \( \epsilon(z) \) be the holomorphic component of an infinitesimal conformal change of coordinates. We then define the conformal charge

\[
Q_\epsilon = \frac{1}{2\pi i} \oint d\epsilon(z)T(z).
\]

(1.94)

Using Eq.(1.90), the conformal Ward identity translates to

\[
\delta_\epsilon \Phi(w) = -[Q_\epsilon, \Phi(w)]
\]

(1.95)

which means that the operator \( Q_\epsilon \) is the generator of conformal transformations.

We expand the energy-momentum tensor according to (1.82):

\[
T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2}L_n \\
\bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n
\]

(1.96)

We also expand the infinitesimal conformal change \( \epsilon(z) \) as

\[
\epsilon(z) = \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n.
\]

(1.97)
Conformal Field Theory

Then expression (1.94) for the conformal charge becomes

\[ Q_\epsilon = \sum_{n \in \mathbb{Z}} \epsilon_n L_n. \quad (1.98) \]

The mode operators \( L_n \) and \( \bar{L}_n \) of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like \( \ell_n \) and \( \bar{\ell}_n \) of Eq.(1.33) are the generators of conformal mappings on the space of functions. Likewise, the generators of \( SL(2, \mathbb{C}) \) in the Hilbert space are \( L_{-1}, L_0 \) and \( L_1 \) (and their antiholomorphic counterparts). In particular, the operator \( L_0 + \bar{L}_0 \) generates the dilations \((z, \bar{z}) \to \lambda(z, \bar{z})\), which are nothing but time translations in radial quantization. Thus, \( L_0 + \bar{L}_0 \) is proportional to the Hamiltonian of the system.

The classical generators of the local conformal transformations obey the algebra (1.34). The quantum generators \( L_n \) obey an identical algebra, except for a central term:

\[
\begin{align*}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_m, \bar{L}_m] &= 0 \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}
\end{align*}
\]

where \( c \) is the central charge of the theory. This is the Virasoro algebra. It may be derived from mode expansion (1.96), the OPE (1.75) and (1.92):

\[
\begin{align*}
[L_n, L_m] &= \frac{1}{(2\pi i)^2} \oint_0 dw w^{m+1} \oint_w d\bar{z} z^{n+1} \left\{ \frac{c/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z+w)} + \text{reg.} \right\} \\
&= \frac{1}{2\pi i} \oint_0 dw w^{m+1} \left\{ \frac{1}{12} c(n+1)n(n-1)w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial T(w) \right\} \\
&= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + 2(n+m)L_{m+n} - \frac{1}{2\pi i} \oint_0 dw(n+m+2)w^{n+m+1}T(w) \\
&= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + (n-m)L_{m+n}
\end{align*}
\]

where, in the third equation, the last term has been integrated by parts. The last equation of (1.99) is demonstrated in exactly the same way, and the second equation of (1.99) follows from the trivial OPE \( T(z)\bar{T}(\bar{w}) \sim 0 \).
1.4 The operator formalism

The Hilbert space

The vacuum state \( |0\rangle \) of the Hilbert space of a conformal field theory must be invariant under global conformal transformations. This means that it must be annihilated by \( L_{-1}, L_0 \) and \( L_1 \) and their antiholomorphic counterparts to fix the ground state energy to zero. This, in turn, can be recovered from the condition that \( T(z)|0\rangle \) and \( \bar{T}(\bar{z})|0\rangle \) are well-defined as \( z, \bar{z} \to 0 \), which implies

\[
L_n|0\rangle = 0 \quad \bar{L}_n|0\rangle = 0 \quad (1.101)
\]

for \( n \geq -1 \), which includes as subcondition the invariance of the vacuum \( |0\rangle \) with respect to global conformal group. It also implies the vanishing of the vacuum expectation value of the energy-momentum tensor:

\[
\langle 0|T(z)|0\rangle = \langle 0|\bar{T}(\bar{z})|0\rangle = 0 . \quad (1.102)
\]

Primary fields, when acting on the vacuum, create asymptotic states, eigenstates of the Hamiltonian. A simple demonstration follows from OPE between \( T(z) \) and a primary field \( \phi(z, \bar{z}) \) of dimensions \( (h, \bar{h}) \), translated into operator language:

\[
[L_n, \phi(w, \bar{w})] = \frac{1}{2\pi i} \oint_w dzz^{n+1}T(z)\phi(w, \bar{w})
\]

\[
= \frac{1}{2\pi i} \oint_w dzz^{n+1} \left[ \frac{h\phi(w, \bar{w})}{(z-w)^2} + \frac{\partial \phi(w, \bar{w})}{z-w} + \text{reg.} \right]
\]

\[
= h(n+1)w^n\phi(w, \bar{w}) + w^{n+1}\partial \phi(w, \bar{w}) \quad (n \geq -1) . \quad (1.103)
\]

The antiholomorphic counterpart of this relation is

\[
[L_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n\phi(w, \bar{w}) + \bar{w}^{n+1}\bar{\partial} \phi(w, \bar{w}) (n \geq -1) . \quad (1.104)
\]

After applying these relations to the asymptotic state

\[
|h, \bar{h}\rangle \equiv \phi(0, 0)|0\rangle , \quad (1.105)
\]

we conclude that

\[
L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle \quad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle . \quad (1.106)
\]

Thus \( |h, \bar{h}\rangle \) is an eigenstate of the Hamiltonian. Likewise, we have

\[
L_n|h, \bar{h}\rangle = 0 \quad \bar{L}_n|h, \bar{h}\rangle = 0 \quad \text{if } n > 0 . \quad (1.107)
\]
1.5 The free boson

This section gives a detailed account of the canonical quantization of the free boson on the cylinder. The mode expansions are obtained, after imposing the appropriate boundary conditions. The mapping from the cylinder to the complex plane is used to define the conformal generators and, in particular, the vacuum energies. Free-field theories are of special importance not only because they can be solved explicitly, but also because they are the building blocks of more complicated models, or can be shown to be equivalent to interesting statistical models.

1.5.1 Canonical Quantization on the cylinder

We let $\varphi(x, t)$ be a free Bose field defined on a cylinder of circumference $L$, i.e. $\varphi(x + L, t) \equiv \varphi(x, t)$. This field may be Fourier expanded as follows:

$$\varphi(x, t) = \sum_n e^{2\pi inx/L} \varphi_n(t)$$

$$\varphi_n(t) = \frac{1}{L} \int dx e^{-2\pi inx/L} \varphi(x, t) . \quad (1.108)$$

In term of the Fourier coefficients $\varphi_n$, the free field Lagrangian

$$\frac{g}{2} \int dx \left[ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right] \quad (1.109)$$

becomes

$$\frac{gL}{2} \sum_n \left[ \varphi_n \dot{\varphi}_{-n} - \left( \frac{2\pi n}{L} \right)^2 \varphi_n \varphi_{-n} \right] . \quad (1.110)$$

The momentum conjugate to $\varphi_n$ is

$$\pi_n = gL \dot{\varphi}_{-n} , \quad [\varphi_n, \pi_m] = i\delta_{nm} \quad (1.111)$$

and the Hamiltonian is

$$H = \frac{1}{2gL} \sum \left[ \pi_n \pi_{-n} + (2\pi ng)^2 \varphi_n \varphi_{-n} \right] . \quad (1.112)$$

We notice that $\varphi_n^\dagger = \varphi_{-n}$, and similarly $\pi_n^\dagger = \pi_{-n}$. Of course, this Hamiltonian represent a sum of decoupled harmonic oscillators, of frequencies $\omega_n = 2\pi |n|/L$. The vanishing
of one of the frequencies \((n = 0)\) is of special importance, since it is a consequence of
the absence of a mass term, which, with the boundary conditions chosen, is equivalent
to conformal invariance.

The usual procedure is to define creation and annihilation operators \(\tilde{a}_n\) and \(\tilde{a}_n^\dagger\):

\[
\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\varphi_n + i\pi_{-n}) \quad (1.113)
\]

such that \([\tilde{a}_n, \tilde{a}_m] = 0\) and \([\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{mn}\); this, of course, does not work for the zero-
mode \(\varphi_0\). Instead of these we shall use the following operators:

\[
a_n = \begin{cases} 
-i\sqrt{n}\tilde{a}_n & (n > 0) \\
-\sqrt{-n}\tilde{a}_n^\dagger & (n < 0)
\end{cases} \quad (1.114)
\]

\[
\tilde{a}_n = \begin{cases} 
-i\sqrt{n}\tilde{a}_n & (n > 0) \\
i\sqrt{-n}\tilde{a}_n^\dagger & (n < 0)
\end{cases} \quad (1.115)
\]

and treat the zero-mode \(\varphi_0\) separately. The associated commutation relations are

\[
[a_n, a_m] = n\delta_{n+m}
\]

\[
[a_n, \tilde{a}_m] = 0
\]

\[
[\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m} \quad . \quad (1.116)
\]

The Hamiltonian is then expressible as

\[
H = \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L}\sum_{n \neq 0}(a_{-n}a_n + \tilde{a}_{-n}\tilde{a}_n) \quad . \quad (1.117)
\]

The commutation relations (1.116) lead to relation

\[
[H, a_{-m}] = \frac{2\pi}{L}ma_{-m} \quad (1.118)
\]

which means that \(a_{-m} (m > 0)\), when applied to an eigenstate of \(H\) of energy \(E\),
produces another wigenstate with energy \(E + 2m\pi/L\).

Since the Fourier modes are

\[
\varphi_n = \frac{i}{n\sqrt{4\pi g}}(a_n - \tilde{a}_{-n}) \quad (1.119)
\]

the mode expansion at \(t = 0\) may be writtena s

\[
\varphi(x) = \varphi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n \neq 0} \frac{1}{n}(a_n - \tilde{a}_{-n})e^{2\pi inx/L} \quad . \quad (1.120)
\]
The time evolution of the operators $\varphi_0$, $a_n$ and $\bar{a}_n$ in the Hisenberg picture follows immediately from the above Hamiltonian.

$$\varphi_0(t) = \varphi_0(0) + \frac{1}{gL}\pi_0 t$$

$$a_n(t) = a_n(0)e^{-2\pi i nt/L}$$

$$\bar{a}_n(t) = \bar{a}_n(0)e^{-2\pi i nt/L} \quad .$$

(1.121)

In terms of constant operators, the mode expansion of the field at arbitrary time is then

$$\varphi(x, t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n e^{2\pi i n(x-t)/L} - \bar{a}_{-n} e^{2\pi i n(x+t)/L}) \quad .$$

(1.122)

If we go over to Euclidean space-time (i.e., replace $t$ by $-i\tau$) and use the conformal coordinates

$$z = e^{2\pi (\tau - ix)/L} \quad \bar{z} = e^{2\pi (\tau + ix)/L}$$

(1.123)

we finally obtain the expansion

$$\varphi(z, \bar{z}) = \varphi_0 - \frac{i}{4\pi g} \pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \quad .$$

(1.124)

We know that $\varphi$ is not a primary field, but that its derivatives $\partial \varphi$ and $\bar{\partial} \varphi$ are. We concentrate on the holomorphic field $\partial \varphi$. From Eq.(1.124) the following expansion follows.

$$i\partial \varphi(z) = \frac{1}{4\pi g} \frac{\pi_0}{z} + \frac{1}{\sqrt{4\pi g}} \sum_{n \neq 0} a_n z^{-n-1} \quad .$$

(1.125)

This expansion coincides with the general conformal mode expansion (1.82). We may introduce two operators $a_0$ and $\bar{a}_0$:

$$a_0 \neq \bar{a}_0 \neq \frac{\pi_0}{\sqrt{4\pi g}}$$

(1.126)

which allow us to include the zero-mode term into sum:

$$i\partial \varphi(z) = \frac{1}{\sqrt{4\pi g}} \sum_n a_n z^{-n-1} \quad .$$

(1.127)

The periodicity condition on the field $\varphi$ is the source of the decoupling between holomorphic and antiholomorphic excitations. Thus, the operators $a_n$ create or destroy "right-moving" excitations, whereas the $\bar{a}_n$ are associated with the "left-moving" excitations.
1.5 The free boson

1.5.2 Vertex Operators

Since the canonical scaling dimension of the boson $\phi$ vanishes, it is possible to construct an infinite variety of local fields related to $\phi$ without introducing a scale, namely the so-called vertex operators:

$$V_\alpha(z, \bar{z}) = e^{i\alpha \varphi(z, \bar{z})} : .$$  \hfill (1.128)

The normal ordering has the following meaning, in terms of the operators appearing in the mode expansion (1.124):

$$V_\alpha(z, \bar{z}) = \exp \left\{ i\alpha \varphi_0 + \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} (a_{-n} z^n + \bar{a}_{-n} \bar{z}^n) \right\} \times$$

$$\times \exp \left\{ \frac{\alpha}{4\pi g} \pi_0 - \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \right\} .$$  \hfill (1.129)

Within each exponential, the different operators commute.

We shall now demonstrate that fields are primary, with holomorphic and antiholomorphic dimensions

$$h(\alpha) = \bar{h}(\alpha) = \frac{\alpha^2}{8\pi g} .$$  \hfill (1.130)

We first calculate the OPE of $\partial \varphi$ with $V_\alpha$:

$$\partial \varphi(z)V_\alpha(w, \bar{w}) = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \partial \varphi(z) : \varphi(w, \bar{w})^n :$$

$$\sim - \frac{1}{4\pi g} \frac{1}{z-w} \sum_{n=1}^{\infty} (i\alpha)^n \frac{1}{(n-1)!} : \varphi(w, \bar{w})^{n-1} :$$

$$\sim - \frac{i\alpha}{4\pi g} \frac{V_\alpha(w, \bar{w})}{z-w} .$$  \hfill (1.131)
Next, we calculate the OPE of $V_\alpha$ with the energy-momentum tensor:

$$T(z)V_\alpha(w, \bar{w}) = -2\pi g \sum_{n=0}^\infty \frac{(i\alpha)^n}{n!} : \partial \varphi(z) \partial \varphi(z) :: \varphi(w, \bar{w})^n :$$

$$\sim -\frac{1}{8\pi g} \sum_{n=2}^\infty \frac{(i\alpha)^n}{(n-2)!} : \varphi(w, \bar{w})^{n-2} :$$

$$+ \frac{1}{z-w} \sum_{n=1}^\infty \frac{(i\alpha)^n}{n!} n : \partial \varphi(z) \varphi(w, \bar{w})^{n-1} :$$

$$\sim \frac{\alpha^2}{8\pi g} \frac{V_\alpha(w, \bar{w})}{(z-w)^2} + \frac{\partial_w V_\alpha(w, \bar{w})}{z-w}.$$  \hspace{1cm} (1.132)

To the $n$-th term in the summation we have applied $2n$ single contractions and $n(n-1)$ double contractions. We have replaced $\partial \varphi(z)$ by $\partial \varphi(w)$ in the last equation since the difference between the two leads to a regular term. It is now clear by the form of this OPE that $V_\alpha$ is primary, with the conformal weight given above. The OPE with $\bar{T}$ has exactly the same form.

In order to calculate the OPE of products of vertex operators, we may use the following relation for a single harmonic oscillator:

$$: e^{A_1} :: a^{A_2} := : e^{A_1 + A_2} : e^{(A_1, A_2)}$$  \hspace{1cm} (1.133)

where $A_i = \alpha_i a + \beta_i a^\dagger$ is some linear combination of annihilation and creation operators. Since a free field is simply an assembl of decoupled harmonic oscillators, the same relation holds if $A_1$ and $A_2$ are linear function of a free field. In particular, we may write

$$: e^{a\varphi_1} :: a^{b\varphi_2} := : e^{a\varphi_1 + b\varphi_2} : e^{ab(\varphi_1, \varphi_2)}.$$  \hspace{1cm} (1.134)

Applied to vertex operators, this relation yields

$$V_\alpha(z, \bar{z})V_\beta(w, \bar{w}) \sim |z-w|^{2\alpha\beta/4\pi g} V_{\alpha+\beta}(w, \bar{w}) + \ldots.$$  \hspace{1cm} (1.135)

In general, the correlator of a string of vertex operators $V_{\alpha_i}$ vanishes unless the sum of the charges vanishes: $\sum_i \alpha_i = 0$.

### 1.5.3 The Fock space

The independence of the Hamiltonian (1.117) on $\varphi_0$ implies that the eigenvalue of $\pi_0$ is a “good” quantum number, which may label different sets of $H$. Since $\pi_0$ commutes
with all the \( a_n \) and \( \bar{a}_n \), these operators cannot change the value of \( \pi_0 \) and the Fock space is built upon a one-parameter family of vacua \(|\alpha\rangle\), where \( \alpha \) is the continuous eigenvalue of \( a_0 = \pi_0/\sqrt{4\pi g} \). As mentioned above, the conformal modes \( a_n \) and \( \bar{a}_n \) are annihilation operators for \( n > 0 \) and creation operators for \( n < 0 \):

\[
a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \quad (n > 0) \quad \text{with} \quad a_0|\alpha\rangle = \bar{a}_0|\alpha\rangle = \alpha|\alpha\rangle .
\]

As we know, the holomorphic energy-momentum tensor is given by

\[
T(z) = -2\pi g \partial \varphi(z) \partial \varphi(z) :
\]

\[z^{-n-m-2} = \frac{1}{2} \sum_{n,m \in \mathbb{Z}} :a_n a_m : \]

which implies

\[
L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} a_m \quad (n \neq 0)
\]

\[
L_0 = \sum_{n>0} a_{-n} a_n + \frac{1}{2} a_0^2
\]

and similarly for the antiholomorphic modes. The Hamiltonian (1.117) may then be written as

\[
H = \frac{2\pi}{L} (L_0 + \bar{L}_0) .
\]

This confirm the role of \( L_0 + \bar{L}_0 \) as Hamiltonian, modulo some multiplicative factor. The mode operators \( a_m \) play a role towards \( L_0 \) similar to \( L_m \), because of the commutation \([L_0, a_{-m}] = ma_{-m}\). This does not mean that \( a_m \) is equivalent to \( L_m \), but rather that its effect on the conformal dimension (the eigenvalue of \( L_0 \)) is the same as that of \( L_m \).

From expression (1.138) we see that the vacuum \(|\alpha\rangle\) has conformal dimension \( \frac{\alpha^2}{8\pi g} \). The elements of the Fock space are, of course, obtained by acting on \(|\alpha\rangle\) with the creation \( a_{-n} \) and \( \bar{a}_{-n} \) (\( n > 0 \)):

\[
a_{n_1} a_{n_2} \ldots a_{n_i} \bar{a}_{n_1} \bar{a}_{n_2} \ldots \quad (n_i, m_j \geq 0) .
\]

These states are eigenstates of \( L_0 \) with conformal dimensions

\[
h = \frac{\alpha^2}{8\pi g} + \sum_j jn_j
\]

\[
\bar{h} = \frac{\alpha^2}{8\pi g} + \sum_j jm_j .
\]
We shall proceed by showing that $\mathcal{V}(0)|0\rangle$ is an eigenstate of $\pi_0$ with eigenvalue $\alpha$, and that $a_n|\alpha\rangle = 0$ for $n > 0$. Now we use the Baker-Hausdorff formula


(1.142)

where $[B, A]$ is assumed to be a constant. If we set $B = \pi_0$ and $A = i\alpha \varphi(z, \bar{z})$, we find

$$[\pi_0, \mathcal{V}_\alpha] = \alpha \mathcal{V}_\alpha \ .$$

(1.143)

This relation, applied at $z = 0$ to the invariant vacuum $|0\rangle$, gives

$$\pi_0 \mathcal{V}_\alpha(0)|0\rangle = \alpha \mathcal{V}_\alpha(0)|0\rangle$$

(1.144)

which is one of the desired elements. The other is obtained by setting $B = a_n$; it follows that

$$[a_n, \mathcal{V}_\alpha(z, \bar{z})] = -\alpha z^n \mathcal{V}_\alpha(z, \bar{z}) \ .$$

(1.145)

At $z = 0$, this relation yields $a_n \mathcal{V}|0\rangle = 0$ when applied on $|0\rangle$. A similar relation holds for $\bar{a}_n \ (n > 0)$.
Chapter 2

Boundary Conformal Field Theory

In any field theory in a domain with a boundary, one needs to consider how to impose a set of consistent boundary conditions. Since Conformal Field Theory is formulated independently of a particular set of fundamental fields and a lagrangian, this must be done in a more general manner. A natural requirement is that the off-diagonal component $T_{∥⊥}$ of the stress tensor parallel/perpendicular to the boundary should vanish. This is called the conformal boundary condition. If the boundary is parallel to the time axis, it implies that there is no momentum flow across the boundary. Moreover, it can be argued that, under the Renormalization Group, any uniform boundary condition will flow into a conformally invariant one. For a given bulk CFT, however, there may be many possible distinct such boundary conditions, and it is one task of BCFT to classify these.

To begin with, take the domain to be the upper half plane, so that the boundary is the real axis. The conformal boundary condition then implies that $T(z) = \bar{T}(\bar{z})$ when $z$ is on the real axis. This has the immediate consequence that correlators of $T$ are those of $\bar{T}$, analytically continued into the lower half plane. The conformal Ward identity now reads

$$
\langle T(z) \prod_j \phi_j(z_j, \bar{z}_j) \rangle = \sum_j \left( \frac{h_j}{(z - z_j)^2} + \frac{1}{z - z_j} \partial z_j \right) \left( \frac{\bar{h}_j}{(\bar{z} - \bar{z}_j)^2} + \frac{1}{\bar{z} - \bar{z}_j} \partial \bar{z}_j \right) \langle \prod_j \phi_j(z_j, \bar{z}_j) \rangle . \tag{2.1}
$$

In radial quantization, in order that the Hilbert spaces defined on different hypersurfaces be equivalent, one must choose semicircles centered on some point on the
boundary, conventionally the origin. The dilatation operator is

\[ D = \frac{1}{2\pi i} \int_S z T(z) dz - \frac{1}{2\pi i} \int_S \bar{z} \bar{T}(\bar{z}) d\bar{z}, \]  

(2.2)

where \( S \) is a semicircle. Using the conformal boundary condition, this can also be written as

\[ D = L_0 = \frac{1}{2\pi i} \int_C z T(z) dz, \]  

(2.3)

where \( C \) is a complete circle around the origin. As before, one may similarly define the \( L_n \), and they satisfy a Virasoro algebra.

Note that there is now only one Virasoro algebra. This is related to the fact that conformal mappings which preserve the real axis correspond to real analytic functions. The eigenstates of \( L_0 \) correspond to boundary operators \( \phi_j(0) \) acting on the vacuum state \( |0\rangle \). It is well-known that in a renormalizable QFT operators at the boundary require a different renormalization from those in the bulk, and this will in general lead to a different set of conformal weights. It is one of the tasks of BCFT to determine these, for a given allowed boundary condition.

However, there is one feature unique to boundary CFT in two dimensions. Radial quantization also makes sense, leading to the same form (2.3) for the dilation operator, if the boundary conditions on the negative and positive real axes are different. As far as the structure of BCFT goes, correlation functions with this mixed boundary condition behave as though a local scaling field were inserted at the origin. This has led to the term “boundary condition changing (bcc) operator”, but it must be stressed that these are not local operators in the conventional sense.

### 2.1 The annulus partition function

Just as consideration of the partition function on the torus illuminates the bulk operator content \( n_{h,\bar{h}} \), it turns out that consistency on the annulus helps classify both the allowed boundary conditions, and the boundary operator content. To this end, consider a CFT in an annulus formed of a rectangle of unit width and height \( \delta \), with the top and bottom edges identified (see Fig. 2.1).

The boundary conditions on the left and right edges, labelled by \( a, b, \ldots \), may be different. The partition function with boundary conditions \( a \) and \( b \) on either edge is denoted by \( Z_{ab}(\delta) \).
2.1 The annulus partition function

One way to compute this is by first considering the CFT on an infinitely long strip of unit width. This is conformally related to the upper half plane (with an insertion of boundary condition changing operators at 0 and \( \infty \) if \( a \neq b \)) by the mapping \( z \to (1/\pi) \ln z \). The generator of infinitesimal translations along the strip is

\[
H_{ab} = \pi D - \pi c/24 = \pi L_0 - \pi c/24 .
\]  

Thus for the annulus

\[
Z_{ab}(\delta) = \text{Tr} \ e^{-\delta H_{ab}} = \text{Tr} \ q^{L_0 - c/24} ,
\]

with \( q \equiv e^{-\pi \delta} \). As before, this can be decomposed into characters

\[
Z_{ab}(\delta) = \sum_h n_{h}^{ab} \chi_h(q) ,
\]

but note that now the expression is linear. The non-negative integers \( n_{ab}^h \) give the operator content with the boundary conditions \((ab)\): the lowest value of \( h \) with \( n_{ab}^h > 0 \) gives the conformal weight of the bcc operator, and the others give conformal weights of the other allowed primary fields which may also sit at this point.

On the other hand, the annulus partition function may be viewed, up to an overall rescaling, as the path integral for a CFT on a circle of unit circumference, being propagated for (imaginary) time \( \delta^{-1} \). From this point of view, the partition function is no longer a trace, but rather the matrix element of \( e^{-H/\delta} \) between boundary states:

\[
Z_{ab}(\delta) = \langle a| e^{-H/\delta} |b \rangle .
\]

Note that \( H \) is the same hamiltonian that appears on the cylinder, and the boundary states lie in the Hilbert space of states on the circle. They can be decomposed into
linear combinations of states in the representation spaces of the two Virasoro algebras \( (\mathcal{V}_h) \), labelled by their lowest weights \( (h, \bar{h}) \).

The conformal boundary condition applied to the circle implies that \( L_n = \overline{L}_{-n} \). This means that any boundary state \( |B\rangle \) lies in the subspace satisfying

\[
L_n |B\rangle = \overline{L}_{-n} |B\rangle.
\]

Moreover, because of the decomposition of \( H \), \( |B\rangle \) is also some linear superposition of states from \( \mathcal{V}_h \otimes \overline{\mathcal{V}}_\bar{h} \). This condition can therefore be applied in each subspace. Taking \( n = 0 \) in (2.8) constrains \( \bar{h} = h \). For simplicity, consider only the diagonal CFTs with \( n_{h,\bar{h}} = \delta_{h,\bar{h}} \). It can then be shown that the solution of (2.8) is unique and has the following form. The subspace at level \( N \) of \( \mathcal{V}_h \) has dimension \( d_h(N) \). Denote an orthonormal basis by \( |h,N;j\rangle \), with \( 1 \leq j \leq d_h(N) \), and the same basis for \( \overline{\mathcal{V}}_\bar{h} \) by \( |\bar{h},\bar{N};\bar{j}\rangle \). The solution to (2.8) in this subspace is then

\[
|h\rangle \equiv \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} |h,N;j\rangle \otimes |\bar{h},\bar{N};\bar{j}\rangle.
\]  

These are called Ishibashi states. Matrix elements of the translation operator along the cylinder between them are simple:

\[
\langle a|h'|e^{-\frac{4\pi i}{\delta}}|h\rangle = \sum_{N,N'=0}^{\infty} \sum_{j=1}^{d_h(N)} \sum_{j'=1}^{d_{h'}(N')} \langle h',N';j'\rangle \otimes \langle h',\bar{N}';\bar{j}'\rangle e^{-\frac{2\pi i}{\delta} (L_0 + \overline{L}_0 - \frac{c}{24})} |h,N;j\rangle \otimes |\bar{h},\bar{N};\bar{j}\rangle
\]

\[
= \delta_{h'\bar{h}} \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} e^{-4\pi(i/\delta)(h-N-(c/24))} = \delta_{h'\bar{h}} \chi_h(e^{-4\pi(i/\delta)}).
\]

The physical boundary states satisfying (2.6), sometimes called the Cardy states, are linear combinations of the Ishibashi states:

\[
|a\rangle = \sum_h \langle h|a\rangle |h\rangle.
\]

Equating the two different expressions (2.6,2.7) for \( Z_{ab} \), and using the linear independence of the characters gives the (equivalent) conditions:

\[
n_{ab}^h = \sum_{h'} S_{h'}^h \langle a|h'\rangle \langle h'|b\rangle,
\]

\[
\langle a|h'\rangle \langle h'|b\rangle = \sum_h S_{h'}^h n_{ab}^h.
\]
2.2 Review of boundary conformal field theory techniques

These are called the Cardy conditions. The requirements that the right hand side of (2.12) should give a non-negative integer, and that the right hand side of (2.13) should factorize in $a$ and $b$, give highly nontrivial constraints on the allowed boundary states and their operator content.

For the diagonal CFTs considered here (and for the nondiagonal minimal models) a complete solution is possible. It can be shown that the elements $S^h_0$ are all non-negative, so one may choose $\langle h| \tilde{0} \rangle = (S^h_0)^{1/2}$. This defines a boundary state

$$|\tilde{0}\rangle \equiv \sum_h (S^h_0)^{1/2} |h\rangle, \quad (2.14)$$

and a corresponding boundary condition such that $n^h_{00} = \delta_{h0}$. Then, for each $h' \neq 0$, one may define a boundary state

$$\langle h| \tilde{h}' \rangle \equiv S^h_{00} / (S^h_0)^{1/2}. \quad (2.15)$$

From (2.12), this gives $n^h_{00} = \delta_{h'0}$. For each allowed $h'$ in the torus partition function, there is therefore a boundary state $|\tilde{h}'\rangle$ satisfying the Cardy conditions. However, there is a further requirement:

$$n^h_{h'h''h'} = \sum_\ell S^h_\ell S^{h'}_\ell S^{h''}_\ell / S^h_0 \quad (2.16)$$

should be a non-negative integer. Remarkably, this combination of elements of $S$ occurs in the Verlinde formula, which follows from considering consistency of the CFT on the torus. This states that the right hand side of (2.16) is equal to the fusion algebra coefficient $N^h_{h'h''}[33]$. Since these are non-negative integers, the consistency of the above ansatz for the boundary states is consistent.

We conclude that, at least for the diagonal models, there is a bijection between the allowed primary fields in the bulk CFT and the allowed conformally invariant boundary conditions. For the minimal models, with a finite number of such primary fields, this correspondence has been followed through explicitly.

2.2 Review of boundary conformal field theory techniques

Here we briefly review the boundary conformal field theory (BCFT) techniques, developed largely by J. Cardy and I. Affleck (see [34] and [35] and references therein), which have been applied to various quantum impurity problems.
Boundary Conformal Field Theory

We consider a general conformal field theory, such as a collection of free bosons or fermions, defined on the 1/2-line, \( x \geq 0 \), with a conformally invariant boundary condition at \( x = 0 \). In general two-point correlation functions will be affected by the boundary. While the correlation function of some operator, \( O(x, \tau) \) may behave as:

\[
\langle O(x, \tau)O(x', \tau') \rangle = \frac{1}{\sqrt{(x-x')^2 + (\tau-\tau')^2}^\Delta},
\]

(2.17)
in the bulk (i.e. in the absence of a boundary or far from the boundary), in the limit \( x, x' \ll |\tau - \tau'| \) it behaves as:

\[
\langle O(x, \tau)O(x', \tau') \rangle \propto \frac{1}{|\tau - \tau'|^{2\Delta_B}}.
\]

(2.18)

Here \( \Delta \) is the bulk scaling dimension of the operator, \( O \) and \( \Delta_B \) is its boundary scaling dimension. In general, \( \Delta_B \) will depend on the boundary conditions. The boundary conditions are always assumed to imply:

\[
P(t, 0) = 0,
\]

(2.19)
where:

\[
P(t, x) \equiv H_R(t - x) - H_L(t + x).
\]

(2.20)

\( P \) is the momentum density and \( H_{L/R} \) are the left and right-moving parts of the Hamiltonian density.

We can determine all boundary scaling dimensions for an arbitrary conformally invariant boundary condition, \( A \), denoted \( \Delta_A \), from the finite size spectrum of the Hamiltonian on a strip of length \( l \) with boundary conditions \( A \) at both ends. This is done by the conformal transformation:

\[
z = le^{\pi w/l},
\]

(2.21)

Here

\[
z = \tau + ix,
\]

(2.22)
covers the infinite half-plane and \( w = u + iv \) the strip with \( 0 < v < l \). The correlation function for two points at the edge of the strip can be obtained by this conformal transformation:

\[
\langle O(u_1)O(u_2) \rangle = \left\{ \frac{\partial z}{\partial w}(u_1)\frac{\partial z}{\partial w}(u_2) \right\}^{\Delta_A}
\] \[ \frac{1}{|z(u_1) - z(u_2)|^2} \]

\[ = \left[ \frac{2l}{\pi} \sinh \frac{\pi}{2l}(u_1 - u_2) \right]^{-2\Delta_A}.
\]

(2.23)
2.2 Review of boundary conformal field theory techniques

As $u_2 - u_1 \to \infty$, this approaches:

$$\langle O(u_1)O(u_2) \rangle \to \left( \frac{\pi}{l} \right)^{2\Delta_A} \exp \left[ -\pi \Delta_A \frac{(u_2 - u_1)}{l} \right]. \quad (2.24)$$

On the other hand, since $u$ is imaginary time, we may evaluate the correlation function on the strip by inserting a complete set of states:

$$\langle O(u_1)O(u_2) \rangle = \sum_n |\langle 0 |O|n \rangle_{AA}|^2 \exp[-E_n(u_2 - u_1)]. \quad (2.25)$$

Here $|0\rangle$ is the groundstate and $|n\rangle$ an arbitrary excited state for the strip Hamiltonian with b.c.’s $A$ at both ends. (The groundstate occurs here since the strip has infinite length in the imaginary time, $u$, direction.) $E_n$ is the energy of the $n^{th}$ state (measured from the groundstate energy). As $u_2 - u_1 \to \infty$, the lowest energy excited state that can be created from the groundstate by the operator $O$ dominates so we conclude that this state has energy:

$$E_1 = \pi \frac{\Delta_A}{l}. \quad (2.26)$$

There is a one-to-one correspondence between boundary operators and the finite size spectrum with the boundary scaling dimensions and finite size energies related by Eq. (2.26).

Another very useful quantity to consider is the partition function $Z_{AB}$ with conformally invariant b.c.’s $A$ and $B$ at the 2 ends of a finite system of length $l$, at inverse temperature, $\beta$. This may be expressed in terms of the finite size spectrum on the strip as:

$$Z_{AB} = tr \exp[-\beta H^{AB}_l] = \sum_n \exp[-\beta E^{AB}_n], \quad (2.27)$$

where the $E^{AB}_n$ are the energies on the strip with b.c.’s $A$ and $B$ at the two ends. This picture is sometimes referred to as “open string channel”. On the other hand, we may interchange space and imaginary time and write instead:

$$Z_{AB} = \langle A | \exp[-\beta H^P_\beta] | B \rangle, \quad (2.28)$$

where $|A\rangle$ and $|B\rangle$ are boundary states that correspond to the b.c.’s $A$ and $B$ respectively. Here $H^P_\beta$ denotes the Hamiltonian with periodic b.c.’s on a ring of length $\beta$. This point of view is sometimes referred to as “closed string channel”. All boundary states obey the condition:

$$\mathcal{P}(x)|A\rangle = 0. \quad (2.29)$$
The eigenstates of $H^P_\beta$ are of the form:

$$E_n^P = \frac{2\pi}{\beta} \Delta_n,$$

(2.30)

where the $\Delta_n$ are bulk scaling dimensions. This follows, similarly to the correspondence between boundary scaling dimensions and boundary energies, from a conformal transformation from the infinite plane to the cylinder of circumference $\beta$.

### 2.2.1 One component free boson

A simple example of boundary dimensions and energies is given by the periodic boson. Thus we consider the bulk Lagrangian density:

$$\mathcal{L} = \frac{g}{4\pi} (\partial_\mu \varphi)^2,$$

(2.31)

and assume that $\varphi(x, t)$ is a periodic variable so that $\varphi$ is identified as

$$\varphi \leftrightarrow \varphi + 2\pi n \ (n \in \mathbb{Z}).$$

(2.32)

These boundary conditions determine the bulk mode expansion for $\varphi(x, t)$ on a circle of circumference $\beta$, $0 < x < \beta$:

$$\varphi(t, x) = \hat{\varphi}_0 + \frac{2\pi}{\beta} \left[ \hat{P}' x + \frac{1}{g} \hat{P} t \right] + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2g}} \left\{ a_n^L \exp \left[ -inx + \frac{2\pi}{\beta} \right] + a_n^R \exp \left[ -inx - \frac{2\pi}{\beta} \right] + \text{h.c.} \right\}.$$

(2.33)

Here: $\hat{P}$ is the momentum operator conjugate to the constant term, $\hat{\varphi}_0$:

$$[\hat{\varphi}_0, \hat{P}] = i,$$

(2.34)

and $\hat{P}'$ is another momentum operator. They both have integer eigenvalues. The integer eigenvalues of $\hat{P}'$ follow from the periodic b.c.’s and the angular nature of $\varphi$:

$$\varphi(\beta) = \varphi(0) + 2\pi n.$$

(2.35)

The integer eigenvalues of $\hat{P}$ follow from the fact that it is conjugate to an angular variable, $\varphi_0$. Explicitly, the wave-function, $\exp[-iP\varphi_0]$, with $P$ an eigenvalue of $\hat{P}$, must be single valued.

$$x_\pm \equiv t \pm x,$$

(2.36)
2.2 Review of boundary conformal field theory techniques

and \( a_n^{L/R} \) are boson creation operators for the left and right moving finite momentum modes. This mode expansion is consistent with the equal-time commutation relations:

\[
\left[ \varphi(x), \frac{\partial \varphi(x')}{\partial t} \right] = \frac{2\pi}{g} i \delta(x - x'), \tag{2.37}
\]

which follow from the normalization of the Lagrangian. We may decompose \( \varphi \) into left and right moving modes:

\[
\varphi(t, x) = \varphi_L(x_+) + \varphi_R(x_-), \tag{2.38}
\]

and then write the dual field:

\[
\theta \equiv g(\varphi_L - \varphi_R). \tag{2.39}
\]

This has the mode expansion:

\[
\theta(t, x) = \hat{\theta}_0 + \frac{2\pi}{\beta} [\hat{P} x + g \hat{P}' t] + \sqrt{\frac{g}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left\{ a_n^L \exp \left[ -inx + \frac{2\pi}{\beta} \right] - a_n^R \exp \left[ -inx - \frac{2\pi}{\beta} \right] + \text{h.c.} \right\}. \tag{2.40}
\]

\( \hat{P}' \) is the momentum conjugate to \( \hat{\theta}_0 \):

\[
[\hat{\theta}_0, \hat{P}'] = i, \tag{2.41}
\]

and we see from this mode expansion that \( \theta(t, x) \) is also an angular variable:

\[
\theta(t, x) \leftrightarrow \theta(t, x) + 2\pi n \quad (n \in \mathbb{Z}). \tag{2.42}
\]

We may equally well write the Lagrangian density in terms of \( \theta \):

\[
\mathcal{L} = \frac{1}{4\pi g} (\partial_\mu \theta)^2. \tag{2.43}
\]

The bulk primary operators are \( \exp \{ i[n \varphi(t, x) + m \theta(t, x)] \} \) with scaling dimension:

\[
\Delta = \frac{n^2}{2g} + \frac{gm^2}{2} \quad (n, m \in \mathbb{Z}). \tag{2.44}
\]

(All descendent operators are simply primaries multiplied by products of multiple derivatives of \( \varphi \). These have dimensions equal to that of the corresponding primary
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plus a positive integer. Inserting the mode expansion of \( \phi \) into the Hamiltonian, gives the finite size spectrum with periodic b.c.'s on a circle of circumference \( \beta \):

\[
H = \frac{2\pi}{\beta} \left[ \frac{\hat{P}^2}{2g} + \frac{\hat{P}'^2}{2g} + \sum_{n=1}^{\infty} n (a_n^L a_n^L + a_n^R a_n^R) \right].
\] (2.45)

We see that the relationship, Eq. (2.30) between the bulk scaling dimensions and finite size energies with periodic b.c.'s is obeyed.

Now consider the Dirichlet (D) b.c. on \( \phi \):

\[
\phi(t, 0) = \phi_0, \quad \text{(a constant, independent of } t),
\] (2.46)

implying:

\[
\frac{\partial \phi}{\partial t}(t, 0) = 0.
\] (2.47)

We may calculate the dimensions of all boundary operators directly by recognizing that the D b.c.:

\[
\phi_R(t, 0) = \phi_0 - \phi_L(t, 0),
\] (2.48)

determines \( \phi_R(t, x) \) (for \( x > 0 \)) as the analytic continuation of \( \phi_L(t, x) \) to the negative \( x \)-axis:

\[
\phi_R(t, x) = \phi_0 - \phi_L(t, -x), \quad (x > 0).
\] (2.49)

The correlation function for \( \phi_L(t, x) \):

\[
\langle \phi_L(t, x) \phi_L(0, 0) \rangle = -\frac{1}{2g} \ln x_+ + \text{constant},
\] (2.50)

is unaffected by the boundary. Thus we see that the boundary operators can be rewritten by the replacements:

\[
\phi(t, 0) \rightarrow \phi_0
\]
\[
\theta(t, 0) \rightarrow 2\phi_L(t, 0) - \phi_0.
\] (2.51)

Thus the non-trivial operators are:

\[
\exp[im\theta(t, 0)] \rightarrow \exp[2im\phi_L(t, 0) - im\phi_0],
\] (2.52)

of scaling dimension:

\[
\Delta_D = gm^2.
\] (2.53)
Note that this is twice the bulk scaling dimension of $\exp[im\theta]$. This factor of 2 arises from the b.c. The bulk scaling dimension of $\exp[im\theta]$ consists of equal contributions of $g m^2/4$ from the left and right factors: $\exp[\pm i\varphi_{L/R}]$. Upon imposing the D b.c. the right factor vanishes and the left factor has an extra factor of 2 in the exponent which quadruples the scaling dimension: $g m^2/4 \to g m^2$.

We may check the general BCFT results on this simple example. Consider the finite size spectrum with the same D b.c. on both ends of the strip of length $l$ (with the same value of $\varphi_0$). The D b.c. essentially sets $\hat{P} = 0$ and $a^R_n = -a^L_n$. The mode expansion of Eq. (2.33) becomes:

$$
\varphi(t, x) \to \varphi_0 + \frac{2\pi}{l} \hat{P}' x + \frac{1}{\sqrt{2g}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} 2 \sin \left( \frac{n\pi x}{l} \right) \left[ a^L_n \exp \left( -\frac{in\pi t}{l} \right) + \text{h.c.} \right].
$$

(2.54)

$\hat{P}'$ must have integer eigenvalues due to the D b.c. and the periodic nature of $\varphi$. Thus the finite size spectrum with D b.c.'s is:

$$
H_{iDD}^l = \frac{\pi}{l} \left[ g \hat{P}'^2 + \sum_{n=1}^{\infty} n a'^L_n a'^L_n \right].
$$

(2.55)

We see that the general relation, Eq. (2.26) between the dimensions of boundary operators and finite size spectrum on a strip with the corresponding boundary conditions is obeyed.

We may also find the corresponding boundary state. This must obey the operator equation:

$$
\varphi(0, x)|D(\varphi_0)\rangle = \varphi_0 |D(\varphi_0)\rangle, \quad \text{(independent of } x). \tag{2.56}
$$

Note that this implies:

$$
\frac{\partial \varphi(0, x)}{\partial x} |D(\varphi_0)\rangle = 0. \tag{2.57}
$$

Note that, compared to the operator boundary condition of Eq. (2.47), $t$ and $x$ have been interchanged. This is a consequence of the interchange of space and time involved in going between the two interpretations of $Z_{AB}$. In the boundary state representation, the boundary corresponds to the circle $\tau = 0$ and Eq. (2.57) is the condition that $\varphi$ (when acting on the boundary state) be constant along the boundary. The D boundary state is:

$$
|D(\varphi_0)\rangle = (2g)^{-1/4} \exp \left[ -\sum_{n=1}^{\infty} a'^L_n a'^R_n \right] \sum_{\rho=-\infty}^{\infty} \exp[-i\rho \varphi_0](0, \rho). \tag{2.58}
$$
Here \((0, P)\) is the eigenstate of \(\hat{P}'\) with eigenvalue 0, the eigenstate of \(\hat{P}\) with (integer) eigenvalue \(P\), and the groundstate of all the harmonic oscillators. The condition Eq. (2.56) follows using the explicit form of the wave-function \((0, P)\):\[\langle \varphi_0 | (0, P) \rangle \propto \exp[iP\varphi_0].\] (2.59)

[We apologize for the confusing, but unfortunately standard, notation here. In Eq. (2.59), \(\varphi_0\) is a co-ordinate whereas in Eq. (2.58), \(\varphi_0\) is a fixed number, corresponding to an eigenvalue of the co-ordinate.] It is straightforward to calculate \(Z_{DD}\) in the boundary state representation:\[\langle D(\varphi_0) | \exp[-lH_\beta^P] | D(\varphi_0) \rangle = (2g)^{-1/2} \frac{1}{\eta(\tilde{q})} \sum_P \exp \left[ -\frac{l2\pi}{2g} \frac{P^2}{\beta} \right].\] (2.60)

Here we have introduced the Dedekind \(\eta\)-function: \[\eta(\tilde{q}) \equiv \tilde{q}^{1/24} \prod_{n=1}^\infty (1 - \tilde{q}^n),\] (2.61)

and the convenient notation: \[\tilde{q} \equiv e^{-\frac{2\pi i}{\beta}}.\] (2.62)

The factor of \(1/\eta(\tilde{q})\) in Eq. (2.60) comes from the oscillator mode factor in the boundary state of Eq. (2.58) and we have included the universal groundstate energy for free bosons:\[E_0 = -\frac{\pi}{6\beta}.\] (2.63)

The sum in Eq. (2.60) can also be written in terms of \(\tilde{q}\):\[Z_{DD} = (2g)^{-1/2} \frac{1}{\eta(\tilde{q})} \sum_P \tilde{q}^{P^2/4g}.\] (2.64)

In order to check that \(Z_{DD}\) indeed gives the correct finite size spectrum with D b.c.’s we need to re-express it in terms of \(q\):\[q \equiv e^{-\frac{\pi \beta}{\beta l}},\] (2.65)
i.e. perform a modular transformation. The modular transformation of the Dedekind \(\eta\)-function is:\[\eta(\tilde{q}) = \sqrt{\frac{\beta}{2l}} \eta(q).\] (2.66)
2.2 Review of boundary conformal field theory techniques

The modular transformation of the sum in Eq. (2.60) can be computed using the Poisson summation formula, i.e. the Fourier transform of the periodic δ-function, δₚ(χ):

$$
\sum_{\pi \in \mathbb{Z}} \tilde{q}^{-P^2/(4g)} = \sum_{P \in \mathbb{Z}} \exp \left[ -\frac{\pi l P^2}{g\beta} \right] = \sqrt{\frac{g\beta}{l}} \sum_{P' \in \mathbb{Z}} \exp \left[ -\frac{\pi P'^2 g\beta}{l} \right] = \sqrt{\frac{g\beta}{l}} \sum_{P' \in \mathbb{Z}} q^{gP'^2}. \tag{2.67}
$$

Inserting Eqs. (2.66) and (2.67) into Eq. (2.60) gives:

$$
Z_{DD} = \frac{1}{\eta(q)} \sum_{P'} q^{gP'^2}. \tag{2.68}
$$

Using the definition of q in Eq. (2.65) and the representation Eq. (2.27) for Z_{AB}, we extract the finite size energies with D b.c.’s at both ends of the strip:

$$
E^{DD} = \frac{\pi}{l}[gP'^2 + \text{integers}], \tag{2.69}
$$

in agreement with Eq. (2.55).

The Neumann (N) b.c. is:

$$
\frac{\partial \phi}{\partial x}(t, 0) = 0. \tag{2.70}
$$

This is equivalent to \( \partial \theta / \partial t = 0 \) or equivalently:

$$
\theta(t, 0) = \theta_0. \tag{2.71}
$$

This implies:

$$
\varphi_R(t, x) = -\theta_0/g + \varphi_L(t, -x), \quad (x > 0). \tag{2.72}
$$

Now the non-trivial primary boundary operators are:

$$
\exp[i n \varphi(t, 0)] \propto \exp[2i n \varphi_L(t, 0) - i n \theta_0/g], \tag{2.73}
$$

of dimension:

$$
\Delta_N = n^2/g. \tag{2.74}
$$
again twice the dimension of the corresponding bulk operator. The mode expansion is:

$$\phi(t, x) \rightarrow \phi_0 + \frac{2\pi}{lg} \hat{P}t + \frac{1}{\sqrt{2g}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} 2\cos \left( \frac{n\pi x}{l} \right) \left[ a_n^L \exp \left( -in\pi t \right) + \text{h.c.} \right],$$

(2.75)

with $\hat{P}$ having integer eigenvalues. The corresponding spectrum can be read from after rewriting the Hamiltonian as:

$$H_{iNN} = \frac{\pi}{g} \left[ \hat{P}^2 + \sum_{n=1}^{\infty} na_n^{L_+} a_n^L \right].$$

(2.76)

The corresponding boundary state, obeying:

$$\theta(0, x)|N(\theta_0)\rangle = \theta_0 |N(\theta_0)\rangle,$$

(2.77)

is:

$$|N(\theta_0)\rangle = \left( \frac{g}{2} \right)^{1/4} \exp \left[ \sum_{n=1}^{\infty} a_n^{L_+} a_n^R \right] \sum_{P'=-\infty}^{\infty} \exp \left[ -iP'\theta_0 \right] |(P', 0)\rangle.$$

(2.78)

### 2.2.2 Multi-component free boson

In some applications we need to consider a multi-component free boson field theory. Let

$$\vec{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_c)$$

(2.79)

be a $c$-component free boson field, with the Lagrangian density

$$\mathcal{L} = \frac{g}{4\pi} (\partial_{\mu}\vec{\varphi})^2.$$

(2.80)

At this point, the theory is just a collection of $c$ free bosons which are independent of each other. However, we will be interested in various possible boundary interactions which couples different components. In fact, we often introduce a multi-dimensional generalization of the periodicity (2.32), which is often called compactification. Sometimes the compactification ties different components together, so that they cannot be regarded as completely independent, even before we introduce an interaction at the boundary.

The general form of the compactification would be given as the identification

$$\vec{\varphi} \leftrightarrow \vec{\varphi} + 2\pi \vec{R},$$

(2.81)
where $\vec{R} \in \Lambda$ for a Bravais lattice $\Lambda$. The different components would be completely independent only if $\Lambda$ is rectangular. For a given Bravais lattice $\Lambda$, we can define a reciprocal lattice $\Lambda^*$ so that

$$\vec{K} \cdot \vec{R} \in \mathbb{Z}$$

(2.82)

for any vectors $\vec{K} \in \Lambda^*$ and $\vec{R} \in \Lambda$. Imposing periodic boundary conditions in the space direction, we can generalize the mode expansion (2.33) to the present case as

$$\vec{\varphi}(t, x) = \hat{\vec{\varphi}}_0 + \frac{2\pi}{\beta} \left[ \vec{R} x + \frac{1}{g} \vec{K} t \right] +$$

$$+ \frac{1}{\sqrt{2g}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left\{ \hat{a}_L^R \exp \left[ -inx + \frac{2\pi}{\beta} \right] + \hat{a}_R^L \exp \left[ -inx - \frac{2\pi}{\beta} \right] + \text{h.c.} \right\}. \quad (2.83)$$

Here we have replaced the zero-mode “momentum” operators with their eigenvalues $\vec{R} \in \Lambda$ and $\vec{K} \in \Lambda^*$. The dual field $\vec{\theta}$ is defined similarly to Eq. (2.39). It has a similar mode expansion to the above, generalizing Eq. (2.40). As a result, $\vec{\theta}$ can be regarded as compactified as

$$\vec{\theta} \leftrightarrow \vec{\theta} + 2\pi \vec{K}, \quad (2.84)$$

where $\vec{K} \in \Lambda^*$.

The vacua of the oscillator modes

$$| (\vec{R}, \vec{K}) \rangle$$

(2.85)

are then labelled by $\vec{K}$ and $\vec{R}$ which are the eigenvalues of $\hat{P}$ and $\hat{P}'$.

Generalizing (2.58), the Dirichlet boundary state corresponding to $\vec{\varphi} = \vec{\varphi}_0$ at the boundary is given as

$$|D(\vec{\varphi}_0)\rangle = (2g)^{-c/4} (V_0(\Lambda))^{-1/2} \sum_{\vec{K} \in \Lambda^*} \exp[-i\vec{K} \cdot \vec{\varphi}_0] |(\vec{0}, \vec{K})\rangle,$$  

(2.86)

where $V_0(\Lambda)$ is the $c$-dimensional volume of the unit cell of the Bravais lattice $\Lambda$, and

$$| (\vec{0}, \vec{K}) \rangle \equiv \exp \left[ -\sum_{n=1}^{\infty} \hat{a}_L^R \cdot \hat{a}_L^R \right] |(\vec{0}, \vec{K})\rangle,$$  

(2.87)

is the bosonic Ishibashi state. The partition function for the cylinder with the same Dirichlet boundary condition at both ends (diagonal cylinder amplitude) is given, gen-
eralizing (2.60), as
\[
Z_{DD}(\tilde{q}) = \langle D(\vec{\varphi}_0)|e^{-iH^D_{\beta}}|D(\vec{\varphi}_0)\rangle = (2g)^{-c/2} \frac{1}{V_0(\Lambda)} \left( \frac{1}{\eta(\tilde{q})} \right)^c \sum_{\vec{R} \in \Lambda^*} \tilde{q}^{R^2/(4g)}.
\] (2.88)

In order to find the spectrum of the boundary operators, we must modular transform Eq. (2.88). For this purpose, the multi-dimensional generalization of Eq. (2.67) is useful:
\[
\left( \frac{1}{\eta(\tilde{q})} \right)^c \sum_{\vec{R} \in \Lambda^*} \tilde{q}^{\frac{1}{2}R^2} = (2g)^{c/2} V_0(\Lambda) \left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{R} \in \Lambda} q^{g\vec{R}^2}.
\] (2.89)

Sometimes it is convenient to renormalize the lattices as $\Lambda^*/\sqrt{g} \rightarrow \Lambda^*$ and $\sqrt{g} \Lambda \rightarrow \Lambda$, to obtain the equivalent expression
\[
\left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{V} \in \Lambda^*} \tilde{q}^{\frac{1}{2}V^2} = 2^{c/2} V_0(\Lambda) \left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{W} \in \Lambda} q^{\vec{W}^2}.
\] (2.90)

Using eq. (2.89), we find
\[
Z_{DD}(q) = \left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{R} \in \Lambda} q^{\vec{R}^2}.
\] (2.91)

In fact, the prefactor $(2g)^{-c/2} \frac{1}{V_0(\Lambda)}$ in the boundary state (2.86), which represents the generally non-integer “ground-state degeneracy” [22], was chosen so that Eq. (2.91) satisfies Cardy’s consistency condition. Namely, the coefficient of Eq. (2.91) must be unity (or integer) to allow the interpretation as in Eq. (2.27). From Eq. (2.91) we can read off the scaling dimensions of the boundary operators with Dirichlet boundary condition as
\[
\Delta_D = g\vec{R}^2 + \text{integer},
\] (2.92)

where $\vec{R} \in \Lambda$ as usual. These are the dimensions of the boundary operators $\exp[i\vec{\varTheta} \cdot \vec{R}]$, consistent with the compactification of Eq. (2.84).

Likewise, the Neumann boundary state corresponding to $\tilde{\varTheta} = \tilde{\varTheta}_0$ at the boundary is given as
\[
|N(\tilde{\varTheta}_0)\rangle = \left( \frac{g}{2} \right)^{c/4} \sqrt{V_0(\Lambda)} \sum_{\vec{R} \in \Lambda} \exp[-i\vec{R} \cdot \tilde{\varTheta}_0]|(\vec{R}, 0\rangle),
\] (2.93)
where

\[
|(\vec{R}, \vec{0})\rangle \equiv \exp \left[ + \sum_{n=1}^{\infty} \vec{a}_{n}^{\dagger} \cdot \vec{a}_{n}^{\dagger} \right] |(\vec{R}, \vec{0})\rangle,
\]

(2.94)
is the bosonic Ishibashi state. The cylinder partition function for the Neumann boundary condition at both ends reads

\[
Z_{NN}(\tilde{q}) = \left( \frac{g}{2} \right)^{c/2} V_{0}(\Lambda) \left( \frac{1}{\eta(\tilde{q})} \right)^{c} \sum_{\vec{R} \in \Lambda} \tilde{q}^{g \vec{R}^{2}/4},
\]

(2.95)
in the closed string channel. Modular transforming using Eq. (2.89) gives the same partition function in the open string channel as

\[
Z_{NN}(q) = \left( \frac{1}{\eta(q)} \right)^{c} \sum_{\vec{K} \in \Lambda^{*}} q^{\vec{K}^{2}/g},
\]

(2.96)
The scaling dimensions of the boundary operators with Neumann boundary conditions are now given by

\[
\Delta_{N} = \frac{\vec{K}^{2}}{g} + \text{integer},
\]

(2.97)
where $\vec{K} \in \Lambda^{*}$. These are the dimensions of the boundary operators $\exp[i \vec{\Phi} \cdot \vec{K}]$, consistent with the compactification of Eq. (2.81).

### 2.3 A problem of compactification

For a free boson CFT, new fields defined as linear combinations of the original field are apparently independent of each other. However, they are not completely independent since the compactification of the new fields intertwine different components. This complication is often ignored in literature and still correct results are obtained in some cases. However, its negligence can lead to erroneous results. The intertwining of new free boson fields in context of boundary CFT was discussed by Wong and Affleck[35] for a quantum impurity problem. There, the compactification of each component of new fields is written explicitly in terms of gluing conditions. Although taking all the gluing conditions into account should lead to a correct result, it becomes increasingly cumbersome for larger number of components.
In a class of two-dimensional critical systems, the groundstate wavefunction is related to a two-dimensional CFT. Namely, the ground state is given as
\[ |\Psi_0\rangle \propto \int \mathcal{D}\phi e^{-S[\phi]/2}|\{\phi\}\rangle, \tag{2.98} \]
for the action \( S[\phi] \) of a CFT.

We consider a groundstate given as eq. (2.98). Let us divide the system into two regions A and B. The entanglement entropy between the two regions is defined by the von Neumann entropy of the subsystem A as
\[ S_E = -\text{Tr} (\rho_A \log \rho_A), \tag{2.99} \]
where
\[ \rho_A = \text{Tr}_B |\Psi_0\rangle\langle \Psi_0| \tag{2.100} \]

We will follow the argument by Fradkin and Moore to derive \( S_E \) for such a system. [36] Namely, the entanglement entropy is rewritten as
\[ S_E = -\left. \frac{\partial \text{Tr} \rho_A^n}{\partial n} \right|_{n=1}. \tag{2.101} \]
In the replica trick, we compute \( \text{Tr} \rho_A^n \) for an integer \( n \) and then make an analytic continuation to arbitrary integer \( n \). For an integer \( n \), we can introduce \( n \) copies of the CFT with the fields \( \phi_1, \phi_2, \ldots, \phi_n \).
\[ \text{Tr} \rho_A^n = \frac{Z_P}{Z_F} \tag{2.102} \]
where \( Z_P \) is the partition function of the \( n \)-component field theory with the condition
\[ \phi_1 = \phi_2 = \ldots = \phi_n \tag{2.103} \]
at the boundary between A and B. \( Z_F \) is the partition function of the same \( n \)-component field theory but without any restriction at the boundary between A and B. Both \( Z_P \) and \( Z_F \) are functions of \( n \), although the dependence is omitted for brevity of the expressions.

Fields with different replica indices are independent, except possibly at the boundary \( \Gamma \) between A and B. Since no coupling is introduced at the boundary in \( Z_F \), we find
\[ Z_F = (z_F)^n, \tag{2.104} \]
2.3 A problem of compactification

where $z_F$ is the partition function of the single component free boson field theory without any restriction at the boundary.

Let us proceed further by changing the basis, taking the linear combinations of the original fields $\phi_j$, as

\[
\varphi_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_j
\]

(2.105)

\[
\varphi_1 = \frac{1}{\sqrt{2}} (\phi_1 - \phi_2)
\]

(2.106)

\[
\cdots
\]

\[
\varphi_{n-1} = \ldots .
\]

It can be argued that, the condition (2.103) does not affect the “center of mass” field $\varphi_0$ and it thus remains free at $\Gamma$. On the other hand, all the other linear combinations $\varphi_j$ with $j > 0$, which correspond to differences among $\phi_k$, obey fixed (Dirichlet) boundary condition at $\Gamma$. The $n - 1$ “difference” fields was then regarded as independent. As a consequence of this argument, it was proposed in Ref. [36] that

\[
Z_P = (z_D)^{n-1} z_F,
\]

(2.107)

where $z_D$ is the partition function for the single component field, with Dirichlet boundary condition at the boundary.

Combining eqs. (2.102),(2.104), and (2.107) leads to

\[
\text{Tr} \rho^A_n = \left( \frac{z_D}{z_F} \right)^{n-1}.
\]

(2.108)

This implies

\[
S_E = - \log \frac{z_D}{z_F} = - \log \frac{z_A z_B}{z_F},
\]

(2.109)

where $z_D^A(z_D^B)$ is the partition function for the single component free boson field theory restricted to the region A (or B), with Dirichlet boundary condition at the boundary between A and B. Eqs. (2.107) and (2.109) are also the basis of the calculations in Ref. [37].

However, in a general CFT, the field $\phi$ is subject to interactions. For example, the two-dimensional critical Ising model correspond to $\phi^4$ field theory with a certain fine-tuning. The field theory with the interaction is generally not invariant under orthogonal
Boundary Conformal Field Theory

transformations of the fields. For example, let us consider the simplest case $n = 2$. The Lagrangian of two identical Ising field theory would read

$$
\mathcal{L}_{\text{(Ising)}}^2 = \sum_{j=1,2} \left( \frac{1}{2} (\partial_{\mu} \phi_j)^2 + \frac{1}{2} m^2 \phi_j^2 + \frac{\lambda}{4} \phi_j^4 \right),
$$

(2.110)

where $m^2$ and $\lambda$ are fine-tuned to make the system critical. Now we introduce the new fields $\varphi_{0,1}$ as in eqs. (2.105) (for $n = 2$) and (2.106). Then, in terms of the new fields,

$$
\mathcal{L}_{\text{(Ising)}}^2 = \sum_{j=1,2} \left( \frac{1}{2} (\partial_{\mu} \varphi_j)^2 + \frac{1}{2} m^2 \varphi_j^2 \right) + \frac{\lambda}{4} \frac{1}{2} (\varphi_0^4 + 6 \varphi_0^2 \varphi_1^2 + \varphi_1^4).
$$

(2.111)

The new fields $\varphi_{0,1}$ are subject to different interaction from the original one. Moreover, different components $\varphi_0$ and $\varphi_1$ are now coupled through the bulk interaction $\varphi_0^2 \varphi_1^2$ and not independent of each other. Thus eq. (2.107) would not hold in general field theories with interactions.

For a free field theory, on the other hand, the change of the basis appears more legitimate, because the theory is also free in terms of the new basis fields. Let us now consider the case of the free boson field theory. This will be relevant for entanglement entropy in the quantum Lifshitz universality class. Let us define the Lagrangian density following the convention in Ref. [29] as

$$
\mathcal{L} = \frac{g}{4\pi} (\partial_{\mu} \phi)^2.
$$

(2.112)

The field $\phi$ is subject to compactification, namely the identification

$$
\phi \sim \phi + 2\pi R,
$$

(2.113)

where $R$ is the compactification radius.

Let us note that, there is unfortunately a large variety of conventions for the free boson field theory, based on different normalizations. In fact, by a renormalization of the field $\phi$, we can fix the value of either coupling constant $g$ or the compactification radius $R$. However, we cannot fix both $g$ and $R$ by the renormalization. This leaves one free parameter, which governs the critical behavior. We can choose either to fix $g$ and consider $R$ as a free parameter, or to fix $R$ and regard $g$ as a free parameter. Both conventions, with various choices of the fixed value, appear in literature.
Employing the replica trick, we consider $n$ component free boson theory, in which each component is independent in the bulk. For the free field case, the Lagrangian density can be also written as

$$\mathcal{L} = \frac{g}{4\pi} \sum_{j=0}^{n-1} (\partial_\mu \phi_j)^2. \quad (2.114)$$

That is, the new fields are governed by the same Lagrangian density as the original fields, and there is no interaction which couples different components. Thus the arguments in Refs. [36, 37] appears to valid. However, even for the free boson field theory, eq. (2.107) does not quite hold because of subtlety in boson compactification.

### 2.3.1 A simple example

In order to illustrate the issue, let us discuss a simple example. We consider a single component free boson theory (2.112) on a rectangle of the size $2L \times \beta$, with the periodic boundary condition on both directions. In other words, the system is defined on a torus. The partition function is given by

$$Z_{\text{simple}} = \frac{1}{\eta(q)} \frac{1}{\eta(\bar{q})} \sum_{n,m=-\infty,\infty} q^{\frac{1}{2} \left( \frac{m^2}{gR^2} + n^2/\sqrt{g}R \right)^2} \bar{q}^{\frac{1}{2} \left( \frac{m^2}{gR^2} - n^2/\sqrt{g}R \right)^2} = \left( \frac{1}{\eta(q)} \right)^2 q^{\frac{1}{2} \left( \frac{m^2}{gR^2} + n^2/\sqrt{g}R \right)}, \quad (2.115)$$

where

$$q = e^{-\pi\beta/L}. \quad (2.116)$$

For a general torus, $q$ is given as $q = e^{2\pi i \tau}$ where $\tau$ is the modulus of the torus which is a complex number, and its complex conjugate $\bar{q}$ is distinguished from $q$. However, here we only consider the rectangular case where $\tau$ is a pure imaginary, and hence $q = \bar{q}$. We have used this fact in the second line of eq. (2.115).

Now we “fold” the system, as shown in Fig. 2.2. After the folding, the system can be regarded as two-component boson field $\phi_1, \phi_2$ defined on a rectangle of size $L \times \beta$. While the periodic boundary condition is still applied on the $\beta$ direction, there are two boundaries which were the folding lines. Thus the system after the folding is topologically a cylinder. At the two boundaries, we impose the condition

$$\phi_1 = \phi_2. \quad (2.117)$$
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Figure 2.2: (a) A single component free boson field theory is defined on the torus of size $2L \times \beta$. (b) The system is folded to a two-component free boson field theory defined on the cylinder with circumference $\beta$ and length $L$.

Namely, if we consider the new fields

$$
\Phi_0 = \frac{\phi_1 + \phi_2}{\sqrt{2}},
\Phi_1 = \frac{\phi_1 - \phi_2}{\sqrt{2}},
$$

(2.118)

(2.119)

$\Phi_1$ obeys the Dirichlet boundary condition $\Phi_1 = 0$, while $\Phi_0$ remains free (obeys the Neumann boundary condition) at the two boundaries.

If we apply the same argument as in Ref. [37], we would obtain

$$
Z_{\text{simple}}(q) = z_{DD}(R, q)z_{NN}(R, q),
$$

(2.120)

where

$$
z_{DD}(R, q) = \frac{1}{\eta(q)} \sum_n q^{gn^2R^2} = \frac{1}{\sqrt{2gR \eta(q)}} \sum_n \tilde{q}^{n^2/(4gR^2)},
$$

(2.121)

is the Dirichlet-Dirichlet amplitude for the single component boson with the compactification radius $R$, and

$$
z_{NN}(R, q) = \frac{1}{\eta(q)} \sum_n q^{n^2/(gR^2)} = \sqrt{\frac{gR}{2}} \frac{1}{\eta(q)} \sum_n \tilde{q}^{gn^2R^2/4},
$$

(2.122)

is the Neumann-Neumann amplitude for the same theory. However, from eqs. (2.121) and (2.122) we can immediately see that eq. (2.120) actually does not hold.
One of the problems is that, in the “closed string channel”, the right-hand side of eq. (2.120) reads

\[ z_{DD}(R, q)z_{NN}(R, q) = \frac{1}{2} \left( \frac{1}{\eta(q)} \right)^2 \sum_{n,m} \tilde{q}^{n^2/(4gR^2)+m^2gR^2/4}. \tag{2.123} \]

This implies that each boundary has the “groundstate degeneracy” (exponential of the boundary entropy) \(1/\sqrt{2}\), for any value of \(R\). This is in contradiction to the fact that the boundary in the present example is just a result of an artificial “folding” along a line in the bulk, and thus should not have any boundary entropy. In fact, the modular invariance of the partition function on the original torus of \(2L \times \beta\) implies

\[ Z_{\text{simple}} = \left( \frac{1}{\eta(q)} \right)^2 \sum_{m,n} \tilde{q}^{\frac{1}{gR^2}(m^2+n^2gR^2)}, \tag{2.124} \]

with the coefficient unity. Interpreted as the amplitude with two boundaries after the folding, this means that the boundary entropy is indeed zero. Thus, there must be something wrong in the assumptions led to eq. (2.120) (and to eq. (2.107) in Ref. [37]) even for the free field.

The problem was that the new fields were implicitly assumed to obey the same compactification as the original fields:

\[ \Phi_j \sim \Phi_j + 2\pi R. \tag{2.125} \]

In fact, while the original fields are compactified independently, a complication is introduced in the compactification by a change of the basis. [35] This can been seen in Fig. 2.3. The compactification in terms of the new fields \(\Phi_{0,1}\) reads

\[ \Phi_0 \sim \Phi_0 + 2\pi n_0 \frac{R}{\sqrt{2}}, \tag{2.126} \]
\[ \Phi_1 \sim \Phi_1 + 2\pi n_1 \frac{R}{\sqrt{2}}, \tag{2.127} \]

where

\[ n_0 \equiv n_1 \mod 2. \tag{2.128} \]

Here \(n_0 \equiv n_1 \equiv 0 \mod 2\) and \(n_0 \equiv n_1 \equiv 1 \mod 2\) correspond respectively to the red and blue sublattice in Fig. 2.3.
Figure 2.3: The compactification lattice for two independent boson fields $\phi_{1,2}$, each of which has compactification radius $R$.

We can also define the new fields $\Theta_{0,1}$, dual fields for $\Phi_{0,1}$. Similarly to the case of $\Phi_{0,1}$, their compactification is given as

\begin{align}
\Theta_0 &\sim \Theta_0 + \frac{2\pi m_0}{\sqrt{2gR}}, \\
\Theta_1 &\sim \Theta_1 + \frac{2\pi m_1}{\sqrt{2gR}},
\end{align}

where

\begin{equation}
m_0 \equiv m_1 \mod 2.
\end{equation}

These considerations imply that the compactification is not independent in terms of the new fields and is subject to “gluing conditions” (2.128) and (2.131) among different fields. This aspect was ignored in Ref. [37]. In the present simple example, we can see that ignoring the gluing conditions leads to a wrong equality (2.120).

This simple example demonstrates the importance of the gluing conditions – namely, that the linear combinations of the compactified fields are not completely independent. In fact, taking the gluing conditions into account, the correct partition function can be reproduced from the boundary CFT.
2.4 BCFT in 1D spinless fermions

The low energy transmission of a 1D interacting electrons through barriers, including resonant tunneling, was analyzed by mapping the problem to a Luttinger liquid where the gapless spin and charge degrees of freedom decouple, interacting with a barrier or a quantum impurity[9, 6, 38]. In the case of spinless fermions scattering off a potential barrier, Kane and Fisher[6, 9] showed that at zero temperature, the charge conductance is zero if the bulk interactions are repulsive and perfect if attractive. More generally, for fermions with spin, the charge and spin conductances depend on two parameters which are related to the bulk interactions in the charge and spin sectors. It was found that there are four possible stable phases whose stability depend on the strength of the bulk interactions: charge and spin with zero or perfect transmission. In addition, there exist unstable phases which have partial conductances separating pairs of the above phases in the region of overlap of the domains where the two phases are stable. These unstable phases were probed perturbatively, using the observation that when the bulk interaction constants approach certain values these fixed points become trivial.

Wong and Affleck[35] used the boundary field theory approach to reproduce the same results of the spinless fermions case and in the more complicated case of spin-1/2 fermions in a somewhat more systematic way. This kind of problem fits into the more general setting where there are one-dimensional gapless degrees of freedom in the bulk coupled to a local potential or impurity degree of freedom. Such systems have been tackled by the boundary critical phenomenon approach in the Kondo problem and in the isotropic spin-1/2 antiferromagnetic Heisenberg chain with an impurity. However, in the Heisenberg chain after a Jordan Wigner transformation, the bulk is composed of interacting spinless fermions and in the Kondo case, free spinful fermions. In general, the system of gapless degrees of freedom coupled to a local degree of freedom is a difficult problem to solve exactly even in 1D. There exists exact solution from the Bethe Ansatz but the Hamiltonian must be fine tuned to become integrable[39]. For a generic situation, the problem can be simplified by asking what the low energy behavior of the system is. At long wavelengths and low energies, the bulk is described by a relativistic (1 + 1)-dimensional field theory with conformal invariance. In the boundary critical phenomenon, the bulk degree of freedom are not integrate out as in[9, 6, 38], but rather at low energies, the effects of local interactions with the barrier or the impurity can be summarized by an effective local condition on the bulk. The boundary condition must renormalize to a fixed point, so that it will be compatible with the bulk conformal
symmetry. At such a boundary fixed point, conformal symmetry in $(1 + 1)$-dimension is powerful enough to give, for example, the finite size spectrum.

In the last chapter we study a new superconducting quantum device and turn the problem into a boundary critical phenomenon, we see that the limits of weak and strong coupled fixed point correspond to the conformally invariant boundary conditions on the bulk.
Chapter 3

Renormalization Group approach to Josephson devices

The Renormalization Group (RG) is a theoretical tool, which among many other available methods in physics, is considered to be one of the most important ideas in quantum field theory (QFT). Besides the RG there is a variety of other methods developed over the years one uses to tackle various problems encountered in QFT, such as perturbation theory or various numerical techniques. Although their usefulness in solving and understanding a large number of problems has been demonstrated, it often happens that the applicability of a particular method is restricted to a specific energy or length scale. Moreover, many problems cannot be investigated due the approximations used within these methods. Hence, there was a need for a method that, at least to some extent, overcomes all these problems. During the recent years the RG method has proven to be one of the most promising tools theorists have at hand. The applicability of RG covers a whole range of physics and physical phenomena that are usually plagued with divergences. Starting with its role as a novel method in QFT, the RG has developed into an uniquely effective technique widely used also in statistical physics, condensed matter theory and all other areas of physics where non-perturbative effects make systematic calculations difficult.

In this chapter we briefly show how techniques of the Renormalization Group are applied to superconducting devices. The first device we review was studied by Hekking and Glazman in 1997 [40]; the model is a thin superconducting loop containing a Josephson junction and a flux \( \Phi \) passing through the loop. The authors show that a finite renormalized Josephson energy can arise because the junction itself affect the
fluctuations of the environment. Simultaneously, the modes of the environment renormalize the plasmon oscillations in the junction. Next, we will see how a one-dimensional Josephson array of small superconducting grains can be mapped into a spin-1/2 XXZ-chain in an external magnetic field and the phase diagram is sketched out. Finally we introduce the device considered in [25], a Josephson junction network consisting of three one-dimensional Josephson junction arrays connected with weak links to a central region.

3.1 A superconducting loop with a Josephson junction

The model considered in Ref.[40] is a superconducting wire of length $L$ and small cross-sectional area $S = a \times a$, which is embedded in a medium with dielectric constant $\varepsilon$. The wire is closed by a Josephson junction to form a loop. The bare Josephson energy of the junction is $E_0^J \equiv \pi \Delta G/(4e^2)$, and its charging energy is $E_C = 4e^2/C$. Here, $G$ and $C$ are the conductance and the capacitance of the junction, respectively. The loop is pierced by a magnetic flux $\Phi = H L^2/(2\Phi_0)$, where $\Phi_0$ is the superconducting flux quantum and $H$ is the magnetic field.

The low energy excitation spectrum of this system can be described by the Lagrangian $\mathcal{L} = K - U$, where

$$K = \int_0^L dx \frac{\hbar^2}{2e_c} [\dot{\varphi}(x)]^2 + \frac{\hbar^2 [\dot{\varphi}(L) - \dot{\varphi}(0)]^2}{2E_C}, \quad (3.1)$$

and

$$U = \int_0^L dx \frac{\hbar^2 n_s S}{8m} \left( \frac{\partial \varphi(x)}{\partial x} - \frac{\Phi}{L} \right)^2 - E_0^J \cos[\varphi(L) - \varphi(0)], \quad (3.2)$$

where $\varphi$ is the phase of the superconducting order parameter. Here, $1/e_c = [8e^2 \ln(R/a)/\varepsilon]^{-1}$ is the characteristic inverse charging energy per unit length of the loop, $R$ is the distance to a metallic screen, $n_s$ is the density of the superconducting condensate, and $m$ is the electron mass.

The first terms on the right hand side of Eqs.(3.1) and (3.2) describe the propagating plasma mode along the loop[41, 42] and correspond to the electrostatic energy stored in the plasmons and to the energy associated with the supercurrent in the wire, respectively.
3.1 A superconducting loop with a Josephson junction

The remaining terms in (3.1) and (3.2) refer to the Josephson junction.

An important length scale in this model is determined by the length $L^*$ at which
the energy of supercurrents in the loop and the Josephson energy of the junction are of
the same order. It is given by

$$L^* = \frac{\hbar^2 n_s S}{4m E_j^0} = \frac{\hbar g v_{pl}}{\pi E_j^0},$$

where $g$ is defined through

$$\frac{1}{g} = \frac{4}{\pi} \frac{mv_{pl}}{\hbar n_s S} = 8 \frac{e^2}{\hbar c} \frac{\lambda_L}{a} \sqrt{\frac{2 \ln (R/a)}{\pi \varepsilon}}. \quad (3.4)$$

Here $e^2/\hbar c \approx 1/137$ is the fine structure constant.

For simplicity, the capacitance $C$ is completely neglected, which is true when

$$\frac{\hbar v_{pl}}{\Delta} \frac{1}{2e_c} \gg 1 \frac{1}{E_C}, \quad (3.5)$$

which correspond to neglecting the last term in comparison with the first one in (3.1)
for all the relevant scales of the phase variation. The term $\frac{\hbar v_{pl}}{\Delta}$ is the smallest part
of the ring involved in the phase fluctuation, $\Delta$ is the superconducting gap in the loop and $v_{pl}$ is the plasma velocity given by

$$v_{pl} = \sqrt{\frac{e_c n_s S}{4m}} = \frac{c a}{2\lambda_L} \sqrt{\frac{2 \ln (R/a)}{\pi \varepsilon}}. \quad (3.6)$$

Here, $c$ is the speed of light and $\lambda_L = \sqrt{mc^2/(4\pi e n_s e^2)}$ is the London penetration depth of the wire.

In the absence of fluctuations, the phase varies linearly with the distance along the
loop; therefore, it is convenient to introduce a new variable, $\chi(x)$, by the relation:

$$\varphi(x) = \varphi_0 \frac{x}{L} + \chi(x), \quad (3.7)$$

where $\varphi_0$ is determined such that the energy $U$ has its minimum at $\chi = 0$. This
condition can be written in a simple form:

$$\sin \varphi_0 = \frac{L^*}{L} (\Phi - \varphi_0) \quad (3.8)$$
Renormalization Group approach to Josephson devices

Using the new variable χ(x) and the angular coordinate θ = 2πx/L, the potential U can be rewritten as

\[
U = E_0^J \frac{2\pi}{\pi} \frac{L^*}{L} \int_0^{2\pi} d\theta \left( \frac{\partial \chi}{\partial \theta} \right)^2 + \frac{L}{2L^*} \sin^2 \varphi_0 - \cos[\chi(2\pi) - \chi(0) + \varphi_0] - [\chi(2\pi) - \chi(0)] \sin \varphi_0 .
\]

(3.9)

The DC Josephson effect at zero temperature is described fully by the Φ-dependence of the ground state energy, \(E_{gr}\), of the system under consideration. Because only the term \(U\) of the energy depends explicitly on \(\Phi\), the persistent current \(J(\Phi) \equiv (2e/\hbar) \partial E_{gr}/\partial \Phi\) can be expressed in terms of the average \(\langle U \rangle\) over the ground-state wave function:

\[
J(\Phi) = \frac{2e}{\hbar} \left( 1 + \frac{L}{L^*} \cos \varphi_0 \right)^{-1} \left\langle \frac{\partial U}{\partial \varphi_0} \right\rangle .
\]

(3.10)

3.1.1 Renormalization of the Josephson energy

In this section we want to analyze the renormalization of the Josephson energy by the charge fluctuation in the framework of the Renomalization Group (RG) approach. Let us restrict our attention to the zero flux phase case, Φ = 0 and then we will recover the general result. The euclidean action \(S\) of the system is obtained by a Wick rotation to imaginary time \(\tau\), and after that we integrate out the fluctuations away from the junction, it reads:

\[
S = \frac{\hbar g}{4\pi} \int \frac{d\omega}{2\pi} |\tilde{\chi}(\omega)|^2 - \int d\tau E_0^J \cos \tilde{\chi}(\theta = 0, \tau) ,
\]

(3.11)

where the fluctuating field is redifined by \(\tilde{\chi}(\theta, \tau) \equiv \chi(\theta, \tau) - \chi(2\pi - \theta, \tau)\). This action was studied by a standard renormalization group method in Ref.[43, 44], introducing a running cut-off energy \(\mu\), and find a flow equation for the dimensionless Josephson coupling energy \(\tilde{E}_J \equiv E_J/\mu\):

\[
\frac{d\tilde{E}_J}{dl} = (1 - 1/g)\tilde{E}_J + \mathcal{O}(\tilde{E}_J^3), \quad dl = -d\mu/\mu .
\]

(3.12)

This equation describes how the Josephson coupling \(E_J\) is renormalized when high-energy degrees of freedom are integrated out.
The Eq.(3.12) can be integrated from the high energy cutoff \( \mu_h = \Delta \) at which \( E_J = E_J^0 \), down to a value \( \mu_l = \hbar v_{pl}/l_0 \), where \( l_0 \) is same characteristic length. As a result, we find

\[
E_J(\mu_l) = E_J^0 \left( \frac{\hbar v_{pl}}{l_0 \Delta} \right)^{1/g}.
\] (3.13)

**The case \( g < 1 \)**

In this case, from Eq.(3.12) it follows that, upon decreasing \( \mu \), the energy \( \bar{E}_J \) flows to zero, Eq.(3.13) remains valid for any possible value of \( l_0 \) and a perturbative analysis can be applied to calculate the Josephson current. In order to do this, let us restrict to the case \( L \ll L^* \). For a short loop only the term with the integral in (3.9), which corresponds to the kinetic energy of the supercurrent along the loop, is important. Substituting Eq.(3.9) into Eq.(3.10), and neglecting terms \( \mathcal{O}(L/L^*) \), we find

\[
J(\Phi) = J_c^0 \left\langle \cos(\chi(2\pi) - \chi(0)) \right\rangle \sin \Phi.
\] (3.14)

To evaluate the average \( \left\langle \cos(\chi(2\pi) - \chi(0)) \right\rangle \) we quantize the fluctuating field \( \chi(\theta) \) in the standard way as follows:

\[
\chi(\theta) = \chi_0 + \frac{1}{\sqrt{g}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos \left( \frac{n\theta}{2} \right) \left[ a_n^\dagger + a_n \right],
\] (3.15)

where operators \( a_n \) satisfy canonical commutation relations. Using (3.15), it is straightforward to evaluate \( \left\langle \cos(\chi(2\pi) - \chi(0)) \right\rangle \) where the average is taken with respect to the quadratic Hamiltonian

\[
H = \sum_{n=1}^{\infty} \frac{\hbar v_{pl} n \pi}{L} \left[ a_n^\dagger a_n + \frac{1}{2} \right].
\] (3.16)

At low temperatures and \( L \gg \hbar v_{pl}/\Delta \), we finally obtain \( J(\Phi) = J_c \sin \Phi \), with a renormalized critical current

\[
J_c = J_c^0 \left( \frac{\hbar v_{pl}}{L \Delta} \right)^{1/g} \left[ \frac{\pi L k_B T / \hbar v_{pl}}{\sinh (\pi L k_B T / \hbar v_{pl})} \right]^{1/g},
\] (3.17)

and \( J_c^0 = 2eE_J^0/\hbar \). The result (3.17) holds for temperatures \( k_B T \ll \Delta \). The classical result \( J(\Phi) = J_c^0 \sin \Phi \) is recovered in the limit \( 1/g \to 0 \), i.e. when the fluctuating field \( \chi = 0 \). This value is smaller from the mean-field result, \( J_c^0 = \pi \Delta G/(2e) \).
Renormalization Group approach to Josephson devices

Quantum fluctuations suppress the maximum Josephson current below its mean-field value $J_0^c$. This suppression depends on the loop length $L$; according to (3.17) $J_c \to 0$ when $L \to \infty$. This is an artefact of the lowest order of perturbation theory, where the effect of the junction on the fluctuations in the attached wire is disregarded completely.

The case $g > 1$

In this case the perturbation theory of the renormalization group break down as soon as $E_J \sim 1$, i.e., when the cutoff energy $\mu$ reaches a value $\mu_l$ which satisfies the condition $\mu_l = E_J(\mu_l)$. As a result $E_J$ is renormalized down to a value $E_{J,\text{eff}}$, and should be determined from the condition of self-consistency,

$$E_{J,\text{eff}}^0 = J_0^c \left( \frac{E_{J,\text{eff}}}{\Delta} \right)^{\frac{1}{g}} , \quad (3.18)$$

which yields

$$E_{J,\text{eff}}^0 = J_0^c \left( \frac{E_{J,\text{eff}}^0}{\Delta} \right)^{\frac{1}{g-1}} . \quad (3.19)$$

The value $E_{J,\text{eff}}^0$ is reached for $l_0 \sim (L^*/g)(\Delta/E_{J,\text{eff}}^0)^{1/(g-1)}$. We thus conclude that if $g > 1$, the result (3.17) at $T = 0$ holds as long as $L \lesssim L^*$; for larger values of $L$, the decrease of the Josephson coupling slows down, and eventually $E_J$ saturates at the value $E_{J,\text{eff}}^0$ given by Eq.(3.19). Further suppression of the Josephson energy is prevented by the fact that the modes $\tilde{\chi}(\omega)$ at frequencies $\omega < E_{J,\text{eff}}^0/\hbar$ are pinned by the Josephson coupling, and hence cannot participate in the renormalization.

When using $\Delta$ as an upper energy cut-off in the derivation of Eqs.(3.13) and (3.19), we assumed the condition (3.5) to be satisfied. In fact, the above treatment remains valid, even if (3.5) is violated, but the weaker condition $L^* \gg C$, holds; in this case, $\Delta$ should be replaced with $E_c$ in (3.13) and (3.19).

To calculate the Josephson current for a large loop with $L \gg L^*$, we have to consider that the solution $\varphi_0(\Phi)$ has discontinuities:

$$\varphi_0 \simeq \frac{L^*}{L} \Phi \quad \text{if} \quad 0 \leq \Phi < \pi ,$$

$$\varphi_0 \simeq 2\pi + \frac{L^*}{L} (\Phi - 2\pi) \quad \text{if} \quad \pi \leq \Phi < 2\pi . \quad (3.20)$$
3.2 One-dimensional Josephson junction array

Because the Josephson energy dominates over the kinetic energy of the supercurrents, the phase $\varphi_0$ remains “pinned” to the minima of the cosine potential. Correspondingly, in the absence of fluctuations the equilibrium persistent current $J(\Phi)$ has cusps,

$$J(\Phi) \simeq J_c^0 \frac{L^*}{L} \Phi \quad \text{if} \quad 0 \leq \Phi < \pi ,$$

$$J(\Phi) \simeq J_c^0 \frac{L^*}{L} (\Phi - 2\pi) \quad \text{if} \quad \pi \leq \Phi < 2\pi .$$

We expect quantum fluctuations (i) to renormalize the bare Josephson energy and thus to suppress the slope of the saw-tooth dependence; (ii) to smear the cusps at $\Phi = (2n + 1)\pi$, as quantum tunneling will remove the degeneracy between pairs of states having the same values of energy but different values of $\varphi_0$.

However, due to the fact that the Josephson energy is not a weak perturbation if $L > L^*$, we have to take its effect on the quantum fluctuations into account. The decrease in $\langle \cos[\chi(2\pi) - \chi(0)] \rangle$ with a growing length of the loop $L$ should saturate when $L$ exceeds the characteristic length $L^*$. The saturation occurs because the Josephson coupling pins the low-frequency modes, thus preventing the logarithmic divergence of the phase fluctuations at the junction.

The result (3.21) for the flux-dependent Josephson current in a loop with $L > L^*$ remains valid for values of flux away from the cusp at $\Phi = \pi$; we just should replace $E^0_j$ by $E_{\text{eff}}^0$. The behavior of $J(\Phi)$ for $\Phi \sim \pi$ is strongly affected by quantum tunneling and, as a result, the cusps in the function $J(\Phi)$ will be smeared. Tunneling between the two macroscopic states characterized by different values of $\varphi_0$ induces a shift $\delta E$ of the ground state energy of the system which can be shown to be smaller than the gap between the degenerate ground state and the first excited state of the loop with the junction. Thus, at zero temperature, we are dealing with an effective two-state system, and the flux dependence of $J(\delta \Phi \equiv \Phi - \pi)$ near $\Phi = \pi$ will be given by [45]

$$J(\delta \Phi) \approx \frac{2eE_J L^*}{\hbar L} \delta \Phi \left\{ 1 - \frac{\pi}{\sqrt{(L^*/L)^2 \delta \Phi^2 + (\delta E/(\pi E_J))^2}} \right\} .$$

In particular, we see that the smearing is characterized by a width $\delta \Phi_s \sim (L/L^*)(\delta E/E_J)$.

### 3.2 One-dimensional Josephson junction array

The device we analyze in this section is a one-dimensional array of Josephson junctions. Let us consider the simplest Hamiltonian describing a one-dimensional JJ-chain
\[ \mathcal{H} = \mathcal{H}_C + \mathcal{H}_J \] where [46]:

\[ \mathcal{H}_C = \frac{E_C}{2} \sum_{j=1}^{L/a} \left( -i \frac{\partial}{\partial \phi_j} - \frac{N}{2} \right)^2 \]

\[ \mathcal{H}_J = -E_J \sum_{j=1}^{L/a} \cos(\phi_j - \phi_{j+1}) \] (3.23)

In Eq. (3.23) \(-i \frac{\partial}{\partial \phi_j}\) is the operator representing the number of Cooper pairs at site \(j\) in the phase representation and, thus, it takes only integer eigenvalues, \(n_j\) and \(J\) is the Josephson coupling energy and \(N\) accounts for the influence of the gate voltage, being \(eN \propto V_g\), (in the absence of charge quantization, \(eN\) is the average grain charge induced by the gate); \(E_C\) is the charging energy of a grain. The sum over \(j\) ranges over the \((L/a)\) sites, with \(L\) being the length of the chain, and \(a\) the intergrain distance; imposing periodic boundary conditions amounts to fix \(\phi_{L/a+j} = \phi_j\).

In the limit \(E_J/E_C \to 0\), the Josephson array is a Mott insulator having a gap of the width, \(E_C\) at almost any \(N\). The exceptions are the discrete points \(N = 2n + 1\), where a grain with charges \(2ne\) and \(2(n + 1)e\) has the same energies. Josephson tunneling lifts this degeneracy. To analyze the properties of the array in the vicinity of the degeneracy points, we can project the Hamiltonian (3.23) on the Fock subspace \(F\) of states with charges of each grain confined to the values \(2ne\) and \(2(n + 1)e\) only.

Resorting to the a well known procedure [47], one may easily construct the effective Hamiltonian \(H_{\text{eff}}\), describing the JJ-chain on the reduced space \(F\) [46]. Let \(P\) be the projector onto \(F\) and \(P_\perp\) be the projector onto the subspace \(F_\perp\), to \(O(J^2/E_C)\), \(H_{\text{eff}}\) takes the form:

\[ H_{\text{eff}} = P(\mathcal{H}_J + \mathcal{H}_C)P + P \left[ \frac{\mathcal{H}_J P_\perp \mathcal{H}_J}{-\frac{9}{16}E_C} \right] P. \] (3.24)

When restricted to \(F\), the operators \(e^{\pm i\phi_j}\) and \(-i \frac{\partial}{\partial \phi_j}\) may be represented with the spin-1/2 operators \(S^z_j\) and \(S^j\), as

\[ PE^{\pm i\phi_j}P = S^\pm_j \quad P \left( -i \frac{\partial}{\partial \phi_j} - n - \frac{1}{2} \right) P = S^z_j. \] (3.25)
From Eq. (3.25), the lowest order in $E_J/E_C$ of the Hamiltonian (3.23) reads:

$$H^{(0)}_C = P(H_C)P = \frac{E_C}{2} \sum_{j=1}^{L/a} (S_j^z - h)^2,$$

$$H^{(0)}_J = P(H_J)P = -\frac{E_J}{2} \sum_{j=1}^{L/a} (S_j^{+}S_{j+1}^{-} + S_j^{-}S_{j+1}^{+}).$$

(3.26)

and the magnetic field $h = (N - 2n - 1)/2$ allows for tuning the system to a degeneracy point by means of the gate voltage.

The Hamiltonian (3.26) can be converted to the conventional tight-binding Hamiltonian for free fermions by means of the Jordan-Wigner transformation [48], and then diagonalized in the plane wave representation:

$$H_0 = \sum_k \epsilon_k a_k^{\dagger} a_k, \quad \epsilon_k = -E_J \cos k - E_C h.$$  

(3.27)

The lowest order in $E_J$ we used is sufficient to find the characteristics of the Mott phase, which occupies the domain $|h| > E_J/E_C$. The complementary phase in this approximation is equivalent to a noninteracting Fermi gas. It is known, however, that even a weak interaction is relevant in one dimension, transforming the gas into a Luttinger liquid. There are two competing mechanisms of interaction between the fermions.

The first one is an intergrain electrostatic interaction leading to an effective repulsion between the Cooper pairs. We will concentrate on the case of a relatively weak intergrain interaction $E_z \ll E_C$, which allows us to use Hamiltonian (3.26) as a starting point. In a realistic system with a gate, the interaction between distant grains is screened out, and following [49] we retain only the nearest-neighbor term

$$\sum_{j=1}^{L/a} E_z S_j^z S_{j+1}^z.$$  

(3.28)

On the other hand, the higher order expansion in $E_J/E_C$ leads to an effective attraction between the Jordan-Wigner fermions. To find the corresponding correction to the low-energy Hamiltonian (3.26), we should take into account the virtual states with energies exceeding $E_C$, calculating the second term in (3.25):

$$H^{(2)} = -\frac{E_J^2}{4E_C} \sum_{j=1}^{L/a} \left( \frac{3}{4} S_j^z S_{j+1}^{z} + S_j^{+} S_{j+1}^{-} S_{j-1}^{+} S_{j+1}^{-} \right).$$

(3.29)
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To account for the contributions coming from intergrain capacitances, it is sufficient to retain only the nearest-neighbor terms [46], since next-to-nearest neighbor hopping terms would give rise to irrelevant operators. The system is thus usefully described by the effective Hamiltonian:

\[ H_{\text{Eff}} = -\frac{E_J}{2} \sum_{j=1}^{L/a} [S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-] - H \sum_{j=1}^{L/a} S_j^z + \Delta \sum_{j=1}^{L/a} S_j^z S_{j+1}^z \]  

(3.30)

with \( H = E_C h \) and \( \Delta = E_z - \frac{3}{16} E_J^2 \) [46, 23].

The Eq.(3.30) is the Hamiltonian for a spin-1/2 XXZ-chain in an external magnetic field \( H \) and in App.B is shown how to perform a bosonization of this model. The anisotropy parameter \( \Delta \) may take positive, as well as negative values, depending on the constructive parameters of the JJ-chain and its sign is crucial for the emergence of a Repulsive Tomonaga-Luttinger (RTL) phase in a JJ-chain.

### 3.2.1 Phase diagram

Now we discuss the properties of the Josephson junctions chain in the Luttinger liquid state, which low-energy properties are described by a quadratic Lagrangian[50, 51],

\[ \mathcal{L} = \frac{g}{2\pi} [v(\partial_x \phi)^2 + v^{-1}(\partial_t \phi)^2]. \]  

(3.31)

Here \( v \) is the velocity of the acoustic excitations in the system (plasmons in our case), and \( g \) determines the long-range behavior of all correlation functions. The phenomenological constant \( g \) can be calculated in terms of the microscopic parameters in some limiting cases.

Deep in the superconducting region, \( E_J \gg E_C \), one can compare Eq.(3.31) with the quadratic Lagrangian derived by expansion of (3.23) in \( \phi \), to find \( g \approx \pi (E_J/E_C)^{1/2} \). In the opposite limit of a relatively weak Josephson coupling, \( E_z \ll E_J \ll E_C \), constant \( g \) is \( \approx 1 - (\lambda/2\pi) \sin k_F \). In addition to these limiting expressions for \( g \), we know its value at the phase boundaries. At the boundary with the Mott phase the interaction is vanishing, which means \( g = 1 \).

To characterize the Luttinger liquid state, we consider a chain of junctions connecting two massive superconducting leads, and containing just one especially weak link, \( E_w \ll E_J \). Deep in the superconducting state (\( g \gg 1 \)) we can neglect with the quantum
fluctuations. The energy of the system consisting of \(L\) junctions of the nominal strength \(E_J\) and one weak link equals:

\[
E(\phi) = \frac{E_J (\phi - \phi_w)^2}{2L} - E_w \cos \phi_w.
\]  

(3.32)

Here \(\phi\) is the phase difference applied to the leads, and \(\phi_w\) is the phase difference across the weak link; the energy (3.32) should be minimized with respect to \(\phi_w\). Quantum fluctuations of phases \(\phi_i\) at finite \(g\) lead to zero-point motion of \(\phi_w\) around some average value \(\bar{\phi}_w\), which depends on the external phase \(\phi\). Averaging over the fluctuations \(\delta\phi_w = \phi_w - \bar{\phi}_w\) results in the energy functional of the form (3.32) with \(E_w\) renormalized by a proper Debye-Waller factor, and \(\bar{\phi}_w\) replaced by \(\bar{\phi}_w\). For a short array, the renormalized Josephson coupling constant is \(E_{\text{eff}}^w \simeq E_w (1/L)^{1/g}\). Energy (3.32) corresponds to two inductors in series. Therefore, the response of the system to the external phase is dominated by the weaker of two elements.

If \(g > 1\) (weaker quantum fluctuations), the renormalized energy \(E_{\text{eff}}^w\) decreases with \(L\) slower than the first term in Eq. (3.32). Consequently, at \(L\) exceeding the crossover length, \(L^* \simeq (E_J/E_w)^{g/(g-1)}\), the energy (3.32) is dominated by the first term with \(\phi_w = 2\pi n\). The Josephson current \(\propto \partial E/\partial \phi\) has a sawtooth dependence on phase, with the amplitude proportional to \(E_J/L\). Thus, the effect of the weak link is “healed” over the distance \(L^*\).

If \(g < 1\), the weak link energy \(E_{\text{eff}}^w\) falls off faster than \(1/L\), and this link remains the weakest of the two inductors connected in series at any \(L\); consequently, \(\bar{\phi}_w = \phi\). Therefore, “healing” does not occur, and at any \(L\) the chain behaves as a single Josephson junction of strength \(E_{\text{eff}}^w\). The corresponding current-phase relation is \(I \sim E_w L^{1/g} \sin \phi\).

The line \(g = 1\) on the phase diagram may be viewed as a phase transition\[9\] between the superconducting and insulating phases in the following sense. In the limit \(L \rightarrow \infty\), the inductance \(L\) of an array with a single weak link is insensitive to the weak link if \(g > 1\) (superconducting state), and \(L \propto L/E_J\). In the insulating state, the scaling of inductance with \(L\) is \(g\)-dependent, \(L \propto L^{1/g}/E_w\), with exponent \(1/g > 1\)[52].

### 3.3 Y-junction of superconducting Josephson chains

Recently, in Ref.\[29, 28\], the transport properties of a Y-junction composed of three quantum wires enclosing a magnetic flux were studied: modeling the wires as Tomonaga-
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Luttinger liquids (TLL), the authors of Ref.[29, 28] were able to show the existence of an attractive fixed point, characteristic of the network geometry of the circuit. A repulsive finite coupling fixed point has been found in Ref.[31], in the analysis of Y-junctions of one-dimensional Bose liquids.

We want introduce in this section another device, the Y-shaped Josephson junction network in Fig.3.1, studied in [25]. The model is made with three finite Josephson junction (JJ) chains ending on one side (inner boundary) with a weak link of nominal strength $\lambda$ and on the other side (outer boundary) by three bulk superconductors held at phases $\varphi_j$ ($j = 1, 2, 3$). The three chains are connected by the weak links to a circular JJ chain $C$, pierced by a dimensionless magnetic flux $f$. For simplicity, we assume that all the junctions have Josephson energies $E_J$ and $\lambda \ll E_J$. The Hamiltonian describing the central region, $H_C$, is given by

$$H_C = \frac{E_c}{2} \sum_{i=1}^{3} \left[-i \frac{\partial}{\partial \varphi_0(i)} - N' \right]^2 - \frac{E_J}{2} \sum_{i=1}^{3} \left[ e^{i[\varphi_0(i) - \varphi_0(i+1) + \frac{1}{2}]f} + e^{-i[\varphi_0(i) - \varphi_0(i+1) + \frac{1}{2}]f} \right], \quad (3.33)$$

where $E_c$ is the charging energy of each grain, $N'$ is the gate voltage applied to the $i$th junction, and $\varphi_0(i)$ ($i = 1, 2, 3; i + 3 \equiv i$) is the phase of the superconducting order parameter at the $i$-th grain in $C$. Following a standard procedure [46, 24, 23], the Hamiltonian in Eq.(3.33) can be presented as

$$H_C = -\tilde{H} \sum_{i=1}^{3} [S_0(i)]^z - \frac{E_J}{2} \sum_{i=1}^{3} \left\{ e^{i\frac{1}{2}[S_0(i)]^+[S_0(i+1)]^-} + e^{-i\frac{1}{2}[S_0(i+1)]^+[S_0(i)]^-} \right\}, \quad (3.34)$$

with $\tilde{H} \propto E_c, [S_0(i)]^z = n_0(i) - N' - \frac{1}{2}$ and $[S_0(i)]^\pm = e^{\pm i\varphi_0(i)}$, where $n_0(i)$ is the total charge at grain $i$ (measured in units of $e^*$).

We require that $C$ is connected to the three finite chains via a charge tunneling Hamiltonian $H_T$, given by

$$H_T = -\lambda \sum_{i=1}^{3} \cos[\varphi_1(i) - \varphi_0(i)]. \quad (3.35)$$

Since $\lambda/E_J \ll 1$, one may resort to a Schrieffer-Wolff transformation [47], to derive an Hamiltonian $H_B$ describing the effective boundary interaction at the inner boundaries of the three chains. Eqs. for the external chains may be written in Tomonaga-Luttinger
Figure 3.1: Y-shaped Josephson junction network: all the junctions are equal to each other and have nominal Josephson energy $E_J$, except for the three ones connecting the central region to the endpoints of the chain, that have nominal energy $\lambda$.

\begin{equation}
H_0 = \sum_{j=1,2,3} \frac{g}{4\pi} \int_0^L dx \left[ \frac{1}{v} \left( \frac{\partial \Phi_j}{\partial t} \right)^2 + v \left( \frac{\partial \Phi_j}{\partial x} \right)^2 \right].
\end{equation}

Since at the outer boundary the three chains are connected to three bulk superconductors at fixed phases $\varphi_j$, the fields $\Phi_j$ must satisfy the Dirichlet boundary conditions

\begin{equation}
\Phi_j(L) = \sqrt{2}[2\pi n_j + \varphi_j],
\end{equation}

where $j = 1, 2, 3$ and $n_j$ are integers. On the inner boundary, the three chains are connected to $C$ via $H_T$: as a result, one should impose here Neumann boundary conditions (i.e., $\frac{\partial \Phi_j(0)}{\partial x} = 0$). For our following analysis, it is most convenient to introduce linear combinations of the plasmon fields, such as $X(x) = \frac{1}{\sqrt{3}} \sum_{j=1}^3 \Phi_j(x)$, $\chi_1(x) = \frac{1}{\sqrt{2}}[\Phi_1(x) - \Phi_2(x)]$, and $\chi_2(x) = \frac{1}{\sqrt{6}}[\Phi_1(x) + \Phi_2(x) - 2\Phi_3(x)]$. Since $E_W$ is of order $\lambda^2/E_J$, one has that $E_W/E_J \ll 1$ and, thus, at $x = 0$, the fields $\chi_1, \chi_2$ also satisfy Neumann boundary conditions. Of course, at the outer boundary, $\chi_1, \chi_2$ satisfy Dirichlet boundary conditions.
In the long wavelength limit, due to Neumann boundary conditions, the equation for the boundary Hamiltonian is

\[ H_B = -2 \tilde{E}_W \sum_{i=1}^{3} : \cos[\vec{\alpha}_i \cdot \vec{\chi}(0) + \gamma] : , \]  

(3.38)

with \( \vec{\alpha}_1 = (1, 0) \), \( \vec{\alpha}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \), \( \vec{\alpha}_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \). The colons (:) in Eq.(3.38) denote normal ordering with respect to the vacuum of the bosonic fields \( \chi_1, \chi_2 \). The effective coupling \( \tilde{E}_W \) is given by \( \tilde{E}_W = \left(\frac{a}{L}\right)^{1/g} E_W \). Eq.(3.38) may be regarded as the bosonic version of the boundary Hamiltonian describing the central region of a Y-junction of three quantum wires, introduced in Ref.[29, 28].

### 3.3.1 The weakly coupled fixed point

Setting \( \tilde{E}_W = 0 \) defines the WFP, where the fields \( \chi_1(x), \chi_2(x) \) obey Dirichlet boundary conditions at the outer boundary and Neumann boundary conditions at the inner boundary. As a result, the mode expansion of \( \chi_i \) is given by

\[ \chi_i(x,t) = \xi_i + \sqrt{\frac{2}{g}} \sum_n \cos \left[ \frac{\pi}{L} \left( n + \frac{1}{2} \right) x \right] \frac{\alpha_i(n)}{n+\frac{1}{2}} e^{-i\frac{\pi}{2} \gamma(n+\frac{1}{2})^2 t} , \]

(3.39)

with \([\alpha_i(n), \alpha_j(n')] = \delta_{ij} (n + \frac{1}{2}) \delta_{n,n'-1,0}, \xi_1 = \mu_1 + 2\pi n_{12}, \xi_2 = \mu_2 + \frac{2}{\sqrt{3}} [2\pi n_{13} - \pi n_{12}], (\mu_1, \mu_2) = ([\varphi_1 - \varphi_2], \frac{2}{\sqrt{3}}[(\varphi_1 - \varphi_3) - (\varphi_1 - \varphi_2)/2]) \), with \( n_{ij} = n_i - n_j \).

The fields \( \chi_i(x) \) are primary field in a boundary conformal field theory, than one sees that the scaling dimension of the boundary interaction in Eq.(3.38) is given by \( h_W(g) = \frac{1}{g} \), and that the dimensionless coupling strength \( G(L) = L \tilde{E}_W \) scales as \( G(L) \sim L^{1-\frac{1}{g}} \).

From the operator product expansion (OPE) between vertex operators

\[ \{ : \exp[i\vec{\alpha}_i \cdot \chi(\tau)] :: \exp[i\vec{\alpha}_j \cdot \chi(\tau')] : \} \rightarrow \tau \approx \left[ \frac{v(\tau - \tau')}{L} \right]^{-\frac{2}{g}} : \exp[-i\vec{\alpha}_k \cdot \vec{\chi}(\tau)] : , \]

(3.40)

with \( i \neq j \neq k \), one gets the second-order renormalization group equations for the complex coupling \( G(L)e^{i\gamma} \) as

\[ \frac{d[G(L)e^{i\gamma}]}{d \ln(\frac{L}{L_0})} = [1 - \frac{1}{g}]G(L)e^{i\gamma} - 2G^2(L)e^{-2i\gamma} , \]

(3.41)
which may be usefully presented as

\[
\frac{dG(\ell)}{d\ell} = \left[1 - \frac{1}{g}\right]G(\ell) + 2\cos(3\gamma)G^2(\ell) \quad (3.42)
\]

\[
\frac{d\gamma}{d\ell} = -2\sin(3\gamma)G^2(\ell) \quad (3.43)
\]

(\ell = \ln \left(\frac{L}{L_0}\right))$. Since Eqs.(3.42,3.43) are periodic under \(\gamma \rightarrow \gamma + \frac{2\pi}{3}\), the resulting phase diagram of the YJJN will present the same periodicity. Also, the phase diagram strongly depends on whether \(g < 1\), or \(g > 1\). Indeed:

1. For \(g < 1\), the linear term in Eq.(3.42) has a negative coefficient and, thus, \(\forall \gamma\), the system is attracted by a fixed point with \(G^* = 0\). Furthermore, Eq.(3.43) shows that the value of \(\gamma\) at the attractive fixed point is \(\gamma^* = \frac{2k\pi}{3}\), if \((2k-1)\pi/3 < \gamma(L_0) < (2k+1)\pi\), while it is \(\gamma^* = \frac{3\pi}{2}\) if \(\gamma(L_0) = (k+1)\pi/3\).

2. For \(g > 1\), Eq.(3.42) has a positive coefficient; as a result, \(G(\ell)\) grows as \(\ell\) increases. Whether \(G^*\) is now finite, or \(\infty\), depends on the values of \(g\) and \(\gamma(L_0)\).

### 3.3.2 The strongly coupled fixed point

The SFP is reached when the running coupling constant \(G\) goes to \(\infty\). The fields \(\chi_j(x), j = 1, 2\), now obey Dirichlet boundary conditions at \(x = 0\). The allowed values of \(\chi_1(0), \chi_2(0)\) are determined by the manifold of the minima of the effective boundary potential (Eq.(3.38)). It is easy to see that:

1. for \((6k-1)\pi/3 < \gamma < (6k+1)\pi/3\), the minima lie on the triangular sublattice A, defined by \((\chi_1(0), \chi_2(0)) = (2\pi n_{12}, \frac{2}{\sqrt{3}}[2\pi n_{13} + \pi n_{12}])\).

2. for \((6k+1)\pi/3 < \gamma < (6k+3)\pi/3\), the minima lie on the triangular sublattice B, given by \((\chi_1(0), \chi_2(0)) = (2\pi n_{12} + \frac{2\pi}{3}, \frac{2}{\sqrt{3}}[2\pi n_{13} + \pi n_{12}])\).

3. for \((6k+3)\pi/3 < \gamma < (6k+5)\pi/3\), the minima lie on the triangular sublattice C, given by \((\chi_1(0), \chi_2(0)) = (2\pi n_{12} - \frac{4\pi}{3}, \frac{2}{\sqrt{3}}[2\pi n_{13} + \pi n_{12}])\).

\(\gamma(L_0)\) is the value of the phase \(\gamma\) at the reference length \(L_0\). It should be noticed that, if \(\gamma(L_0) = (2k+1)\pi/3\), \(\gamma\) does not scale with \(L\).
Figure 3.2: Points on the three triangular sublattices A, B and C: at $\gamma = (2k + 1) \pi/3$, the energies of two sublattices are degenerate and the minima of the boundary potential span a honeycomb lattice, whose sites are connected by W-instanton trajectories, shorter than the V-instanton trajectories, connecting sites on the same sublattice. The black honeycomb is an elementary cell of the lattice of the minima emerging at $\gamma = \pi/3$.

At $\gamma = (6k + 1) \pi/3$, $\gamma = (6k + 3) \pi/3$, $\gamma = (6k + 5) \pi/3$, the two sublattices A and B, B and C, and C and A become degenerate in energy, respectively.

For $\gamma \neq (2k + 1) \pi/3$, the minima of the boundary potential span only one of the sublattices A, B and C. In this case, the leading boundary perturbation at the inner boundary is given by a linear combination of the dual vertex operators $\tilde{V}_1^\pm$, $\tilde{V}_2^\pm$, and $\tilde{V}_3^\pm$, defined in terms of the dual fields as

$$\tilde{V}_j^\pm =: \exp \left[ \pm i2 \sqrt{\frac{2}{3}} \tilde{\rho}_j \cdot \tilde{\psi}(0) \right] : (j = 1, 2, 3) , \quad (3.44)$$

with $\tilde{\rho}_1 = (0, 1)$, $\tilde{\rho}_2 = (\sqrt{3}/2, -1/2)$, $\tilde{\rho}_3 = (-\sqrt{3}/2, -1/2)$: they describe instanton trajectories connecting two sites in one of the triangular sublattices A, B or C ("V-instantons"). The two-point correlation function of the dual boundary vertices is given by
3.3 Y-junction of superconducting Josephson chains

\[
\left\langle \tilde{V}_j^{\pm}(\tau)\tilde{V}_i^{\mp}(\tau') \right\rangle \propto \delta_{j,i} \left[ e^{\frac{\pi v}{L}} - e^{\frac{\pi v}{L}} \right]^{-\frac{g}{4}},
\]

and, thus, the scaling dimension of \( \tilde{V}_j^{\pm}(\tau), j = 1, 2, 3 \), is given by \( h_S(g) = \frac{4g}{3} \). As a result, the SFP is stable for \( g > \frac{3}{4} \) and for \( \gamma \neq (2k + 1)\pi/3 \). Thus, for \( \frac{3}{4} < g < 1 \) and for \( \gamma \neq (2k + 1)\pi/3 \), both the WFP, and the SFP are stable and, accordingly, the phase diagram allows for a repulsive FFP. For \( g > 1 \) and for \( \gamma \neq (2k + 1)\pi/3 \), the SFP is the only IR stable fixed point.

### 3.3.3 The Finite Fixed Point

For \( \gamma = (2k + 1)\pi/3 \), the sets of minima belonging to two sublattices have the same energy. As a result, the eigenvalues of the zero-mode operators lie all on a honeycomb lattice obtained by merging two triangular sublattices, as sketched in Fig.3.2 for \( \gamma = \pi/3 \), at which point the sublattices A and B merge into a honeycomb lattice. The leading perturbation near the Dirichlet fixed point contains, now, operators representing “shorter” jumps between neighboring minima on the honeycomb lattice (“W-instantons”).

Following Ref.[53], one may describe these instantons by introducing an isospin operator \( \vec{\tau} \), acting on a pertinent two-component spinor: an \( \uparrow \)-spinor is associated to a minimum lying on sublattice A and a \( \downarrow \) spinor to a minimum lying on sublattice B. As a result, the leading boundary perturbation may now be written as

\[
H_B = -\xi \sum_{i=1}^{3} \left\{ \tau^+ W_i^\dagger(\tau) + \tau^- W_i(\tau) \right\},
\]

with \( W_j(\tau) =: \exp \left[ \frac{2}{3} i \vec{\alpha}_j \cdot \vec{\psi}(\tau) \right] \) and \( \xi \sim E_J - E_W \). Since the boundary interaction contains the isospin operators \( \vec{\tau} \), the relevant OPEs are obtained by combining the multiplication rules for the isospin operators

\[
\tau^z \tau^\pm = \pm \tau^\pm, \quad \tau^\pm \tau^\mp = 1 \pm \tau^z,
\]

with the OPEs of the bosonic vertex operators.
\[
\mathcal{E}_\pm[\mp \frac{2}{3} \boldsymbol{\alpha}_j, \bar{\psi}(\tau)] = e^{\mp \frac{2}{3} i \cdot \boldsymbol{\alpha}_j \cdot \bar{\psi}(\tau)} ,
\]

\[
\mathcal{E}_\pm[\mp \frac{2}{3} i \cdot \boldsymbol{\alpha}_j \cdot \bar{\psi}(\tau') ] = e^{\mp \frac{2}{3} i \cdot \boldsymbol{\alpha}_j \cdot \bar{\psi}(\tau')} .
\]

\[
\tau' \rightarrow \tau - \approx \left[ \pi v (\tau - \tau') \right] - \frac{4g}{9} \left[ 1 \pm \frac{2}{3} (\tau - \tau') \beta_j \cdot \frac{\partial \bar{\psi}(\tau)}{\partial \tau} \right].
\] (3.48)

Terms proportional to \( \frac{\partial \bar{\psi}(\tau)}{\partial \tau} \), which could be generated to second-order in \( \xi \), are suppressed by the condition \( \sum_{j=1}^{3} \beta_j = 0 \). As a result, higher-order contributions to the \( \beta \)-function of the running coupling strength \( \zeta = L \xi \) only appears to order \( \zeta^3 \). The RG equation for \( \zeta \) is then given by

\[
\frac{d \zeta}{d \ell} = [1 - h_F(g)] \zeta - 2 \zeta^3 .
\] (3.49)

For \( \gamma = \pi/3 \) the scaling dimension of the boundary interaction, \( h_F(g) \), gets renormalized as

\[
\frac{dh_F(g)}{d \ell} = -h_F(g) \zeta^3 .
\] (3.50)

For a small enough value of \( \zeta \), the renormalization of \( h_F(g) \) may be safely neglected, since it appears only to the third-order in \( \zeta \), and one may substitute \( h_F(g) \) in Eq.(3.49) with its bare value \( \frac{4g}{9} \). Thus, the leading boundary perturbation at the SFP is irrelevant for \( g > 9/4 \), while it is relevant for \( g < 9/4 \). As a result, for \( \gamma = \pi/3 \), there is a range of values of \( g \) -namely, \( 1 < g < 9/4 \)- where neither the WFP, or the SFP, are stable. The flow diagram then implies the existence of a FFP in the phase diagram.

### 3.4 The perturbative Renormalization Group approach

To conclude this chapter we shall derive, in this section, the scaling laws for boundary interaction operators.

Let us consider a one-dimensional system described by the Euclidean action

\[
S = S_0 - \sum_{\ell=1}^{M} \lambda_{\ell} \int_{0}^{\beta} d \tau \Phi_{\ell}(\tau) ,
\] (3.51)
3.4 The perturbative Renormalization Group approach

where \( S_0 \), the action for \( K \) independent massless Klein-Gordon fields, \( \{ \phi_j \} \), with pertinent boundary conditions, is given by:

\[
S_0 = \frac{g}{4\pi} \sum_{j=1}^{K} \int_0^\beta d\tau \int_0^L dx \left[ \frac{1}{v_{pl}} \left( \frac{\partial \phi_j}{\partial \tau} \right)^2 + v_{pl} \left( \frac{\partial \phi_j}{\partial \tau} \right)^2 \right].
\]

(3.52)

The boundary operators \( \{ \Phi_\ell(\tau) \} \) are functionals of the fields \( \{ \phi_j \} \) with scaling dimension \( h_\ell \). As a consequence, for \( \tau \rightarrow \tau' \), one gets

\[
G_\ell(\tau - \tau') = \langle \Phi_\ell(\tau) \Phi_\ell'(\tau') \rangle_0 \propto \delta_{\ell,\ell'} \frac{1}{[2\pi v_{pl}(\tau - \tau')/L]^{2h_\ell}},
\]

(3.53)

where \( \langle \ldots \rangle_0 \) denotes averaging with respect to the "free" action \( S_0 \), while \( L \) is the "size" of the system.

It is well known that, in order to introduce scale invariant interaction terms, one needs to define dimensionless coupling constants, given by \( g_\ell(a) = \lambda_\ell a^{1-h_\ell} \), where \( \lambda_\ell \) is the coupling constant appearing in Eq.(3.51), and \( a \) is a short distance cutoff. To first order in the coupling strengths, the renormalization group equations are given by

\[
\frac{dg_\ell(a)}{d\ln(a/a_0)} = [1 - h_\ell] g_\ell(a).
\]

(3.54)

Higher order corrections to the renormalization group equations come from nontrivial short-distance fusion rules of the \( \Phi_\ell \)'s, which are represented by the OPEs

\[
\Phi_\ell(\tau) \Phi_\ell'(\tau') \approx_{\tau' \rightarrow \tau} \sum_k \frac{C_{\ell \ell' \ell''}}{|\tau - \tau'|^{h_\ell + h_\ell' - h_\ell''}} \Phi_\ell''(\tau).
\]

(3.55)

If the OPE’s coefficients \( C_{\ell \ell' \ell''} \) are different than zero, the scaling equations get contributions which are of second order in the couplings. To derive them, one should start from the partition function

\[
Z = Z_0 \left\langle \exp \left[ \sum_{\ell=1}^{M} g_\ell(a) \int_0^\beta d\tau a^{-1+b_\ell} \Phi_\ell(\tau) \right] \right\rangle_0.
\]

(3.56)

Upon introducing the cutoffs \( a \) and \( L \) and by expanding Eq.(3.56) up to the third-order...
Renormalization Group approach to Josephson devices

in the running coupling strengths, one gets

\[ \frac{Z}{Z_0} \approx 1 + \sum_{\ell, \ell' = 1}^{M} g_\ell g_{\ell'} \int_0^{L/v_{pl}} d\tau \int_0^{L/v_{pl}} d\tau' a^{-1+h_\ell} a^{-1+h_{\ell'}} \theta(\tau - \tau' - \frac{a}{v_{pl}}) \Phi_\ell(\tau) \Phi_{\ell'}(\tau')_0 + \]

\[ + \sum_{\ell, \ell', \ell'' = 1}^{M} g_\ell g_{\ell'} g_{\ell''} \int_0^{\frac{L}{v_{pl}}} d\tau \int_0^{L/v_{pl}} d\tau' \int_0^{L/v_{pl}} d\tau'' a^{-1+h_\ell} a^{-1+h_{\ell'}} a^{-1+h_{\ell''}} \times \]

\[ \times \theta(\tau - \tau' - \frac{a}{v_{pl}}) \theta(\tau' - \tau'' - \frac{a}{v_{pl}}) \langle \Phi_\ell(\tau) \Phi_{\ell'}(\tau') \Phi_{\ell''}(\tau'') \rangle_0 . \tag{3.57} \]

If one rescales the cutoff, \( a \to (1 + \epsilon) a \), with \( \epsilon \ll 1 \), additional contributions to the coupling constant renormalization arise, as a result of configurations where the arguments of two fields lie between \( a \) and \( (1 + \epsilon) a \). The ensuing renormalization is derived by means of the identity

\[ \theta(\tau - \tau' - \frac{a}{v_{pl}}) \theta(\tau' - \tau'' - \frac{a}{v_{pl}}) \approx \theta(\tau - \tau' - \frac{a}{v_{pl}}) \theta(\tau' - \tau'' - \frac{a}{v_{pl}}) + \]

\[ - \epsilon \frac{a}{v_{pl}} \left[ \delta(\tau - \tau' - \frac{a}{v_{pl}}) \theta(\tau' - \tau'' - \frac{a}{v_{pl}}) + \theta(\tau - \tau' - \frac{a}{v_{pl}}) \delta(\tau' - \tau'' - \frac{a}{v_{pl}}) \right] . \tag{3.58} \]

Thus, the third-order contribution to Eq.(3.57) renomalizes the second-order term with

\[ 2\epsilon \sum_{\ell', \ell'' = 1}^{M} g_{\ell'} g_{\ell''} \sum_{\ell}^{M} [g_\ell C_{\ell', \ell''}] \int_0^{L/v_{pl}} d\tau' \int_0^{L/v_{pl}} d\tau'' \theta(\tau' - \tau'' - \frac{a}{v_{pl}}) \times \]

\[ \times a^{-1+h_{\ell'}} a^{-1+h_{\ell''}} \Phi_{\ell'}(\tau') \Phi_{\ell''}(\tau'') . \tag{3.59} \]

By setting \( \epsilon = \ln(a/a_0) \) in Eq.(3.56), it is straightforward, but tedious, to derive explicit expression for the nonlinear terms appearing in the RG equations for the running coupling strengths; the final result is

\[ \frac{dg_\ell(a)}{d\ln(a/a_0)} = [1 - h_\ell] g_\ell(a) + \sum_{\ell', \ell'' = 1}^{M} C_{\ell', \ell''} g_{\ell'}(a) g_{\ell''}(a) . \tag{3.60} \]

The RG equations in Eq.(3.60) may be rewritten as

\[ \frac{dg_\ell}{d\ln(a/a_0)} - \frac{\partial C[\{g\}]}{\partial g_\ell} = 0 \tag{3.61} \]
where \( C[\{g\}] \) is given by

\[
C[\{g\}] = \frac{1}{2} \sum_{\ell=1}^{N} (1 - h_\ell) g_\ell^2 + \frac{1}{3} \sum_{\ell,\ell',\ell''=1}^{M} C_{\ell,\ell',\ell''} g_\ell g_{\ell'} g_{\ell''} .
\] (3.62)

The RG fixed points coincide with the extrema of the function \( C[\{g\}] \), that is, with the set of values of \( \{g\}, \{g^*\} \), such that

\[
\frac{\partial C[\{g^*\}]}{\partial g_\ell} = 0 , \quad \forall \ell = 1, \ldots, M .
\] (3.63)

Due to its properties, \( C \) may also be identified with the "boundary entropy" of the system [22].

We would like to point out that the thermodynamics limit may be achieved either by sending either \( a \) or \( L \) to \( \infty \). In fact, sending \( a \) to \( \infty \) amounts to cut off high momenta contributions, which amounts to send the size of the system to \( \infty \). This implies that scaling may be realized either using \( a/a_0 \), or \( L/L_* \), as scaling parameters.
Chapter 4

A Josephson Junctions network

In this chapter we introduce a new device of Josephson junction network proposed in [54, 55]. We study our JJ-network by relying onto a perturbative approach in $\mathcal{H}_B$, basically suggested by the apparent smallness of the boundary coupling strength, given in Eq.(4.23), with respect to the main reference energy scale $E_J$. Within the perturbative framework, we shall discuss how to manipulate and detect the state of $S_G$ by means of an appropriate Josephson current pattern, induced across the JJ-network by connecting the outer boundary of the leads to three bulk superconductors at fixed phases $\{\phi_j\}, (j = 1, 2, 3)$. Finally, by pertinently resorting to a renormalization group (RG) approach for determining the flow of the running boundary coupling strengths, we will eventually be able to check the consistency of the perturbative approach against interaction of $T$ with low-energy, long-wavelength collective plasmon excitations of the leads.

4.1 The tetrahedron-shaped network

In this section, we shall describe the device, depicted in Fig.4.1, and derive in detail its low-energy, long-wavelength description, as a 1+1 dimensional field theory, with a pertinent boundary interaction describing the central region. In building the boundary field theory (BFT) description of the network, we will proceed piecewisely: first of all here we introduce the external chains (the “leads”), then we will discuss the central region $T$, depicted in detail in Fig. 4.2, and finally, we will show how to formally connect $T$ to the rest of the network. The three leads will be realized, as in Sec.3.2, with three one-dimensional Josephson junction chains, each one set at Coulomb blockade, with
A Josephson Junctions network

Figure 4.1: The device. A central region $\mathbf{T}$, depicted below, made with four superconducting grains (corresponding to six quantum Josephson junctions) tuned near by the degeneracy between two charge eigenstates, is connected to three leads, made with one-dimensional Josephson junction arrays and driven with three bulk superconductors held at fixed phases $\varphi_1, \varphi_2, \varphi_3$.

Josephson energy $E_J$ and charging energy $E_c$, and $E_c/E_J \gg 1$, and with the gate voltage $V_g$ acting onto each junction tuned near by the degeneracy point between the states with $\mathcal{N}$, or $\mathcal{N} + 1$ Cooper pairs at each junction. In this regime, each lead may be described by a one-dimentional spin-1/2 chain, so that, if each chain is made out of $L$ sites, the corresponding Hamiltonian is given by

$$H_{\text{Leads}} = -\frac{E_J}{2} \sum_{a=1,2,3} \sum_{j=0,L-1} \left\{ S_{a,j}^+ S_{a,j+1}^- + S_{a,j+1}^+ S_{a,j}^- \right\} + E^z \sum_{a=1,2,3} \sum_{j=0}^{L-1} S_{a,j}^z S_{a,j+1}^z, \quad (4.1)$$

with $S_{a,j}^+ = \mathbf{P}_G e^{i \phi_{a,j}} \mathbf{P}_G^\dagger$, and $S_{a,j}^z = \mathbf{P}_G \left[ -i \frac{\partial}{\partial \phi_{a,j}} - V_g \right] \mathbf{P}_G^\dagger$, $\phi_{a,j}$ being the phase of the superconducting order parameter at site-$j$ of the $a$-chain, and $\mathbf{P}_G$ being the projector onto the subspace of the Hilbert space with only either $\mathcal{N}$, or $\mathcal{N} + 1$ Cooper pairs at each superconducting grain, and $E^z$ being the charge repulsion strength between nearest-neighboring junctions. At low-energy, long wavelength, the Hamiltonian in Eq.(4.1) can be pertinently traded for that of a one-dimensional spinless Luttinger liquids, given by

$$H_{\text{LL}} = \frac{g}{4\pi} \sum_{i=1}^3 \int_0^L dx \left[ \frac{1}{u} \left( \frac{\partial \Phi_i}{\partial t} \right)^2 + u \left( \frac{\partial \Phi_i}{\partial x} \right)^2 \right], \quad (4.2)$$
4.1 The tetrahedron-shaped network

Figure 4.2: Detailed view of the central region T. The junctions are assumed to have the same Josephson energy $E_J$, while $f_1, f_2, f_3$ represent the applied fluxes.

where $\Phi_a$ describes collective plasmon modes of the leads, while the Luttinger parameters $g$ and $u$ are given by

$$g = \frac{\pi}{2(\pi - \arccos(\frac{\Delta}{2}))}, \quad u = a E_J \left[ \frac{\pi}{2} \sqrt{1 - \left(\frac{\Delta}{2}\right)^2} \right], \quad \Delta = \left( \frac{E^z}{E_J} - \frac{3}{16} \frac{E_J}{E_c} \right) \quad (4.3)$$

and $a$ is the lattice step.

We shall work out the effective Hamiltonian for the central region T, $\mathcal{H}_T$. The central region is composed of six small Josephson junction, joined to each other as displayed in Fig.4.1. Each junction is assumed to lie within charging regime, that is, its charging energy $E_c$ is assumed to be $\gg E_J$, where $E_J$ is the Josephson energy. In this regime, we will resort to an effective spin-1/2 representation for $\mathcal{H}_T$, by trading each junction for a quantum spin-1/2 variable $S^{(i)}_0$. Assuming that each elementary triangle in Fig.4.1 is pierced by a (dimensionless) magnetic flux $f$, one obtains the effective Hamiltonian

$$\mathcal{H}_T = -H \sum_{i=0}^{3} (S^{(i)}_0)^2 - \frac{E_J}{2} \left[ \sum_{i=1}^{3} \left( e^{i f} (S^{(i)}_0)^+ (S^{(i+1)}_0)^- + e^{-i f} (S^{(i+1)}_0)^+ (S^{(i)}_0)^- \right) + \sum_{i=1}^{3} (S^{(i)}_0)^+ (S^{(0)}_0)^- + (S^{(0)}_0)^+ (S^{(i)}_0)^- \right] \quad (4.4)$$
In Eq.(4.4), $S^{(i)}_0$, $i = 1, 2, 3$, denotes an effective spin-1/2 operator lying at the three side sites of $T$, while $S^{(0)}_0$ lies on the central site. $E_J$ is the nominal Josephson energy of the junctions, while the effective magnetic field $H$ corresponds to a slight uniform charge bias off degeneracy, at each site. $H_T$ in Eq.(4.4) can be exactly diagonalized for a generic value of the applied flux $f$. In particular, in the following we shall label the energy eigenstates by means of two quantum numbers: the former ones refer to the total spin momentum of the states, the latter ones to the $z$-component of the total spin. The states, together with the corresponding energy eigenvalues, are given by (notice that the spin labels correspond to sites 0,1,2,3, respectively; we shall set $s = \sqrt{1 + 3\sin^2 f}$ and $t = \sqrt{3 + \cos^2 f}$ henceforth):

- $m = 2$: a fully polarized spin-2 state:

  \[ |2\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle \]

  \[ \varepsilon_2 = -2H \] (4.5)

- $m = 1$: four states with $m = 1$, given by

  \[ |1, 1\rangle = \frac{1}{\sqrt{2t(t - \cos f)}} [(-\cos f + t) |\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle] \]

  \[ \varepsilon_{1,1}(f) = -H + \frac{E_J}{2} \left(-\cos f - \sqrt{3 + \cos^2 f}\right) \] (4.6)

  \[ |1, 2\rangle = \frac{1}{\sqrt{2t(t + \cos f)}} [(-\cos f - t) |\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle] \]

  \[ \varepsilon_{1,2}(f) = -H + \frac{E_J}{2} \left(-\cos f + \sqrt{3 + \cos^2 f}\right) \] (4.7)

  \[ |1, 3\rangle = \frac{1}{\sqrt{3}} \left[ |\downarrow\downarrow\uparrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\uparrow\uparrow\downarrow\rangle \right] \]

  \[ \varepsilon_{1,3}(f) = -H + \frac{E_J}{2} \left(\cos f - \sqrt{3}\sin f\right) \] (4.8)

  \[ |1, 4\rangle = \frac{1}{\sqrt{3}} \left[ |\downarrow\downarrow\uparrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle \right] \]

  \[ \varepsilon_{1,4}(f) = -H + \frac{E_J}{2} \left(\cos f + \sqrt{3}\sin f\right) \] (4.9)
4.1 The tetrahedron-shaped network

• $m = 0$: six states with $m = 0$, given by

\[
|0, 1\rangle = \frac{1}{\sqrt{6s(s - \sqrt{3} \sin f)}} \left[ (|\downarrow\downarrow\uparrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\downarrow\uparrow\uparrow\downarrow\rangle) + \\
+ (\sqrt{3} \sin f - s) (|\uparrow\uparrow\downarrow\downarrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\downarrow\uparrow\downarrow\rangle + e^{-i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle) \right]
\]

\[
\varepsilon_{0,1}(f) = \frac{E_J}{2} \left( \cos f - \sqrt{1 + 3 \sin^2 f} \right) \tag{4.10}
\]

\[
|0, 2\rangle = \frac{1}{\sqrt{6s(s - \sqrt{3} \sin f)}} \left[ (\sqrt{3} \sin f - s) (|\downarrow\downarrow\uparrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\downarrow\uparrow\uparrow\downarrow\rangle) + \\
+ (|\uparrow\uparrow\downarrow\downarrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\downarrow\uparrow\downarrow\rangle + e^{-i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle) \right]
\]

\[
\varepsilon_{0,2}(f) = \varepsilon_{0,1}(f) \tag{4.11}
\]

\[
|0, 3\rangle = \frac{1}{\sqrt{6s(s + \sqrt{3} \sin f)}} \left[ (|\downarrow\downarrow\uparrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\downarrow\uparrow\uparrow\downarrow\rangle) + \\
+ (\sqrt{3} \sin f + s) (|\uparrow\uparrow\downarrow\downarrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\downarrow\uparrow\downarrow\rangle + e^{-i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle) \right]
\]

\[
\varepsilon_{0,3}(f) = \frac{E_J}{2} \left( \cos f + \sqrt{1 + 3 \sin^2 f} \right) \tag{4.12}
\]

\[
|0, 4\rangle = \frac{1}{\sqrt{6s(s + \sqrt{3} \sin f)}} \left[ (\sqrt{3} \sin f + s) (|\downarrow\downarrow\uparrow\uparrow\rangle + e^{i\frac{2\pi}{3}} |\downarrow\uparrow\downarrow\uparrow\rangle + e^{-i\frac{2\pi}{3}} |\downarrow\uparrow\uparrow\downarrow\rangle) + \\
+ (|\uparrow\uparrow\downarrow\downarrow\rangle + e^{i\frac{2\pi}{3}} |\uparrow\downarrow\uparrow\downarrow\rangle + e^{-i\frac{2\pi}{3}} |\uparrow\uparrow\downarrow\uparrow\rangle) \right]
\]

\[
\varepsilon_{0,4}(f) = \varepsilon_{0,3}(f) \tag{4.13}
\]

\[
|0, 5\rangle = \frac{1}{\sqrt{6}} \left[ ((|\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle) - (|\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle) \right]
\]

\[
\varepsilon_{0,5}(f) = -E_J (\cos f - 1) \tag{4.14}
\]

\[
|0, 6\rangle = \frac{1}{\sqrt{6}} \left[ ((|\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle) + (|\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle) \right]
\]

\[
\varepsilon_{0,6}(f) = -E_J (\cos f + 1) \tag{4.15}
\]

• $m = -1$: four states with $m = -1$, given by

\[
|-1, 1\rangle = \frac{1}{\sqrt{2t(t - \cos f)}} \left[ (-\cos f + t) |\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle \right]
\]
\[ \varepsilon_{-1,1}(f) = H + \frac{E_J}{2} \left( -\cos f - \sqrt{3 + \cos^2 f} \right) \] (4.16)

\[ |{-1, 2}\rangle = \frac{1}{\sqrt{2t(t + \cos f)}} \left[ (-\cos f - t) |\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\rangle \right] \]

\[ \varepsilon_{-1,2}(f) = H + \frac{E_J}{2} \left( -\cos f + \sqrt{3 + \cos^2 f} \right) \] (4.17)

\[ |{-1, 3}\rangle = \frac{1}{\sqrt{3}} \left[ |\uparrow\uparrow\downarrow\downarrow\rangle + e^{-i \frac{2\pi}{3}} |\downarrow\downarrow\uparrow\downarrow\rangle + e^{i \frac{2\pi}{3}} |\downarrow\downarrow\downarrow\uparrow\rangle \right] \]

\[ \varepsilon_{-1,3}(f) = H + \frac{E_J}{2} \left( \cos f - \sqrt{3} \sin f \right) \] (4.18)

\[ |{-1, 4}\rangle = \frac{1}{\sqrt{3}} \left[ |\downarrow\uparrow\uparrow\downarrow\rangle + e^{i \frac{2\pi}{3}} |\downarrow\downarrow\uparrow\downarrow\rangle + e^{-i \frac{2\pi}{3}} |\downarrow\downarrow\downarrow\uparrow\rangle \right] \]

\[ \varepsilon_{-1,4}(f) = H + \frac{E_J}{2} \left( \cos f + \sqrt{3} \sin f \right) \] (4.19)

- **m = -2**: this is a fully polarized spin state, as well

\[ |{-2}\rangle = |\downarrow\downarrow\downarrow\downarrow\rangle \]

\[ \varepsilon_{-2} = 2H \] (4.20)

### 4.2 The charge hopping Hamiltonian

As derived in previous section, for \( \frac{\pi}{2} < f < \frac{3\pi}{2} \), the groundstate of \( H_\mathcal{T} \) is twofold degenerate, which makes it possible to define and the effective spin-1/2 operator \( S_\mathcal{G} \), acting on the corresponding two-dimensional subspace. As one eventually attempts to employ \( \mathcal{T} \) as an effective two-state quantum system, it is important to operate on \( \mathcal{T} \) by connecting it to pertinently designed leads. The “charge hopping” between \( \mathcal{T} \) and the leads is realized by means of three Josephson junctions, of nominal strength \( \lambda \), connecting the endpoints of the leads to the outer sites of \( \mathcal{T} \). The corresponding Hamiltonian is then given by [25]

\[ H_\lambda = \lambda \sum_{i=1}^{3} \left( (S_0^{(i)})^+ e^{-i \frac{\theta_0^{(0)}}{\sqrt{2}}} + (S_0^{(i)})^- e^{i \frac{\theta_0^{(0)}}{\sqrt{2}}} \right) \] (4.21)

Using Eq.(4.21) and the spectrum of \( \mathcal{T} \), one may resort to a Schrieffer-Wolff (SW) transformation (see App. C), which yields an effective boundary Hamiltonian involving...
4.2 The charge hopping Hamiltonian

Figure 4.3: The spectrum of $\mathcal{H}_T$ as a function of the flux $f$. We can see that for $\frac{\pi}{2} < f < \frac{3\pi}{2}$ the ground state is doubly degenerate ($\varepsilon_{0,2}(f) = \varepsilon_{0,1}(f)$) and for $f = \pi$ it has an energy gap from higher states of $\sim E_J$.

only the low-energy degrees of freedom of $T$. This transformation is a kind of onestep renormalization process in which the fluctuations of higher energy states are integrated out.

4.2.1 The effective Boundary Hamiltonian

In our model, when $f = \pi$, the ground state for the unperturbed Hamiltonian $\mathcal{H}_T$ is given by the states $|0, 1\rangle \equiv |\uparrow\rangle$ and $|0, 2\rangle \equiv |\downarrow\rangle$ (or a linear combination of them) with energy $\varepsilon = -E_J$. Using the Schrieffer-Wolff transformation, the resulting interacting boundary Hamiltonian is given by

$$\mathcal{H}_B = 2E_1 \sum_i \cos \left[ \frac{\Phi_i(0) - \Phi_{i+1}(0)}{\sqrt{2}} \right] + 4E_2 S_G^z \sum_i \cos \left[ \frac{\Phi_i(0) - \Phi_{i+1}(0)}{\sqrt{2}} + \frac{\pi}{2} \right] +$$

$$+ 4E_3 S_G^z \sum_j \cos \frac{2}{3} \pi(j - 2) \cos \left[ \frac{\Phi_j(0) - \Phi_{j+1}(0)}{\sqrt{2}} \right] +$$
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\[ +4E_3S_G^z \sum_j \sin \frac{2}{3}\pi(j-2) \cos \left[ \frac{\Phi_j(0) - \Phi_{j+1}(0)}{\sqrt{2}} \right], \quad (4.22) \]

where the coupling constants are given by:

\[ E_1 = \frac{E_J\lambda^2}{3(E_J^2 - 4H^2)} \; , \]
\[ E_z = \frac{2H\lambda^2}{\sqrt{3}(E_J^2 - 4H^2)} \; , \]
\[ E_3 = \frac{2E_J\lambda^2}{3(E_J^2 - 4H^2)} \; , \quad (4.23) \]

As it is evident from the formulas reported above, the effective Hamiltonian \( \mathcal{H}_B \) is, as expected, to second order in parameter expansion \( \lambda \). Additional terms must be added to this Hamiltonian to account for the breaking of the degeneracy of the ground state. There are two contributions to be taken into account:

- when the fluxes are not exactly \( f = \pi \), from SW transformation an extra term appear

\[ +B_{\|}S^z \quad (4.24) \]

where

\[ B_{\|} = -24\sqrt{3}\lambda^2H(f - \pi) \frac{E_J^2}{E_J^2} ; \quad (4.25) \]

- another term may arise in the case in which the three fluxes \( f_j, (j = 1, 2, 3) \) are not anymore equal to each other, we assume that the three fluxes are splitted according to \( f_j \rightarrow f + \delta_j \). Thus, we expand \( H_T[\{f + \delta_i\}] \) as

\[ H_T[\{f + \delta_i\}] \approx H_T[\{f\}] - i\frac{E_J}{2} e^{if} \sum_{i=1}^{3} [\delta_i(S_0^{(i)})^+(S_0^{(i+1)})^-] + \text{h.c.} + \mathcal{O}[[\delta_i^2]] \quad . (4.26) \]

To compute the matrix elements of \( H_T[\{f + \delta_i\}] - H_T[\{f\}] \) within the subspace spanned by \( |\uparrow\rangle, |\downarrow\rangle \), we need the following matrix elements:
4.2 The charge hopping Hamiltonian

\begin{align}
\langle \sigma | \sum_{i=1}^{3} [\delta_i (S_0^{(i)} + (S_0^{(i+1)})^{-}) | \sigma \rangle \propto \sin(f) \sum_{i=1}^{3} \delta_i \\
\langle \sigma | \sum_{i=1}^{3} [\delta_i (S_0^{(i)} + (S_0^{(i+1)})^{-}) | \bar{\sigma} \rangle \propto \sin(f) \left( e^{i \frac{\pi}{3} \delta_1 - \delta_2 + e^{-i \frac{\pi}{3} \delta_3} \right) ,
\end{align}

(4.27)

with $\sigma = \uparrow, \downarrow$, and $\bar{\uparrow} = \downarrow, \bar{\downarrow} = \uparrow$. According to Eqs.(4.27), one sees that, while the twofold groundstate degeneracy takes place within an interval of values of $f$, for the particular value $f = \pi$, all the leading corrections to the energies (including, of course, degeneracy-breaking terms), only appear to second order in the phases $\delta_i$. This resembles the mechanism that “protects” the degeneracy between the states of the tetrahedral qubit proposed in Refs.[56, 57], where it is evidenced that, because of the fact that $T$ responds to fluctuations in the applied flux only to second order in the fluctuations, the twofold groundstate degeneracy is robust against both displacements in the applied gate voltages (represented by $H$), as the twofold degenerate groundstate is made out of two singlets, acting on which $(S_0)^z = 0$, and to displacements in the applied fluxes, as discussed above. To second-order in the phases $\{\delta_i\}$, one finds that two combinations of $| \uparrow \rangle$ and $| \downarrow \rangle$ become the new eigenvalues that splits in energy and their expression and energies are given by:

\begin{align}
| \uparrow' \rangle &= \frac{1}{\sqrt{2}} (| \uparrow \rangle + e^{i \xi} | \downarrow \rangle) \\
| \downarrow' \rangle &= \frac{1}{\sqrt{2}} (| \uparrow \rangle - e^{i \xi} | \downarrow \rangle) \\
\varepsilon_{\uparrow'} &= -E_J (1 + \frac{\delta_1^2 + \delta_2^2 + \delta_3^2}{12}) + \Delta \\
\varepsilon_{\downarrow'} &= -E_J (1 + \frac{\delta_1^2 + \delta_2^2 + \delta_3^2}{12}) - \Delta ,
\end{align}

(4.28)

with

\begin{align}
e^{i \xi} &= \frac{e^{i \frac{\pi}{3} \delta_1^2 + \delta_2^2} + e^{-i \frac{\pi}{3} \delta_3^2}}{\sqrt{(\delta_1^2 - \delta_2^2) + (\delta_2^2 - \delta_3^2) + (\delta_3^2 - \delta_1^2)}} \\
\Delta &= \frac{E_J}{12} \sqrt{2} (\delta_1^2 - \delta_2^2)^2 + (\delta_2^2 - \delta_3^2)^2 + (\delta_3^2 - \delta_1^2)^2 .
\end{align}

(4.29)
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From Eq.(4.28,4.29), one sees that, to second order, a “transverse magnetic field contribution” appears for the effective spin \{\ket{\uparrow}, \ket{\downarrow}\}, given by

\[
H_\perp = B_\perp \cos(\xi) S_G^x + B_\perp \sin(\xi) S_G^y , \quad (B_\perp = -\Delta)
\]

(4.30)

Thus, apparently the twofold groundstate degeneracy of \( \mathcal{T} \) survives up to second order in the parameter displacement of the degenerate point. Now, in order for implementing the system as an effective two-state quantum device, one has to achieve quite a high (and robust) quantum coherence between the degenerate states against fluctuations. As we will discuss in the next section, low-energy plasmon modes of the leads may strongly renormalize the parameters of \( \mathcal{H}_B \), thus letting the twofold degeneracy break down. However, as we will show in the following, fluctuations may also drive the system towards interacting fixed points of the boundary interaction renormalization group flow. In particular, for pertinently chosen parameters, it is possible to let the system operate near by a finite-coupling fixed point of its phase diagram, where an effective two-level quantum device emerges, whose degeneracy is not spoiled out, but rather protected by quantum fluctuations.

4.3 Analysis of the system near by the weakly coupled fixed point

In this section, we study our JJ-nework by relying onto a perturbative approach in \( \mathcal{H}_B \), basically suggested by the apparent smallness of the boundary coupling strength, given in Eq.(4.23), with respect to the main reference energy scale \( E_J \). Within the perturbative framework, we shall discuss how to manipulate and detect the state of \( S_G \) by means of an appropriate Josephson current pattern, induced across the JJ-network by connecting the outer boundary of the leads to three bulk superconductors at fixed phases \{\( \varphi_j \), \( j = 1, 2, 3 \)\}. Finally, by pertinently resorting to a renormalization group (RG) approach for determining the flow of the running boundary coupling strengths, we will eventually be able to check the consistency of the perturbative approach against interaction of \( \mathcal{T} \) with low-energy, long-wavelength collective plasmon excitations of the leads.
4.3 Analysis of the system near by the weakly coupled fixed point

4.3.1 Josephson currents near by the WFP

A Josephson current pattern, induced across the network by connecting the outer boundary of the leads to three bulk superconductors at fixed phases \(\{\varphi_j\} \), may be used to either manipulate, or probe, the state of the twolevel system, emerging at \(T\). As only static phase differences are applied to the network, the Josephson current may be derived within imaginary time path integral formalism. The key object is the imaginary time (Euclidean) action for \(T\) connected to the three leads, \(S_{B}^{(I)}\). From Eq.(4.22), one obtains

\[
S_{B}^{(I)} = 2E_1 \int_0^\beta d\tau \sum_j \cos (\vec{\alpha}_j \cdot \vec{\chi}(\tau)) + 4E_2 \int_0^\beta d\tau S_G^z \sum_j \cos \left(\vec{\alpha}_j \cdot \vec{\chi}(\tau) + \frac{\pi}{2}\right) + \\
+ 4E_3 \int_0^\beta d\tau S_G^x \sum_j \cos \left(\frac{2}{3}\pi(j-2)\right) \cos (\vec{\alpha}_j \cdot \vec{\chi}(\tau)) + \\
+ 4E_3 \int_0^\beta d\tau S_G^y \sum_j \sin \left(\frac{2}{3}\pi(j-2)\right) \cos (\vec{\alpha}_j \cdot \vec{\chi}(\tau)) ,
\]

(4.31)

with \(\beta = (k_B T)^{-1}\), \(\vec{\chi}(\tau) = \vec{\chi}(0, \tau)\), and the fields \(\chi_1(x, \tau) = \frac{1}{\sqrt{2}}[\Phi_1(x, \tau) - \Phi_2(x, \tau)]\), \(\chi_2(x, \tau) = \frac{1}{\sqrt{6}}[\Phi_1(x, \tau) + \Phi_2(x, \tau) - 2\Phi_3(x, \tau)]\). Notice that, according to a standard notation [57, 25], we have defined

\[
\vec{\alpha}_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \quad \vec{\alpha}_2 = \left( \begin{array}{c} -1/2 \\ \sqrt{3}/2 \end{array} \right) , \quad \vec{\alpha}_3 = \left( \begin{array}{c} -1/2 \\ -\sqrt{3}/2 \end{array} \right),
\]

(4.32)

to evidence that the “center of mass” field \(\Phi(x, \tau) = \frac{1}{\sqrt{2}}[\Phi_1(x, \tau) + \Phi_2(x, \tau) - \Phi_3(x, \tau)]\) decouples from \(S_{B}^{(I)}\), which corresponds to charge conservation at \(T\) [57, 25]. By keeping only the two fields that interact of the boundary, \(\chi_1, \chi_2\), one obtains that the “free” action \(S_0\) will be given by \(S_0 = S_{\text{Lead}}[\vec{\chi}] + S_S[\Phi, \Theta]\), with

\[
S_{\text{Lead}}[\vec{\chi}] = \frac{g}{4\pi} \int_0^\beta d\tau \int_0^L dx \left[ \frac{1}{u} \left( \frac{\partial \vec{\chi}}{\partial \tau} \right)^2 + u \left( \frac{\partial \vec{\chi}}{\partial x} \right)^2 \right],
\]

(4.33)

while \(S_S\) describes the dynamics of the spin-1/2 variable \(S_G\) [54, 55]. In terms of the angular variables \(\Phi(\tau), \Theta(\tau)\), \(S_S\) is given by
A Josephson Junctions network

\[ S_{\Sigma}[\Phi, \Theta] = -\frac{i}{2} \int_0^\beta d\tau \frac{d\Phi(\tau)}{d\tau} [1 - \cos(\Theta(\tau))] + \]
\[ -\frac{1}{2} \int_0^\beta d\tau \{ B_\parallel \cos(\Theta(\tau)) + B_\perp \sin(\Theta(\tau)) \cos(\Phi(\tau) - \xi) \} . \quad (4.34) \]

Assuming, now, that the three leads are connected to three bulk superconductors at fixed phases \( \{\varphi_1, \varphi_2, \varphi_3\} \), one finds that the Josephson currents along the three leads of the JJ-network, are given by \[25\],

\[ I_1 = -\frac{e^*}{\beta} \left\{ \sqrt{2} \frac{\partial \ln Z}{\partial \mu_1} - \sqrt{6} \frac{\partial \ln Z}{\partial \mu_2} \right\} \]
\[ I_2 = -\frac{e^*}{\beta} \left\{ -\sqrt{2} \frac{\partial \ln Z}{\partial \mu_1} - \sqrt{6} \frac{\partial \ln Z}{\partial \mu_2} \right\} \]
\[ I_3 = -\frac{e^*}{\beta} \sqrt{6} \frac{\partial \ln Z}{\partial \mu_2} , \quad (4.35) \]

where the relative phases \( \mu_1, \mu_2 \) are defined as \( \mu_1 = (\varphi_1 - \varphi_2)/\sqrt{2} \), and \( \mu_2 = (\varphi_1 + \varphi_2 - 2\varphi_3)/\sqrt{6} \), while \( Z \) is the partition function of the network, given by

\[ Z = Z_0 \langle \mathbf{T} e^{-S_{\Sigma}} \rangle_{(0)} . \quad (4.36) \]

In Eq.(4.36), we have set

\[ Z_0 = \int \prod_{i=1,2} \mathcal{D}\chi_i \int \mathcal{D}\Omega e^{-S_{\text{Lead}}[\vec{\chi}]-S_{\Sigma}[\Phi, \Theta]} , \quad (4.37) \]

while \( \langle \ldots \rangle_{(0)} \) denotes averaging with respect to the “statistical weight” defined by \( e^{-S_{\text{Lead}}[\vec{\chi}]-S_{\Sigma}[\Phi, \Theta]} \), and \( \Omega \) is the solid angle on the unit sphere.

To compute \( Z \), one has to sum over the oscillating modes of the fields \( \chi_j \). In order to do so, one needs to choose the pertinent boundary conditions for \( \chi_1(x, \tau), \chi_2(x, \tau) \) at \( x = 0 \). These are dictated by energy conservation and are given by
4.3 Analysis of the system near by the weakly coupled fixed point

\[ \frac{u g}{2\pi} \frac{\partial \bar{\chi}(0, \tau)}{\partial x} = 2 \tilde{E}_1 \sum_j \tilde{\alpha}_j \sin[\tilde{\alpha}_j \cdot \bar{\chi}(0, \tau)] + 
+ 4 \tilde{E}_2 \sum_j \tilde{\alpha}_j \mathbf{S}_j \cdot \sin \left[ \tilde{\alpha}_j \cdot \bar{\chi}(0, \tau) + \frac{\pi}{2} \right] + 
+ 4 \tilde{E}_3 \sum_j \tilde{\alpha}_j \mathbf{S}_j \cdot \cos \left( \frac{2\pi}{3} (j - 2) \right) \sin \left[ \tilde{\alpha}_j \cdot \bar{\chi}(0, \tau) \right] + 
+ 4 \tilde{E}_3 \sum_j \tilde{\alpha}_j \mathbf{S}_j \cdot \sin \left( \frac{2\pi}{3} (j - 2) \right) \sin \left[ \tilde{\alpha}_j \cdot \bar{\chi}(0, \tau) \right], \quad (4.38) \]

where, in order to account for normal ordering of boundary interaction operators, the boundary interaction strengths have been redefined as \( \tilde{E}_\ell = \left( \frac{a}{L} \right)^{\frac{1}{2}} E_\ell , \) \( (\ell = 1, z, 3) \) [23, 25]. Consistently with the boundary conditions reported in Eq.(4.38), one implements the weak coupling assumption on the boundary interaction by setting Neumann boundary conditions for the plasmon fields at \( x = 0 \), that is, \( \frac{\partial \chi_1(0, \tau)}{\partial x} = \frac{\partial \chi_2(0, \tau)}{\partial x} = 0 \). At the outer boundary, \( x = L \), the fields are connected to three bulk superconductors at fixed phases. This is accounted for by assuming Dirichlet-like boundary conditions at \( x = L \), that is, \( \chi_1(L, \tau) = \mu_1, \chi_2(L, \tau) = \mu_2 \). Accordingly, the fields \( \chi_j(x, \tau) \) take the mode expansion

\[ \chi_j(x, t) = \mu_j + \sqrt{2} \sum_{n \in \mathbb{Z}} \cos \left[ \frac{\pi}{L} \left( n + \frac{1}{2} \right) x \right] \frac{\alpha_j(n)}{n + \frac{1}{2}} e^{-\frac{\pi}{L} \left( n + \frac{1}{2} \right) \mu_\tau}, \quad (4.39) \]

with the algebra between the oscillator modes being given by

\[ [\alpha_i(n), \alpha_j(m)] = \delta_{i,j} \delta_{n+m-1,0} \left( n + \frac{1}{2} \right). \quad (4.40) \]

Since we intend to use the currents/applied phases as “gauges” to manipulate the state of the spin-1/2 variable \( \mathbf{S}_G \), it is convenient to define “spin-1/2 current operators”, that is, operators that act onto the two-dimensional Hilbert space of \( \mathbf{S}_G \) and give the correct value of the current, when \( \mathbf{S}_G \) is averaged over, as well. From the derivation of this section, it is straightforward to see that the required operators are given by
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\[ I_1 = -\frac{e^*}{\beta} \left\{ \sqrt{2} \frac{\partial \ln Z_S}{\partial \mu_1} - \sqrt{6} \frac{\partial \ln Z_S}{\partial \mu_2} \right\}, \]

\[ I_2 = -\frac{e^*}{\beta} \left\{ -\sqrt{2} \frac{\partial \ln Z_S}{\partial \mu_1} - \sqrt{6} \frac{\partial \ln Z_S}{\partial \mu_2} \right\}, \]

\[ I_3 = -\frac{e^*}{\beta} \sqrt{6} \frac{\partial \ln Z_S}{\partial \mu_2}, \] (4.41)

with

\[ Z_S = \int \prod_{i=1,2} D\chi_j e^{-S_{\text{Lead}} - s_{\text{B}} - s_S}. \] (4.42)

Relying over the weak coupling assumption (which will be spelled out in detail in the next subsection), one may sum over the plasmon modes \( \chi_j \) within a mean-field like approach. In particular, because of the mode expansion in Eq.(4.39), one gets

\[ \langle \cos[\vec{\alpha}_j \cdot \vec{\chi}(\tau)] \rangle = \cos[\vec{\alpha}_j \cdot \vec{\mu}], \quad \langle \sin[\vec{\alpha}_j \cdot \vec{\chi}(\tau)] \rangle = \sin[\vec{\alpha}_j \cdot \vec{\mu}]. \] (4.43)

Thus, one readily sees that, resorting to the mean-field approximation amounts to trade \( \vec{\chi}(\tau) \) for the applied phase differences \( \vec{\mu} \). As a result, one obtains

\[ -\frac{e^*}{\beta} \frac{\partial \ln Z_S}{\partial \vec{\mu}} = 2e^* E_1 \sum_j \alpha_j \sin[\vec{\alpha}_j \cdot \vec{\mu}] + 4e^* E_z \sum_j \alpha_j S^z_G \cos[\vec{\alpha}_j \cdot \vec{\mu}] + \]

\[ + 4e^* E_3 \sum_j \alpha_j S^x_G \cos \left( \frac{2\pi}{3}(j - 2) \right) \sin[\vec{\alpha}_j \cdot \vec{\mu}] + \]

\[ + 4e^* E_3 \sum_j \alpha_j S^y_G \sin \left( \frac{2\pi}{3}(j - 2) \right) \sin[\vec{\alpha}_j \cdot \vec{\mu}]. \] (4.44)

There is two possible ways of interpreting Eq.(4.44):

- one may regard the applied phases (and the induced currents) as a probe of the two-level state (which may be set by acting upon it with the external fields \( B_{||}, B_{\perp} \)). For instance, assuming that the system lies within either one of the eigenstates of \( S^x, |\pm\rangle_z \), and computing the average values of the spin operators as outlined in the next section, from Eq.(4.44) one gets
4.3 Analysis of the system near by the weakly coupled fixed point

\[
\left\langle \frac{\partial \ln Z_S}{\partial \vec{\mu}} \right\rangle_S = 2 \sum_j \bar{\alpha}_j \left\{ \bar{E}_1 \sin [\bar{\alpha}_j \cdot \vec{\mu}] \pm \bar{E}_z \cos [\bar{\alpha}_j \cdot \vec{\mu}] \right\} , \tag{4.45}
\]

where the average is computed over the spin coordinates. The corresponding current pattern may be derived from Eq.(4.41) and from Eq.(4.45): clearly, a current measurement discriminates between the two eigenstates $|\pm\rangle_z$. The same procedure may be applied to probing a generic state of $S_G$, obviously as long as the modification in the state induced by the application of the phases $\vec{\mu}$ is negligible.

- When no other fields are applied to $S_G$ (that is, when $B_\parallel = B_\perp = 0$), the phases (and, of course, the currents) themselves may be regarded as defining an effective applied field $\mathbf{B}$, whose components are given by

\[
\begin{align*}
\mathbf{B}^x &= -4\bar{E}_3 \sum_j \cos \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\bar{\alpha}_j \cdot \vec{\mu}] , \\
\mathbf{B}^y &= -4\bar{E}_3 \sum_j \sin \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\bar{\alpha}_j \cdot \vec{\mu}] , \\
\mathbf{B}^z &= 4\bar{E}_z \sum_j \sin[\bar{\alpha}_j \cdot \vec{\mu}] . \tag{4.46}
\end{align*}
\]

In this case, the phases may be used to drive the state of the two-level system, just as one does when acting with a local magnetic field onto a true spin-1/2 variable.

In fact, though, apparently, we eventually got an emerging two-level quantum system in a pertinent effective magnetic field, our derivation strongly relies onto the “mean field-like” approach, which is basically justified by the consistency of the perturbative approach against renormalization effects due to interaction with the plasmon degrees of freedom of the leads. These effects will be investigated in detail in the following. Eventually, we will be able to map out the whole phase diagram of our system, finding out for which parameter range the network operates within a regime where fluctuations enforce, rather than suppressing, the stability of the emerging two-level quantum device.
4.3.2 Renormalization group flow near by the weakly coupled fixed point

In this section we shall implement the Renormalization Group (RG) approach to infer how, and to what extent, the results we obtained in the previous subsection keep preserved against renormalization of the boundary coupling strengths due to the coupling with the plasmon modes of the leads. To derive the RG equations for the running coupling strengths, we use of a boundary version of the RG approach to perturbed conformal field theories, developed by Cardy [34]. To begin, let us rewrite $S_B^{(I)}$ in Eq.(4.31) as

$$S_B^{(I)} = \sum_j \int_0^\beta d\tau : e^{i\vec{\alpha}_j \cdot \vec{\chi}(\tau)} : \left\{ E_1 + i2\bar{E}_z S^z_G(\tau) + 2\bar{E}_3 \cos \left[ \frac{2\pi}{3} (j - 2) \right] S^x_G(\tau) + 2\bar{E}_3 \sin \left[ \frac{2\pi}{3} (j - 2) \right] S^y_G(\tau) \right\} \right) + \sum_j \int_0^\beta d\tau : e^{-i\vec{\alpha}_j \cdot \vec{\chi}(\tau)} : \left\{ E_1 - i2\bar{E}_z S^z_G(\tau) + 2\bar{E}_3 \cos \left[ \frac{2\pi}{3} (j - 2) \right] S^x_G(\tau) + 2\bar{E}_3 \sin \left[ \frac{2\pi}{3} (j - 2) \right] S^y_G(\tau) \right\} ,$$

where the colons $: \ldots :$ denote normal ordering with respect to the vacuum of the bosonic theory [23]. Since here we are only interested in the renormalization of the running boundary coupling strengths, for simplicity we assume free boundary conditions for the plasmon fields of the leads, which correspond to Neumann boundary conditions at both boundaries for the fields $\chi_1, \chi_2$, that is

$$\frac{\partial \chi_j(0, \tau)}{\partial x} = \frac{\partial \chi_j(L, \tau)}{\partial x} = 0 \quad (j = 1, 2) \quad . \quad (4.47)$$

According to Eqs.(4.47), $\chi_1, \chi_2$ now take the mode expansion [58]

$$\chi_j(x, \tau) = \frac{1}{\sqrt{g}} \left[ \phi_j + \frac{2\pi i u \tau}{L} \tilde{P}_j \right] + \frac{i}{\sqrt{2} g} \left\{ \sum_{n \neq 0} \frac{a_j(n)}{n} \cos \left[ \frac{2\pi n x}{L} \right] e^{-\frac{2\pi n u \tau}{L}} \right\} , \quad (4.48)$$

with $[\phi_j, \tilde{P}_j'] = i\delta_{jj'}$, and $[\alpha_j(n), \alpha_j'(m)] = n\delta_{jj'} \delta_{n+m,0}$. The RG equations for the dimensionless boundary coupling strengths $G_1(L) = L\bar{E}_1, G_z(L) = L\bar{E}_z, G_3(L) = L\bar{E}_3$ to second order in the boundary coupling themselves, are derived by considering the
nonzero operator product expansions (O.P.E.) between the vertex operators: \(e^{\pm iv\vec{a}_j \chi(\tau)}\). They are given by

\[
\{ : \exp [\pm i\vec{a}_i \cdot \chi(\tau)] :: \exp [\pm i\vec{a}_j \cdot \chi(\tau') ] : \} \tau \rightarrow \tau^- \approx \left[ \frac{v(\tau - \tau')}{L} \right]^{-\frac{2}{g}} : \exp [\mp i\vec{a}_k \cdot \chi(\tau)] : ,
\]

(4.49)

with \(i \neq j \neq k\). The O.P.E. for the spin-1/2 operators may be derived starting from the following correlation functions:

- **One-spin correlation functions:**

\[
\langle S^x_G \rangle_{(0)} = \frac{1}{2} \sin(\phi) \sin(\theta) , \quad \langle S^y_G \rangle_{(0)} = \frac{1}{2} \cos(\phi) \sin(\theta) , \quad \langle S^z_G \rangle_{(0)} = \frac{1}{2} \cos(\theta) ,
\]

(4.50)

with the angles \(\phi\) and \(\theta\) are defined by

\[
\cos(\phi) = B_x / \sqrt{B^2_x + B^2_y} \quad \sin(\phi) = B_y / \sqrt{B^2_x + B^2_y}
\]

\[
\cos(\theta) = B_\parallel / \sqrt{B^2_\parallel + B^2_\perp} \quad \sin(\theta) = B_\perp / \sqrt{B^2_\parallel + B^2_\perp}
\]

and \(B_\perp = \sqrt{B^2_x + B^2_y}\).

- **Two-spin correlation functions:**

\[
\langle \mathbf{T}_r [S^x_G(\tau)S^x_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \sin^2(\phi) \cos^2(\theta) e^{-2\lambda|\tau - \tau'|} + \sin^2(\phi) \sin^2(\theta) \right]
\]

\[
\langle \mathbf{T}_r [S^y_G(\tau)S^y_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos^2(\phi) \sin^2(\theta) + \left[ \cos^2(\phi) \cos^2(\theta) + \sin^2(\phi) \right] e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^z_G(\tau)S^z_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos^2(\theta) + \sin^2(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^x_G(\tau)S^y_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \sin(\phi) \cos(\phi) \sin^2(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) - \cos(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^y_G(\tau)S^x_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \sin(\phi) \cos(\phi) \sin^2(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) + \cos(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^x_G(\tau)S^z_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos(\phi) \sin(\theta) \cos(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) - i \sin(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^y_G(\tau)S^z_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos(\phi) \sin(\theta) \cos(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) + i \sin(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]

\[
\langle \mathbf{T}_r [S^z_G(\tau)S^z_G(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos(\phi) \sin(\theta) \cos(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) + i \sin(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]
\]
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$$\langle T_{\tau} [S_{G}^{z}(\tau)S_{G}^{x}(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \cos(\phi) \sin(\theta) \cos(\theta) (1 - e^{-2\lambda|\tau - \tau'|}) + i \sin(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]$$

$$\langle T_{\tau} [S_{G}^{z}(\tau)S_{G}^{y}(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \sin(\phi) \sin(\theta) \cos(\theta) (1 + e^{-2\lambda|\tau - \tau'|}) - i \cos(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]$$

$$\langle T_{\tau} [S_{G}^{x}(\tau)S_{G}^{z}(\tau')] \rangle_{(0)} = \frac{1}{4} \left[ \sin(\phi) \sin(\theta) \cos(\theta) (1 + e^{-2\lambda|\tau - \tau'|}) - i \cos(\phi) \sin(\theta) e^{-2\lambda|\tau - \tau'|} \right]$$

(4.51)

with $\lambda = \sqrt{B^2_\parallel + B^2_\perp}$.

When $B_\parallel = B_\perp = 0$, Eqs.(4.50,4.51) simply reduce to the equations

$$S^a_G S^b_G = \frac{1}{2} i \epsilon^{abc} S^c_G .$$

(4.52)

As a first consequence, we readily find out that no $B$ field is induced up to second-order in the running couplings. Indeed, the leading correction to the boundary action induced to second order in the boundary couplings is given by

$$\delta S^{(II)}_B = - \sum_{k=1,2,3} \int_0^\beta d\tau : e^{-i\vec{a}_k \cdot \vec{\chi}(\tau)} : \left\{ \mathcal{E}_1 + \mathcal{E}_z S^z_G + \mathcal{E}_x S^x_G + \mathcal{E}_y S^y_G \right\} + \text{h.c.} ,$$

(4.53)

where $i, j, k = 1, 2, 3$, or cyclic permutations and

$$\mathcal{E}_1 = \bar{E}_1^2 - \bar{E}_z^2 + \bar{E}_3^2 \cos \left[ \frac{2\pi}{3} (i - j) \right] ,$$

$$\mathcal{E}_z = 2i \left[ 2\bar{E}_1 \bar{E}_z - \bar{E}_3^2 \sin \left[ \frac{2\pi}{3} (i - j) \right] \right] ,$$

$$\mathcal{E}_x = 2 \left[ \bar{E}_1 \bar{E}_3 \left( \cos \left( \frac{2\pi}{3} (j - 2) \right) + \cos \left( \frac{2\pi}{3} (i - 2) \right) \right) + \bar{E}_z \bar{E}_3 \left( \sin \left( \frac{2\pi}{3} (j - 2) \right) - \sin \left( \frac{2\pi}{3} (i - 2) \right) \right) \right] ,$$

$$\mathcal{E}_y = 2 \left[ \bar{E}_1 \bar{E}_3 \left( \sin \left( \frac{2\pi}{3} (j - 2) \right) + \sin \left( \frac{2\pi}{3} (i - 2) \right) \right) + \bar{E}_z \bar{E}_3 \left( - \cos \left( \frac{2\pi}{3} (j - 2) \right) - \cos \left( \frac{2\pi}{3} (i - 2) \right) \right) \right] .$$

(4.54)

Since Eq.(4.47) implies $\langle e^{\pm i\vec{a}_k \cdot \vec{\chi}(\tau)} \rangle = 0$, $\forall k = 1, 2, 3$, one readily gets that, to second order, no $B$-field is induced. Based on symmetry arguments, it is straightforward
4.3 Analysis of the system near by the weakly coupled fixed point

to infer that $B$ keeps equal to zero to any order in the boundary coupling strengths. Moreover, from Eqs.\((4.53)\) and \((4.54)\) one readily derives the RG equations for the boundary coupling strengths, which are given by

\[
\frac{dG_1(L)}{d\ln\left(\frac{L}{L_0}\right)} = \left[1 - \frac{1}{g}\right] G_1(L) - 2 \left[ G_1^2(L) - G_2^2(L) - \frac{1}{2} G_3^2(L) \right] \equiv \beta_1(G_1, G_2, G_3) \quad (4.55)
\]

\[
\frac{dG_2(L)}{d\ln\left(\frac{L}{L_0}\right)} = \left[1 - \frac{1}{g}\right] G_2(L) + 4 G_1(L) G_2(L) \equiv \beta_2(G_1, G_2, G_3) \quad (4.56)
\]

\[
\frac{dG_3(L)}{d\ln\left(\frac{L}{L_0}\right)} = \left[1 - \frac{1}{g}\right] G_3(L) + 2 G_1(L) G_3(L) \equiv \beta_3(G_1, G_2, G_3) \quad . \quad (4.57)
\]

From the sign of the linear term in the $\beta$-functions in Eqs.\((4.57)\), one sees that the boundary couplings are all irrelevant/marginal/relevant, according to whether $g < 1$, $g = 1$, or $g > 1$. Thus, when $g < 1$, a perturbative approach, as the one implemented in subsection 4.3.1, is certainly reliable to study the properties of the system. At variance, when $g > 1$ the boundary interaction provides a relevant perturbation. To infer the low-energy behavior of the system in this case, in the next section we employ a nonperturbative analysis of the JJ-network near by a strongly coupled fixed point. Finally, we notice that, for $g = 1$ and $G_1(L_0) = 0$, the RG equations in Eq.\((4.57)\) coincide with the RG equations for the boundary coupling in an anisotropic Kondo model \[59]\]. Thus, it is natural to assume that, for these values of the parameters, our network emulates a Kondo spin with anisotropic couplings to the itinerant electrons. A richer structure emerges, if one assumes a nonzero “bare” magnetic field $B_0$. It is likely that phenomena like the “pairing of Cooper pairs” studied in Ref. \[60]\) might arise in this case.

In the next section, we shall start from Eqs.\((4.57)\) to infer the main features of the strongly coupled fixed point and, possibly, its stability against the leading boundary perturbation allowed by system’s symmetry properties.
4.4 Analysis of the system near by the strongly coupled fixed point

From the RG equations in Eq.(4.57), one sees that all the boundary couplings are relevant, for \( g > 1 \). To analyze the large-scale behavior of the system in this case, one assumes that, as the size of the system exceeds the “healing length” \( L_* \), the running boundary couplings flow all the way down to infinity. Accordingly, the boundary conditions for the fields \( \vec{\chi}(0,t) \) will be fixed at a minimum of \( \mathcal{H}_B \) (Dirichlet boundary condition), and the leading boundary interaction at such a strongly coupled fixed point (SFP) will be built as combination of operators describing tunneling between neighboring minima of the boundary Hamiltonian (instantons).

4.4.1 The lattice of the minima at the strongly coupled fixed point

To analyze the behavior of the system at the SFP, we start by assuming that \( \chi_1(0), \chi_2(0) \) take values corresponding to a minimum of \( \mathcal{H}_B \). In order to do so, we follow the approach developed in Ref.[29], by representing the spin-1/2 operator \( S_G \) as a 2×2 matrix and, accordingly, rewriting \( \mathcal{H}_B \) in a matrix form as

\[
\mathcal{H}_B = \begin{pmatrix}
V_1(\vec{\chi}(0)) + V_2(\vec{\chi}(0)) + B_\parallel(f) & V_2(\vec{\chi}(0)) - iV_g(\vec{\chi}(0)) \\
V_2(\vec{\chi}(0)) + iV_g(\vec{\chi}(0)) & V_1(\vec{\chi}(0)) - V_2(\vec{\chi}(0)) - B_\parallel(f)
\end{pmatrix}.
\]  

In Eq.(4.58) the term \( B_\parallel(f) \) takes into account a possible deviation from the point \( f = \pi \) and it is given by Eq.(4.25), while the functions \( V_i(\vec{\chi}(0)) \) correspond to the terms in \( \mathcal{H}_B \) that multiply the \( i \)th component of the spin \( S_G \). From Eq.(4.22), one finds that they are given by

\[
V_1[\vec{\chi}] = 2\tilde{E}_1 \sum_j \cos[\vec{\alpha}_j \cdot \vec{\chi}] ,
\]

\[
V_x[\vec{\chi}] = 2\tilde{E}_3 \sum_j \cos \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\vec{\alpha}_j \cdot \vec{\chi}] ,
\]

\[
V_y[\vec{\chi}] = 2\tilde{E}_3 \sum_j \sin \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\vec{\alpha}_j \cdot \vec{\chi}] ,
\]
4.4 Analysis of the system near by the strongly coupled fixed point

Figure 4.4: Lattice of the minima of $\mathcal{H}_B$. In the box: the triangle whose vertex can be made degenerate with a proper choice of the external phases.

\[ V_z[\vec{\chi}] = -2\bar{E}_z \sum_j \sin[\vec{\alpha}_j \cdot \vec{\chi}] . \] (4.59)

The next step is to determine the eigenvalues of $\mathcal{H}_B$ as a function of $\chi_1(0)$ and $\chi_2(0)$: these are given by

\[
\Lambda_1(\vec{\chi}(0)) = V_1 - \sqrt{V_x^2 + V_y^2 + (V_z + B_\parallel)^2} \quad (4.60)
\]
\[
\Lambda_2(\vec{\chi}(0)) = V_1 + \sqrt{V_x^2 + V_y^2 + (V_z + B_\parallel)^2} . \quad (4.61)
\]

Clearly, for any value of $\vec{\chi}$, one gets $\Lambda_1 \leq \Lambda_2$. Thus, the minima of $\mathcal{H}_B$ are identified with the minima of $\Lambda_1(\vec{\chi}(0))$. We find a Kagome lattice of minima (in the $\chi_1(0), \chi_2(0)$-plane), which we display in Fig.4.4. To determine the polarization of $\mathbf{S}_G$ corresponding to each minimum, we compute the $\mathbf{V}^i$’s at the minima and set the polarization of $\mathbf{S}_G$ so to minimize the corresponding boundary energy. Finally, in order to take into account the effects of the applied phases, as well, we assume Dirichlet boundary conditions for
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\( \chi(x, \tau) \) at both boundaries. Accordingly, the plasmon fields take the mode expansion

\[
\chi_j(x, \tau) = \xi_j + \sqrt{\frac{2}{g}} \left\{ (L - x) \frac{\pi}{L} P_j - \sum_{n \neq 0} \sin \left( \frac{\pi n x}{L} \right) \frac{\alpha_j}{n} e^{-\frac{\pi}{2} n \nu \tau} \right\}, \tag{4.62}
\]

with the spectrum of the zero-mode eigenvalues being fixed by the boundary conditions and given by

\[
(p_{1l}, p_{2l}) = \sqrt{2g} \left( -\frac{\mu_1}{2\pi} + n_{12} + \epsilon_l, -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}} (2n_{13} - n_{12} + \delta_l) \right), \tag{4.63}
\]

\( n_{12}, n_{13} \) being relative integers. The index \( l \) discriminates among the "red" (R), the "green" (G), and the "blue" (B) sublattices in Fig.4.4, and \( \epsilon_R = 1, \epsilon_G = 1/2, \epsilon_B = 1/2; \delta_R = 0, \delta_G = -1/2, \delta_B = 1/2 \), while the constants \( \xi_1, \xi_2 \) cancel the terms \( \propto \mu_1, \mu_2 \) in \( \chi_1(0), \chi_2(0) \), respectively. The associated spin polarization is uniform throughout each one of the three sublattices, and the spin component of the corresponding states are given by

\[
|R\rangle = \left( \cos \frac{\theta_f}{2} |\uparrow\rangle + e^{i\frac{\pi}{3}} \sin \frac{\theta_f}{2} |\downarrow\rangle \right), \tag{4.64a}
\]

\[
|G\rangle = \left( \cos \frac{\theta_f}{2} |\uparrow\rangle + e^{-i\frac{\pi}{3}} \sin \frac{\theta_f}{2} |\downarrow\rangle \right), \tag{4.64b}
\]

\[
|B\rangle = \left( \cos \frac{\theta_f}{2} |\uparrow\rangle + e^{i\pi} \sin \frac{\theta_f}{2} |\downarrow\rangle \right), \tag{4.64c}
\]

where \( \cos \frac{\theta_f}{2} = \frac{\sqrt{16 + B^2 - B_\parallel^2}}{\sqrt{16 + B^2 - B_\parallel^2} \sqrt{16 + B^2}} \) and \( \sin \frac{\theta_f}{2} = \frac{1}{\sqrt{16 + B^2 - B_\parallel^2 \sqrt{16 + B^2}}} \). At first order in \( B_\parallel \) (i.e. at first order in \( (f - \pi) \)) one obtains:

\[
\theta_f \approx \frac{\pi}{2} + \frac{B_\parallel}{4}. \tag{4.65}
\]

Due to the nontrivial zero-mode spectrum, the energy of each field configuration takes a zero-mode contribution that is quadratic in \( \vec{p} \). Inserting the solution in Eq.(4.62) into the noninteracting Hamiltonian \( H_{LL} \), one finds the zero-mode contribution to the total energy over each sublattice, as listed below:
4.4 Analysis of the system near by the strongly coupled fixed point

- **R-points:**

  Zero-mode eigenvalues:

  \[ (p_1, p_2)_R = \sqrt{2g} \left( -\frac{\mu_1}{2\pi} + n_{12} + 1, -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12}) \right) \]

  Zero-mode contribution to the total energy:

  \[ E_R^{(0)}_{n_{12}, n_{13}}(\vec{\mu}) = \frac{\pi ug}{L} \left[ \left( -\frac{\mu_1}{2\pi} + n_{12} + 1 \right)^2 + \left( -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12}) \right)^2 \right] \]

- **G-points:**

  Zero-mode eigenvalues:

  \[ (p_1, p_2)_G = \sqrt{2g} \left( -\frac{\mu_1}{2\pi} + n_{12} + 1, -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12} - \frac{1}{2}) \right) \]

  Zero-mode contribution to the total energy:

  \[ E_G_{n_{12}, n_{13}}(\vec{\mu}) = \frac{\pi ug}{L} \left[ \left( -\frac{\mu_1}{2\pi} + n_{12} + 1 \right)^2 + \left( -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12} - \frac{1}{2}) \right)^2 \right] \]

- **B-points:**

  Zero-mode eigenvalues:

  \[ (p_1, p_2)_B = \sqrt{2g} \left( -\frac{\mu_1}{2\pi} + n_{12} + \frac{1}{2}, -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12} + \frac{1}{2}) \right) \]

  Zero-mode contribution to the total energy:

  \[ E_B_{n_{12}, n_{13}}(\vec{\mu}) = \frac{\pi ug}{L} \left[ \left( -\frac{\mu_1}{2\pi} + n_{12} + \frac{1}{2} \right)^2 + \left( -\frac{\mu_2}{2\pi} + \frac{1}{\sqrt{3}}(2n_{13} - n_{12} + \frac{1}{2}) \right)^2 \right] \]
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It is quite remarkable that the zero-mode energy spectrum reported in Eqs. (4.67, 4.69, 4.71) displays, for pertinent choices of the applied phases $\mu_1, \mu_2$, a threefold degeneracy between the three sites lying at the vertices of a single triangle in Fig. 4.4. This is quite important, in view of practical applications of our results, as we shall discuss later, while, in the next section, we shall derive in detail the leading boundary perturbation allowed by system’s symmetries at the SFP.

### 4.5 A two level quantum system at Finite Fixed Point

To derive the leading boundary perturbation at the SFP, we begin by neglecting the finite-size zero-mode contributions to the total energy calculated above, and construct the corresponding boundary Hamiltonian in terms of phase-slip operators representing instanton trajectories connecting nearest neighboring sites on the lattice in Fig. 4.4. To do so, we introduce the field operators “dual” of $\chi_1(x,t), \chi_2(x,t), \Theta_1(x,t), \Theta_2(x,t)$, which are related to $\chi_1, \chi_2$ by means of the cross-derivative relations [25]

\begin{equation}
\frac{1}{ug} \frac{\partial \Theta_1(x,t)}{\partial t} = \frac{\partial \Phi_1(x,t)}{\partial x}, \quad \frac{g}{u} \frac{\partial \Phi_1(x,t)}{\partial t} = \frac{\partial \Theta_1(x,t)}{\partial x} .
\end{equation}

In particular, Eqs. (4.72) imply that $\Theta_{1,2}(x,t)$ obey Neumann boundary conditions at $x = 0$ and at $x = L$. Accordingly, their mode expansion is given by

\begin{equation}
\Theta_j(x,t) = \sqrt{2g} \left\{ \frac{\theta_0^j}{L} + \frac{\pi vt}{L} P_j + i \sum_{n \neq 0} \cos \left[ \frac{\pi nx}{L} \right] \frac{\tilde{\alpha}_n^j}{n} e^{-i \frac{\pi}{L} \nu t} \right\} ,
\end{equation}

with the mode algebra

\begin{equation}
[P_i, \theta_0^j] = -i \delta_{i,j} , \quad [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n \delta_{n+m,0} \delta_{i,j} .
\end{equation}

A quantum tunneling between two adjacent minima lying on the lattice shown in Fig. 4.4 obviously involves a quantum jump between different zero mode eigenstates. The quantum phase slip operators corresponding to the allowed quantum jumps, $W_i$,
4.5 A two level quantum system at Finite Fixed Point

$W_i^\dagger (i = 1, 2, 3)$, are given by:

\[
W_1^\dagger = \exp \left[ \frac{i}{\sqrt{3}} \vec{\rho}_1 \cdot \vec{\Theta}(0) \right]: \quad = \quad = \\
W_2^\dagger = \exp \left[ \frac{i}{\sqrt{3}} \vec{\rho}_2 \cdot \vec{\Theta}(0) \right]: \quad = \quad = \\
W_3^\dagger = \exp \left[ \frac{i}{\sqrt{3}} \vec{\rho}_3 \cdot \vec{\Theta}(0) \right]: \quad = \quad = ,
\]

(4.75)

with the vectors $\vec{\rho}_i$ are such that $\sqrt{2g_3/\sqrt{3}}$ is the "distance" between nearest neighboring eigenvalues of $\vec{P}$, as determined by the commutation relations

\[
[\vec{P}, W_i] = -\sqrt{\frac{2g}{3}} \vec{\rho}_i W_i, \quad [\vec{P}, W_i^\dagger] = \sqrt{\frac{2g}{3}} \vec{\rho}_i W_i^\dagger ,
\]

(4.76)

which yield

\[
\vec{\rho}_1 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \quad \vec{\rho}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \vec{\rho}_3 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}.
\]

(4.77)

For the sake of clarity, in Eq.(4.75) we have represented in color the quantum jumps between eigenvalues of $\vec{P}$ corresponding to the operators $W_i^\dagger$; the action of the hermitean conjugate operators $W_i$ may be simply represented by reversing the arrows. Since an actual tunneling event between minima of $\mathcal{H}_B$ must necessarily involve a pertinent action on the space of the spin-1/2 operator $S_G$ states, each quantum phase slip operator has to be multiplied by the spin operator that pertinently maps the state $\{|i\rangle, i = 1, \ldots, 3\}$ onto $|i \pm 1\rangle (4 \equiv 1)$. A possible choice for the appropriate operator $S$ is to set $S = \left\{ e^{2i\pi} [I + S_G]^2 + [I - S_G]^2 \right\}$, which implies

\[
|G\rangle = S|R\rangle, \quad |B\rangle = S|G\rangle, \quad |R\rangle = S|B\rangle 
\]

(4.78)

as well as

\[
|R\rangle = S^\dagger |G\rangle, \quad |G\rangle = S^\dagger |R\rangle, \quad |B\rangle = S^\dagger |B\rangle.
\]

(4.79)

Taking into account the symmetries of the system at the SFP, we construct the leading boundary perturbation at the SFP as

\[
\tilde{\mathcal{H}}_B = -Ye^{i\gamma} \{ W_1^\dagger S_1 + W_2^\dagger S_2 + W_3^\dagger S_3 \} + \text{h.c.} ,
\]

(4.80)
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with $Y, \gamma$ calculated in section 4.8. The scaling dimension of the operators $\{W_i, W_i^\dagger\}$ is $\frac{g}{3}$; accordingly, the running coupling strength for the dual boundary coupling is defined as $\zeta(L) = L^{1-\frac{g}{3}} e^{i\gamma} = y(L) e^{i\gamma}$. Having acknowledged that the $\zeta$ is, in general, a complex coupling strength, the corresponding renormalization group equation may be derived exactly in the same way as two of us have done for a $Y$-junction of JJ-chains near by the weakly coupled fixed point [25]. In particular, resorting again to the imaginary-time formulation, from Eq.(4.80) one obtains the euclidean dual boundary action near by the strongly coupled fixed point:

$$\tilde{S}_B = -Y e^{i\gamma} \int_0^\beta d\tau \left\{ S(\tau)[W_1(\tau) + W_2(\tau) + W_3(\tau)] + h.c. \right\}, \quad (4.81)$$

where we have defined $W_j(\tau) = e^{-\frac{1}{3}i \cdot \vec{\rho} \cdot \vec{\Theta}(\tau)}$, with $\Theta(\tau) = \Theta(0, i\tau)$. Following again the standard approach to constructing the $\beta$-functions for the boundary coupling strengths [34, 25], we consider the O.P.E.s

$$[W_1^\dagger(\tau) S(\tau)] [W_2^\dagger(\tau') S(\tau')] \approx_{\tau' \rightarrow \tau} \left| \frac{u(\tau - \tau')}{L} \right|^{\frac{2g}{3}} [W_3(\tau) S(\tau)], \quad (4.82)$$

plus cyclic permutations. From Eq.(4.82), one readily gets the second-order renormalization group equation for the running coupling $\zeta(L)$, which is given by

$$\frac{d\zeta(L)}{d \ln \left( \frac{L}{L_0} \right)} = \left( 1 - \frac{g}{3} \right) \zeta(L) - 2 e^{-2i\gamma} \zeta^2(L), \quad (4.83)$$

where $L_0$ is, again, a reference length scale. Eq.(4.83) is equivalent to the system of real differential equations for the real parameters $y(L), \gamma(L)$, given by

$$\frac{dy(L)}{d \ln \left( \frac{L}{L_0} \right)} = \left( 1 - \frac{g}{3} \right) y(L) - 2 \cos(3\gamma) y^2(L)$$

$$\frac{d\gamma(L)}{d \ln \left( \frac{L}{L_0} \right)} = 2 \sin(3\gamma(L)) y^2(L). \quad (4.84)$$

From Eqs.(4.84) we can see that the renormalization of the phase $\gamma$ takes place only to second order in the boundary couplings, but the value $\gamma = k\pi/3$ ($k$ integer) corresponds to a fixed line in the $y\gamma$-plane under renormalization group trajectories. In factin our
Figure 4.5: Phase diagram in the strip in the $g,f$-plane corresponding to $0 \leq f \leq 2\pi$: the weakly-coupled phase (corresponding to Neumann-(N)-boundary conditions at the inner boundary) is stable for $g < 1$ and for any $f$; the strongly-coupled phase (corresponding to Dirichlet-(D)-boundary conditions at the inner boundary) is stable for $g > 3$ and for any $f$, as well as for $1 < g < 3$ and $f \not\in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$; for $1 < g < 3$ and $\frac{\pi}{2} \leq f \leq \frac{3\pi}{2}$ a novel phase opens, corresponding to a stable finite coupling fixed point (FFP). The phase diagram for $f \not\in [0,2\pi]$ is obtained by periodically extending the picture we provide here.

system $\gamma = \pi/3$, which implies that $\gamma$ is not renormalized along RG trajectories: a result that is quite important, in order to assess the robustness of the emerging two-level quantum system that we are going to discuss in the next section.

The main phase diagram of our JJ-network may be inferred by considering together the renormalization group equations near by the weakly coupled fixed point, as derived in section 4.3, and the ones near by the strongly coupled fixed point, Eqs.(4.84). As displayed in Fig.4.5, there is a window of values of $g$, that is, $1 < g < 3$, within which neither the WFP, or the SFP, is infrared stable. As a consequence, in this window of values of $g$, the infrared behavior of the JJ-network will be driven by a finite coupling fixed point (FFP) of the phase diagram, at which a new two-level quantum system emerges, this time robust against quantum fluctuations, whose properties we are going to discuss in the next section.
4.6 The emerging two-level quantum system

In the previous section we have shown that a stable FFP emerges in the phase diagram of our system for $1 < g < 3$ and $\frac{\pi}{2} \leq f \leq \frac{3\pi}{2}$. The emergence of the FFP is due to the relevance of boundary operators representing tunneling events between neighboring minima of the Kagome lattice in Fig.4.4. As we are going to show in this section, the renormalization of the instanton tunneling strength $Y$, due to interaction with the plasmon modes of the leads, enforces the twofold degeneracy of the emerging two-level quantum system, when this is operated near by the FFP, thus making it to be a candidate for operating as a stable two-level quantum system.

Since a real device has a finite size ($L$), as discussed in subsection 4.4.1, the infinite degeneracy at the SFP with respect to the eigenvalues of the zero-mode operators, is removed by finite-size effects, that is, by the zero-mode contributions to the total energy. As a function of the applied phases, the zero-mode energy associated to each eigenvalue $\vec{p}$ is given by

$$E^{(0)}_{\{R,G,B\},n_{12},n_{13}}[\vec{p}]=\frac{\pi u}{2L}[\vec{p}(n_{12},n_{13})]^2.$$  \hspace{1cm} (4.85)

From Eqs. (4.67,4.69,4.71), one sees that, for a appropriate choice of the phases $\vec{\mu}$, the three zero-mode energies associated to the vertices of a triangle as the one drawn in the box of Fig.4.4, can be made degenerate with each other. To explicitly show this, let us set $\mu^*_1 = \frac{4\pi}{3}, \mu^*_2 = 0$. For such a choice of the applied phases, one gets $E^{(0)}_{R,0,0}[\vec{p}^*] = E^{(0)}_{G,0,0}[\vec{p}^*] = E^{(0)}_{B,0,0}[\vec{p}^*]$. Thus, if one restrict himself to the three-dimentional subspace $\mathcal{F}$ of the Hilbert space spanned by the states $|R,0,0\rangle, |G,0,0\rangle, |B,0,0\rangle$ (which, for simplicity, henceforth we shall refer to simply as $|R\rangle, |G\rangle, |B\rangle$), an effective low-energy description of the system may be provided by projecting the Hamiltonian onto $\mathcal{F}$, thus getting the $3\times3$ Hamiltonian matrix $H_{\text{Restr}}$, given by

$$H_{\text{Restr}} = \frac{\pi gu}{9L} I + \begin{bmatrix} 0 & -Ye^{i\gamma} & -Ye^{-i\gamma} \\ -Ye^{-i\gamma} & 0 & -Ye^{i\gamma} \\ -Ye^{i\gamma} & -Ye^{-i\gamma} & 0 \end{bmatrix},$$  \hspace{1cm} (4.86)

While one may realize that $H_{\text{Restr}}$ has one eigenvalue that is twofold degenerate by directly diagonalizing the corresponding matrix, such a degeneracy may be directly related to the symmetries of the degenerate triangle by considering the two operators acting on $\mathcal{F}$ whose matrix representations are given by...
4.6 The emerging two-level quantum system

\[
A_1 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & e^{-i\frac{2\pi}{3}} \\
0 & e^{i\frac{2\pi}{3}} & 0
\end{bmatrix}.
\] (4.87)

While \(A_1\) corresponds to a cyclic shift of the states \(|R\rangle, |G\rangle, |B\rangle\), \(A_2\) realizes the mirror inversion about the triangle height passing through the vertex corresponding to the \(|R\rangle\)-state. Though, as expected, when \(\gamma = \pi/3\) both \(A_1\) and \(A_2\) commute with \(H_{\text{Restr}}\), \([H_{\text{Restr}}, A_1] = [H_{\text{Restr}}, A_2] = 0\), they do not commute with each other: \([A_1, A_2] \neq 0\). This is enough to ensure that the spectrum of \(H_{\text{Restr}}\) must contain at least one degenerate eigenvalue. To explicitly see it, let us consider the eigenvalue equation associated to \(H_{\text{Restr}}\). It reads

\[-E^3 + 3Y^2E - 2\cos(3\gamma)Y^3 = 0.\] (4.88)

The nonzero phase \(\gamma(= \pi/3)\) implies that there is a twofold degenerate eigenvalue \(E = -Y\), and a nondegenerate eigenvalue \(E = 2Y\). The corresponding eigenstates are given by

\[
|\text{\(-Y\rangle}_1\rangle = \frac{1}{\sqrt{3}}[|R\rangle + |G\rangle + |B\rangle] \\
|\text{\(-Y\rangle}_2\rangle = \frac{1}{\sqrt{3}}[|R\rangle + e^{-i\frac{2\pi}{3}}|G\rangle + e^{i\frac{2\pi}{3}}|B\rangle] \\
|2Y\rangle = \frac{1}{\sqrt{3}}[|R\rangle + e^{i\frac{2\pi}{3}}|G\rangle + e^{-i\frac{2\pi}{3}}|B\rangle].\] (4.89)

To rewrite \(H_{\text{Restr}}\) in the basis of the states reported in Eq.(4.89), one has to rotate it to \(\tilde{H}_{\text{Restr}} = P^\dagger H_{\text{Restr}} P\), where the matrix \(P\) is given by

\[
P = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \\
1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}}
\end{bmatrix}.
\] (4.90)

Thus, one obtains

\[
P^\dagger H_{\text{Restr}} P = \frac{\pi gu}{9L} \begin{bmatrix}
-Y & 0 & 0 \\
0 & -Y & 0 \\
0 & 0 & 2Y
\end{bmatrix}.
\] (4.91)

Similarly, rotating \(A_2\), one gets
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\[
P^t A_2 P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

(4.92)

which clarifies the meaning of \( A_2 \) as the operator that swaps the degenerate states \(|-Y\rangle_1, |-Y\rangle_2\) with each other. Eq.(4.91) shows that another twofold degenerate groundstate has now emerged in the system driven near by the FFP. In fact, taking into account the fluctuations amounts to substitute \( Y \) with the running coupling strength, that is \( \sim \left( \frac{L}{L_0} \right)^{-\frac{q}{2}} \). Instantons are a relevant perturbation and the renormalization effects due to interaction with the plasmon modes of the leads do enforce them. As a result, at odds to what happens near by the WFP, making the system large stabilizes the level structure resulting from Eq.(4.91), thus generating a stable and robust two-state quantum system. The phases \( \vec{\mu} \) and the applied flux \( f \) may be used as external control parameters, to drive the state of the emerging two-level quantum system (2LQS), as we are going to show in the next subsection.

### 4.7 Manipulation of the two-level system

In this section, we discuss how to manipulate the effective 2LQS emerging at the FFP by properly acting upon it with the applied phases and the flux \( f \). Let us assume that the phase are tuned near by, but not exactly at, the triple degeneracy point, \( \mu_1^*, \mu_2^* \), that is, \( |\mu_1 - \mu_1^*|/\pi \ll 1, |\mu_2 - \mu_2^*|/\pi \ll 1 \). In this case, the low-energy effective Hamiltonian \( H_{\text{Restr}}[\vec{\epsilon}] \) in the basis \(|R\rangle, |G\rangle, |B\rangle\) is modified to

\[
H_{\text{Restr}}[\vec{\epsilon}] = \left\{ \frac{\pi g u}{L} \left[ \frac{1}{9} + \frac{\epsilon_1^2}{4\pi^2} \right] \right\} I + \begin{bmatrix} -\frac{2\pi g u}{L} \frac{\epsilon_1}{3} & -Ye^{i\frac{\pi}{3}} & -Ye^{-i\frac{\pi}{3}} \\ -Ye^{-i\frac{\pi}{3}} & \frac{\pi g u}{L} \left[ \frac{\epsilon_1}{3} + \frac{\epsilon_2}{\sqrt{3}\pi} \right] & -Ye^{i\frac{\pi}{3}} \\ -Ye^{i\frac{\pi}{3}} & -Ye^{-i\frac{\pi}{3}} & \frac{\pi g u}{L} \left[ \frac{\epsilon_1}{3} + \frac{\epsilon_2}{\sqrt{3}\pi} \right] \end{bmatrix},
\]

(4.93)

where we have set \( \epsilon_i = \mu_i - \mu_i^*, i = 1, 2 \). For \( \epsilon_1 = \epsilon_2 = 0 \). Clearly, for \( \epsilon_1 = \epsilon_2 = 0 \), \( H_{\text{Restr}}[\vec{\epsilon}] \) reduces back to the Hamiltonian in Eq.(4.86), whose eigenstates are listed in Eq.(4.89). To rewrite \( H_{\text{Restr}}[\vec{\epsilon}] \) in the basis of the states reported in Eq.(4.89), one has to rotate it to \( \tilde{H}_{\text{Restr}}[\vec{\epsilon}] = P^t H_{\text{Restr}}[\vec{\epsilon}] P \). Thus, one obtains

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4.7 Manipulation of the two-level system

\[ P^\dagger H_{\text{Restr}}[\vec{e}]P = \left\{ \frac{\pi gu}{L} \left[ \frac{1}{9} + \frac{\vec{e}^2}{4\pi^2} \right] \right\} I + \begin{bmatrix} -Y & a - ib & a + ib \\ a + ib & -Y & a - ib \\ a - ib & a + ib & 2Y \end{bmatrix} , \]  

(4.94)

with \( a = -\frac{1}{3} \frac{\pi gu}{L} \frac{\epsilon_1}{2\pi} \), \( b = -\frac{\pi gu}{L} \frac{\epsilon_2}{3\pi} \). From Eq.(4.94), one readily writes down the restricted two-level Hamiltonian, by simply keeping the matrix elements of \( P^\dagger H_{\text{Restr}}[\vec{e}]P \) that involve the low energy twofold degenerate ground state. The result is

\[ H_{\text{Eff}} = \left\{ \frac{\pi gu}{L} \left[ \frac{1}{9} + \frac{\vec{e}^2}{4\pi^2} \right] - Y \right\} I_2 + \begin{bmatrix} 0 & a - ib \\ a + ib & 0 \end{bmatrix} . \]  

(4.95)

From Eq.(4.95), one sees that, besides the “trivial” term proportional to the identity matrix, the matrix \( H_{\text{Eff}} \) can be rewritten as \( H_{\text{Eff}} = b_x \sigma^x + b_y \sigma^y \), with \( b_x = a \), and \( b_y = b \). In fact, while, as evidenced in Eq.(4.95), the phases \( \vec{\mu} \) induce an effective magnetic field \( \vec{b} \) lying within the \( xy \)-plane, in order to keep full control on the state of the emerging two-level quantum system, one has to add an additional \( z \)-component to the effective magnetic field. This is achieved by introducing a slight breaking of the degeneracy between the \( | \uparrow \rangle \)- and the \( | \downarrow \rangle \)-states of the localized spin \( S_G \), which may be realized by turning on a nonzero \( B_\parallel \), as illustrated in Eq.(4.25). Having a nonzero \( B_\parallel \) corresponds to an additional contribution to \( H_{\text{Restr}} \), given by

\[ \delta H_{\text{Restr}} = -\frac{B_\parallel}{4} Y \begin{bmatrix} 0 & z & z^* \\ z^* & 0 & z \\ z & z^* & 0 \end{bmatrix} , \]  

(4.96)

with \( z = \frac{\pi}{3} e^{-i\frac{\pi}{6}} \). When transforming to the basis of the eigenstates in Eq.(4.89), one obtains

\[ P^\dagger \delta H_{\text{Restr}} P = -\frac{\sqrt{3}}{4} B_\parallel Y \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \]  

(4.97)

From Eq.(4.97) one readily sees that, when restricting again to the two low-energy states only, one obtains the effective Hamiltonian

\[ H_{\text{Eff}}[\vec{b}] = \left\{ \frac{\pi gu}{L} \left[ \frac{1}{9} + \frac{\vec{e}^2}{4\pi^2} \right] - Y \right\} I_2 + \vec{b} \cdot \vec{\sigma} , \]  

(4.98)
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with \( b_x = a, \ b_y = b, \) and \( b_z = -\sqrt{3}/4B_\parallel Y. \) The Hamiltonian \( H_{\text{Eff}}[\vec{b}] \) is the standard Hamiltonian for a spin in an external magnetic field, whose components may be fully controlled from outside, by even applying a modulation in time to the phases and to the flux \( f. \)

The emerging two-level quantum system may used to implement standard quantum information protocols. For instance, a NOT port may be realized by following the procedure outlined in Ref.[56]. In particular, one may modulate in time \( \epsilon_2, \) so that \( \epsilon_2(t) = \nu \sin(\omega_0 t). \) This results in an effective \( \vec{b} \) field given by \( \vec{b} = (b_x, \tilde{b} \sin(\omega_0 t), b_z), \) with \( b_x, b_z \) constant and \( \tilde{b} = \frac{\pi \nu}{3L} \nu. \) The instantaneous eigenvalues of \( H_{\text{Eff}}[\vec{b}(t)] \) are given by \( \pm \Lambda(t) = \pm \sqrt{b_x^2 + b_z^2 + \tilde{b}^2 \sin^2(\omega_0 t)}, \) while the corresponding adiabatic eigenstates are given by

\[
| - \Lambda(t) \rangle = e^{i\xi(t)} \left\{ \cos \left( \frac{\Theta(t)}{2} \right) | - Y \rangle_1 + \sin \left( \frac{\Theta(t)}{2} \right) e^{-i\Phi(t)} | - Y \rangle_2 \right\} \\
| \Lambda(t) \rangle = e^{i\xi(t)} \left\{ -\sin \left( \frac{\Theta(t)}{2} \right) | - Y \rangle_1 + \cos \left( \frac{\Theta(t)}{2} \right) e^{-i\Phi(t)} | - Y \rangle_2 \right\} , \quad (4.99)
\]

with \( \cos(\Theta) = -b_z/\Lambda(t) \) and \( \Phi(t) = \arg(-b_x - i\tilde{b} \sin(\omega_0 t)). \) The phases \( \xi(t) \) are chosen so to satisfy the parallel transport condition

\[
\langle \Lambda(t) \left| \frac{d}{dt} \right| \Lambda(t) \rangle = \langle -\Lambda(t) \left| \frac{d}{dt} \right| - \Lambda(t) \rangle = 0 \quad . \quad (4.100)
\]

Preparing, at \( t = 0, \) the system in the state \( | - \Lambda(0) \rangle, \) after twice a period \( 2T = 4\pi \omega_0, \) the relative phases between \( | - Y \rangle_1 \) and \( | - Y \rangle_2 \) will be varied by

\[
\Delta \Phi = -\frac{1}{2} \int_0^{4\pi/\omega_0} dt \dot{\Phi}(t) [1 - \cos(\Theta(t))] \quad . \quad (4.101)
\]

Setting \( b_z = 0 \) and \( \nu = b_x, \) one finds \( \Delta \Phi = \pi, \) thus changing the relative sign between the two states, as it is supposed to happen for a NOT port.

Finally, to detect the state of the system one may, as it was done for the system near by the WFP, look at the current pattern arising across the network when biasing the device off the triple-degeneracy point. Indeed, within the restricted subspace \( \mathcal{F} \) the current operators are given by

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4.8 Instanton amplitude

\[ I_{1,F} = e^* \frac{\partial P^* H_{\text{Restr}}[\epsilon] P}{\partial \epsilon_1} = \frac{e^* g u}{2\pi L} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

\[ I_{2,F} = e^* \frac{\partial P^* H_{\text{Restr}}[\epsilon] P}{\partial \epsilon_2} = \frac{e^* g u}{2\pi L} \epsilon_2 I + i \frac{e^* g u}{3L} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \].  

From Eq.(4.102), one eventually obtains

\[ 1\langle -Y | I_{1,F} | -Y \rangle_1 = \frac{e^* g u}{2\pi L} \epsilon_1 = 2\langle -Y | I_{1,F} | -Y \rangle_2 \]  

\[ 1\langle -Y | I_{1,F} | -Y \rangle_2 = -\frac{e^* g u}{6L} \]  

\[ 1\langle -Y | I_{2,F} | -Y \rangle_1 = \frac{e^* g u}{2\pi L} \epsilon_2 = 2\langle -Y | I_{2,F} | -Y \rangle_2 \]  

\[ 1\langle -Y | I_{2,F} | -Y \rangle_2 = i \frac{e^* g u}{3L} \]  

For a general qubit state \( |\alpha\rangle = \cos \frac{\theta}{2} | -Y \rangle_1 + e^{i\phi} \sin \frac{\theta}{2} | -Y \rangle_2 \), the expectation value of the current operators are

\[ \langle \alpha | I_{1,F} | \alpha \rangle = -\frac{e^* g u}{6L} \frac{1}{2} \sigma_x \rangle_\alpha + \frac{e^* g u}{2\pi L} \epsilon_1 \]  

\[ \langle \alpha | I_{2,F} | \alpha \rangle = -\frac{e^* g u}{3L} \frac{1}{2} \sigma_y \rangle_\alpha + \frac{e^* g u}{2\pi L} \epsilon_2 \]  

From Eqs.(4.41) and from Eq.(4.108), it is straightforward to evaluate the current pattern that identifies either one of the degenerate states \( | -Y \rangle_1 , | -Y \rangle_2 \).

4.8 Instanton amplitude

Here, we will derive the modulus \( Y \) and the phase \( \gamma \) of the instanton tunneling amplitudes \( \mathcal{Y} \).

In view of the mode expansion for the field with Dirichlet boundary conditions at both boundaries, the instanton trajectory may be formalized as an imaginary time evolution in the zero mode term in Eq.(4.62), corresponding to a tunneling event between nearest neighboring sites on the minima lattice, that is, \( \vec{P} = \vec{P}(\tau) \). Accordingly, computing the “bulk” Euclidean action for the field \( \chi \)
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\[ S^{(0)} = \frac{g}{4\pi} \int_0^\beta d\tau \int_0^L dx \left[ \frac{1}{u} \left( \frac{\partial \bar{\chi}}{\partial \tau} \right)^2 + u \left( \frac{\partial \bar{\chi}}{\partial x} \right)^2 \right] , \quad (4.109) \]

yields

\[ S^{(0)} = \frac{1}{2} \int_0^\beta d\tau \left[ M \left( \frac{d\bar{\chi}(\tau)}{d\tau} \right)^2 + M\omega^2 (\bar{\chi}(\tau) - \bar{\xi})^2 \right] + \ldots \equiv S^{(0)}[\bar{\chi}(\tau)] + \ldots , \quad (4.110) \]

with \( \bar{\chi}(\tau) = \bar{\chi}(0, \tau) \), \( M = Lg/(6\pi u) \), and \( M\omega^2 = ug/(2\pi L) \). The ellipses in Eqs.(4.109) and (4.110) corresponds to interactions between instantons, mediated by oscillations of the plasmon bulk modes, which we shall neglect henceforth. The coupling between the \( \bar{\chi} \)-modes and the spin degrees of freedom takes place via the boundary hamiltonian in Eq.(4.22), which one may rewrite as

\[ H_B^{(i)} = B[\bar{\chi}(\tau)] \cdot S_G(\tau) + B_0[\bar{\chi}(\tau)] , \quad (4.111) \]

with

\[ B_0[\bar{\chi}(\tau)] = 2E_1 \sum_{j=1}^3 \cos[\bar{\alpha}_j \cdot \bar{\chi}(\tau)] , \]

\[ B_x[\bar{\chi}(\tau)] = 2E_3 \sum_{j=1}^3 \cos \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\bar{\alpha}_j \cdot \bar{\chi}(\tau)] , \]

\[ B_y[\bar{\chi}(\tau)] = 2E_3 \sum_{j=1}^3 \sin \left[ \frac{2\pi}{3} (j - 2) \right] \cos[\bar{\alpha}_j \cdot \bar{\chi}(\tau)] , \]

\[ B_z[\bar{\chi}(\tau)] = 2E_2 \sum_{j=1}^3 \cos[\bar{\alpha}_j \cdot \bar{\chi}(\tau) + \frac{\pi}{2}] + B_\| . \quad (4.112) \]

Focusing onto the “threefold degenerate” point \((\mu_1^*, \mu_2^*) = \left( \frac{4\pi}{3}, 0 \right)\), the instanton trajectories of interest lie along the sides of the triangle whose vertices coincide with the \( R, G, B \) points defined by \( n_{12} = n_{13} = 0 \). At the \( R, G, B \)-vertex, one respectively gets

\[ \bar{\chi}_R = 2\pi(1, 0) , \quad \bar{\chi}_G = 2\pi \left( \frac{1}{2}, -\frac{1}{\sqrt{3}} \right) , \quad \bar{\chi}_B = 2\pi \left( \frac{1}{2}, \frac{1}{\sqrt{3}} \right) \quad (4.113) \]
Instanton amplitude

Any instanton path will run between two of the points in the $\vec{\chi}$-configuration space listed in Eq.(4.113). Moreover, because of the identities

$$\vec{\alpha}_2 = R\left(\frac{2\pi}{3}\right) \vec{\alpha}_1 \quad \text{and} \quad \vec{\alpha}_3 = R\left(\frac{4\pi}{3}\right) \vec{\alpha}_1,$$

one also gets that the trajectory flowing from $B$ to $R$ may be obtained by acting with $R\left(\frac{2\pi}{3}\right)$ on the trajectory described going from $G$ to $B$, while the trajectory flowing from $R$ to $G$ may be obtained by acting with $R\left(\frac{4\pi}{3}\right)$ on the same trajectory. Thus, it is enough to compute only one tunneling amplitude, for example between $G$ and $B$. To do so, let us parametrize such an instanton path as $(\chi_1(\tau), \chi_2(\tau)) = (\pi, 2\pi\sqrt{3} \sigma(\tau))$, with $\sigma(0) = -\frac{1}{2}$, and $\sigma(\beta) = \frac{1}{2}$. The corresponding Euclidean action will be given by

$$S[\sigma] = \int_0^\beta d\tau \left\{ \frac{M\pi^2}{6} [\dot{\sigma}^2 + \omega^2 \sigma^2] - 2E_1 - 2E_3 \sqrt{4 - 3\sin^2(\pi\sigma)} \right\}.$$

The term $\propto \sigma^2$ is basically constant, along the instanton path. Thus, the actual value of $S[\sigma]$ may, in principle, be computed by determining the zero-action solution in the “inverted potential”, $\sigma_{\text{Inst}}(\tau)$, as

$$\frac{M\pi^2}{6} \sigma^2_{\text{Inst}}(\tau) \approx 2E_3 \left( \sqrt{4 - 3\sin^2(\pi\sigma_{\text{Inst}}(\tau))} + 1 \right),$$

and, then, by computing $\bar{S} = S[\sigma_{\text{Inst}}]$. However, as we discussed in the main text of the paper, the fugacity $Y = e^{-\bar{S}}$ is strongly renormalized by interaction with the collective plasmon modes of the bulk, so that it eventually scales as $Y(L) \sim L^{-\frac{2}{3}}$ and, accordingly, $e^{-\bar{S}} \sim Y(L = L_0)$ may be just taken as a phenomenological parameter.

The tunneling amplitudes has also a phase that comes from a topological term due to the spin [61]. To derive it, one may consider that the probability amplitude for the system to remain in the same state (for example $R$), is not only given only by the instanton/anti-instanton contributions, but also by the loops around the three degenerate states. The action of such a loop will have an extra topological term that
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comes from the trajectory of the spin state during the loop. For a generic spin state
$|\alpha(\tau)\rangle = \cos \frac{\theta(\tau)}{2} |\uparrow\rangle + e^{i\phi(\tau)} \sin \frac{\theta(\tau)}{2} |\downarrow\rangle$ the topological term can be written as:

$$S_{\text{Top}} = \frac{i}{2} \int_0^T d\tau \dot{\phi}(\tau) \left( 1 + \cos \theta(\tau) \right).$$

(4.118)

Thus the Euclidean action for a loop will be given by:

$$S_{\text{loop}}[\vec{\chi}(\tau); \phi(\tau); \theta(\tau)] = S_{\text{Top}}[\phi, \theta] + 3S_{\text{inst}}.$$

(4.119)

Writing the spin states along the instanton paths and substituting the instanton coor-
dinates $\vec{\chi}(\tau)$ into Eq.(4.58), one finds the corresponding lowest energy eigenstate. For
small $B_\parallel$, one obtains:

$$|\alpha(\tau)\rangle = \cos \frac{\theta_f}{2} |\uparrow\rangle + e^{i\gamma(\tau)} \sin \frac{\theta_f}{2} |\downarrow\rangle$$

(4.120)

where $\theta_f$ is defined in Eq.(4.64) and $\gamma(\tau) = \arg [B_x[\vec{\chi}(\tau)] + IB_y[\vec{\chi}(\tau)]]$. Since $\theta$ remains
constant, the evaluation of $S_{\text{Top}}$ yields

$$S_{\text{Top}} = i\pi (1 + \cos \theta_f) \approx i\pi (1 - \frac{B_\parallel}{4}),$$

(4.121)

that is, the phase contribution to the amplitude of loop tunneling. Assuming that the
three instanton tunneling are equivalent, one naturally assign to each tunnel amplitude
a third of the total phase. Thus, the single instanton tunneling amplitude eventually
comes out to be given by

$$Y = Y e^{i\frac{\pi}{3} - \frac{B_\parallel}{4\pi}}.$$

(4.122)
Conclusions

In this work of thesis we introduced and examine in detail a new superconducting device of Josephson junction network. It consists of three leads, realized with three one-dimensional Josephson junction chains, connected to a central region of four superconducting grain, joined to each other to form a tetrahedral-like structure. Performing a Schrieffer-Wolff transformation, an effective boundary Hamiltonian for the leads is obtained. In two different regimes, the weak and the strong ones, energy conservation provide us boundary condition for the fields describing the leads. Here we see the importance of choosing to connect such a central region to leads of Josephson junction chain: the leads are described by Luttinger liquids in 1+1 dimension, i.e. a free massless bosonic theory, therefore, the whole Hamiltonian describing the system is a conformal field theory for the bulk with boundary conditions conformally invariant. The analysis of the system is performed using the Renormalization Group approach.

We want remark that in the weak coupling regime the whole picture of the two-level quantum system appears to be fragile, as soon as fluctuations are taken into account. Indeed, while, as $g < 1$, the perturbative analysis may coherently be applied, the rather weak value of the coupling constants that determine the effective magnetic field acting on the two-level system does not provide a robust and stable control parameter for the two-level quantum state. At variance, when $g > 1$, the whole perturbative scenario breaks down, and, accordingly, one has to resort to an alternative approach, in order to study the large-scale dynamics of the JJ-network.

In the strong coupling regime, the basic protection mechanism is suggested by the scaling equations for the instanton amplitudes, from which one sees that fluctuations, due to low-energy collective plasmon modes of the leads, make the running coupling $g(L)$ scale as $L^{1-g/2}$. As a result, as long as $1 < g < 3$ (and $\frac{\pi}{2} \leq f \leq \frac{3\pi}{2}$, in order to ensure twofold degeneracy of the groundstate of Hamiltonian for the central region), instantons are a relevant perturbation, which lets the system operate near by a finite coupling fixed
point in the boundary phase diagram. Fluctuations of the values of the applied phases $\vec{\mu}$ yield marginal perturbations. From our analysis, it can be seen that fluctuations in the applied phases are unable to spoil the twofold ground state degeneracy of the emerging system at the SFP, provided that an efficient parameter renormalization simultaneously makes the instanton operators to be a relevant perturbation.

As a result, a new two-level quantum system emerge, this time robust against quantum fluctuations, and it may used to implement standard quantum information protocols. It manipulation is possible via the tuning of the fluxes and a measurement of the Josephson currents identifies either one of the degenerate state (or a combination of them).
Appendix A

Josephson effect in superconductive junctions

An important quantum effects that may occur on a macroscopic scale is the so-called Josephson effect\cite{62}. Discovered first in Josephson junctions made with superconducting materials, it was observed later in systems of superfluid helium and Bose gases. For the Bose gas, in fact, has been demonstrated the existence of a Josephson effect when the cold atoms are confined in a potential double-hole. The cold atoms in appropriate periodic potentials exhibit properties similar to networks of Josephson junctions.

It is possible to follow several different approaches in order to obtain the basic Josephson relations. We discuss a very simple derivation, due to Feynman\cite{63, 64, 65}, which is based on a “two level system” picture. This approach, despite its simplicity, offers a powerful key for the understanding of the peculiar features of Josephson phenomena.

Let us consider the tunneling structure superconductor-barrier-superconductor. We call $\psi_R$ ($\psi_L$) the pair wave function for the right (left) superconductor. Each superconducting electrode can be described by a single quantum state and $\psi$’s can be regarded as macroscopic wave functions, so that $|\psi|^2$ represents the actual Cooper pair density $\rho$. Following the notation of Rogovin and Scully\cite{66} we indicate with the ket $|R\rangle$ ($|L\rangle$) the base state for the right (left) superconductor. Then

$$
\langle L| \psi^*_L \psi_L |L\rangle = |\psi_L|^2 = \rho_L
$$
$$
\langle R| \psi^*_R \psi_R |R\rangle = |\psi_R|^2 = \rho_R .
$$

If we now take into account the weak coupling existing between the two superconduc-
Josephson effect in superconductive junctions

\[ S_L \quad \text{and} \quad S_R \]

\[ \psi_L \quad \text{and} \quad \psi_R \]

Figure A.1: Schematic of a Josephson junction. \( S_L \) and \( S_R \) are the left and right superconductors. \( \psi_L \) and \( \psi_R \) are the left and right pair wavefunctions.

...tors, “transitions” between the two states \(|R\rangle\) and \(|L\rangle\) can occur. This coupling is essentially related to the finite overlap of the two pair wave functions \( \psi_L \) and \( \psi_R \). This situation is schematically depicted in Fig.A. A state vector of this two base state system can be described as

\[ |\psi\rangle = \psi_R |R\rangle + \psi_L |L\rangle \]  \hspace{1cm} (A.2)

That is, the particle can be either in a “left” or “right” state with amplitude \( \psi_L \) or \( \psi_R \) respectively. The time evolution of the system is described by the Schrödinger equation.

\[ i\hbar \frac{\partial |\psi\rangle}{\partial t} = \mathcal{H} |\psi\rangle \]  \hspace{1cm} (A.3)

with the Hamiltonian given by

\[ \mathcal{H} = \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_T \]  \hspace{1cm} (A.4)

where \( \mathcal{H}_L = E_L |L\rangle \langle L| \) and \( \mathcal{H}_R = E_R |R\rangle \langle R| \) are relative to the impertubed states \(|L\rangle\) and \(|R\rangle\), while

\[ \mathcal{H}_T = K [ |L\rangle \langle R| + |R\rangle \langle L| ] \]  \hspace{1cm} (A.5)

is the term of interaction (tunneling Hamiltonian) between the two states. \( E_L \) and \( E_R \) are the ground state energies of the two superconductors. \( K \) is the coupling amplitude of the state system which gives a measure of the coupling interaction between the two superconductors and depend on the specific junction structure (electrode geometry, tunneling barrier, etc.). In absense of a vector potential \( A \), the quantity \( K \) can be assumed to be real.
Considering the projections on the two base states, Eq.(A.3) can be written in terms of amplitudes as

\[ i \hbar \frac{\partial \psi_R}{\partial t} = E_R \psi_R + K \psi_L , \]
\[ i \hbar \frac{\partial \psi_L}{\partial t} = E_L \psi_L + K \psi_R . \]

(A.6)

The energy terms, for two isolated superconductor, are given by \( E_R = 2\mu_r \) and \( E_L = 2\mu_L \) where \( \mu_R \) and \( \mu_L \) are the two chemical potentials. If we consider a d.c. potential difference \( V \) across the junction these chemical potentials are shifted by an amount \( eV \) and consequently it is \( E_L - E_R = 2eV \). We can choose the zero of the energy halfway between the two values on the right and on the left, so that

\[ i \hbar \frac{\partial \psi_R}{\partial t} = -eV \psi_R + K \psi_L , \]
\[ i \hbar \frac{\partial \psi_L}{\partial t} = eV \psi_L + K \psi_R . \]

(A.7)

We can substitute for \( \psi_L \) and \( \psi_R \) the expressions

\[ \psi_L = \rho_L^{1/2} e^{i\varphi_L} \quad \text{and} \quad \psi_R = \rho_R^{1/2} e^{i\varphi_R} , \]

(A.8)

where \( \varphi_{R/L} \) is the phase common to all the particles in each superconductor. Separating real and imaginary terms in each equation we get

\[ \frac{\partial \rho_L}{\partial t} = \frac{2}{\hbar} K \sqrt{\rho_L \rho_R} \sin \varphi \]
\[ \frac{\partial \rho_R}{\partial t} = -\frac{2}{\hbar} K \sqrt{\rho_L \rho_R} \sin \varphi \]

(A.9)

\[ \frac{\partial \varphi_L}{\partial t} = \frac{K}{\hbar} K \sqrt{\frac{\rho_L}{\rho_R}} \cos \varphi + \frac{eV}{\hbar} \]
\[ \frac{\partial \varphi_R}{\partial t} = \frac{K}{\hbar} K \sqrt{\frac{\rho_L}{\rho_R}} \cos \varphi - \frac{eV}{\hbar} \]

(A.10)

where \( \varphi \) now is

\[ \varphi = \varphi_L - \varphi_R . \]

(A.11)

The pair current density \( J \) is given by

\[ J \equiv \frac{\partial \rho_L}{\partial t} = -\frac{\partial \rho_R}{\partial t} \]

(A.12)
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and, therefore, from Eqs.(A.9) and (A.10) it follows that

$$J = \frac{2K}{\hbar} \sqrt{\rho_L \rho_R} \sin \varphi .$$

(A.13)

If we assume $\rho_L = \rho_R = \rho_1$ where $\rho_1$ is constant, Eq.(A.13) becomes

$$J = J_1 \sin \varphi$$

(A.14)

where $J_1 = 2K/\hbar \rho_1$.

Let us note that although $\rho_R$ and $\rho_L$ are considered constant their time derivative $J$ is not zero. There is no contradiction if we take into account the presence of the current source which continuously replaces the pairs tunneling across the barrier. These feeding currents are not included in the equations; however, this would not change the expressions of the tunneling pair current density\[67].

From the two equations (A.10) it follows.

$$\frac{\partial \varphi}{\partial t} = \frac{2eV}{\hbar} .$$

(A.15)

Equations (A.14) and (A.15) are the constitutive relations of the Josephson effect. Assuming $V = 0$ the phase difference $\varphi$ results, from Eq.(A.15), to be constant not necessarily zero, so that from Eq.(A.14) a finite current density with a maximum value $J_1$ can flow through the barrier with zero voltage drop across a junction. This is the essence of the d.c. Josephson effect.

We observe that the existence of a supercurrent (current at $V = 0$) suggests that the josephson effect can be regarded qualitatively as an extension of the superconductive properties over the whole structure including the barrier. In the bulk superconductor the current is related to the gradient of its phase; in the Josephson junction the pair current is related to the phase difference between the two coupled superconductors by Eq.(A.14).

If we apply a constant voltage $V \neq 0$, it follows by integration of (A.15) that the phase $\varphi$ varies in time as $\varphi = \varphi_0 + (2e/\hbar)Vt$ and therefore there appears an alternating current

$$J = J_1 \sin \left( \varphi_0 + \frac{2e}{\hbar}Vt \right)$$

(A.16)

with a frequency $\omega = 2\pi \nu = 2eV/\hbar$. This is called a.c. Josephson effect.

Let us observe that the derivation of (A.14) and (A.15), although referred to a tunneling junction, can hold for other kinds of weak links between superconductors.
In fact the parameter of the specific structure are essentially included in the coupling factor $K$, which can be assumed not necessarily as a “tunneling” interaction term.
Appendix B

Bosonization of a one-dimensional Josephson Junction array

Here, we will employ the standard fermionization and re-bosonization procedure, that is, we express the lattice spin operators in terms of spinless Jordan-Wigner lattice operators and, then, take the continuum limit of fermionic version of the model. Finally, we will bosonize the resulting theory. This approach to XXZ-model has been largely discussed in literature [68], also in the context of the effective spin-1/2 model of one-dimensional Josephson arrays [46].

To fermionize the lattice spin-1/2 operators, one introduces lattice Jordan-Wigner (JW) fermions, $a_i$ and $a_i^\dagger$, obeying the usual anticommutation relations:

$$\{a_i, a_j^\dagger\} = \delta_{ij}. \quad (B.1)$$

The correspondence between bosonic spin operators and JW-fermions take place via the nonlocal mapping

$$S_j^z = a_j^\dagger a_j - \frac{1}{2}$$
$$S_j^+ = a_j^\dagger e^{i\pi \sum_{l=1}^{j-1} a_l^\dagger a_l}. \quad (B.2)$$

In terms of the fermionic operator, the projected Hamiltonian (3.30) is given by

$$\mathcal{H}_{JW} = \mathcal{H}_K + \mathcal{H}_P - H \sum_{j=1}^{L/a} a_j^\dagger a_j$$
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\[- \frac{E_J}{2} \sum_{j=1}^{L/a} [a_j^+ a_{j+1} + a_{j+1}^+ a_j] + \Delta \sum_{j=1}^{L/a} \left( a_j^+ a_j - \frac{1}{2} \right) \left( a_{j+1}^+ a_{j+1} - \frac{1}{2} \right) - H \sum_{j=1}^{L/a} a_j^+ a_j. \] (B.3)

The hopping term \( \mathcal{H}_K \) is readily diagonalized by resorting to the Fourier components of \( a_j, a_k \),

\[ a_j = \frac{1}{\sqrt{L/a}} \sum_k a_k e^{ikj}, \] (B.4)

leading to:

\[ \mathcal{H}_K = \sum_k \epsilon(k) a_k^+ a_k; \quad (\epsilon(k) = -E_J \cos(ka) - H) \] (B.5)

If \( |H| < E_J \), the Fermi surface is disconnected and consists of two isolated points at \( \pm k_F \), with \( k_F = \arccos(H/E_J) \). Keeping only the excitations about the Fermi points with momenta \( k \) such that \( |k \pm k_F| \leq \Lambda \), one obtains:

\[ \mathcal{H}_K \approx \sum_{|k-k_F| \leq \Lambda} \epsilon(k) a_k^+ a_k + \sum_{|k+k_F| \leq \Lambda} \epsilon(k) a_k^+ a_k \approx \]

\[ \approx E_J \sin(ak_F) \sum_{|p| \leq \Lambda} \sin(pa) a_L^+(p)a_L(p) + \]

\[ -E_J \sin(ak_F) \sum_{|p| \leq \Lambda} \sin(pa) a_R^+(p)a_R(p), \] (B.6)

with:

\[ a_L(p) \equiv a_{p+k_F}; \quad a_R(p) \equiv a_{p-k_F} \quad (|p| \leq \Lambda). \] (B.7)

For \( \Lambda \ll k_F \), one may define the continuum chiral fields \( \psi_{L/R}(x) \) as

\[ \frac{a_j}{\sqrt{2\pi a}} \approx e^{ik_Fx_j} \psi_L(x_j) + e^{-ik_Fx_j} \psi_R(x_j), \] (B.8)

with \( x_j = ja \); one gets, then

\[ \mathcal{H}_K \approx E_J \sin(k_Fa) \sum_{|p| \leq \Lambda} p[a_L^+(p)a_L(p) - a_R^+(p)a_R(p)] \]

\[ = -iv_F \int_0^L dx \left[ \psi_L^+(x) \frac{d\psi_L(x)}{dx} - \psi_R^+(x) \frac{d\psi_R(x)}{dx} \right] \] (B.9)
where the Fermi velocity is given by \( v_F = 2\pi E_j \sin(ak_F) \).

Eq.(B.9) is, of course, the effective low-energy theory of the hopping Hamiltonian \( \mathcal{H}_K \); the cutoff \( \Lambda \) will be specified later.

The dynamics of the fermionic fields \( \psi_L \) and \( \psi_R \) in the Heisenberg representation, is described by

\[
\psi_L(x,t) = \psi_L(x-v_Ft) = \frac{1}{\sqrt{L}} \sum_p e^{i p (x-v_Ft)} \psi_L(p) \\
\psi_R(x,t) = \psi_R(x+v_Ft) = \frac{1}{\sqrt{L}} \sum_p e^{i p (x+v_Ft)} \psi_R(p),
\]

and the equal time anticommutation relations are given by

\[
\{\psi_L(p), \psi_L^\dagger(p')\} = \delta_{p,p'} \\
\{\psi_R(p), \psi_R^\dagger(p')\} = \delta_{p,p'} \\
\{\psi_R(p), \psi_L^\dagger(p')\} = 0.
\]

Since \( \psi_L^\dagger(p) \) with \( p > 0 \) creates positive energy left-handed states, while \( \psi_R^\dagger(p) \) creates positive energy right-handed states if \( p < 0 \), the “Fermi sea” fermionic ground state is defined as

\[
|FS\rangle = \prod_{p<0} [\psi_L^\dagger(p) \psi_R^\dagger(-p)] |0\rangle \quad (\psi_L(p)|0\rangle = \psi_R(p)|0\rangle = 0) .
\]

Thus, by choosing \( \Lambda = 1/(4a) \), one gets:

\[
\langle FS|2\pi a[ e^{-2ik_Fx_j} \psi_L^\dagger(x_j) \psi_R(x_j) + e^{2ik_Fx_j} \psi_R^\dagger(x_j) \psi_L(x_j) ]|FS\rangle = 0
\]

and

\[
\langle FS|a[\psi_L^\dagger(x_j) \psi_L(x_j) + \psi_R^\dagger(x_j) \psi_R(x_j)]|FS\rangle = \frac{1}{2} ;
\]

thus, \( S^z_j \) is normal ordered respect to \( |FS\rangle \), i.e.,

\[
S^z_j = 2\pi a \left[ : \psi_L^\dagger(x_j) \psi_L(x_j) : + : \psi_R^\dagger(x_j) \psi_R(x_j) : \right] + \\
+ 2\pi a \left[ : \psi_L^\dagger(x_j) \psi_R(x_j) : e^{-2ik_Fx_j} + : \psi_R^\dagger(x_j) \psi_L(x_j) : e^{2ik_Fx_j} \right].
\]
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where $::$ denotes normal ordering.

Using fermionic coordinates, one should now evaluate the Ising-Néel interaction $H_P$ as

\[
H_P \equiv \Delta \sum_{j=1}^{N} S_j^z S_{j+1}^z \equiv \Delta \sum_{j=1}^{L/a} (a_j^+ a_j - \frac{1}{2}) (a_{j+1}^+ a_{j+1} - \frac{1}{2}) \approx \approx (4\pi^2 a \Delta) \int_0^L dx_j \left\{ : \psi_L^\dagger(x_j) \psi_L(x_j) : + : \psi_R^\dagger(x_j) \psi_R(x_j) : + e^{-2ikF x_j} \psi_L^\dagger(x_j) \psi_R(x_j) + e^{2ikF x_j} \psi_R^\dagger(x_j) \psi_L(x_j) \right\} 
\]

\[
\times \left[ : \psi_L^\dagger(x_{j+1}) \psi_L(x_{j+1}) : + : \psi_R^\dagger(x_{j+1}) \psi_R(x_{j+1}) : + e^{-2ikF x_{j+1}} \psi_L^\dagger(x_{j+1}) \psi_R(x_{j+1}) + e^{2ikF x_{j+1}} \psi_R^\dagger(x_{j+1}) \psi_L(x_{j+1}) \right] \right\} = H_P^{(1)} + H_P^{(2)},
\]

where

\[
H_P^{(1)} = (4\pi^2 a \Delta) \int_0^L dx_j \left\{ : \psi_L^\dagger(x_j) \psi_L(x_j) : \psi_L^\dagger(x_{j+1}) \psi_L(x_{j+1}) : + : \psi_R^\dagger(x_j) \psi_R(x_j) : \psi_R^\dagger(x_{j+1}) \psi_R(x_{j+1}) : + : \psi_L^\dagger(x_j) \psi_L(x_j) : \psi_R^\dagger(x_{j+1}) \psi_R(x_{j+1}) : + : \psi_R^\dagger(x_j) \psi_R(x_j) : \psi_L^\dagger(x_{j+1}) \psi_L(x_{j+1}) : \right\},
\]

and

\[
H_P^{(2)} = (4\pi^2 a \Delta) \int_0^L dx_j \left[ \psi_L^\dagger(x_j) \psi_R(x_j) e^{-2ikF x_j} + \psi_R^\dagger(x_j) \psi_L(x_j) e^{2ikF x_j} \right] \times \left\{ \psi_L^\dagger(x_{j+1}) \psi_R(x_{j+1}) e^{-2ikF x_{j+1}} + \psi_R^\dagger(x_{j+1}) \psi_L(x_{j+1}) e^{2ikF x_{j+1}} \right\}.
\]

While evaluating $H_P^{(1)}$ is rather straightforward, since it contains only normal-ordered fermionic left- and right- densities, evaluating $H_P^{(2)}$ is a little bit more involved, due to “crossed” $L - R$-interaction. In fact, at any $k_F$, momentum conservation selects the pertinent contribution to Eq.(B.18), given by

\[
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\]
\[ H_P^{(2)} = (4\pi^2 a \Delta) \int_0^L dx_j \left[ e^{2ik_Fa} \psi_L^\dagger(x_j)\psi_R(x_j)\psi_R^\dagger(x_{j+1})\psi_L(x_{j+1}) + 
\right. \\
+ e^{-2ik_Fa} \psi_R^\dagger(x_j)\psi_L(x_j)\psi_L^\dagger(x_{j+1})\psi_R(x_{j+1}) \right] , \quad (B.19) \]

where “fast oscillating” terms have been neglected [69]. To normal order \( H_P^{(2)} \), one may rewrite it as

\[ H_P^{(2)} = -(4\pi^2 a \Delta) e^{2ik_Fa} \int_0^L dx \left[ \psi_L^\dagger(x)\psi_L(x + a) : + \frac{i}{2\pi a} \right] \left[ \psi_R^\dagger(x + a)\psi_R(x) : + \frac{i}{2\pi a} \right] + \\
- (4\pi^2 a \Delta) e^{-2ik_Fa} \int_0^L dx \left[ \psi_L(x + a)\psi_L(x) : - \frac{i}{2\pi a} \right] \left[ \psi_R(x)\psi_R(x + a) : - \frac{i}{2\pi a} \right] , \quad (B.20) \]

which, for \( a \to 0 \), becomes

\[ H_P^{(2)} = -2(4\pi^2 a \Delta) \cos(2k_Fa) \int_0^L dx : \psi_L^\dagger(x)\psi_L(x) :: \psi_R^\dagger(x)\psi_R(x) : + \\
+ 4\pi \Delta \sin(2k_Fa) \int_0^L dx \left[ \psi_L^\dagger(x)\psi_L(x) : + \psi_R^\dagger(x)\psi_R(x) : \right] + \\
- i4\pi a \Delta \cos(k_Fa) \int_0^L dx \left[ \psi_L^\dagger(x) \frac{d\psi_L(x)}{dx} - \psi_R^\dagger(x) \frac{d\psi_R(x)}{dx} \right] . \quad (B.21) \]

The various terms in Eq.\((B.21)\) may be interpreted as follows:

- **A shift in the chemical potential:**

  \[ 4\pi \Delta \sin(2k_Fa) \int_0^L dx [: \psi_L^\dagger(x)\psi_L(x) : + \psi_R^\dagger(x)\psi_R(x) :] , \quad (B.22) \]

  which is accounted for by simply redefining \( k_F \) through the equation

  \[ -(aE_J) \cos(k_Fa) + 2\Delta \sin(2k_Fa) = H \; ; \quad (B.23) \]

- **A \( L - R \) interaction term:**

  \[ -2(4\pi^2 a \Delta) \cos(2k_Fa) \int_0^L dx : \psi_L^\dagger(x)\psi_L(x) :: \psi_R^\dagger(x)\psi_R(x) : , \quad (B.24) \]
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that adds up to a similar term coming from $H^{(1)}_P$, giving

$$2(4\pi^2 a\Delta)[1 - \cos(2k_F a)] \int_0^L dx : \psi_L^\dagger(x)\psi_L(x) :: \psi_R^\dagger(x)\psi_R(x) : ; \quad (B.25)$$

- A renormalization of the Fermi velocity given by

$$- i4\pi a\Delta \cos(k_F a) \int_0^L dx \left[ \psi_L^\dagger(x) \frac{d\psi_L(x)}{dx} - \psi_R^\dagger(x) \frac{d\psi_R(x)}{dx} \right]. \quad (B.26)$$

Using the bosonization rules [69], the fermionic Hamiltonian $H^{f}_{\text{Eff}}$ may be written in bosonic coordinates as

$$H^b = \frac{v_F + g_2}{4\pi} \int_0^L dx \left[ \left( \frac{\partial \phi_L}{\partial x} \right)^2 + \left( \frac{\partial \phi_R}{\partial x} \right)^2 \right] + \frac{2g_4}{4\pi} \int_0^L dx \left[ \frac{\partial \phi_L}{\partial x} \frac{\partial \phi_R}{\partial x} \right], \quad (B.27)$$

where $g_2 = g_4 = 4\pi(a\Delta)[1 - \cos(2k_F a)]$.

One may readily see that $H^b$ corresponds to the Hamiltonian for a free, massless, real bosonic field $\Phi$ in 1+1 dimensions, which is described by the Hamiltonian

$$H[\Pi, \Phi] = \frac{v}{4\pi} \int_0^L dx \left[ \frac{4\pi^2}{g} \Pi^2 + g \left( \frac{\partial \Phi}{\partial x} \right)^2 \right], \quad (B.28)$$

where the momentum conjugate to $\Phi$ is

$$\Pi = \frac{2\pi}{g} \frac{\partial \Phi}{\partial t}. \quad (B.29)$$

Upon defining two independent chiral fields, $\phi_L^g$ and $\phi_R^g$, as

$$\frac{\partial \phi_L^g(x - vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[ \frac{2\pi}{\sqrt{g}} \Pi \sqrt{g} \frac{\partial \Phi}{\partial x} \right],$$
$$\frac{\partial \phi_R^g(x + vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[ -\frac{2\pi}{\sqrt{g}} \Pi \sqrt{g} \frac{\partial \Phi}{\partial x} \right], \quad (B.30)$$

one immediately sees that

$$H[\Pi, \Phi] \rightarrow H[\phi_L^g, \phi_R^g] = \frac{v}{4\pi} \int_0^L dx \left[ \left( \frac{\partial \phi_L^g}{\partial x} \right)^2 + \left( \frac{\partial \phi_R^g}{\partial x} \right)^2 \right], \quad (B.31)$$
which, when expressed in terms of \( \phi_L^g \equiv \phi_L \) and \( \phi_R^g \equiv \phi_R \), yields Eq.(B.27), provided that

\[
v = \sqrt{(v_F + g_2)^2 - g_4^2} \quad ; \quad g = \sqrt{\frac{v_F + g_2 + g_4}{v_F + g_2 - g_4}} . \tag{B.32}
\]

Thus, the correlation functions of all the operators depending on \( \phi_L \) and \( \phi_R \) may be evaluated by the replacements

\[
\phi_L - \phi_R = \sqrt{g} [\phi_L^g - \phi_R^g] \quad , \quad \phi_L + \phi_R = \sqrt{\frac{1}{g}} [\phi_L^g + \phi_R^g] \quad , \tag{B.33}
\]

with \( \phi_L^g, \phi_R^g \) free, chiral bosonic fields.
Appendix C

Schrieffer-Wolff transformation

To see how the Schrieffer-Wolff transformation works, let us consider a generic Hamiltonian
\[ H = H_1 + \lambda V \]  
(C.1)
where \( \lambda \) is an expansion parameter. Here
\[ H_1 = \begin{pmatrix} H_l & 0 \\ 0 & H_h \end{pmatrix} \]  
(C.2)
is diagonal in the low energy \( (H_l) \) and the high energy \( (H_h) \) subspaces, whereas the “hopping term”
\[ V = \begin{pmatrix} 0 & V^\dagger \\ V & 0 \end{pmatrix} \]  
(C.3)
provides the off-diagonal matrix elements between these two subspaces. The idea of the Schrieffer Wolff transformation is to carry out a canonical transformation that returns an Hamiltonian to block-diagonal form:
\[ U \begin{pmatrix} H_l & \lambda V^\dagger \\ \lambda V & H_h \end{pmatrix} U^\dagger = \begin{pmatrix} H^* & 0 \\ 0 & H' \end{pmatrix} . \]  
(C.4)
This is a “renormalized” Hamiltonian, and the block-diagonal part of this matrix \( H^* \) in the low energy subspace provide an effective Hamiltonian for the low energy physics. If we set \( U = e^S \), then \( U^\dagger = U^{-1} = e^{-S} \) (which implies \( S^\dagger = -S \) is anti-hermitian). Writing \( S \) as a power series in \( \lambda \),
\[ S = \lambda S_1 + \lambda^2 S_2 + \ldots , \]  
(C.5)
then by using the identity, \( e^A e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \ldots \), Eq.(C.4) can also be expanded in powers of \( \lambda \) as follows

\[
e^S (H_1 + \lambda \mathcal{V}) e^{-S} = H_1 + \lambda (\mathcal{V} + [S_1, H_1]) + \lambda^2 \left( \frac{1}{2} [S_1, [S_1, H_1]] + [S_1, \mathcal{V}] + [S_2, H_1] \right) + \ldots . \tag{C.6}
\]

Since \( \mathcal{V} \) is not diagonal, by requiring

\[
[S_1, H_1] = -\mathcal{V}, \tag{C.7}
\]

we can eliminate all off-diagonal components to leading order in \( \lambda \). To second order

\[
e^S (\mathcal{H}_T + \lambda \mathcal{V}) e^{-S} = \mathcal{H}_T + \lambda^2 \left( \frac{1}{2} [S_1, \mathcal{V}] + [S_2, \mathcal{H}_T] \right) + \ldots . \tag{C.8}
\]

Since \([S_1, \mathcal{V}]\) is block-diagonal, we can satisfy (C.4) to second order by requiring \( S_2 = 0 \), so that to this order, the effective Hamiltonian has the form

\[
H^* = H_I + \lambda^2 H_{\text{int}} \tag{C.9}
\]

where, introducing a projector operator \( \mathcal{P}_l \) into the low energy subspace,

\[
H_{\text{int}} = \frac{1}{2} \mathcal{P}_l [S_1, \mathcal{V}] \mathcal{P}_l + \ldots \tag{C.10}
\]

is an interaction term induced by virtual fluctuation into the high-energy manifold. Writing

\[
S = \begin{pmatrix} 0 & -s^\dagger \\ s & 0 \end{pmatrix} \tag{C.11}
\]

and substituting into Eq.(C.7), we obtain \( V = -s H_I + H_h s \). Now since \((H_I)_{ab} = E^I_a \delta_{ab}\) and \((H_h)_{ab} = E^h_a \delta_{ab}\) are diagonal, it follows that

\[
s_{ab} = \frac{V_{ab}}{E^h_a - E^I_b}, \]

\[
s^\dagger_{ab} = \frac{V^\dagger_{ab}}{E^h_a - E^I_b}. \tag{C.12}
\]

From Eq.C.12, we obtain

\[
(H_{\text{int}})_{ab} = -\frac{1}{2} (V^\dagger s + s^\dagger V)_{ab} = \frac{1}{2} \sum_{c \in |h|} \left[ \frac{V^\dagger_{ac} V_{cb}}{E^I_a - E^I_c} + \frac{V_{ac}^\dagger V_{cb}}{E^I_c - E^I_b} \right]. \tag{C.13}
\]
Bibliography


Bibliography


