Dynamical Aspects of Non-Abelian Vortices

Thesis for Doctor of Philosophy

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Abstract

It is the aim of this thesis to investigate the moduli spaces and the low-energy effective theories of the BPS non-Abelian vortices in supersymmetric gauge theories with $U(1) \times SU(N)$, $U(1) \times SO(2M)$, $U(1) \times SO(2M+1)$ and $U(1) \times USp(2M)$ gauge groups. The Goddard-Nuyts-Olive-Weinberg (GNOW) dual group emerges from the transformation properties of the vortex solutions under the original, exact global symmetry group acting on the fields of the theory in the color-flavor locking phase. The moduli spaces of the vortices turn out to have the structure of those of quantum states, with sub-moduli corresponding to various irreducible representations of the GNOW dual group of the color-flavor group. We explicitly construct the vortex effective world-sheet action for various groups and winding numbers, representing the long-distance fluctuations of the non-Abelian orientational moduli parameters. They are found to be two-dimensional sigma models living on appropriate coset spaces, depending on the gauge group, global symmetry, and on the winding number. The mass-deformed sigma models are then constructed from the vortex solutions in the corresponding unequal-mass theories, and they are found to agree with the two-dimensional models obtained from the Scherk-Schwarz dimensional reduction. The moduli spaces of higher-winding BPS non-Abelian vortices in $U(N)$ theory are also investigated by using the Kähler quotient construction, which clarifies considerably the group-theoretic properties of the multiply-wound non-Abelian vortices. Certain orbits, corresponding to irreducible representations are identified; they are associated with the corresponding Young tableau.
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Chapter 1

Introduction

Solitons play an important role in a wide range of contemporary physics, from condensed-matter physics and fluid dynamics to cosmology and elementary particle physics. A soliton, without a pretense of rigorous definition, is an object whose energy density is well localized in space and which does not dissipate in time, occurring in many non-linear differential equations of interest in physics. Topological solitons, such as kinks (domain walls), vortices, monopoles, and higher-dimensional textures occur in many field theory models with non-trivial vacuum, as classical solutions of field equations of motion.

The last decade has witnessed a remarkable advance in our understanding of soliton excitations of non-Abelian variety, i.e., solitons possessing moduli of solutions connected by continuous, in general non-Abelian, symmetry transformations. The problem of solitons of non-Abelian variety is believed to have many interesting applications in different branches of physics, for instance, in the condensed-matter physics. A particularly important problem in which these development can have possible relevance, is the quark confinement. The central issue involved here is to understand the non-Abelian gauge dynamics in a strong-coupling regime. Although the relevant theory for the real-world problem of the quark confinement is the so-called quantum chromodynamics ($SU(3)$ Yang-Mills gauge theory) with various light quark flavors, it appears to be rather useful to investigate systematically different gauge groups and flavors (and different amount of supersymmetry), as a guide to understand the nature and types of strong-interaction dynamics in a given theory. For instance there has been a remarkable progress in our understanding of the quantum-mechanical behavior of magnetic monopoles, made possible by the discovery of
the exact Seiberg-Witten solutions for \( N = 2 \) supersymmetric gauge theories as well as by other important results such as Seiberg’s electromagnetic duality in \( N = 1 \) supersymmetric gauge theories. The attempts to understand better some of the phenomena found there (such as the occurrence of non-Abelian monopoles as the infrared degrees of freedom) have eventually led to the discovery of non-Abelian vortices (2003), a problem which had remained unsolved since the work of Nielsen-Olesen. The remarkable development which followed in our understanding of solitons of non-Abelian variety seems to give a renewed vigor in our efforts of understanding the nature at a deep, fundamental level.

The present thesis is devoted to the discussion of what we believe to be some of the key contributions in these development. Concretely, we discuss the moduli spaces and the low-energy effective theories of the BPS non-Abelian vortices in supersymmetric gauge theories with various gauge groups. The Goddard-Nuyts-Olive-Weinberg (GNOW) dual group emerges from the transformation properties of the vortex solutions under the original, exact symmetry group acting on the fields of the theory in the color-flavor locking phase. The study of general class of gauge theories allows us to study related group-theoretic issues quite clearly. The moduli spaces of the vortices in the \( SO(2M) \) and \( USp(2M) \) theories turn out to be isomorphic to those of quantum particle states in spinor representations of \( SO(2M) \) and \( SO(2M + 1) \) groups, respectively. For the minimum vortex in \( U(1) \times SO(2M) \) or \( U(1) \times USp(2M) \) gauge theories, we explicitly construct the effective low-energy action: it is a two-dimensional sigma model living on the Hermitian symmetric space \( SO(2M)/U(M) \) or \( USp(2M)/U(M) \), respectively. The effective action of some higher-winding vortices in \( U(N) \) and \( SO(2M) \) theories are also obtained. The mass-deformed sigma models are then constructed from the vortex solutions in the corresponding unequal-mass theories, and they are found to agree with the two-dimensional models obtained from the Scherk-Schwarz dimensional reduction.

By using the Kähler-quotient construction, the moduli spaces of higher-winding BPS non-Abelian vortices in \( U(N) \) theory are investigated, our attention being focused on the group-theoretic properties of multi-vortex solutions. The moduli space of vortices with a definite winding-number contains in it various sub-moduli representing irreducible orbits, transforming within which according to some irreducible representations of the GNOW dual group. These sub-moduli of winding-number \( k \) vortices are nicely represented by a Young tableau made of \( k \) boxes, and the correspondence with the standard composition-decomposition rule of the products of two or more objects in the fundamental representation
(i.e., of $SU(N)$ group) is quite nontrivial, due to very special properties of higher winding vortex solutions of coincident centers.

The Kähler potentials corresponding to the irreducible moduli spaces can be determined from the symmetry consideration, but whenever the comparison is possible, the results agree with what is found from the direct determination of the effective actions discussed in the first part of the thesis. These results are important in view of the connection between the vortex solutions and the monopole solutions.

1.1 Abelian Higgs model

We begin with the Abelian Higgs (AH) model as a warm-up. The AH model is an important model in quantum field theory. First it is the relativistic extension of the Ginzburg-Landau (GL) model which describes some basic aspects of the phenomenon of superconductivity. The existence of non-trivial minima shows that the gauge symmetry (in this case, the $U(1)$ electromagnetic gauge group) is spontaneously broken; it provides the simplest possible example of the mass generation mechanism – the Higgs mechanism. It also has non-trivial topological solutions: the Abrikosov-Nielsen-Olesen (ANO) vortex [3, 4].

1.1.1 Higgs Mechanism

The Abelian Higgs model Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\mathcal{D}_\mu \phi)^* \mathcal{D}^\mu \phi - \frac{\lambda}{4} (|\phi|^2 - \nu^2)^2,$$

where $\phi$ is a complex scalar field, $\lambda$ is the self-interacting constant of $\phi$. $F_{\mu\nu}$ is the gauge fields strength, which is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$\lambda$ is assumed to be positive to guarantee that the energy is bounded. The covariant derivative is defined as

$$\mathcal{D}_\mu \equiv \partial_\mu + ie A_\mu,$$

where $e$ is the Abelian coupling constant. The Lagrangian has a global $U(1)$ symmetry, since it is invariant under $\phi' = e^{i\alpha} \phi$. Also, the Lagrangian is invariant under a local gauge
symmetry,
\[ \phi(x) \rightarrow \phi'(x) = e^{i\alpha(x)}\phi(x), \quad A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x). \] (1.1.4)

In superconductivity, the field \( \phi \) is the order parameter, which is used to describe phase transition. With \( \phi^4 \) term, the system has second order phase transition. In quantum field theory, \( \phi^4 \) is the renormalized self-interacting term which is the highest order in four dimension.

Let us consider the classical vacuum of the system. The potential is minimal when the Higgs field satisfies \( \langle |\phi|^2 \rangle = \upsilon^2 \), or \( \phi = \upsilon e^{i\beta} \). This ground state exhibits a vacuum degeneracy. The manifold of the vacuum is isomorphic to \( S^1 \) in the complex \( \phi \) plane. As will be discussed, this is the source of the non-trivial topological excitation of the Abelian Higgs model.

The complex scalar field \( \phi \) can be decomposed as \( \phi = \phi_1 + i\phi_2 \), \( \phi_1 \) and \( \phi_2 \) are two real component fields. The vacuum manifold still has \( U(1) \) symmetry. However the \( U(1) \) symmetry is spontaneously broken when we choose a special vacuum. For instance, let us choose \( \langle \phi_1 \rangle = \upsilon, \langle \phi_2 \rangle = 0 \). A new field, which is a small perturbation around the chosen vacuum, can be defined as \( \phi' \equiv \phi_1' + i\phi_2' = \phi - \upsilon \). The kinetic term for the scalar field becomes

\[
(D_\mu \phi)^* D^\mu \phi = (\partial_\mu \phi_1' - eA_\mu \phi_2')^2 + (\partial_\mu \phi_2' + eA_\mu \phi_1')^2
+ e^2 \upsilon^2 A_\mu A^\mu + 2e\upsilon A^\mu (\partial_\mu \phi_2' + eA_\mu \phi_1').
\] (1.1.5)

The first term in the second line can be explained as the mass term for \( A_\mu \). The mixing term \( 2e\upsilon A^\mu \partial_\mu \phi_2' \) can be eliminated by the Unitary gauge. Now we define \( \phi \) as

\[
\phi(x) = [\upsilon + \rho(x)] \exp \left[ i \frac{\theta(x)}{\upsilon} \right], \quad (1.1.6)
\]

when only the linear term is truncated in expansion, \( \rho(x) \) and \( \theta(x) \) are the same with \( \phi_1 \) and \( \phi_2 \), respectively. Taking the unitary gauge, we obtain

\[
\phi^U(x) = \exp \left[ -i \frac{\theta(x)}{\upsilon} \right] \phi(x) = \upsilon + \rho(x), \quad B_\mu = A_\mu + \frac{1}{e\upsilon} \partial_\mu \theta(x).
\] (1.1.7)

The Lagrangian of (1.1.1) can be written as

\[
\mathcal{L} = -\frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + e^2 \upsilon^2 B_\mu B^\mu + \partial_\mu \rho \partial^\mu \rho - \lambda \upsilon^2 \rho^2
+ e^2 B_\mu B^\mu \rho^2 + 2e^2 \upsilon B_\mu B^\mu \rho - \frac{\lambda}{4} \rho^4 - \lambda \upsilon \rho^3.
\] (1.1.8)
1.1 Abelian Higgs model

It is clear that $\theta(x)$ field disappears in the Lagrangian, which is called the massless Goldstone boson. The real field $\rho$ and vector field $B_\mu$ become massive, their masses are

$$m_s = \sqrt{2\lambda} v, \quad m_g = \sqrt{2} e v,$$  \hspace{1cm} (1.1.9)

respectively. However, the total number of the degrees of the freedom does not change. Before spontaneously symmetry breaking, we have one massless complex scalar $\phi$ (two real components) and one massless gauge boson $A_\mu$ (with only two polarization states). After symmetry breaking, we have one massive real scalar field $\rho$ and one massive gauge boson $B_\mu$ (with three polarization states). The Goldstone boson $\theta$ is “eaten” by the gauge boson $A_\mu$, which is called “Abelian Higgs mechanism”.

1.1.2 Topological excitations

Switching off the gauge fields in Lagrangian (1.1.1), we obtain

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi),$$  \hspace{1cm} (1.1.10)

where $\phi$ is defined on two spacial dimension plane with coordinates $\vec{x} = r(\cos \varphi, \sin \varphi)$. The vacuum manifold $\phi = ve^{i\alpha}$ is isomorphic to $S^1$, a non-trivial map from the spatial infinity circle to the manifold of vacuum exists due to the homotopy group

$$\pi_1(S^1) = \mathbb{Z}.$$  \hspace{1cm} (1.1.11)

Since $\phi$ must be single valued, such function must obey $\alpha(\varphi + 2\pi) = \alpha(\varphi) + 2\pi n$, where $n$ is the winding number of the map. We want to find the finite energy solution of the model in (1.1.10), such Ansatz can be made

$$\phi(r, \varphi) = v f(r)e^{in\varphi},$$  \hspace{1cm} (1.1.12)

where $n$ is the winding number, $f(r)$ is a profile function which satisfies boundary conditions: $f(\infty) = 1$ and $f(0) = 0$, respectively. However the energy of the system is divergent logarithmically,

$$\int d^2 x |\partial_\mu \phi|^2 \sim 2\pi v^2 n^2 \int_0^\infty \frac{dr}{r}.$$  \hspace{1cm} (1.1.13)

Here comes the Derrick’s theorem [5]: there are no finite-energy time-independent solitons which are localized in more than one dimension with scalar fields only $^1$. In order to have

$^1$The theorem only excludes time independent solutions, and no conclusion for time-dependent case.
static solitons \(^2\), we must switch on the gauge fields. Requiring that

\[
\lim_{r \to \infty} D_i \phi \sim 0, \quad (1.1.14)
\]

the Ansatz for gauge field \(A_i\) can be written as

\[
A_i = \frac{\epsilon^{ij} x_j}{er^2} [n - a(r)], \quad (1.1.15)
\]

the boundary conditions for \(a(r)\) are

\[
a(\infty) = n, \quad a(0) = 0, \quad (1.1.16)
\]

which eliminates the singularity in the origin.

The static energy of the soliton becomes

\[
E = 2\pi \nu^2 \int_0^\infty \rho d\rho \left[ \frac{(a')^2}{\rho^2} + (f')^2 + \frac{a^2 f^2}{\rho^2} + \frac{1}{2} \beta^2 (f^2 - 1)^2 \right], \quad (1.1.17)
\]

where \(\rho\) is the dimensionless radial coordinate \(\rho \equiv \sqrt{2e}\nu r\), and \(\beta\) is defined as \(\beta \equiv \lambda/2e^2\). It is easy to obtain the equations of motion:

\[
\begin{align*}
 f'' + \frac{f'}{\rho} - \frac{a^2}{\rho^2} f - \beta^2 f (f^2 - 1) &= 0, \\
 a'' - \frac{a'}{\rho} - af^2 &= 0.
\end{align*} \quad (1.1.18, 1.1.19)
\]

The solution of these equations are known as the Nielsen-Olesen string or Abrikosov-Nielsen-Olesen (ANO) vortex \([3, 4]\).

When we apply the Abelian Higgs model to the superconductor, the profile functions \(a\) and \(f\) correspond to the penetration depth and the coherent length, and furthermore they are controlled by the inverse of gauge boson and scalar masses respectively. \(\beta\) is the Ginzburg-Landau parameter which is important for the stability of classical soliton solutions \([6]\). When the critical value \(\beta = 1\) is chosen, the vortices are stable and do not interact, which are called BPS vortices.

For non-BPS vortices, \(\beta \neq 1\), there are interactions between them. An overall conclusion is that the particle with the lowest mass dominates the interaction at large distance. When \(\beta > 1\), corresponding to type II superconductor, the gauge bosons have the lowest mass.

\(^2\)Since the static configuration is a safe solution when equations of motion solution is considered, it is the minimal extreme energy of the system.
The gauge boson fields tend to repel, so such vortices will repel each other. A vortex with winding number $n > 1$ is unstable and will decay into $n$ separated vortices with minimal winding. If $\beta < 1$, corresponding to type I superconductor, the scalar field dominates, and the vortices tend to attract each other. Therefore vortices in type I superconductors will attract each other and form a vortex with high winding numbers (or, a region of normal state). A similar argument basically also applies to the cases of non-Abelian vortices, even though the interactions among non-Abelian vortices non-trivially depend on the internal orientations, see [7, 8, 9].

Abelian topological excitations have applications in superfluid or superconductor; non-Abelian topological excitations may help us in understanding various other aspects of physics, including possibly the high temperature superconductor, and the mechanism of quark confinement.

### 1.2 Non-Abelian Confinement

#### 1.2.1 Monopoles

In a pioneering work by Dirac [10], a magnetic monopole with a singular string is introduced to the Abelian electrodynamics. The existence of even one monopole would explain the quantized electric charges, with the well-known quantization condition

$$e g = 2\pi n, \quad n \in \mathbb{Z} \tag{1.2.1}$$

where $n$ is an integer, $e$ and $g$ are electric and magnetic charge, respectively. One significant indication of quantization condition is the electric-magnetic duality, which implies the interchange of weak and strong coupling constants.

However, the Dirac monopole is not part of the spectrum of standard QED. Moreover, we can not define a local field theory which contains both the electrons and monopoles on the classical level. The situation changes dramatically if the Abelian electrodynamics theory is only a part of grand unification theory, where the $U(1)$ subgroup is embedded in a non-Abelian gauge group of higher rank. In 1974, 't Hooft and Polyakov independently found regular monopole solutions in Georgi-Glashow model, where the $SO(3)$ symmetry is broken to $U(1)$ by the Higgs mechanism [11, 12]. In this theory the field contents is just
the $SU(2)$ Yang-Mills Higgs model with scalar fields in the adjoint representation. Taking the gauge $A_0 = 0$, the energy of a static field configuration has the expression

$$E = \int d^3x \left[ \frac{1}{4}(F_{ij}^a)^2 + \frac{1}{2}(D_\mu \phi^a)^2 + V(\phi) \right],$$ (1.2.2)

where

$$V(\phi) = \frac{\lambda}{4} (\phi^a \phi^a - \nu^2)^2.$$ (1.2.3)

The integral in Eq.(1.2.2) can converge only if the potential $V(\phi)$ disappears at large distance: the scalar field approaches the value at the minimum of the potential. In particular, the problem becomes somewhat simpler when $\lambda = 0$ but the boundary condition $|\phi^a|^2 \to \nu^2$ is imposed nonetheless. In this so-called Bogomol’nyi-Prasad-Sommerfield (BPS) limit, the field equation becomes first order

$$F_{ij}^a = \epsilon_{ijk} D_k \phi^a,$$ (1.2.4)

and monopole solutions are known analytically [6, 13].

The vacuum manifold is defined to be the set of $\phi$ which minimizes the potential, i.e.,

$$\phi^a \phi^a = \nu^2,$$ (1.2.5)

which forms a $S^2$ sphere in the isotopic space. A configuration of $\phi$ which satisfies Eq.(1.2.5) is $\phi^a = \nu \delta_{3a}$, spontaneously breaks the original gauge group $G = SO(3)$. However, the system has an unbroken subgroup $H = SO(2) \sim U(1)$, the energy is invariant under such Abelian gauge transformation. The field component remaining massless is the $U(1)$ gauge field identified as the photon. Using the ’t Hooft-Polyakov Ansatz, the field strength asymptotes

$$F_{ij}^a = \epsilon_{ijk} \frac{r_k}{r^3} \phi^a,$$ (1.2.6)

where $e$ is the gauge coupling constant. We can define a Compton wavelength which is of the order $\frac{1}{e \nu}$ for the ’t Hooft-Polyakov monopoles. At short distance, or inside the monopole, all field are excited, the system has a dynamics of the gauge symmetry $G$. In the outside, only gauge symmetry $H$ remains. The ’t Hooft-Polyakov monopole has no singularities, contrary to the Dirac monopole. As the unbroken subgroup $H$ is Abelian in this case, the ’t Hooft-Polyakov monopole is an Abelian monopole.
Even though the initial enthusiasm about the ’t Hooft-Polyakov monopoles as physical objects has somewhat faded after the failed search in the cosmological context (relic monopoles from the cosmological phase transition and grand unified symmetry breaking), the interest in magnetic monopole solutions in gauge theories has been forcefully revived by ’t Hooft and Mandelstam [14, 15]. According to the ’t Hooft-Mandelstam mechanism the quark confinement is equivalent to a dual Meissner effect: the quarks are confined by the Chromo-electric string. The ground state of QCD would be characterized by the condensation of these monopoles. This is dual of what happens in the ordinary superconductor where the electrically charged Cooper pairs are condensed and in which the monopoles (if introduced in the theory) are confined by the Abelian ANO vortex.

When the gauge symmetry breaking

\[ G \rightarrow H. \]  

(1.2.7)

involves a non-Abelian group \( H \) as the unbroken gauge group, we expect the monopole solutions to form a degenerate set, connected by some continuous, non-Abelian group transformations. The true nature of these “non-Abelian monopoles” have so far eluded our complete grasp for various subtle reasons, in spite of considerable efforts dedicated to this problem by many authors. Although it is not the purpose of this thesis to discuss in detail these issues (which is mainly dedicated to the study of non-Abelian vortices), let us make a few general remarks which are relevant to our thesis work, and which ultimately may lead to the resolution of the old puzzle of the non-Abelian monopoles.

The gauge field strength looks asymptotically as

\[ F_{ij} = \epsilon_{ijk} \frac{r_k}{r^3} (\beta \cdot \mathbf{H}), \]  

(1.2.8)

in an appropriate gauge, where \( \mathbf{H} \) are the diagonal generators of \( H \) in the Cartan subalgebra; \( \beta \) may be named as the “magnetic charge”, as these constant vectors characterize each monopole solution. Goddard et. al. showed that the magnetic charges are classified according to the weight vectors of the group \( \tilde{H} \), which is dual with respect to \( H \) [16]. The quantization condition is written as

\[ 2 \beta \cdot \alpha \in \mathbb{Z}, \]  

(1.2.9)

where \( \alpha \) denotes a root vector of \( H \). The solution to this condition (1.2.9) is well known in group theory, \( \beta = \sum_i n_i \beta_i^* \), i.e., \( \beta_i^* \) is any weight vector of \( \tilde{H} \), generated by the dual root
This leads to the Goddard-Nuyts-Olive-Weinberg (GNOW or simply GNO) conjecture, i.e., the monopoles form multiplets of the dual group $\tilde{H}$, and transform as such. Indeed, it is not difficult to construct the set of monopole solutions semi-classically, which are degenerate and whose multiplicity appears to match correctly that of various representations of the dual group $\tilde{H}$. For groups $U(N)$ and $SO(2M)$, the dual group is the same group, i.e.,

$$U(N) \longleftrightarrow U(N), \quad SO(2M) \longleftrightarrow SO(2M).$$

(1.2.11)

For groups $SU(N)$, $SO(2M + 1)$, and $USp(2M)$, we have instead

$$SU(N) \longleftrightarrow \frac{SU(N)}{Z_N}, \quad SO(2M + 1) \longleftrightarrow USp(2M).$$

(1.2.12)

The non-Abelian monopoles are essentially constructed by embedding the ’t Hooft-Polyakov monopoles in various broken $SU(2)$ subgroups [17, 18]. Also the multi-monopole moduli spaces and moduli metrics have been constructed [19].

In view of the presence of an unbroken gauge group $H$, it is on the other hand quite natural to regard these degenerate monopole solutions as something related to each other by the group $H$ itself. However, there are well-known difficulties with such a concept. The first difficulty is the topological obstruction [20, 21, 22, 23, 24]: in the presence of fundamental monopole, there are no globally well-defined set of generators isomorphic to the unbroken gauge group $H$. In the simplest case of the symmetry breaking

$$SU(3) \xrightarrow{\langle \phi \rangle} SU(2) \times U(1),$$

(1.2.13)

the above implies that no monopoles with charges $(2,1^*)$ can exist, where the asterisk indicates a dual magnetic charge.

The second difficulty is related to the fact that certain bosonic zero modes around the monopole solution, corresponding to $H$ gauge transformations, are non-normalizable [20, 25, 1]. For the model in (1.2.13), the monopole solutions can exist both in $(1,3)$ and $(2,3)$ subspaces, they are degenerate and may be related by the unbroken gauge group $SU(2)$. The zero-modes corresponding to the unbroken $SU(2)$ gauge group are however non-normalizable (behaving as $r^{-1/2}$ asymptotically), meaning that such a transformation requires an infinite amount of energy.
Both the above difficulties concern the transformation properties of the monopoles under the subgroup $H$. Actually the GNO duality (conjecture) states that the monopoles are to transform under $\tilde{H}$, and not under $H$. The GNO duality, being a natural generalization of electromagnetic duality, implies that the action of the group $\tilde{H}$ is a nonlocal field transformation group, if seen in the original electric description. The explicit form of such non-local transformations is not known in the case of the non-Abelian GNO duality, and this is one of the reasons why the concept of non-Abelian monopoles has remained somewhat mysterious up to today.

Moreover, the quantum dynamics of monopoles is rather hard to analyze; it suffices to remember the difficulties in quantizing fields in the monopole backgrounds beyond one-loop approximation, even for the Abelian monopoles. In particular, due to the non-Abelian nature of the gauge groups involved, the concept of a dual group multiplet is well-defined only when $\tilde{H}$ interactions are weak. This means that in order to study such weakly-coupled dual gauge groups, the original $H$ theory must be arranged in a strong-coupling regime, e.g., in a confining phase, which renders the task a formidable one.

Fortunately, supersymmetric theories tell us something important about the non-Abelian monopoles and confinement. The breakthrough in this sense came from the work by Seiberg and Witten in $\mathcal{N} = 2$ supersymmetric gauge theories, in which the full quantum-mechanical results of strong-interaction dynamics have been obtained in terms of weakly-coupled dual magnetic variables [26, 27]. They have indeed shown that the condensation of Abelian monopoles takes place near the monopole point in the moduli space of the theory, once $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = 1$ by the addition of a mass term of the adjoint matter fields.

More interestingly, non-Abelian monopoles are found to appear as the infrared degrees of freedom in the so-called $r$ vacua of softly broken $\mathcal{N} = 2$, $SU(N)$ gauge theory with matter hypermultiplets [28, 29, 30, 31]. Upon soft breaking to $\mathcal{N} = 1$, the condensation of monopoles gives rise to confinement and at the same time to dynamical symmetry breaking. In $\mathcal{N} = 2$ SQCD with $N_f$ flavors, light non-Abelian monopoles with $SU(r)$ dual gauge group appear for $r \leq N_f/2$ only [28, 29]. Such a fact clearly displays the dynamical properties of the monopoles: the effective low-energy gauge group must be either infrared free or conformal invariant under the renormalization group, such that these entities can emerge as recognizable low-energy degrees of freedom [29, 30, 32]. The Jackiw-Rebbi mechanism
which endows quantum-mechanically the monopoles with flavor multiplet degeneracy is fundamental for this phenomenon to occur [33, 34]. This then is a clear evidence of the existence of fully quantum-mechanical non-Abelian monopoles, in spite of a somewhat murky situation as regard to our understanding of the semiclassical non-Abelian monopoles, reviewed above. Note that the exact flavor symmetry and dynamical aspects play a crucial role for such objects to exist, that is an aspect which has not been taken into account in the straightforward approach to quantize the semi-classical non-Abelian monopoles. Nevertheless, there must be ways to comprehend the essential features of non-Abelian monopoles, starting from the semi-classical analysis of these soliton-like excitations. It is this line of reasoning which has eventually led to the discovery of non-Abelian vortices: the vortices and monopoles are indeed closely related to each other, as the following argument show.

1.2.2 Monopole-vortex complex

In the discussion of non-Abelian duality, the phase of the system is an important ingredient. The discussion of the non-Abelian monopoles in the Coulomb phase of \( H \) and \( \tilde{H} \) leads to the difficulties already mentioned. If the original electric system is in strongly-coupled, confinement phase, the dual theory \( \tilde{H} \) would be in Higgs phase, the non-Abelian multiplet structure of the monopoles would be lost. The best we can do seems to be that of studying the \( H \) system in the Higgs phase, such that the dual gauge system \( \tilde{H} \) is in an unbroken, confinement phase. The \( H \) gauge system in a Higgs phase \( H \longrightarrow 1 \) generate the vortex solutions at low energy scale, the monopoles are confined by these vortices. As we shall see in the next section a crucial ingredient for not losing the non-Abelian multiplet structure of monopoles (in \( \tilde{H} \)) comes from the exact color-flavor locking symmetry in the original theory.

In summary, we are led to consider spontaneous symmetry breaking of the gauge group occurring at two hierarchically separate mass scales [31, 35, 36, 37]

\[
G \xrightarrow{v_1} H \xrightarrow{v_2} 1, \tag{1.2.14}
\]

where \( v_1 \gg v_2 \). At the first stage of symmetry breaking which occurs at a high energy scale \( v_1 \), monopoles quantized according to \( \pi_2(G/H) \) are generated. At the low-energy scale \( v_2 \), the subgroup \( H \) is completely broken, and vortices with the flux quantized according to
\( \pi_1(H) \) appear.

The monopoles and vortices appearing in such a system are related to each other by the exact sequence

\[
\cdots \rightarrow \pi_2(G) \rightarrow \pi_2(G/H) \rightarrow \pi_1(H) \rightarrow \pi_1(G) \rightarrow \cdots
\]

(1.2.15)

of the homotopy groups which classify these solitons. An exact sequence means that the kernel of the map at any point of the chain is equal to the image of the preceding map. Physically, non-trivial \( \pi_1(G) \) classifies the Dirac monopoles; \( \pi_2(G/H) \) classifies the regular \('t\) Hooft-Polyakov monopoles. See Figure 1.1. For any compact Lie group \( G \), we have

\[
\pi_2(G) = 1.
\]

The exact sequence implying the isomorphism

\[
\pi_1(G) \sim \pi_1(H)/\pi_2(G/H),
\]

(1.2.16)

then shows that the regular monopoles correspond to the kernel of the map \([38]\)

\[
\pi_1(H) \xrightarrow{\sim} \pi_1(G).
\]

(1.2.17)

These relations explain what might otherwise look as an apparent paradox: when the smaller VEV \( v_2 \) is taken into account, there should be no topologically stable \('t\) Hooft-Polyakov monopoles, because \( \pi_2(G) = 1 \). Another, related paradox is that the vortices of the low-energy theory, whose the flux is quantized according to \( \pi_1(H) \), must disappear in the full theory, if \( \pi_1(G) = 1 \).

Such an apparent paradox is not actually a bad news; the situation becomes meaningful when we consider the high-energy breaking \( G \rightarrow H \) and the low-energy breaking \( H \rightarrow 1 \),
in the context of the full theory. The massive regular monopoles are in fact confined by the vortices in the low-energy theory. This conjecture can be verified by counting the magnetic flux [37], i.e., the flux of the minimal monopole agrees precisely with the total magnetic flux of the single minimal vortex. In cases $\pi_1(G)$ is non-trivial the situation becomes slightly more elaborate. For instance, $SO(3) \rightarrow U(1) \rightarrow 1$, which is precisely the model considered by 't Hooft in his pioneering paper on monopoles. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, the monopoles can be classified into two classes, the regular 't Hooft-Polyakov monopoles with magnetic charge $2n$ ($n = 1, 2, \ldots$) times the Dirac unit, correspond to the trivial element of $\mathbb{Z}_2$, while the monopoles with magnetic charge are singular Dirac monopoles.

The hierarchical scales allow us to study these configurations as approximate solutions, as the motion of massive monopoles at the extremes of the vortex can be at first neglected. Of course, a monopole-vortex-anti-monopole configuration cannot be stable because the shorter the vortex is, the smaller the energy of the whole configuration becomes. This does not mean that it is incorrect to consider such configurations. Quite the contrary, they may be perfectly well stabilized dynamically, e.g., with the endpoints rotating with speed of light, or after the radial modes are properly quantized. We believe that the real-world mesons are quark-string-anti-quark bound states of this sort.

When the $H$ group is non-Abelian, we can generalize the 't Hooft-Mandelstam mechanism to the regime of non-Abelian confinement. It turns out that such a scenario is naturally realized in $\mathcal{N} = 2$ supersymmetric theories with quark hypermultiplets. The role of the flavor symmetry is fundamental. When $H$ gauge group is broken by the VEVs of squarks, an exact global diagonal color-flavor symmetry group $H_{C+F}$ remains exact, and continuous non-Abelian flux moduli arise as $H_{C+F}$ is broken by individual vortex configurations. What the consideration of the monopole-vortex complex teaches us is the fact that the transformation law of the non-Abelian monopoles follows from that of non-Abelian vortices. As the orientational zero-modes of the non-Abelian vortices (here we exclude the size moduli associated with the so-called semi-local vortices; we are interested only in the internal, orientational moduli) are normalizable, there is no conceptual problem in their quantization, in contrast to the non-normalizable $H$ gauge modes around the semi-classical monopoles, the problem we have discussed in Section 1.2.1.
1.3 \( U(N) \) Non-Abelian vortices

A breakthrough on non-Abelian vortices during the last several years has allowed us to study the monopoles in the Higgs phase; monopole confinement by the vortices can elegantly be analyzed in the context of \( N = 2 \) supersymmetric models [39, 40, 41, 42].

Let us start with the \( U(N) \) non-Abelian vortices. The Lagrangian of the simplest theory in which non-Abelian vortices arise reads [43]

\[
\mathcal{L} = \text{Tr} \left[ -\frac{1}{4\epsilon^2} F_{\mu\nu} F^{\mu\nu} + \left| D_\mu q \right|^2 - \frac{e^2}{2} \left( qq^\dagger - \nu^2 1_{N_c} \right)^2 \right]. \tag{1.3.1}
\]

The gauge group of the system is \( U(N_c) \), the matter field \( q \) is written in an \( N_c \times N_f \) color-flavor matrix form, \( qq^\dagger \) in the potential is an \( N_c \times N_c \) matrix. For \( N_c > N_f \), an unbroken \( U(N_c - N_f) \) gauge group exist, while for \( N_c \leq N_f \), the gauge group is broken completely and the theory is in a Higgs phase. In order to have vortex solutions of desired types, we set \( N_c = N_f = N \). The key to non-Abelian vortices lies in the particular symmetry breaking pattern. When \( q \) takes vacuum expectation value (VEV), \( \langle q \rangle = \text{diag}(\nu, \nu, \ldots, \nu) \), the gauge group \( U(N_c) \) and flavor group \( SU(N_f) \) is broken to

\[
U(N_c) \times SU(N_f) \rightarrow SU(N)_{c+f}, \tag{1.3.2}
\]

and the system is in a so-called color-flavor locked phase.

The overall \( U(1) \subset U(N_c) \) is broken in the vacuum which ensures the existence of vortex solutions. Let the vortex lie in the \( x_3 \) direction, and the coordinate of the transverse plane is \( z = x_1 + ix_2 \). We consider only the static configuration energy on the \( z \) plane, so we turn to Hamiltonian expression. The tension is calculated to be

\[
T = \int dx_1 dx_2 \left\{ \left| (D_1 - iD_2) q \right|^2 + \frac{1}{2\epsilon^2} \left( B_3^2 - e^2 (qq^\dagger - \nu^2 1_N) \right) - B_3 \nu^2 \right\} \geq 2\pi\nu^2 k. \tag{1.3.3}
\]

where \( k \) is the winding number, and \( B_3 = F_{12} \) is the magnetic field strength. The last term \( 2\pi\nu^2 k \) is the bounded tension term of this system, if the BPS equations are satisfied

\[
B_3 = e^2 \left( qq^\dagger - \nu^2 1_N \right), \quad D_5 q = 0. \tag{1.3.4}
\]

Setting \( N = 1 \), this is the Abelian Higgs model, which has the ANO vortex solution. For generic \( N > 1 \), we have non-Abelian vortices. The solutions to Eq. (1.3.4) can be obtained,
which read

\[
B_3 = \begin{pmatrix} B^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad q = \begin{pmatrix} q^* \\ v \\ \vdots \\ v \end{pmatrix}.
\]

(1.3.5)

The \(B^*\) and \(q^*\) can be embedded in any diagonal entry of the matrices via Weyl permutation. As such an embedding (i.e., each solution) breaks the exact color-flavor symmetry, a sort of Nambu-Goldstone zero-modes appear, which however can be excited only along the vortex length and in time (that is, in the vortex world-sheet) as the \(SU(N)_{c+f}\) symmetry remains exact in the bulk. As a consequence, these modes manifest themselves as a two-dimensional \(\mathbb{C}P^{N-1}\) sigma model:

\[
\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)}.
\]

(1.3.6)

The orientational zero-modes are the defining character of the non-Abelian vortices. We can also construct other solutions by simply acting on it by global transformations, i.e.,

\[
B_3 \rightarrow U B_3 U^\dagger, \quad q \rightarrow U q U^\dagger,
\]

(1.3.7)

in which \(U\) parameterizes \(\mathbb{C}P^{N-1}\) model and \(U \in SU(N)_{c+f}\).

This is in contrast to certain solutions called non-Abelian vortices in the literature, where the vortex flux is actually always oriented in a fixed direction in the Cartan subalgebra. In such solutions no orientational zero-modes exist, which means that they are actually Abelian [44].

The non-Abelian vortex solutions are characterized by the weight vectors of the dual group, due to the quantization condition

\[
\vec{\alpha} \cdot \vec{\alpha} \in \mathbb{Z}, \quad \vec{\alpha} \in \Lambda_{\text{root}},
\]

(1.3.8)

where \(\vec{\alpha}\) are the root vectors of \(H\) (see below). This is formally identical to the GNOW duality quantization condition for the monopoles (1.2.9). As a result, studying the vortices will help us understand the problems of non-Abelian monopoles.

A few comments are given about the remarkable progress made on non-Abelian vortices during the last decade. When BPS non-Abelian monopoles are embedded in a supersymmetric gauge theory, the monopoles transform under the group dual to \(H\) in a tensor
representation of rank determined by the corresponding element in $\pi_1(H)$ [31]. The moduli matrix formalism explicitly identifies a well-defined set of the vortex moduli parameters, which also helps to study the collision of non-Abelian vortices (the cosmic string) [45]. The moduli space of vortices in the $U(N)$ gauge theory was found [46], the transformation law for the orientational modes is the same as that of a $\mathbb{CP}^{N-1}$ sigma model.

One development concerned the higher-winding vortices [47]; for instance the manifold of the $k = 2$ $U(2)_{c+f}$ vortex moduli space was found to contain an $A_1$-type singularity $\mathbb{Z}_2$ on a patch with one winding number on each flavor index. The $\mathcal{N} = 2$ $U(1) \times SO(2M)$ supersymmetric gauge theories are also found to have both vortex and monopole solutions [48], the moduli space of vortices have $2^{M-1} \otimes \bar{2}^{M-1}$ patches, which gives us the hints about the GNOW dual property of the vortices. The correspondence between non-Abelian monopoles and vortices are established through homotopy maps and flux matching, as has already been noted. Non-Abelian vortex solutions with arbitrary gauge groups (including $U(N), U(1) \times SO(N)$ and $U(1) \times USp(2M)$) have been constructed [49]. A detailed study of vortices with gauge groups $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ has been given in [47, 50], the structure of the moduli spaces turns out to be much richer than the $U(N)$ case. The moduli space metric has been found recently for well-separated vortices [51, 52] using a generic formula for the Kähler potential on the moduli space [53].

When the number of flavors $N_f$ is bigger than the number of colors $N_C$, there are semi-local vortices with non-renormalizable size moduli. The stability of semi-local non-Abelian vortices has been discussed. The local vortices (i.e., ANO type vortices) have been found to be stable under a wide class of non-BPS perturbations [9]. The interactions between non-BPS non-Abelian vortices have been studied, which were found to non-trivially depend on the internal orientations [8], giving rise to various types of (e.g., $I^*$ and type $\Pi^*$) superconductors.

Another direction involved the Chern-Simons non-Abelian vortices [54]. In some of Abelian Chern-Simons vortex, the magnetic fields near the vortices core were found to invert the direction as compared to the rest of the vortex, according to the fine-tuned coupling constants.

Also, there occur vortices with nontrivial substructures in transverse plane, in diverse Abelian or non-Abelian Higgs models. One of the causes for such “fractional vortices” [55]
is the presence of orbifold singularities in the vacuum moduli. Another possible cause is deformed geometry of the sigma-model lump (map from the transverse plane $S^2$ to the 2-cycle in the vacuum moduli space). Both of these types of fractional vortices reflect the presence of nontrivial vacuum moduli, over which vortices with their own moduli space are constructed. Such a double structure of moduli spaces characterizes many of the non-Abelian vortices we are going to study in this thesis.

1.4 Organization of the work

The present thesis is organized as follows. In Chapter 2 we review and discuss the properties of the non-Abelian vortices in more detail, in a wide class of supersymmetric gauge theories. The construction of the BPS vortex solutions in generic gauge groups of the form $G' \times U(1)$ is given and their transformation properties are discussed carefully. The cases of $SU(N)$, $SO(N)$, and $USp(2M)$ models are discussed in detail, and the emergence of GNOW duality is illustrated.

Chapter 3, Chapter 4 and Chapter 5 constitute the central part of this thesis, in which our main original contributions are expounded. In Chapter 3, we focus our attention to the effective world-sheet action of the non-Abelian orientational zero-modes of our vortex solutions. In the background of non-Abelian vortices, the Ansatz for the gauge fields components in the vortex transverse plane are determined by the finite energy condition. However, the time and string direction components of the gauge fields are not known. We put forward an appropriate Ansatz for gauge field in time and string direction. With all these Ansatz, we integrate over the two spacial dimensions (the string transverse plane), and reduce the four-dimensional action to a two-dimensional (time and string direction) sigma model. We found the effective world-sheet action to be supersymmetric sigma models in appropriate target spaces, specific to the gauge group considered. We find some remarkable universal feature in this reduction which applies to all cases discussed, in $SU$, $SO$, and $USp$ gauge theories. Essentially the same method can be used to analyze the world-sheet action of some higher winding vortices. This part is based on the work [56].

In Chapter 4, we generalize the discussion of the previous Chapter to the unequal mass cases. An appropriate Ansatz for the adjoint scalars will be given. We integrate two terms related to adjoint fields in the bulk four-dimensional theory, and find them to be a
massive term in two-dimensions. This massive term is the potential term for the sigma model. The massive sigma models have kink solutions, which are the confined monopoles on the vortex string. Three concrete examples are discussed, the $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models, the $CP^{N-1}$ sigma models, and the quadratic surface $Q^{2M-2}$ sigma models. Some high winding cases in the last chapter can also be applied, for instance, the completely anti-symmetric $k$ winding vortices. We also check the result from Scherk-Schwarz dimensional reduction [57, 58], which calculates the mass term from kinetic term directly. We found that the our integration and SS dimension reduction result in the same mass potential term. This part of thesis is based on an unpublished paper [59].

Chapter 5 is devoted to the group-theoretic study of the $U(N)$ non-Abelian vortices with $N$ fundamental Higgs fields. The moduli space of single vortex can be parameterized by the homogeneous coordinates of $CP^{N-1}$, which belongs to the fundamental representation of the global $SU(N)$ symmetry. The moduli space of winding-number $k$ vortices is decomposed into the sum of irreducible representations each of which is associated with a Young tableau made of $k$ boxes, in a way somewhat similar to the standard group composition rule of $SU(N)$ multiplets. The Kähler potential is determined in each moduli subspace, corresponding to an irreducible $SU(N)$ orbit of the highest-weight configuration. This part of work is based on the work [60]. The conclusion of this thesis is given in Chapter 6.
Chapter 2

Non-Abelian vortex solutions in supersymmetric gauge theories

In this chapter, we start with the bosonic truncation of the Lagrangian of the $\mathcal{N} = 2$ supersymmetric field theory, the fermion part can easily be taken into account by supersymmetry [61]. The symmetry breaking of the system is presented, the Bogomol’nyi completions show that monopole is confined in the Higgs phase. Deforming the massive degrees of freedom, we construct the non-Abelian vortex solutions with generic gauge group. The master equations, the quantization condition, and the orientational zero-modes are discussed in details, and the vortex transformation properties under the action of color-flavor exact symmetry group are studied. The moduli space of vortices are shown to be isomorphic to the GNOW dual groups, as exemplified by the concrete $SO$ and $USp$ cases. The construction of $\mathcal{N} = 2$ supersymmetric gauge field theories is briefly reviewed in Appendix A.
2.1 The bulk theory

The action of the bosonic truncation of the $\mathcal{N} = 2$ supersymmetric gauge theory in 4-dimension can be written as

$$S = \int d^4x \left\{ \text{Tr} \left[ -\frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{i\theta}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{2}{e^2} (\partial_{\mu}\phi)^\dagger \partial^\mu \phi \right. \right.$$ 

$$+ \frac{2}{g^2} (D_{\mu}\hat{\phi})^\dagger D^\mu \hat{\phi} + D_{\mu}q(D^\mu q)^\dagger + (D_{\mu}\tilde{q})^\dagger D^\mu \tilde{q} \left. \right] - V_D - V_F \right\},$$

(2.1.1)

in which the $D$ and $F$ potential terms read

$$V_D = \frac{g^2}{2} \text{Tr} \left( -\frac{2}{g^2} t^a [\hat{\phi}^\dagger, \hat{\phi}] + qq^\dagger t^a - \hat{q}^\dagger \tilde{q}^a \right)^2 + \frac{e^2}{2} \text{Tr} \left( qq^\dagger t^0 - \hat{q}^\dagger \tilde{q}^0 \right)^2,$$

(2.1.2)

$$V_F = 2e^2 \text{Tr} \left( (qq^0) + \sqrt{2\mu_0} \phi^0 \right)^2 + 2g^2 \sum_{a=1}^{\dim(G)-1} \left| \text{Tr}(qq^a) + \sqrt{2\hat{\mu}} \phi^a \right|^2$$

$$+ 2 \sum_{i=1}^{N_f} \left| (\phi + \phi + \frac{1}{\sqrt{2}} m_i \tilde{q}_i) \right|^2 + 2 \sum_{i=1}^{N_f} \left| (\phi + \phi + \frac{1}{\sqrt{2}} m_i) q_i \right|^2,$$

(2.1.3)

where $m_i$ is the bare mass of squark $q_i$. The parameters $\mu_0$ and $\hat{\mu}$, which softly breaks the supersymmetry to $\mathcal{N} = 1$, are the masses of the adjoint scalar field $\phi^0$ and $\hat{\phi}$, respectively. When $\phi^0$ and $\hat{\phi}$ take their VEVs, the related terms are named as Fayet-Iliopoulos(FI) F-terms, which trigger the quark condensation. The gauge group of the system is a type of

$$G = U(1) \times G',$$

(2.1.4)

in which $G'$ denotes the generic non-Abelian group, including $SU$, $SO$ and $USp$. The Abelian and the non-Abelian gauge fields and gauge field strengths are denoted as

$$A_\mu = A_\mu^0 t^0 + A_\mu^a t^a,$$

$$F_{\mu\nu} \equiv F_{\mu\nu}^0 t^0, \quad F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a,$$

$$\tilde{F}_{\mu\nu} \equiv \tilde{F}_{\mu\nu}^a t^a, \quad \tilde{F}_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a - f^{abc} A_{\mu}^b A_{\nu}^c.$$  

(2.1.5)

We also explicitly distinguish the Abelian gauge coupling constant $e$ and the non-Abelian gauge coupling constant $g$. The generators of the generic gauge group are normalized as

$$\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}, \quad \text{Tr}(t^0)^2 = \frac{1}{2}.$$  

(2.1.6)

---

1. The metric convention is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
2. The comparison between D-term and F-term is presented in appendix A3.
2.1 The bulk theory

The right forms of these generators depend on the gauge group $G$. The dual gauge field strength $\tilde{F}^{\mu \nu}$ is defined as

$$\tilde{F}^{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \tilde{F}_{\rho \sigma}.$$  \hspace{1cm} (2.1.7)

The adjoint scalar fields $\phi$ and $\hat{\phi}$ belong to the adjoint representation, their covariant derivative has the following forms

$$\phi \equiv \phi^\alpha t^\alpha, \quad \hat{\phi} \equiv \hat{\phi}^a t^a,$$

$$D_\mu \phi = \partial_\mu \phi, \quad D_\mu \hat{\phi} = \partial_\mu \hat{\phi} + i [A_\mu, \hat{\phi}].$$ \hspace{1cm} (2.1.8)

Finally, we have the squark fields $q$ and its partner $\tilde{q}$, which are in fundamental and anti-fundamental representation respectively. The covariant derivative is defined as

$$D_\mu q = \partial_\mu q + i A_\mu q.$$ \hspace{1cm} (2.1.9)

In order to construct the approximate monopole and vortex solutions, we will only consider the VEVs of adjoint scalars and squarks. In the vacuum, the $q$ and $\tilde{q}$ satisfy that

$$q \equiv \tilde{q}^\dagger,$$ \hspace{1cm} (2.1.10)

which are color-flavor matrices. $N_c$ and $N_f$ are set to be equal, so semi-local defect is not discussed in this thesis. The non-Abelian scalar fields $\hat{\phi}$ and $\hat{\phi}^\dagger$ are commute,

$$[\hat{\phi}^\dagger, \hat{\phi}] = 0.$$ \hspace{1cm} (2.1.11)

Hence the D-term potential $V_D$ is set to be zero throughout.

The mass parameters is fine-tuned in the following manner. Let us first assume that all the masses of squark take the same value. However, the formula of mass matrix depends on the request of constructing color-flavor locking phase. For the $SU(N)$ case, we choose the masses such that

$$M = \text{diag}(m_1, m_2, \cdots, m_N) = m \frac{1}{\sqrt{N}} 1_N.$$ \hspace{1cm} (2.1.12)

So there is a global $SU(N)$ symmetry for the squark fields. For the $SO(2M)$ and $USp(2M)$ case, we write the mass matrix as

$$M = \text{diag}(m_1, m_2, \cdots, m_M, -m_1, -m_2, \cdots, -m_M).$$ \hspace{1cm} (2.1.13)
where \( m_1 = m_2 = \cdots = m_M \). It is convenient to keep the mass matrix form for the purpose that the theory applies to most general \( r \) vacua case \([62]\).

The VEVs of the adjoint scalar fields \( \phi^0 \) and \( \tilde{\phi}^a \) are written as

\[
\langle \phi \rangle = -\frac{1}{\sqrt{2}} M, \quad \langle \tilde{\phi} \rangle = 0.
\]

When the adjoint scalar fields are fixed to their VEVs, the F-term potential \( V_F \) can be written as

\[
V_F = 2g^2 |\text{Tr}(qq^\dagger t^a)|^2 + 2e^2 |\text{Tr}(qq^\dagger t^0) - \xi|^2,
\]

in which \( \xi \equiv \sqrt{2}\mu_0 m \). Here we are limited to the \( SU(N) \) case, other cases can be studied in a similar way. This vacuum of the leading order potential determines the VEV of the squark field

\[
\langle q \rangle = \langle \tilde{q} \rangle \equiv \frac{\upsilon}{\sqrt{N}} 1_N.
\]

The \( \upsilon \) is introduced here satisfying \( \upsilon^2 = \xi \sqrt{2N} \), we will use it for calculating profile functions of the BPS equations.

Now we define that

\[
v_1 \equiv -m, \quad v_2 \equiv \sqrt{2m\mu_0},
\]

the hierarchical symmetry breaking in Eq.(1.2.14) is realized if

\[
|m| \gg |\mu_0| \gg \Lambda, \quad \therefore |v_1| \gg |v_2|.
\]

The relative value of \( v_1 \) and \( v_2 \) determine the sizes of the monopole and vortex solutions, and the size of monopoles is much smaller than the transverse size of vortices. In the particle spectrum, the adjoint scalar fields are heavier than the gauge bosons, they are decoupled in the Higgs phase.

The \( \theta \) term in the bulk theory can be ignored, because we discuss only the static solutions and choose the gauge fixing condition \( A_0 = 0 \). In the following, we will rescale \( q, \tilde{q} \) and \( \xi \) for the convenience of calculation\(^3\), i.e.

\[
q \rightarrow \frac{1}{\sqrt{2}} q, \quad \tilde{q} \rightarrow \frac{1}{\sqrt{2}} \tilde{q}, \quad \xi \rightarrow \frac{1}{2} \xi.
\]

\(^3\)In theory with FI D-term, the Anzatz \( \tilde{q} = 0 \) leads to the same vortices theory but different physics.
2.1 The bulk theory

Taking $\partial_0 = 0$, the time independent Hamiltonian reads

$$\mathcal{H} = \text{Tr} \left[ \frac{1}{\epsilon^2} (B_n)^2 + \frac{1}{g^2} (\dot{B}_n)^2 + \frac{2}{\epsilon^2} |\partial_n \phi|^2 + \frac{2}{g^2} |D_n \dot{\phi}|^2 + |D_n q|^2 + e^2 |X^0 t^0 - \xi t^0|^2 + g^2 |X^a t^a|^2 + |\sqrt{2} (\phi + \dot{\phi}) q + q M|^2 \right],$$

(2.1.20)

in which

$$X^0 \equiv \text{Tr}(qq^\dagger t^0), \quad X^a \equiv \text{Tr}(qq^\dagger t^a).$$

(2.1.21)

The magnetic field strengths are defined as

$$B_n = \frac{1}{2} \varepsilon^{nij} F_{ij}, \quad \dot{B}_n = \frac{1}{2} \varepsilon^{nij} \dot{F}_{ij}.$$  

(2.1.22)

The Bogomol’nyi completion for this system appear to be

$$\mathcal{H} = \text{Tr} \left[ \frac{1}{\epsilon^2} |B_i - \sqrt{2} \partial_i \phi|^2 + \frac{1}{g^2} |\dot{B}_i - \sqrt{2} \partial_i \dot{\phi}|^2 + |D_i q + i D_2 q|^2 + \frac{1}{\epsilon^2} |B_3 - \sqrt{2} \partial_3 \phi - e^2 (X^0 t^0 - \xi t^0)|^2 + \frac{1}{g^2} |\dot{B}_3 - \sqrt{2} \partial_3 \dot{\phi} - g^2 (X^a t^a)|^2 + |D_3 q + \sqrt{2} (\phi + \dot{\phi}) q + q M|^2 - \partial_3 \{ \sqrt{2} q^\dagger (\phi + \dot{\phi}) q + q^\dagger q M \} - i \varepsilon^{ij} \partial_i (q^\dagger D_j q) + 2 \sqrt{2} \partial_3 \phi \xi t^0 - 2 F_{12} \xi t^0 + \frac{2 \sqrt{2}}{\epsilon^2} \partial_n (B_n \phi) + \frac{2 \sqrt{2}}{g^2} \partial_n (\dot{B}_n \dot{\phi}) \right].$$

(2.1.23)

where the $n$ denotes spatial indices $n = 1, 2, 3$, and $i, j = 1, 2$. We choose $x^3$ as the direction of the vortex. The bound energy density is written as

$$E = \sqrt{2} \partial_3 \phi^0 \xi - F_{12} \xi + \frac{\sqrt{2}}{\epsilon^2} \partial_n (B_n^0 \phi^0) + \frac{\sqrt{2}}{g^2} \partial_n (\dot{B}_n^0 \dot{\phi}^0) - \partial_3 \left[ \sqrt{2} q^\dagger (\phi + \dot{\phi}) q + q^\dagger q M \right] - i \varepsilon^{ij} \partial_i [q^\dagger D_j q].$$

(2.1.24)

The exact soliton solutions of kinks, vortices, monopoles all are contained in Eq. (2.1.23). The mass of kinks, the flux of vortices, and the mass of monopoles are given by $\sqrt{2} \partial_3 \phi^0 \xi$, $-F_{12} \xi$, $\frac{\sqrt{2}}{\epsilon^2} \partial_n (B_n^0 \phi^0)$, and $\frac{\sqrt{2}}{g^2} \partial_n (\dot{B}_n^0 \dot{\phi}^0)$ respectively. The first order Bogomol’nyi equations are

$$B_i = \sqrt{2} \partial_i \phi, \quad B_3 = \sqrt{2} \partial_3 \phi + e^2 (X^0 t^0 - \xi t^0),$$

$$\dot{B}_i = \sqrt{2} \partial_i \dot{\phi}, \quad \dot{B}_3 = \sqrt{2} \partial_3 \dot{\phi} + g^2 (X^a t^a),$$

$$D_i q = 0, \quad D_3 q = - \sqrt{2} (\phi + \dot{\phi}) q - q M.$$  

(2.1.25)

It is difficult to give the analytic solutions, but the system can be studied when the deformation is considered, see ref. [40].
Non-Abelian vortex solutions in supersymmetric gauge theories

2.2 Generic Non-Abelian vortices

Due to the hierarchical symmetry breaking type, the adjoint scalar fields become massive in low energy scale. After integrating out these massive degrees of freedom, we have the light fields in the infrared limit. The Lagrangian density of the system becomes

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + D_\mu q (D^\mu q)^\dagger - e^2 |X^0 t^0 - \xi t^0|^2 - g^2 |X^a t^a|^2 \right]. \quad (2.2.1)$$

Here the Lagrangian is limited to the 2 + 1 dimension. Now only the vortex solutions remain. This model is the most general non-Abelian BPS vortices with an arbitrary gauge group [49]. Performing the Bogomol'nyi completion, the tension which is independent of time and $x^3$-direction reads

$$T = \text{Tr} \int d^2 x \left[ \frac{1}{e^2} |F_{12} - e^2 (X^0 t^0 - \xi t^0)|^2 + \frac{1}{g^2} |\tilde{F}_{12} - g^2 X^a t^a|^2 \right] + 4 |\tilde{D} q|^2 - 2 \xi F_{12} t^0 \geq -\xi \int d^2 x F_{12}^0, \quad (2.2.2)$$

where the complex coordinates $z = x^1 + i x^2$ parameterize the transverse plane of the vortices, and $\tilde{D} \equiv \frac{1}{2} (D_1 + i D_2)$. Other useful objects are

$$\begin{align*}
\partial &= \frac{1}{2} (\partial_1 - i \partial_2), \\
\bar{\partial} &= \frac{1}{2} (\partial_1 + i \partial_2), \\
\bar{A} &= \frac{1}{2} (A_1 + i A_2), \\
A &= \frac{1}{2} (A_1 - i A_2). \quad (2.2.3)
\end{align*}$$

The BPS equations of the system become

$$\begin{align*}
\tilde{D} q &= 0, \quad (2.2.4) \\
F_{12}^0 &= e^2 [\text{Tr} (qq^\dagger t^0) - \xi], \quad (2.2.5) \\
\tilde{F}_{12}^a &= g^2 \text{Tr} (qq^\dagger t^a). \quad (2.2.6)
\end{align*}$$

Eqs. (2.2.4)-(2.2.6) hold for classical Lie groups $G'$. When the profile functions are to be calculated, the matrix form of the BPS equations is useful. For $SU(N)$ group, we have

$$F_{12} = \frac{g^2}{2} \left[ qq^\dagger - \frac{1}{N} \text{Tr} (qq^\dagger) \right], \quad (2.2.7)$$

For $SO$ and $USp$, we have

$$\tilde{F}_{12} = \frac{g^2}{4} \left[ qq^\dagger - J^T (qq^\dagger)^T J \right], \quad (2.2.8)$$
2.2 Generic Non-Abelian vortices

in which $J$ is the rank-2 invariant tensor

$$J = \begin{pmatrix} 0_M & 1_M \\ \epsilon 1_M & 0_M \end{pmatrix}, \quad J = \begin{pmatrix} 0_M & 1_M & 0 \\ 1_M & 0_M & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (2.2.9)$$

where $\epsilon = 1$ for $SO(2M)$; while $\epsilon = -1$ for $USp(2M)$. The second $J$ in Eq. (2.2.9) applies for $SO(2M + 1)$.

Introducing an $N$ by $N$ matrix $S(z, \bar{z})$, it takes a value in the complexification $G^C$ of $G$,

$$S(z, \bar{z}) = S_e(z, \bar{z})S'(z, \bar{z}),$$  \hspace{1cm} (2.2.10)$$

with $S_e \in U(1)^C$ and $S' \in G'^C$. The Ansatz for the gauge field is denoted as the Maurer-Cartan form:

$$\bar{A} = -iS^{-1}(z, \bar{z})\bar{\partial}S(z, \bar{z}).$$  \hspace{1cm} (2.2.11)$$

The Eq. (2.2.4) can be solved by

$$q = S^{-1}(z, \bar{z})q_0(z),$$  \hspace{1cm} (2.2.12)$$

where $q_0$ is holomorphic in $z$, the so-called moduli matrix $[46]$. The $q_0(z)$ contains all the information of the vortex solutions: the positional and internal orientational parameters.

It is convenient to construct a gauge invariant quantity $\Omega = SS^\dagger$, where the $U(1)$ part is $\Omega_e = S_eS^\dagger_e$, and the $G'$ part is $\Omega' = S'S'^\dagger$. The gauge field strength can be expressed by

$$F_{12}^{\alpha\beta} = 2\bar{\partial}(\Omega_e \partial\Omega_e^{-1}),$$  \hspace{1cm} (2.2.13)$$

$$S'F_{12}^{a\alpha}S'^{-1} = 2\bar{\partial}(\Omega' \partial\Omega'^{-1}).$$  \hspace{1cm} (2.2.14)$$

The Eq. (2.2.5) and (2.2.6) can be cast in the form

$$\partial\bar{\partial} \log \Omega_e = -\frac{e^2}{4N}1_N \left[\Omega_e^{-1}\text{Tr}(\Omega_0\Omega'^{-1}) - v^2\right],$$  \hspace{1cm} (2.2.15)$$

$$\bar{\partial}(\Omega' \partial\Omega'^{-1}) = \frac{g^2}{2}\Omega_e^{-1}\text{Tr}(\Omega_0\Omega'^{-1}t^a)t^a,$$  \hspace{1cm} (2.2.16)$$

in which $\Omega_0 \equiv q_0q_0^\dagger$, we call these equations: master equations. For $SU(N)$, Eq. (2.2.16) is expressed as

$$\bar{\partial}(\Omega' \partial\Omega'^{-1}) = \frac{g^2}{4}\Omega_e^{-1}\left[\Omega_0\Omega'^{-1} - \frac{1}{N}\text{Tr}(\Omega_0\Omega'^{-1})\right].$$  \hspace{1cm} (2.2.17)$$

For $SO$ and $USp$, Eq. (2.2.16) is expressed as

$$\bar{\partial}(\Omega' \partial\Omega'^{-1}) = \frac{g^2}{8}\Omega_e^{-1}\left[\Omega_0\Omega'^{-1} - J^T(\Omega_0\Omega'^{-1})^TJ\right].$$  \hspace{1cm} (2.2.18)$$
2.2.1 Quantization condition

The boundary condition for the color-flavor matrix \( q \) reads

\[
q|_{z \to \infty} = S'^{-1}S^{-1}(q_0),
\]

(2.2.19)

with \( S_e \in U(1) \) and \( S' \in G' \). The vortex configuration has axial symmetry, which leads that \( S_e \) and \( S' \) depend on polar angle \( \theta \). The matter fields \( q \) should be single-valued on the boundary, so \( S_e \) and \( S' \) have the following periodicity

\[
S_e(\theta + 2\pi) = e^{i2\pi\nu}S_e(\theta), \quad S'(\theta + 2\pi) = e^{-i2\pi\nu}1_N S'(\theta).
\]

(2.2.20)

From the fact that \( S'(\theta + 2\pi) \) and \( S'(\theta) \) are the elements of \( G' \), the \( e^{i2\pi\nu}1_N \) belongs to the center of \( G' \)

\[
e^{i2\pi\nu}1_N = e^{i2\pi k/n_0}1_N, \quad k \in \mathbb{Z}_+, \]

where \( n_0 \) is the greatest common divisor, which depends on \( G' \). This last fact shows that a minimal vortex solution can be constructed [48] by letting the scalar field wind (far from the vortex axis) by an overall \( U(1) \) phase rotation with angle \( \frac{2\pi}{n_0} \), and completing (or canceling) it by a winding \( \pm \frac{2\pi}{n_0} \) in each and all of \( U(1) \) center in \( G' \). The value of \( n_0 \) for the classical non-Abelian vortexes is the combination of \( U(1) \) and simple Lie group \( G' \). Without the \( U(1) \) part, the homotopy group is trivial, and the system can have \( \mathbb{Z}_k \) vortex strings, which are not our aim. Without \( G' \), only Abelian vortexes are presented. An overlap between \( U(1) \) and \( G' \) occurs in order to satisfy the boundary condition, which we count two times in the group manifold. The true group manifold should be

\[
G = \frac{U(1) \times G'}{\mathbb{Z}_{n_0}},
\]

(2.2.21)

in which \( \mathbb{Z}_{n_0} \) is the center of the group \( G' \). The topological characterization of the vortex configuration is given by the map: \( S^1 \to G \), which is pictured in Figure 2.1
The homotopy group reads

\[ \pi_1 \left( \frac{U(1) \times G'}{\mathbb{Z}_{n_0}} \right) \cong \mathbb{Z}, \]  

which corresponds to the winding number. Note that the aforementioned representation of \( G' \) is the fundamental representation, this representation can be generalized to representations with higher dimensions, which will vary the center of \( G' \).

Because any generator of \( G' \) is conjugate to at least one generator of the Cartan sub-algebra, the boundary condition \((2.2.19)\) for \( q \) can be rewritten as

\[ q\big|_{z \to \infty} = g \cdot \exp \left[ i \left( \frac{k}{n_0} \mathbf{1}_N + \vec{\nu} \cdot \mathbf{H} \right) \theta \right] \cdot g^{-1} \cdot \langle q_0 \rangle, \quad g \in G^{\text{HC}} \tag{2.2.23} \]

in which \( \mathbf{H} \) denotes the Cartan sub-algebra. The single-valued condition results in the quantization condition for the coefficients \( \vec{\nu} \), i.e.,

\[ \frac{k}{n_0} + \vec{\nu} \cdot \vec{\mu}^{(a)} \in \mathbb{Z}_{\geq 0}, \tag{2.2.24} \]

in which \( a \) denotes the representation dimension of \( G' \). The \( \vec{\nu} \) marks the “patch”, which means a specific solution of \( q_0(z) \). Subtracting pairs of adjacent weight vectors, one arrives at the quantization condition which we have already obtained in the introduction

\[ \vec{\nu} \cdot \vec{\alpha} \in \mathbb{Z}, \quad \vec{\alpha} \in \Lambda_{\text{root}}. \tag{2.2.25} \]

The general \( \vec{\nu} \) which satisfies the last quantization condition Eq. \((2.2.25)\) should be an element of the co-weight lattice,

\[ \vec{\nu} \in \Lambda_{\text{cowt}}. \tag{2.2.26} \]
Eq. (2.2.26) is formally identical to the well-known GNOW quantization condition for the monopoles [16, 17]. Here the co-weight vector $\vec{\nu}$ is called “the vortex weight vector”. If two co-weight vectors $\vec{\nu}, \vec{\nu}' \in \Lambda_{\text{cowt}}$ satisfy Eq. (2.2.24), and the difference between them satisfy

$$\vec{\nu} - \vec{\nu}' \in \Lambda_{\text{cort}},$$  

(2.2.27)

namely $\vec{\nu}$ and $\vec{\nu}'$ belong to same equivalence class $[\vec{\nu}] = [\vec{\nu}']$. Two configurations (corresponding to vortex weight vectors $\vec{\nu}$ and $\vec{\nu}'$) with the same winding number $k$ can be continuously deformed into each other. Given the vortex weight vector $\vec{\nu}$, we still have invariant subgroup depending on $\vec{\nu}$, which will be interpreted in next section.

A redundancy exists for the solution (2.2.12). The master equations are invariant under the following transformations

$$(q_0, S) \rightarrow (V_e(z)V'(z)q_0, V_e(z)V'(z)S), \quad V_e(z) \in \mathbb{C}^*, \; V'(z) \in G'^{\mathbb{C}},$$

(2.2.28)

with $V(z)$ being holomorphic in $z$. For $SU(N)$ case, $V(z) \in GL(N)$. We call this the $V$-equivalence relation. Two configurations connected by the $V$ transformation is the same in physics, with the difference similar to gauge fixing conditions.

For $SU(N)$ case, the set of all the solutions $q_0(z)$ to the master equations is denoted as

$$\tilde{\mathcal{M}} = \{ q_0(z) \mid \det q_0(z) = \mathcal{O}(z^k) \},$$

(2.2.29)

and for $SO$ and $USp$, we have

$$\tilde{\mathcal{M}} = \{ q_0(z) \mid q_0^T(z) J q_0(z) = z^{\frac{2k}{n_0}} J + \mathcal{O}(z^{\frac{2k}{n_0}-1}) \}.$$  

(2.2.30)

The moduli space of the non-Abelian vortices is defined to be

$$\mathcal{M} = \frac{\tilde{\mathcal{M}}}{\{ V(z) \}}.$$  

(2.2.31)

An index theorem gives the dimension of the moduli space with given winding number $k$ [63, 64, 50], which is

$$\dim_{\mathbb{C}}(\mathcal{M}_{G',k}) = \frac{k}{n_0} N^2.$$  

(2.2.32)

where $N_c = N_F = N$ denote the color and flavor of the system.
2.2 Generic Non-Abelian vortices

Based on the consideration of the strong coupling limit and the index theorem, we assume that there exist unique solutions to master equations [49]. If BPS conditions are saturated, the tension becomes simply as

\[ T = 2\pi v^2\nu, \quad \nu = \frac{1}{\pi} \int d^2x \partial \partial \log \Omega. \]  

(2.2.33)

where \( \nu = \frac{k}{n_0} \) is the \( U(1) \) winding number of the vortex.

2.2.2 Orientational zero-modes

First, let us review the symmetry of the system. In the Coulomb phase, our model owns the gauge group \( G \) and flavor symmetry \( G_F \). The squark field \( q \) transforms as

\[ q \rightarrow VqU, \quad V \in G, \quad U \in G_F. \]  

(2.2.34)

After Spontaneous symmetry breaking, the vacuum Eq.(2.1.16) is invariant if the transformation has the form

\[ \tilde{U} \langle q \rangle \tilde{U}^{-1} = \langle q \rangle, \quad \tilde{U} \in G_{C+F}. \]

The preserved symmetry is a global color-flavor group, i.e. \( G \times G_F \rightarrow G_{C+F} \). For instance of \( SU(N) \), we have the symmetry breaking as in Eq. (1.3.2), which is known as the color-flavor locked phase in the high-density QCD literature [65]. However, the existence of non-Abelian vortices will break the color-flavor symmetry. Consider for instance a particular BPS solution of \( SU(N) \) vortices

\[ q = \text{diag} (q^{\text{ANO}}, v, \cdots, v), \quad A_\mu = \text{diag} (A_\mu^{\text{ANO}}, 0, \cdots, 0), \]  

(2.2.35)

where \( q^{\text{ANO}} \) and \( A_\mu^{\text{ANO}} \) are the fields with winding factors, which are similar to the well-known ANO vortex solution. Clearly, the solution breaks \( SU(N)_{C+F} \) down to \( SU(N-1) \times U(1) \) and therefore the corresponding Nambu-Goldstone zero-modes, which we call internal orientational modes, appear on the vortex and parameterized the coset \( \frac{SU(N)}{SU(N-1) \times U(1)} \cong \mathbb{C}P^{N-1} \), whose size (Kähler class) is given by \( 4\pi/g^2 \) [41, 66, 67, 68].

Other vortex solutions can be obtained, with the winding factors \( q^{\text{ANO}} \) on arbitrary diagonal entries of \( q \), and also \( A_\mu^{\text{ANO}} \) is embedded on the corresponding diagonal entry of \( A_\mu \). One kind of configuration is named as “one patch”, which is characterized by one weight vector in Eq. (2.2.25). In total there are \( N \) patches for the single \( U(N) \) vortices.
Now we interpret the orientational moduli in more general cases. For axially symmetric vortices with winding number \( k \), the moduli matrix \( q \) can be written as
\[
q(z) = (q_0) g \, z^{k/n_0} 1_N + \vec{\nu} \cdot \mathbf{H} \, g^{-1}.
\] (2.2.36)

For a fixed vortex weight vector \( \vec{\nu} \in \Lambda_{\text{cont}} \), the orientational zero modes is parameterized by the group element \( g \in G' \). However, two different group elements \( g, g' \) give the same moduli parameters if they are related as
\[
g' = g \, h, \quad h \in \mathbb{L}_\vec{\nu},
\] (2.2.37)

where \( \mathbb{L}_\vec{\nu} \) is the Levi subgroup defined by
\[
\mathbb{L}_\vec{\nu} = \{ h \in G' \mid \vec{\nu} \cdot h \, \mathbf{H} \, h^{-1} = \vec{\nu} \cdot \mathbf{H} \}.
\] (2.2.38)

So the orientational moduli space for the BPS vortex with weight \( \vec{\nu} \) is written as
\[
\mathcal{M}_{\text{orientation}}^{\vec{\nu}} = \frac{G'}{\mathbb{L}_\vec{\nu}}.
\] (2.2.39)

The generator of \( \mathbb{L}_\vec{\nu} \) for weight vector \( \vec{\nu} \) contains
\[
\mathbf{H}, \quad \frac{1}{2} (E_\alpha + E_{-\alpha}), \quad \frac{1}{2i} (E_\alpha - E_{-\alpha}),
\] (2.2.40)
in which \( \alpha \) satisfies \( \vec{\nu} \cdot \alpha = 0 \). One most convenient way to mark the Levi subgroup is using the painted Dynkin diagram. Also there are Weyl subgroup which permutes the entries of weight vector \( \vec{\nu} \), and so change the “patches”. However, we can always find a highest weight vector (whose entries are ordered) which corresponds one “irreducible orbit”, other weight vectors obtained by Weyl reflection belong to the same orbit. The moduli space of non-Abelian vortices are composed of distinguished orbits, which is important to construct the representation of such space and to manifest the GNOW duality.

In total, the moduli matrix \( q \) can be decomposed as in Eq. (2.2.36), the diagonal matrix \( z^{k/n_0} 1_N + \vec{\nu} \cdot \mathbf{H} \) part and the moduli zero modes part. The question is how to write the moduli zero modes part corresponding to the irreducible orbit. Principally, we should exclude the Levi subgroup part, which can be done by using \( V \)-transformation. In the following, some concrete cases are discussed.

In the beginning we count the number of patches of the single vortices. There are \( N \) patches in the \( SU(N) \) case, and the orientational zero-modes are the \( \mathbb{C}P^{N-1} \) sigma model. For \( SO(2M) \) and \( USp(2M) \) cases, the squark field \( q \) has the form
\[
q_0(z)^{\vec{\mu}} = \text{diag}(z^{k_1}, \ldots, z^{k_M}, z^{-k_1}, \ldots, z^{-k_M}),
\] (2.2.41)
while for $SO(2M + 1)$,
\[
q_0(z) = \text{diag}(z^{k_1^+}, \ldots, z^{k_M^+}, z^{k_1^-}, \ldots, z^{k_M^-}, z^k),
\]
(2.2.42)
where $k_a^\pm = \nu \pm \bar{\mu}^a$, defined in (2.2.24). By advantage of the co-weight vectors, the total number of patches can be counted. The rank of $SO(2M)$ and $USp(2M)$ is $M$, so the weight $\bar{\mu}$ is a vector with $M$ entries. For single ($k = 1$) vortices, every entries is $\frac{1}{2}$ or $-\frac{1}{2}$, no other constraints exist. So the number of patches are $2^M$ for single $SO(2M)$ and $USp(2M)$ vortices.

Further on, the $SO(2M + 1)$ vortices has $3^M$ patches. Since the rank of $SO(2M + 1)$ is $M$, but the entries of $\bar{\mu}$ have three values to choose, i.e., 1, 0, and $-1$. This also agrees with the case of $k = 2$ $SO(2M)$ vortices, we will give all the irreducible orbits for $SO(2M + 1)$ vortices in next section.

One special patch of $SO(2M)$ and $USp(2M)$ vortices is written as
\[
q_0(z) = \begin{pmatrix} z 1_M & 0 \\ 0 & 1_M \end{pmatrix},
\]
(2.2.43)
In this patch, the vortices of generic orientation (in the local coordinate patch) was constructed in Ref. [49] and is simply expressed by
\[
q_0(z) = \begin{pmatrix} z 1_M & 0 \\ 0 & 1_M \end{pmatrix} U \sim \begin{pmatrix} z 1_M & 0 \\ B & 1_M \end{pmatrix},
\]
(2.2.44)
where $\sim$ denotes that we have used an appropriate $V$-transformation. The matrix $U \in G'$ is the color-flavor rotation [49],
\[
U = \begin{pmatrix} 1_M & -B^1 \\ 0 & 1_M \end{pmatrix} \begin{pmatrix} \sqrt{1_M + B^1 B} & 0 \\ 0 & \sqrt{1_M + B B^1} \end{pmatrix} \begin{pmatrix} 1_M & 0 \\ B & 1_M \end{pmatrix},
\]
(2.2.45)
where $B$ is an $M$-by-$M$ symmetric (antisymmetric) matrix for $SO(2M)$ ($USp(2M)$) case.

### 2.3 The vortex transformations: GNOW duality

Based on the quantization condition Eq. (2.2.25) and Moduli space analysis, a conjecture for the moduli space of vortices is put forward, that is \textit{The moduli space of non-Abelian
Non-Abelian vortex solutions in supersymmetric gauge theories

\[ G' \quad | \quad \tilde{G}' \]

| \( SU(N) \) | \( SU(N)/\mathbb{Z}_N \) |
| \( SO(2M) \) | \( SO(2M) \) |
| \( USp(2M) \) | \( SO(2M + 1) \) |
| \( SO(2M + 1) \) | \( USp(2M) \) |

Table 2.2: Pairs of dual groups

\( G' \) vortices is isomorphic to the representation of dual group \( \tilde{G}' \). That can be checked explicitly.

As seen in Table 2.2, the dual group of \( SU(N) \) is \( SU(N)/\mathbb{Z}_N \), the moduli space of \( SU(N) \) vortices belong to the same group, no evident difference can be found. We will study the irreducible representation of \( SU(N) \) vortices in chapter 5, which shows that the moduli space is indeed a irreducible representation of \( SU(N)/\mathbb{Z}_N \). However, for \( USp(2M) \), \( SO(2M) \) and \( SO(2M + 1) \) cases, the dual property is manifest.

### 2.3.1 \( SO(2M) \) case

Let us start with the minimal vortices \( SO(2M) \times U(1) \) in Eq.(2.2.43). In the case of \( SO(2M) \times U(1) \) theory, the minimal vortex solutions fall into two distinct classes \([48, 49]\) which do not mix under the \( SO(2M) \) transformations of the original fields. These observations suggest that the vortices transform according to spinor representations of the GNOW dual of \( SO(2M) \) i.e. as two \( 2^{M-1} \) dimensional spinor representations of \( Spin(2M) \).

First, we analyze the variation in the moduli space. In order to discuss the general group transformation properties of our moduli space using the holomorphic matrix \( q_0 \), we should fix a “coordinate patch” and consider the transformation under \( G_{C+F} \). we concentrate on the positive-chirality \( k = 1 \) vortex in a patch where the moduli matrix \( q_0 \) is already given in Eq.(2.2.44) \([50]\)

\[
q_0(z) = \begin{pmatrix}
1_M & 0 \\
B & 1_M
\end{pmatrix},
\]

(2.3.1)

Its moduli \( V \) has the coordinates \( B \), which is an \( M \) by \( M \) antisymmetric matrix. The
2.3 The vortex transformations: GNOW duality

The generator of $SO(2M)$ is given by

$$\delta U = \begin{pmatrix} \alpha & \beta_A \\ -\beta_A^t & -\alpha^T \end{pmatrix},$$

(2.3.2)

where $\alpha^\dagger = -\alpha$, and $\beta$ is antisymmetric $N \times N$ complex matrix. They are the standard ($i$ times) $SO(2M)$ generators, written in the $SU(M)$ basis. $\delta U$ is anti-Hermitian. As noted in Section 2.2, the $V$ transformation does not change physics. We act the infinitesimal transformation (2.3.2) on the moduli matrix (2.3.1) from the right, and then use the $V$ transformation to pull it back to the original form but with a different coordinate $B' = B + \delta B$,

$$\delta q_0(z) = q_0(z) \cdot \delta U + \delta V \cdot q_0(z),$$

(2.3.3)

in which

$$\delta q_0(z) = \begin{pmatrix} 0 & 0 \\ \delta B & 0 \end{pmatrix}.$$  

(2.3.4)

The local moduli coordinate $B$ transforms as

$$\delta B = \alpha^T \cdot B + B \cdot \alpha - B \cdot \beta_A \cdot B - \beta_A^t.$$  

(2.3.5)

We denote the moduli space as $\mathcal{V}$. The $\delta V$ matrix is holomorphic in $z$ and can be solved by Eq. (2.3.3), which reads

$$\delta V = \begin{pmatrix} \beta_A \cdot B \cdot \alpha & -z \cdot \beta_A \\ 0 & -B \cdot \beta_A + \alpha^T \end{pmatrix}.$$  

(2.3.6)

As required by $V$-equivalence, this $\delta V$ is a generator of the complexified $SO(2M)$ group, which satisfies

$$\delta V^T \cdot J + J \cdot \delta V = 0, \quad J = \begin{pmatrix} 0 & 1_M \\ 1_M & 0 \end{pmatrix}.$$  

(2.3.7)

Later, we will show that the transformation Eq. (2.3.5) is isomorphic to the transformation of a generic spinor state under a general infinitesimal $SO(2M)$ transformation.

Transformation around any other point $P$ is generated by the conjugation

$$R \begin{pmatrix} 0 & -B'^t \\ B' & 0 \end{pmatrix} R^{-1},$$

(2.3.8)

where $R$ is a finite $SO(2M)$ transformation of the form of Eq. (2.2.45), bringing the origin of the moduli space to $P$. 
Second, we analyze the transformation property in the spinor representation space. The spinors can be represented by using a system made of $M$ spin-$\frac{1}{2}$ subsystems: $|s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_M\rangle$. The $SO(2M)$ generators $\Sigma_{ij}$ in the spinor representation can be expressed in terms of the (anti-commuting) creation and annihilation operators $a_i, a_i^\dagger$ in the well-known fashion [69] (see Appendix B). The $k$-th annihilation operators acts as

$$a_k = \frac{1}{2} \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_- \otimes \1_{M-k} \1_{k-1}, \quad k = 1, 2, \ldots, M,$$

while $\tau_-$ is replaced by $\tau_+$ in $a_k^\dagger$.

We map the special vortex configurations and the spinor states as follows:

$$\left(\pm, \cdots, \pm\right) \sim |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_M\rangle, \quad |s_j\rangle = \downarrow\rangle \text{ or } \uparrow\rangle.$$ (2.3.10)

In particular, the $(+ + \ldots +)$ vortex solution described by Eq. (3.1.2) is mapped to the all-spin-down state

$$\left(\ldots +\right) \sim |\ldots \downarrow\rangle.$$ (2.3.11)

An infinitesimal transformation of this spinor state is given by

$$S = e^{i \omega_{ij} \Sigma_{ij}} = 1 + \sum_{i,j=1}^{M} (\omega_{ij} - \omega_{M+i,M+j} - i \omega_{i,M+j} - i \omega_{M+i,j}) a_i^\dagger a_j^\dagger + \ldots,$$ (2.3.12)

as the operators $a_j$ annihilate the state $|\ldots \downarrow\rangle$. There is thus a one-to-one correspondence between the vortex transformation law (3.1.11) and the spinor transformation law, under the identification

$$B_{ij} = \sum_{i,j=1}^{M} (\omega_{ij} - \omega_{M+i,M+j} - i \omega_{i,M+j} - i \omega_{M+i,j}),$$ (2.3.13)

which are indeed generic antisymmetric, complex $M \times M$ matrices.

Infinitesimal transformations around any other spinor state ($|P\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_M\rangle$) are generated by the conjugation

$$S \left(B_{ij}^* a_i^\dagger a_j^\dagger\right) S^{-1},$$ (2.3.14)

where $S \in Spin(2M)$ transforms the origin (2.3.11) to $|P\rangle$.

We conclude that the connected parts of the vortex moduli space are isomorphic to the orbits of spinor states: they form two copies of $SO(2M)/U(M)$. 
2.3 The vortex transformations: GNOW duality

2.3.2 $USp(2M)$ case

The consideration in the case of the $USp(2M)$ vortices is analogous. The (abstract) $SO(2M + 1)$ spinor generators can be expressed in terms of the annihilation and creation operators as in Appendix B. We map the $USp(2M)$ vortex solutions and $SO(2M + 1)$ spinor states as in Eq. (2.3.10), with the origin of the moduli spaces identified as before, i.e. as in Eq. (2.3.11).

Both in the vortex and the spinor moduli spaces, in contrast to the $SO(2M)$ case, there is no conserved chirality now: all of the $2^M$ special vortex solutions (spinor states) are connected by $USp(2M)$ ($SO(2M + 1)$) transformations. Infinitesimal transformations of the $USp(2M)$ vortices around the origin are generated by a complex, symmetric matrix $B$, Eq. (3.1.11). On the other hand, the $SO(2M + 1)$ spinors transform as in Eq..(B7)-(B8): the origin $|\downarrow \ldots \downarrow\rangle$ is transformed by

$$ S = e^{iω_αβΣ_αβ+ιω_γ,2M+1Σ_γ,2M+1} = 1 + \beta_{ij} a_i^a a_j^a + d_i a_i^a + O(\omega^2) , \quad (2.3.15) $$

they describe the coset $SO(2M + 1)/U(M)$. The map between the $USp(2M)$ vortex transformation law and the $SO(2M + 1)$ spinor transformation law is then

$$ (\beta_{ij}, d_i) \leftrightarrow B , \quad (2.3.16) $$

that is, the infinitesimal neighborhoods of the origin of the vortex and spinor moduli spaces are mapped to each other by the identification of the local coordinates

$$ \beta_{ij} = -\beta_{ji} = B_{ij} \quad (i > j) ; \quad d_i = B_{ii} . \quad (2.3.17) $$

Both for the vortex and for the spinors, transformations around any other point are generated by the conjugation analogous to eqs. (2.3.8), (2.3.14) with appropriate modifications ($B_{\text{anti}} \rightarrow B_{\text{sym}}$; $\beta_{ij} a_i^a a_j^a \rightarrow \beta_{ij} a_i^a a_j^a + d_i a_i^a$). Under such a map, the vortex transformations in the moduli space ($USp(2M)/U(M)$) are mapped to the orbits of the spinor states, $SO(2M + 1)/U(M)$.
2.3.3 \( SO(2M+1) \) case

As discussed previously, the single \( SO(2M+1) \) vortices is similar to \( k = 2 \) \( SO(2M) \) vortices. First we mark all the patches corresponding irreducible orbits, that is

\[
\begin{pmatrix}
2 & 0 \\
2 & 0 \\
\vdots & \vdots \\
2 & 0 \\
2 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 0 \\
2 & 0 \\
\vdots & \vdots \\
2 & 0 \\
0 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 0 \\
2 & 0 \\
\vdots & \vdots \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 0 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
1 & 1
\end{pmatrix}.
\tag{2.3.18}
\]

These correspond to different \( SO(2M+1)C_{+F} \) orbits, which correspond to the cosets

\[
\mathcal{M}_{\text{orbit}} = \frac{SO(2M+1)}{U(r) \times SO(2(M-r)+1)},
\tag{2.3.19}
\]

where \( r \) is the number of \( (2,0) \) pairs. The internal moduli spaces with different \( r \)'s are not connected by the action of \( SO(2M+1) \), and there is no chirality as in \( SO(2M) \) case.

In order to verify that the irreducible orbits correspond indeed to various the representations of \( USp(2M) \), all these orbits should be considered. Here we choose one particular orbit corresponding to a special solution \[48\]

\[
\begin{pmatrix}
2 & 0 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1
\end{pmatrix}.
\tag{2.3.20}
\]

Permuting the \( (2,0) \) pair with another \( (1,1) \) pair, one finds \( 2N \) special solutions, this fact suggests that these solutions belong to a vector \( 2N \) representation of \( USp(2M) \). This is the minimal non-trivial irreducible representation. The moduli matrix for this patch is \[50\]

\[
q_0(z) = \begin{pmatrix}
z^2 & 0 & 0 & 0 \\
-z\vec{C}_2^T & z\mathbf{1}_{M-1} & 0 & 0 \\
A(z) & \vec{C}_1 & 1 & \vec{C}_2 & C_2 \\
-z\vec{C}_1 & 0 & 0 & z\mathbf{1}_{M-1} & 0 \\
-zC_2 & 0 & 0 & 0 & z,
\end{pmatrix},
\tag{2.3.21}
\]

where

\[
A = -\frac{1}{2}(\vec{C}_1 \cdot \vec{C}_2^T + \vec{C}_2 \cdot \vec{C}_1^T + C_2^2).
\]
The moduli matrix is determined by $\vec{C}_1, \vec{C}_2$ and $C_2$, which have only $4M - 2$ parameters.

By the aforementioned procedure, we make a variation of the coordinate, and pull back by the $V$-transformation. The local moduli coordinates $\vec{C}_1, \vec{C}_2$ and $C_2$ transform as (See Appendix C)

$$\delta \vec{C}_1 = \vec{\beta}_{12}, \quad (2.3.22)$$
$$\delta \vec{C}_2 = - \vec{\alpha}_{12}, \quad (2.3.23)$$
$$\delta C_2 = \xi. \quad (2.3.24)$$

Our recipe for the verification of duality property is that, the transformation property of the moduli parameters near a point is the same with transformation property of objects transforming under the dual group. We construct the irreducible representations of the $USp(2M)$ group. Locally, there is a vector state corresponding to the patch Eq. (2.3.20), which reads

$$|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \iff \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}, \quad (2.3.25)$$

Letting a $USp(2M)$ generator to act on this state, we obtain

$$\delta U_{USp} |\phi\rangle = \begin{pmatrix} \sigma & \tau_s \\ -\tau_s^T & -\sigma^T \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (2.3.26)$$

Here we use different symbols $\sigma$ and $\tau$ to prevent confusion. The effective parameters in $\delta U_{USp}$ are $\sigma_{21}, \sigma_{31}, \ldots, \sigma_{M1}$, $\tau_{11}, \tau_{21}, \ldots, \tau_{M1}$. The neighborhood of the manifold around
this state is generated by the above parameters, and it has dimension $4M - 2$. Comparing the results with Eq. (C6), we find a one-to-one map

$$
\begin{align*}
(\sigma_2, \sigma_3, \cdots, \sigma_M) & \Leftrightarrow -\alpha_{12}^* \\
(\tau_2, \cdots, \tau_M) & \Leftrightarrow \beta_{12}^* \\
\tau_1 & \Leftrightarrow \xi
\end{align*}
$$

This argument is based on the transformation property around the origin, we can also construct a global transformation to move to a generic point. Thus in this special patch, there is a one-to-one correspondence between the $SO(2M + 1)$ vortex solutions and the quantum states in a $USp(2M)$ vector representation. We leave the proof of duality for more general class of solutions in Eq.(2.3.18) to future work.

## 2.4 Conclusion

In this chapter, we have introduced the bulk $\mathcal{N} = 2$ supersymmetric field theory. FI F-term is used to trigger the condensation of squarks. By integrating out the massive degrees of freedom, the system has the non-Abelian vortices solutions. The BPS equations were given, the quantization condition and the orientational zero-modes were discussed in detail. The quantization condition and index theorem suggest strongly these vortex solutions to possess transformation properties under the GNOW dual group. By analyzing concretely several lowest-winding vortex solutions in $SO$ and $USp$ theories, we have explicitly shown that the moduli space of these vortex solutions are indeed isomorphic to the moduli space of the quantum particle states, transforming under the GNOW duals of the underlying $SO(2M)$, $USp(2M)$, or $SO(2M + 1)$ color-flavor symmetry group.
Chapter 3

Effective world-sheet action of non-Abelian vortices

Non-Abelian vortices possess exact, continuous non-Abelian moduli. These continuous zero-modes arise from the breaking (by the soliton vortex) of an exact color-flavor diagonal symmetry of the system under consideration. The structure of their moduli, the varieties and group-theoretic properties of these modes as well as their dynamics, and the dependence of all these on the details of the theory such as the matter content and gauge groups, turn out to be surprisingly rich. In spite of quite an impressive progress made in the last several years, the full implication of these theoretical development is as yet to be seen. In the present chapter, we turn our attention to the low-energy dynamics of non-Abelian vortex. In particular our aim is to construct the low-energy effective action describing the fluctuations of the orientational moduli parameters on the vortex world-sheet, generalizing the results found several years ago in the context of $U(N)$ models [39, 41, 66].

3.1 $k=1$ minimal winding models

For concreteness and for simplicity, we start our discussion with the case of the $SO(2M) \times U(1)$ and $USp(2M) \times U(1)$ theories, although our method is quite general.
3.1.1 \(SO(2M)\) and \(USp(2M)\) vortices

By choosing the plus sign for all of the \(U(1)^N \subset G'\) factors, one finds a solution of the form\(^1\)

\[
q = \begin{pmatrix} e^{i\theta} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix}_M = \frac{e^{i\theta} \phi_1(r) + \phi_2(r)}{2} \frac{1_{2M}}{1_{2M}} + \frac{e^{i\theta} \phi_1(r) - \phi_2(r)}{2} T ,
\]

\[
A_i = \frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [(1 - f(r)) 1_{2M} + (1 - f_{NA}(r)) T] , \tag{3.1.2}
\]

where

\[
T = \text{diag} (1_M, -1_M) , \tag{3.1.3}
\]

and \(z, r, \theta\) are cylindrical coordinates. The appropriate boundary conditions are

\[
\phi_{1,2}(\infty) = \frac{v}{\sqrt{2M}} , \quad f(\infty) = f_{NA}(\infty) = 0 ,
\]

\[
\phi_1(0) = 0 , \quad \partial_r \phi_2(0) = 0 , \quad f(0) = f_{NA}(0) = 1 . \tag{3.1.4}
\]

By going to singular gauge,

\[
q \rightarrow \text{diag} (e^{-i\theta} 1_M, 1_M) q , \tag{3.1.5}
\]

the vortex takes the form

\[
q = \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix}_M = \frac{\phi_1(r) + \phi_2(r)}{2} \frac{1_{2M}}{1_{2M}} + \frac{\phi_1(r) - \phi_2(r)}{2} T ,
\]

\[
A_i = -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [f(r) 1_{2M} + f_{NA}(r) T] ; \tag{3.1.6}
\]

in this gauge the whole topological structure arises from the gauge-field singularity along the vortex axis. The BPS equations (2.2.4)-(2.2.5) for the profile functions are given (in both gauges) by

\[
\partial_r \phi_1 = \frac{1}{2r} (f + f_{NA}) \phi_1 , \quad \partial_r \phi_2 = \frac{1}{2r} (f - f_{NA}) \phi_2 , \tag{3.1.7}
\]

\[
\frac{1}{r} \partial_r f = \frac{e^2}{2} \left( \phi_1^2 + \phi_2^2 - \frac{v^2}{M} \right) , \quad \frac{1}{r} \partial_r f_{NA} = \frac{g^2}{2} \left( \phi_1^2 - \phi_2^2 \right) . \tag{3.1.8}
\]

\(^1\)It is convenient to work with the skew-diagonal basis for the \(SO(2M)\) group, i.e. the invariant tensors are taken as

\[
J = \begin{pmatrix} 0 & 1_M \\ \epsilon 1_M & 0 \end{pmatrix} , \tag{3.1.1}
\]

where \(\epsilon = \pm\) for \(SO(2M)\) and \(USp(2M)\) groups, respectively.
3.1 k=1 minimal winding models

The above is a particular vortex solution with a fixed \((+ + \ldots +)\) orientation. As the system has an exact \(SO(2M)_{C+F}\) or \(USp(2M)_{C+F}\) color-flavor diagonal (global) symmetry, respected by the vacuum (2.1.16), which is broken by such a minimum vortex, the latter develops “orientational” zero-modes. Degenerate vortex solutions can indeed be generated by color-flavor \(SO(2M)\) (or \(USp(2M)\)) transformations

\[
q \rightarrow U q U^{-1}, \quad A_i \rightarrow U A_i U^{-1},
\]

as

\[
q = U \begin{pmatrix} \phi_1(r) 1_M & 0 \\ 0 & \phi_2(r) 1_M \end{pmatrix} U^{-1} = \frac{\phi_1(r) + \phi_2(r)}{2} 1_{2M} + \frac{\phi_1(r) - \phi_2(r)}{2} U T U^{-1},
\]

\[
A_i = -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [f(r) 1_{2M} + f_{NA}(r) U T U^{-1}] , \quad i = 1, 2.
\]

Actually, the full \(SO(2M)\) (or \(USp(2M)\)) group does not act on the solution, as the latter remains invariant under \(U(M) \subset SO(2M)\) (or \(USp(2M)\)). Only the coset \(SO(2M)/U(M)\) (or \(USp(2M)/U(M)\)) acts non-trivially on it, and thus generates physically distinct solutions.\(^2\) An appropriate parameterization of the coset, valid in a coordinate patch including the above solution, has been known for some time (called the reducing matrix) [70, 49],

\[
U = \begin{pmatrix} 1_M & -B^\dagger \\ 0 & 1_M \end{pmatrix} \begin{pmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1_M & 0 \\ B & 1_M \end{pmatrix} = \begin{pmatrix} X^{-\frac{1}{2}} & -B^\dagger Y^{-\frac{1}{2}} \\ BX^{-\frac{1}{2}} & Y^{-\frac{1}{2}} \end{pmatrix},
\]

where the matrices \(X\) and \(Y\) are defined by

\[
X \equiv 1_M + B^\dagger B, \quad Y \equiv 1_M + BB^\dagger,
\]

in terms of an \(M \times M\) complex matrix \(B\), being antisymmetric for \(SO(2M)\) and symmetric for \(USp(2M)\). Note that the matrix (3.1.11) indeed satisfies the defining properties the two groups

\[
U^{-1} = U^\dagger, \quad U^\dagger J U = J,
\]

\(^2\)As was studied in detail in Ref. [49], the vortex moduli space in \(SO(2M)\) (or \(USp(2M)\)) theories is a non-trivial complex manifold, requiring at least \(2^{M-1}\) (or \(2^M\)) local coordinate neighborhoods (patches). The moduli space structure is actually richer, as these vortices possess semi-local moduli (related to the size and shape moduli) as well, besides the orientational moduli under consideration here, even with the minimum number of flavors needed for a color-flavor locked phase, in contrast to the original \(U(N)\) model. Here we consider only the orientational moduli related to the exact symmetry of the system.
with the respective invariant tensor (3.1.1). The matrix $B$ parameterizes the “Nambu-Goldstone” modes of symmetry breaking (by the vortex)

$$SO(2M) \rightarrow U(M) \quad \text{or} \quad USp(2M) \rightarrow U(M) ,$$

(3.1.14)

and the number of independent parameters in $B$, $M(M-1)$ or $M(M+1)$, correctly matches the (real) dimension of the coset $SO(2M)/U(M)$ or $USp(2M)/U(M)$. The following identities turn out to be useful below:

$$BX^m = Y^m B , \quad X^m B\dagger = B\dagger Y^m , \quad [X^m, B\dagger B] = 0 , \quad [Y^m, BB\dagger] = 0 .$$

(3.1.15)

In the next section we shall allow for a $(x^3, x^0)$ dependence in $B$ and determine the effective action for these degrees of freedom.

As the orientational modes considered in Eq. (3.1.10) represent exact Nambu-Goldstone-like zero-modes, nothing can prevent them from fluctuating in the space-time, from one point to another, with an arbitrarily small expenditure of energy. However, they are not genuine Nambu-Goldstone modes, as the vacuum itself is symmetric under $SO(2M)_{C+F}$ or $USp(2M)_{C+F}$: they are massive modes in the 4-dimensional space-time bulk. They propagate freely only along the vortex-axis and in time. To study these excited modes we set the moduli parameters $B$ to be (quantum) fields of the form

$$B = B(x^\alpha) , \quad x^\alpha = (x^3, x^0) .$$

(3.1.16)

When this expression is substituted into the action $\int d^4x L$, however, one immediately notes that

$$\sum_{\alpha=0,3} \left[ \sum_{f=1}^{2M} |\partial_\alpha q_f|^2 + \sum_{i=1,2} \frac{1}{2g_i^2} |F_{i\alpha}|^2 \right] ,$$

(3.1.17)

leads to an infinite excitation energy, whereas one knows that the system must be excitable without mass gap (classically).\(^3\)

The way how the system reacts to the space-time dependent change of the moduli parameters, can be found by an appropriate generalization of the procedure adopted earlier for the vortices in $U(N)$ theories. A key observation [39]–[66] is to introduce non-trivial

\(^3\)Whereas in the far infrared, we expect that either the world-sheet effective sigma model will by quantum effects develop a dynamic mass gap (as the $\mathbb{C}P^{N-1}$ model) or end up in a conformal vacuum – a possibility for $SO,USp$ theories [29].
gauge field components, $A_\alpha$, to cancel the large excitation energy from (3.1.17). A naïve guess would be

$$A_\alpha = -i \rho(r) U^{-1} \partial_\alpha U, \quad (3.1.18)$$

with $U$ of Eq. (3.1.11) and some profile function $\rho$. This however does not work. The problem is that even though

$$i U^{-1} \partial_\alpha U = i \left( X^{-\frac{1}{2}} B^\dagger \partial_\alpha B X^{-\frac{1}{2}} - \partial_\alpha X^\frac{1}{2} X^{-\frac{1}{2}} - X^{-\frac{1}{2}} \partial_\alpha B^\dagger Y^{-\frac{1}{2}} Y^{-\frac{1}{2}} - \partial_\alpha X^{-\frac{1}{2}} X^{-\frac{1}{2}} - X^{-\frac{1}{2}} \partial_\alpha B^\dagger Y^{-\frac{1}{2}} - \partial_\alpha Y^{-\frac{1}{2}} Y^{-\frac{1}{2}} \right), \quad (3.1.19)$$

certainly is in the algebra $g'$ of $G'$, it in general contains the fluctuations also in the $U(M)$ directions (massive modes). To extract the massless modes, we first project it on directions orthogonal to the fixed matter-field orientation, Eq. (3.1.6), that is

$$i \left( U^{-1} \partial_\alpha U - T U^{-1} \partial_\alpha U T \right) = i \left( \begin{array}{cc} 0 & -X^{-\frac{1}{2}} \partial_\alpha B^\dagger Y^{-\frac{1}{2}} \\ Y^{-\frac{1}{2}} \partial_\alpha B^\dagger Y^{-\frac{1}{2}} & 0 \end{array} \right), \quad (3.1.20)$$

such that $\text{Tr} \left[ U^{-1} \partial_\alpha U \right] q^0 = 0$, where $q^0$ indicates the vortex (3.1.6). As the quark fields fluctuate in the $SO(2M)$ (or $USp(2M)$) group space, we must keep $A_\alpha$ orthogonal to them. The appropriate Ansatz then is

$$A_\alpha = i \rho(r) U \left( U^{-1} \partial_\alpha U \right) \perp U^{-1}, \quad \alpha = 0, 3, \quad (3.1.21)$$

together with $q$ and $A_i$ of Eq. (3.1.10). One sees that the following orthogonality conditions

$$\text{Tr} \{ A_\alpha \} = 0, \quad \text{Tr} \{ A_\alpha U T U^{-1} \} = 0, \quad \text{Tr} \{ A_\alpha \partial_\alpha (U T U^{-1}) \} = 0 \quad (3.1.22)$$

are satisfied: the first two hold by construction; the third can easily be checked. The constant BPS tension is independent of the vortex orientation; the excitation above it arises from the following terms of the action

$$\text{Tr} \{ \mathcal{D}_\alpha q \}^2 = - \left[ \frac{\rho^2}{2} \left( \phi_1^2 + \phi_2^2 \right) + (1 - \rho) (\phi_1 - \phi_2)^2 \right] \text{Tr} \left[ (U^{-1} \partial_\alpha U) \right]^2, \quad (3.1.23)$$

$$\frac{1}{g^2} \text{Tr} F_{\alpha \beta}^2 = - \frac{1}{g^2} \left[ (\partial_t \rho)^2 + \frac{1}{r^2} \partial_\alpha \rho^2 \right] \text{Tr} \left[ (U^{-1} \partial_\alpha U) \right]^2, \quad (3.1.24)$$

where we have used the identity

$$\text{Tr} \left( \partial_\alpha (U T U^{-1}) \right)^2 = - \text{Tr} \left( U^{-1} \partial_\alpha U - T U^{-1} \partial_\alpha U T \right)^2 = - 4 \text{Tr} \left[ (U^{-1} \partial_\alpha U) \right]^2. \quad (3.1.25)$$
By using Eq. (3.1.20) one arrives at the world-sheet effective action

\[ S_{1+1} = 2\beta \int dtdz \text{tr} \left\{ X^{-1} \partial_0 B^\dagger Y^{-1} \partial_0 B \right\} = 2\beta \int dtdz \text{tr} \left\{ (1_M + B^\dagger B)^{-1} \partial_0 B^\dagger (1_M + BB^\dagger)^{-1} \partial_0 B \right\}, \quad (3.1.26) \]

where

\[ \beta = \frac{2\pi}{g^2 I}, \quad (3.1.27) \]

and the trace \( \text{tr} \) acts on \( M \times M \) matrices. Even though the sigma-model metric reflects the specific symmetry breaking patterns of the system under consideration, the coefficient \( I \) turns out to be universal, and indeed is formally identical to the one found for the \( U(N) \) model \(^4\)

\[ I = \int_0^\infty dr r \left[ \left( \partial_r \rho \right)^2 + \frac{1}{r^2} f_{\text{NA}}^2 (1 - \rho)^2 + g^2 \rho^2 \left( \phi_1^2 + \phi_2^2 \right) + g^2 (1 - \rho) \left( \phi_1 - \phi_2 \right)^2 \right]. \quad (3.1.28) \]

The equation of motion for \( \rho \) minimizing the coupling constant \( \beta \) (the Kähler class) of the vortex world-sheet sigma model can be solved accordingly by \([41, 66]\)

\[ \rho = 1 - \frac{\phi_1}{\phi_2}, \quad (3.1.29) \]

as can be checked by a simple calculation making use of the BPS equations for the profile functions \( \phi_{1,2}, f_{\text{NA}} \). The integral \( I \) turns out to be a total derivative

\[ I = \int_0^\infty dr \partial_r \left( f_{\text{NA}} \left[ \left( \frac{\phi_1}{\phi_2} \right)^2 - 1 \right] \right), \quad (3.1.30) \]

and by using the boundary conditions (3.1.4) the final result is

\[ I = f_{\text{NA}}(0) = 1. \quad (3.1.31) \]

The action found in Eq. (3.1.26) is precisely that of the \((1 + 1)\)-dimensional sigma model on Hermitian symmetric spaces \( SO(2M)/U(M) \) and \( USp(2M)/U(M) \) \([70, 71]\).\(^5\)

The metric is Kählerian, with the Kähler potential given by

\[ K = \text{tr} \log \left( 1_M + BB^\dagger \right) , \quad g_{IJ} = \frac{\partial^2 K}{\partial B^I \partial B^J}, \quad (3.1.32) \]

\(^4\)In that case the effective sigma model has a \( \mathbb{CP}^{N-1} \) target space \([41, 66]\); see Subsec. 3.1.2 below.

\(^5\)In Ref. [72], these \( \text{NL}\sigma\text{Ms} \) on Hermitian symmetric spaces were obtained from supersymmetric gauge theories by gauging a symmetry big enough to absorb all quasi-Nambu-Goldstone bosons (which are contained in mixed-type multiplets) and hence obtain a compact manifold parameterized by only pure-type multiplets.
3.1 k=1 minimal winding models

<table>
<thead>
<tr>
<th>moduli space $\mathcal{M}$</th>
<th>$\chi(\mathcal{M})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(2M)$</td>
<td>$2^{M-1}$</td>
</tr>
<tr>
<td>$U(M)$</td>
<td>$2^M$</td>
</tr>
<tr>
<td>$USp(2M)$</td>
<td>$N$</td>
</tr>
<tr>
<td>$U(N)$</td>
<td>($N \choose k$)</td>
</tr>
<tr>
<td>$\mathbb{C}P^{N-1}$</td>
<td>$2^M$</td>
</tr>
<tr>
<td>$Gr_{N,k}$</td>
<td>$SO(2M)\times SO(2M-2)$</td>
</tr>
<tr>
<td>$Q^{2M-2}$</td>
<td>$2^{M-1}$</td>
</tr>
</tbody>
</table>

Table 3.1: Number of quantum vacua for the relevant vortices under consideration which is given by the Euler characteristic $\chi$.

where $I, J = \{(i, j) = 1, \ldots, M \mid i \leq j \}$.

In the context of $\mathcal{N} = 2$ supersymmetric models, the low-energy effective vortex action is a two-dimensional, $\mathcal{N} = (2, 2)$ supersymmetric sigma model [71]:

$$S_{\text{susy}}^{1+1} = 2\beta \int dt dz \, d^2\theta \, d^2\bar{\theta} \, K(B, \bar{B})$$

in terms of the Kähler potential Eq. (3.1.32), where $B$ now is a matrix chiral superfield ($\bar{B}$ anti-chiral superfield containing $B^\dagger$). The $\beta$-functions for these sigma models have been determined in [71]. In the supersymmetric case, the number of quantum vacua is given by the Euler characteristic of the manifold on which the world-sheet action lives [73, 74], which can be found in the mathematical literature [75] and we show the relevant numbers in Table 3.1.

### 3.1.2 $U(N)$ vortices and the $\mathbb{C}P^{N-1}$ sigma model

For the fundamental (i.e. of the minimum winding) vortex of the $U(N)$ model discussed by Shifman et. al. [41, 66], the vortex Ansatz is very similar to Eq. (3.1.2) except for changes in the field Ansatz and accordingly the reducing matrix $U$:

$$q = \begin{pmatrix} e^{i\theta} \phi_1(r) & 0 \\ 0 & e^{i\theta} \phi_2(r) \end{pmatrix} = \frac{e^{i\theta} \phi_1(r) + \phi_2(r)}{2} 1_N + \frac{e^{i\theta} \phi_1(r) - \phi_2(r)}{2} T, \quad (3.1.34)$$

$$A_i = \epsilon_{ij} \frac{ax^j}{y^2} \left[ \frac{1}{N} (1 - f(r)) \, 1_N + \frac{1}{2} (1 - f_{\text{NA}}(r)) \left( T - \frac{2 - N}{N} 1_N \right) \right], \quad (3.1.35)$$
where $T$ is defined to be
\[
T = \begin{pmatrix}
1 & 0 \\
0 & -1_{N-1}
\end{pmatrix}.
\]

Meanwhile the boundary conditions are
\[
\begin{align*}
\phi_{1,2}(\infty) &= \frac{v}{\sqrt{N}}, \\
f(\infty) &= f_{NA}(\infty) = 0, \\
\phi_1(0) &= 0, \\
\partial_r \phi_2(0) &= 0, \\
f(0) &= f_{NA}(0) = 1.
\end{align*}
\]

The BPS equations are written as
\[
\begin{align*}
\partial_r \phi_1 &= \frac{1}{N_T} \phi_1[f + (N - 1)f_{NA}], \\
\partial_r f &= \frac{e^2}{2} \left( \phi_1^2 + (N - 1)\phi_2^2 - v^2 \right), \\
\partial_r \phi_2 &= \frac{1}{N_T} \phi_2 (f - f_{NA}), \\
\partial_r f_{NA} &= \frac{g^2}{2} \left( \phi_1^2 - \phi_2^2 \right).
\end{align*}
\]

The unitary transformation $U$ (the reducing matrix) giving rise to vortices of generic orientation has the same form as in Eq. (3.1.11), except that the matrix $B$ is now an $(N - 1)$-component column-vector
\[
B = \begin{pmatrix}
b_1 \\
\vdots \\
b_{N-1}
\end{pmatrix},
\]
while $B^\dagger$ is correspondingly a row-vector;
\[
X = 1 + B^\dagger B, \\
Y = 1_{N-1} + BB^\dagger,
\]
are a scalar and an $(N - 1) \times (N - 1)$ dimensional matrix, respectively. Going through the same steps as in Sec. 3.1.1, the effective world-sheet action in this case is exactly given by Eq. (3.1.26), including the normalization integral of Eqs. (3.1.27)-(3.1.31), with these replacement. $B = (b_1, \ldots, b_{N-1})^T$ represent the standard inhomogeneous coordinates of $\mathbb{C}P^{N-1}$.

In order to find the relation between the $N$-component complex unit vector $n$ used by Gorsky et al. [66] and our $B$ matrix, note that
\[
\begin{align*}
\frac{1}{N} U \begin{pmatrix}
-(N - 1) & 0 \\
0 & 1_{N-1}
\end{pmatrix} U^{-1} &= \frac{1}{N} 1_N - n n^t, \\
\Rightarrow 
n n^t &= U \begin{pmatrix}
1 & 0 \\
0 & 0_{N-1}
\end{pmatrix} U^{-1} = \begin{pmatrix}
X^{-1} & X^{-1}B^\dagger \\
BX^{-1} & BX^{-1}B^\dagger
\end{pmatrix},
\end{align*}
\]
which allows us to identify

\[ n = \left( \begin{array}{c} X^{-\frac{1}{2}} \\ BX^{-\frac{1}{2}} \end{array} \right). \]  

By the identification (3.1.43), our Ansatz (3.1.21) is seen to be equal, after some algebra, to

\[ A_\alpha = i \rho(r) \left[ \partial_\alpha n n^\dagger - n \partial_\alpha n^\dagger - 2 n n^\dagger (n^\dagger \partial_\alpha n) \right], \]  

which is the one proposed in Ref. [66].

The effective action for \( \mathbb{C}P^{N-1} \) in Ref. [66] reads

\[ S_{1+1} = 2 \beta \int dt dz \left\{ \partial_\alpha n^\dagger \partial_\alpha n + (n^\dagger \partial_\alpha n)^2 \right\}. \]  

(3.1.45)

Now we substitute Eq.(3.1.43) into the action, it is easy to obtain that

\[ S_{1+1} = 2 \beta \int dt dz \left\{ \partial_\alpha B^\dagger \partial_\alpha B + \frac{(B^\dagger \partial_\alpha B)(\partial_\alpha B^\dagger B)}{(1 + B^\dagger B)^2} \right\}. \]  

(3.1.46)

This action Eq.(3.1.46) reduces to Eq. (3.1.26). Our result thus goes some way towards clarifying the meaning of the seemingly arbitrary Ansatz (3.1.44) (or better, an Ansatz found by a brilliant intuition, but that cannot easily be applied to other theories) used in Ref. [66].

### 3.2 Higher winding models

#### 3.2.1 Completely symmetric \( k \)-winding vortices in the \( U(N) \) model

Next let us consider the orientational moduli of the coincident \( k \)-winding vortex in the \( U(N) \) model [35, 47, 76, 69]. We consider a vortex solution of a particular, fixed orientation given by

\[ q := \left( \begin{array}{c} e^{ik\theta} \phi_1(r) \\ 0 \end{array} \right), \quad T = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]  

\[ A_i = \epsilon_{ij} \frac{e^j}{r^2} \left[ \frac{1}{N} (k - f(r)) \mathbf{1}_N + \frac{1}{2} (k - f_{NA}(r)) \left( T - \frac{2 - N}{N} \mathbf{1}_N \right) \right], \]  

(3.2.1)  

(3.2.2)
with the boundary conditions
\[
\phi_{1,2}(\infty) = \frac{v}{\sqrt{N}}, \quad f(\infty) = f_{NA}(\infty) = 0,
\phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = k.
\] (3.2.3)

The BPS equations of the profile function is the same with that of \(\mathbb{C}P^{N-1}\) sigma model.
Being a composition of \(k\) vortices of minimum winding in the same orientation, it is obvious that the vortex (3.2.1) transforms under the totally symmetric representation:

\[\begin{array}{ccccccc}
& & & & \square & & \\
& & & & \mathcal{S}& & \\
& k
\end{array}\]

of the color-flavor \(SU(N)_{C+F}\) group.

The construction of the effective vortex action in this case is almost identical to that in the preceding subsection, in particular the reducing matrix acting non-trivially on the vortex is the same as in the single \(U(N)\) vortex case, see Eqs. (3.1.39)-(3.1.40). The effective vortex action is the same \(\mathbb{C}P^{N-1}\) model (3.1.26). The only difference is in the value of the gauge profile functions at the vortex core, Eq. (3.2.3). As a consequence the coefficient (the coupling strength) in front of the action (3.1.26) (see Eq. (3.1.30)) is now given by
\[
\beta = \frac{2\pi}{g^2 I}, \quad I = f_{NA}(0) = k.
\] (3.2.4)

### 3.2.2 Completely antisymmetric \(k\)-winding vortices in the \(U(N)\) model

Consider now a \(k\)-vortex (with \(k < N\)) of the form
\[
q := \begin{pmatrix}
\epsilon^i \phi_1(r) 1_k & 0 \\
0 & \phi_2(r) 1_{N-k}
\end{pmatrix}, \quad T = \begin{pmatrix}
1_k & 0 \\
0 & -1_{N-k}
\end{pmatrix},
\]
\[
A_i = \epsilon_{ij} \frac{e^j}{r^2} \left[ \frac{k}{N} (1 - f(r)) \, 1_N + \frac{1}{2} (1 - f_{NA}(r)) \left( T - \frac{2k - N}{N} 1_N \right) \right],
\] (3.2.5)

with the following boundary conditions
\[
\phi_{1,2}(\infty) = \frac{v}{\sqrt{N}}, \quad f(\infty) = f_{NA}(\infty) = 0,
\phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = 1.
\] (3.2.7)
3.2 Higher winding models

The BPS equations for this $U(N)$ model are calculated to be

$$r \partial_r \phi_1 = \phi_1 \left( \frac{k}{N} f + \frac{N-k}{N} f_{\text{NA}} \right), \quad \frac{1}{r} \partial_r f = \frac{e^2}{2} \left( \phi_1^2 + \frac{N-k}{N} \phi_2^2 - \frac{v^2}{k} \right), \quad (3.2.8)$$

$$r \partial_r \phi_2 = \frac{k}{N} \phi_2 (f - f_{\text{NA}}), \quad \frac{1}{r} \partial_r f_{\text{NA}} = \frac{g^2}{2} \left( \phi_1^2 - \phi_2^2 \right). \quad (3.2.9)$$

It is invariant under an $SU(k) \times SU(N-k) \times U(1) \subset SU(N)_{C+F}$ subgroup, showing that it belongs to the completely antisymmetric $k$-th tensor representation:

$$\begin{align*}
\begin{array}{c}
\vdots \\
\end{array}
\end{align*}$$

The color-flavor transformations $U$ acting non-trivially on it belong to the coset

$$G_{r N, k} = \frac{SU(N)}{SU(k) \times SU(N-k) \times U(1)}, \quad (3.2.10)$$

and is again of the standard form of the reducing matrix, Eq. (3.1.11), but now the matrix $B$ is a $(N-k) \times k$ complex matrix field, whose elements are the local coordinates of the Grassmannian manifold. The effective action – the world-sheet sigma model – is then simply given by Eq. (3.1.26) with the standard normalization, Eqs. (3.1.27)-(3.1.31) and the Kähler potential is then given by Eq. (3.1.32).

3.2.3 Higher-winding vortices in the $SO(2M)$ model

Let us now consider doubly-wound vortex solutions in the $SO(2M) \times U(1)$ system. They fall into distinct classes of solutions which do not mix under the $SO(2M)$ transformations.
of the original fields [48]; they are:

\[
\begin{pmatrix}
  n_1^+ & n_1^- \\
  n_2^+ & n_2^- \\
  \vdots & \vdots \\
  n_{M-1}^+ & n_{M-1}^- \\
  n_M^+ & n_M^- \\
\end{pmatrix}
\begin{pmatrix}
  2 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\
  2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\] (3.2.11)

These correspond to different \(SO(2M)_{C+F}\) orbits, living in coset spaces \(SO(2M)/[U(M-\ell) \times SO(2\ell)]\), where \(\ell\) is the number of \((1, 1)\) pairs. Analogously vortices with \(k \geq 3\) can be constructed. As was explained in Ref. [48], the argument that the minimum vortices transform as two spinor representations implies that the \(k = 2\) vortices (3.2.11) transform as various irreducible antisymmetric tensor representations of \(SO(2M)_{C+F}\), appearing in the decomposition of products of two spinors [69]:

\[
2^{M-1} \otimes 2^{M-1} \quad \text{or} \quad 2^{M-1} \otimes \overline{2^{M-1}},
\] (3.2.12)

where the spinors of different chiralities are distinguished by the bar. For instance, the last configuration of Eq. (3.2.11) is a singlet, the second last is the \(2\mathbf{M}\) representation, and so on.

The effective action of the

\[
\begin{pmatrix}
  2 & 0 \\
  \vdots & \vdots \\
  2 & 0 \\
\end{pmatrix}
\] (3.2.13)

vortex (the first of Eq. (3.2.11)) has the same form as that found for the fundamental vortices in Sec. 3.1.1: a sigma model in the target space \(SO(2M)/U(M)\). The normalization constant in front is however different: it is now given by

\[
\beta = \frac{2\pi g}{T}, \quad T = f_{\text{NA}}(0) = 2.
\] (3.2.14)

---

6Here we use the notation of [48]. \(n_i^\pm = \frac{1}{2} \pm M_i \in \mathbb{Z}\), where \(\frac{1}{2}\) is the winding in the overall \(U(1)\); \(M_i\) is the winding number of the \(i\)-th Cartan \(U(1)\) factor. \(M_i \in \mathbb{Z}/2\) are quantized in half integers [48, 49]. In this notation the fundamental vortex of Eq. (3.1.2) is simply \(\begin{pmatrix}
  1 & 0 \\
  \vdots & \vdots \\
  1 & 0 \\
\end{pmatrix}\).
As a last nontrivial example, let us consider the vortex solutions belonging to the second last group of (3.2.11). The orientational modes of the vortex now live in the coset space

\[ SO(2M)/[SO(2) \times SO(2M - 2)] , \tag{3.2.15} \]
a real Grassmannian space. The construction of the reducing matrix in this case is slightly more elaborated, but has already been done by Delduc and V alent [70].

The Ansatz for this vortex can be written as

\[ q = \begin{pmatrix} e^{i\theta} \phi_0(r) & 0 & 0 \\ 0 & e^{i2\theta} \phi_1(r) & 0 \\ 0 & 0 & \phi_2(r) \end{pmatrix} \]
\[ = e^{i\theta} \phi_0 1_{2M} + \frac{1}{2} (e^{i2\theta} \phi_1 + \phi_2 - 2e^{i\theta} \phi_0) T_1 + \frac{1}{2} (e^{i2\theta} \phi_1 - \phi_2) T_2 \, , \]
\[ A_i = \epsilon_{ij} \frac{x_j}{r^2} [(1 - f) 1_{2M} + (1 - f_{NA}) T_2] \, , \tag{3.2.16} \]

where the relevant matrices are

\[ T_1 \equiv \begin{pmatrix} 0_{2M-2} & 1 \\ 1 & 1 \end{pmatrix} \, , \quad T_2 \equiv \begin{pmatrix} 0_{2M-2} & 1 \\ 1 & -1 \end{pmatrix} \, , \tag{3.2.17} \]

and the following relations are useful

\[ T_1^2 = T_1 \, , \quad T_2^2 = T_1 \, , \quad T_1 T_2 = T_2 T_1 = T_2 \, . \tag{3.2.18} \]

We will also need the BPS equations for this vortex

\[ \partial_r \phi_0 = \frac{1}{r} f \phi_0 \, , \quad \frac{1}{r} \partial_r f = \frac{e^2}{4M} (2(M - 1)\phi_0^2 + \phi_1^2 + \phi_2^2 - v^2) \, , \tag{3.2.19} \]
\[ \partial_r \phi_1 = \frac{1}{r} (f + f_{NA}) \phi_1 \, , \quad \frac{1}{r} \partial_r f_{NA} = \frac{g^2}{4} (\phi_1^2 - \phi_2^2) \, , \tag{3.2.20} \]
\[ \partial_r \phi_2 = \frac{1}{r} (f - f_{NA}) \phi_2 \, , \tag{3.2.21} \]

with the following boundary conditions

\[ \phi_{0,1,2}(\infty) = \frac{v}{\sqrt{2M}} \, , \quad f(\infty) = f_{NA}(\infty) = 0 \, , \]
\[ \phi_0(0) = \phi_1(0) = 0 \, , \quad \partial_r \phi_2(0) = 0 \, , \quad f(0) = f_{NA}(0) = 1 \, . \tag{3.2.22} \]
We have furthermore made a basis change such that the invariant rank-two tensor of $SO(2M)$ is

$$J = \begin{pmatrix} 1_{2M-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.2.23)$$

The Ansatz for the gauge fields $A_{0,3}$ is still given by Eq. (3.1.21), however the reducing matrix is now [70]:

$$U = \begin{pmatrix} \sqrt{1_{2M-2} - EE^\dagger} & E \\ -E^\dagger \sqrt{1 - E^\dagger E} \end{pmatrix}, \quad (3.2.24)$$

where

$$E \equiv \frac{\sqrt{2}}{D} \left( \varphi \; \bar{\varphi} \right), \quad (3.2.25)$$

$$D \equiv \sqrt{1 + 2\varphi^\dagger \varphi + (\varphi^T \varphi) (\bar{\varphi}^\dagger \bar{\varphi})}. \quad (3.2.26)$$

$E$ is a $(2M - 2) \times 2$-dimensional matrix and $\varphi$ is a $(2M - 2)$-dimensional column vector, while the following matrix expressions are essential for the calculation

$$\sqrt{1 - E^\dagger E} = \frac{1}{D} \begin{pmatrix} 1 & -\varphi^\dagger \bar{\varphi} \\ -\varphi^T \bar{\varphi} & 1 \end{pmatrix}, \quad (3.2.27)$$

$$\sqrt{1_{2M-2} - EE^\dagger} = 1_{2M-2} - \frac{(1 + D) (\varphi \varphi^\dagger + \bar{\varphi} \bar{\varphi}^T) + (\varphi^T \varphi) \bar{\varphi} \varphi^\dagger + (\varphi^\dagger \bar{\varphi}) \varphi \bar{\varphi}^T}{D (1 + \varphi^\dagger \varphi + D)}. \quad (3.2.28)$$

Now we will follow the recipe of Sec. 3.1.1, by going to the singular gauge and rotating with the color-flavor rotation $U$ of Eq. (3.2.24)

$$q = \phi_0 1_{2M} + \frac{1}{2} (\phi_1 + \phi_2 - 2\phi_0) UT_1 U^{-1} + \frac{1}{2} (\phi_1 - \phi_2) UT_2 U^{-1},$$

$$A_i = -\epsilon_{ij} \frac{x_j}{r^2} \left[ f 1_{2M} + f_N UT_2 U^{-1} \right], \quad (3.2.29)$$

from which together with the Ansatz (3.1.21) and

$$T = 1_{2M} - 2T_1 = \begin{pmatrix} 1_{2M-2} \\ -1 \\ -1 \end{pmatrix}, \quad (3.2.30)$$
we can calculate the contributions
\[ \text{Tr} \mid D_\alpha q \mid^2 = - \left[ (1 - \rho) \left[ (\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 \right] + \frac{\rho^2}{2} \left( 2\phi_0^2 + \phi_1^2 + \phi_2^2 \right) \right] \]
\[ \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_1 X_\alpha \right], \quad (3.2.31) \]
\[ \frac{1}{g^2} \text{Tr} F_{\mu\alpha}^2 = - \frac{2}{g^2} \left[ (\partial_r \rho)^2 + \frac{1}{r^2 f_{\text{NA}}^2} (1 - \rho)^2 \right] \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_1 X_\alpha \right], \quad (3.2.32) \]
where \( X_\alpha \equiv U^{-1} \partial_\alpha U \) and we have used the following non-trivial relations
\[ \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_1 X_\alpha \right] = \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_2 X_\alpha \right], \quad \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_2 X_\alpha \right] = 0. \quad (3.2.33) \]

Let us use the notation
\[ X_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix}. \quad (3.2.34) \]

The first relation of Eq. (3.2.33) can be proved by showing that \( D_\alpha \) is indeed diagonal, while the second relation can be proved by showing that \( B_\alpha \tau^3 C_\alpha \) is antisymmetric, and hence traceless.

The following trace can be rewritten as
\[ \text{Tr} \left[ (I_{2M} - T_1) X_\alpha T_1 X_\alpha \right] = \frac{1}{8} \text{Tr} [X_\alpha - TX_\alpha T]^2 = \frac{1}{2} \text{Tr} [(X_\alpha)_\perp]^2. \quad (3.2.35) \]

After the dust settles one finds the effective world-sheet action
\[ S_{1+1} = 2\beta \int dt \, dz \, 4 \left\{ \partial_\alpha \varphi^\dagger \partial_\alpha \varphi + 2 \mid \varphi^\dagger \partial_\alpha \varphi \mid^2 \right\} + \frac{2}{1 + 2\varphi^\dagger \varphi + \mid \varphi^\dagger \varphi \mid^2} \left\{ \partial_\alpha \varphi^\dagger \partial_\alpha \varphi + (\varphi^\dagger \partial_\alpha \varphi) (\varphi^\dagger \partial_\alpha \varphi) \right\} , \quad (3.2.36) \]
where
\[ \beta = \frac{2\pi}{g^2 I} , \quad (3.2.37) \]
and the normalizing integral now reads
\[ I = \int_0^\infty dr \, r \left[ (\partial_r \rho)^2 + \frac{1}{r^2 f_{\text{NA}}^2} (1 - \rho)^2 
+ \frac{g^2 \rho^2}{2} (1 - \rho) \left[ (\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 \right] + \frac{g^2 \rho^2}{4} \left( 2\phi_0^2 + \phi_1^2 + \phi_2^2 \right) \right] . \quad (3.2.38) \]
The boundary conditions for \( \rho(r) \) are
\[
\rho(0) = 1, \quad \rho(\infty) = 0 ,
\]
while its equation of motion is simply
\[
\frac{1}{r} \partial_r (r \partial_r \rho) + \frac{1}{r^2} f_{NA}^2 (1 - \rho) + \frac{g^2}{4} \left[ (\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 \right] - \frac{g^2 \rho}{4} (2\phi_0^2 + \phi_1^2 + \phi_2^2) = 0 .
\]
(3.2.40)

It is non-trivial to find a solution to this non-linear equation. To find the solution, the crucial point is the non-trivial relation
\[
\phi_0^2 = \phi_1 \phi_2 .
\]
(3.2.41)

By using this relation, the solution can be expressed in several different forms, which however can be seen all to be equivalent to each other:
\[
\rho = 1 - \frac{\phi_0}{\phi_2} = 1 - \frac{1}{2} \left( \frac{\phi_1}{\phi_0} + \frac{\phi_0}{\phi_2} \right) = 1 - \frac{\phi_0 (\phi_1 + \phi_2)}{\phi_0^2 + \phi_2^2} ,
\]
(3.2.42)

To prove the relation (3.2.41), we combine the BPS-equations as follows
\[
\partial_r \log \left( \frac{\phi_0^2}{\phi_1 \phi_2} \right) = 0 ,
\]
(3.2.43)

from which it follows that this ratio is a constant. This constant is given by the boundary conditions and hence is equal to one.

Now we can plug the result into the normalizing integral and by using the BPS equations again, we find that the integral reduces to
\[
\mathcal{I} = \int_0^\infty dr \partial_r \left( f_{NA} \left[ \left( \frac{\phi_0}{\phi_2} \right)^2 - 1 \right] \right) = f_{NA}(0) = 1 .
\]
(3.2.44)

The action (3.2.36) is exactly that of the (1 + 1)-dimensional sigma model on the Hermitian symmetric space \( SO(2M)/[SO(2) \times SO(2M - 2)] \) [70]. It has a Kähler metric: the Kähler potential is given by
\[
K = \log \left( 1 + 2\phi^\dagger \phi + |\varphi^T \varphi|^2 \right) .
\]
(3.2.45)
3.3 Conclusion

In this chapter, the low-energy vortex effective action is constructed in a wide class of systems in a color-flavor locked vacuum, which generalizes the results found earlier in the context of $U(N)$ models. It describes the weak fluctuations of the non-Abelian orientational moduli on the vortex world-sheet. For instance, for the minimum vortex in $SO(2M) \times U(1)$ or $USp(2M) \times U(1)$ gauge theories, the effective action found is a two-dimensional sigma model living on the Hermitian symmetric spaces $SO(2M)/U(M)$ or $USp(2M)/U(M)$, respectively. Applied to the benchmark $U(N)$ model our procedure reproduces the known $\mathbb{C}P^{N-1}$ world-sheet action; our recipe allows us to obtain also the effective vortex action for some higher-winding vortices in $U(N)$ and $SO(2M)$ theories.
Chapter 4

Mass-deformed effective theory

When non-Abelian vortices are the host solitons, the supersymmetric sigma model with twisted mass exhibit kinks, which are found to be the confined monopoles [40, 41]. In this chapter, we will extend the non-linear sigma models to the case of the mass-deformed theories. As in the $\mathbb{C}P^{N-1}$ model, a potential will be induced on the sigma model. We will make use of the Ansatz for the adjoint scalar field to induce the effective potential for the non-linear sigma model. We also check the result from Scherk-Schwarz dimensional reduction [57, 58], which allows us to calculate the mass term from the kinetic term directly.

4.1 The $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models

Abelian monopoles can occur as confined junctions (kinks) on vortices [41], which will produce a “shallow” potential for the world sheet action of vortices. In the bulk theory, the monopole solutions originate from the symmetry broken by adjoint scalars. So we need to consider the contribution from the adjoint field in the bulk. There are two terms related to adjoint fields in the bulk Hamiltonian, which are written as

$$S_{\text{adj}} = \int d^4x \text{Tr} \left[ \frac{2}{g^2} |D_i \phi|^2 + 2|\phi + \phi_q + qM|^2 \right]. \quad (4.1.1)$$

where we rescale the mass matrix $M \to \sqrt{2}M$. 
4.1.1 The universal Ansatz

Let us consider the case of $G' = SO(2M)$ or $G' = USp(2M)$. The mass matrix can be written with the form $M = \text{diag}(\hat{m}, -\hat{m})$ with $\hat{m} = \text{diag}(m_1, \cdots, m_M)$. Our Ansatz for the adjoint field reads

$$\phi = 0,$$

$$\hat{\phi} = -\frac{1}{2} \left[ (1 + b(r)) M + (1 - b(r)) UTU^{-1} M UTU^{-1} \right],$$

with $b(r)$ a radial profile function subject to the boundary conditions $b(\infty) = 1$ and $b(0) = 0$. While the matrices $U$ are the reducing matrix given in Eq.(3.1.11). The Ansatz for the squark $q$, gauge field $A_1, A_\alpha$ are also given in Sec.(3.1.1). Being ready with all these materials, we can calculate the two terms in (4.1.1), which are written as

$$\frac{2}{g^2} \text{Tr} \left| \mathcal{D}_r \hat{\phi} \right|^2 = \frac{2}{g^2} \left[ (\partial_r b)^2 + \frac{1}{r^2} b^2 f_{NA}^2 \right] \text{Tr}[M_\perp^2],$$

$$2 \text{Tr}[ (\phi + \hat{\phi}) q + q M]^2 = 2 \left[ \frac{(1 - b)^2}{2} (\phi_1^2 + \phi_2^2) + b(\phi_1 - \phi_2)^2 \right] \text{Tr}[M_\perp^2].$$

where

$$\text{Tr}[M_\perp^2] = \text{Tr} \left[ X^{-1} \{ \hat{m}, B^\dagger \} Y^{-1} \{ \hat{m}, B \} \right].$$

Now the action (4.1.1) becomes

$$S_{adj} = 2\beta_2 \int d^2 x \text{Tr} \left[ X^{-1} \{ \hat{m}, B^\dagger \} Y^{-1} \{ \hat{m}, B \} \right].$$

The integration is along the direction of vortex string and time. $\beta_2$ in an integration over the transverse plane of vortices, which are written as

$$\beta_2 \equiv \frac{2\pi}{g^2} \int dr \left\{ (\partial_r b)^2 + \frac{1}{r^2} f_{NA}^2 b^2 + g^2 \left[ \frac{(1 - b)^2}{2} (\phi_1^2 + \phi_2^2) + b(\phi_1 - \phi_2)^2 \right] \right\}. $$

Minimizing the action for the function $b(r)$ in the radial direction $r$, we obtain the equation of motion for $b(r)$

$$b'' + \frac{b'}{r} - \frac{f_{NA}}{r^2} b + \frac{g^2}{2} (\phi_1^2 + \phi_2^2) (1 - b) - \frac{g^2}{2} (\phi_1 - \phi_2)^2 = 0,$$

which can be solved by $b = \phi_1/\phi_2$. In sec.(3.1.1), we have already solved the equation of motion for $\rho$. Comparing these two equations, there is a simple non-trivial relation between $\rho$ and $b$, i.e.

$$\rho + b = 1.$$
4.1 The $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models

This relation holds for arbitrary $r$.

Plugging $b = \phi_1/\phi_2$ back into the action, we obtain

$$\beta_2 = \frac{2\pi}{g^2} \int dr \partial_r \left( f_{NA} \left[ \left( \frac{\phi_1}{\phi_2} \right)^2 - 1 \right] \right) = \frac{2\pi}{g^2} f_{NA}(0) = \frac{2\pi}{g^2}.$$  (4.1.10)

Now the mass term reads $2\beta_2 \text{Tr} \left[ X^{-1} \left\{ \hat{m}, B^\dagger \right\} Y^{-1} \left\{ \hat{m}, B \right\} \right]$, in agreement with Eq. (4.1.31).

4.1.2 The complete low-energy effective action

The $\theta$-term in the bulk also contributes to the world sheet effective action of the sigma model. Referring to the Ansatz in Eq. (3.1.6) and Eq. (3.1.21), the $\theta$-term reads that

$$i\theta \frac{16\pi^2}{16\pi^2} \text{Tr} \left[ \hat{F}_\mu \hat{F}^{\mu} \right] = -\frac{\theta}{2\pi} I_\theta \int dt dz \epsilon^{\alpha\beta} \text{Tr} \left[ X^{-1} [\partial_\alpha B^\dagger Y^{-1} \partial_\beta B] \right],$$  (4.1.11)

in which $I_\theta$ is a function with analytical solution

$$I_\theta = -\int_0^\infty dr \left[ (2 - 2\rho) f_{NA} \frac{d\rho}{dr} + (2\rho - \rho^2) \frac{df_{NA}}{dr} \right].$$  (4.1.13)

$$= -\int_0^\infty dr \frac{d}{dr} [2f_{NA} - \rho^2 f_{NA}].$$  (4.1.14)

With the boundary conditions in Eq. (3.1.4), we can calculate

$$I_\theta = 1.$$  (4.1.15)

Collecting all the terms, now we have the total low-energy effective action for the system

$$S_{1+1} = \frac{4\pi}{g^2} \int dt dz \text{tr} \left[ X^{-1} [\partial_\alpha B^\dagger Y^{-1} \partial_\beta B + X^{-1} \left\{ \hat{m}, B^\dagger \right\} Y^{-1} \left\{ \hat{m}, B \right\} \right]$$

$$-\frac{\theta}{2\pi} \int dt dz \epsilon^{\alpha\beta} \text{tr} \left[ X^{-1} [\partial_\alpha B^\dagger Y^{-1} \partial_\beta B] \right].$$  (4.1.16)

This is the exact $1+1$ dimension sigma model with twisted mass. Introducing the complex coupling constants,

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}.$$  (4.1.17)
The Lagrangian is can be written in a compact form

\[ \mathcal{L} = \mathcal{S} \text{Tr} \left[ \tau \left( Y^{-1} \partial_- B X^{-1} \partial_+ B^\dagger \right) \right], \quad \partial_- \equiv \partial_t - \partial_Z, \quad \partial_+ \equiv \partial_t + \partial_Z. \tag{4.1.18} \]

The 1 + 1 dimension supersymmetric model in superfield form can be constructed easily, by promoting the field \( B \) to a chiral superfield.

### 4.1.3 Scherk-Schwarz dimension reduction

It is necessary to introduce Scherk-Schwarz dimension reduction method to calculate the mass term [57, 58]. We calculate the \( \mathbb{C}P^1 \) sigma model as a warm up. The Kähler potential for the \( \mathbb{C}P^1 \) model is

\[ K = 2 \beta \log \left( 1 + |b|^2 \right), \tag{4.1.19} \]

giving rise to the sigma model

\[ \mathcal{L} = 2 \beta \frac{\partial_\alpha b \partial_\bar{\alpha} b^\dagger}{(1 + |b|^2)^2}, \tag{4.1.20} \]

with \( \alpha = 0, 3 \) in Euclidean notation. We can generate the twisted mass potential on the sigma model by using the following method which was first put forward in Ref. [68].

First we have to use a global symmetry of the system under consideration. In the case of the \( \mathbb{C}P^1 \) model, there exists a global phase transformation

\[ b \rightarrow e^{i\alpha} b, \tag{4.1.21} \]

which leaves the Lagrangian (4.1.20) invariant. Next, we add a fictitious (spatial) dimension to that of the sigma model fields

\[ b(t, z) \rightarrow b(t, z, \vartheta), \tag{4.1.22} \]

which we compactify on a circle of radius \( R \) according to the well-known Scherk-Schwarz dimensional reduction [57, 58] as opposed to the trivial dimensional reduction. Now we can expand the field in modes as

\[ b(t, z, \vartheta) = e^{im\vartheta} \sum_{n=-\infty}^{\infty} b_n(t, z) e^{im\vartheta}, \tag{4.1.23} \]
and then use the prescription of Ref. [57], i.e. to send the compactification radius to zero, \( R \to 0 \), yielding
\[
b(t, z, \vartheta) = e^{im\vartheta}b_0(t, z) . \tag{4.1.24}\]
Plugging this field back into the Lagrangian (4.1.20) leaves us with the following mass-deformed theory [41]
\[
\mathcal{L} = \frac{2\beta}{(1 + |b|^2)^2} \left( \partial_\alpha b \partial_\alpha b^\dagger + m^2 |b|^2 \right) , \tag{4.1.25}\]
where we have dropped the suffix. Note that the theory is known to have two vacua. The description we have used here uses the inhomogeneous coordinates on \( \mathbb{C}P^1 \) and hence we need two patches to describe the theory. On each patch the vacuum is seen to be given by \( b = 0 \), which corresponds to \( b \to \infty \) in the other patch. Hence we have checked (trivially) that the number of vacua found is indeed 2, in accord with the known result.

Let us now apply the Scherk-Schwarz dimension reduction technique to the sigma models on the target spaces \( SO(2M)/U(M) \) and \( USp(2M)/U(M) \). We will treat them on the same footing in the following. For \( SO(2M)/U(M) \) the field \( B^T = -B \) is an anti-symmetric matrix valued field while for \( USp(2M)/U(M) \) it is symmetric \( B^T = B \). The Kähler potential reads (see (3.1.32))
\[
K = 2\beta \text{Tr} \log \left( \mathbf{1}_M + BB^\dagger \right) , \tag{4.1.26}\]
which leads to the Lagrangian (3.1.26)
\[
\mathcal{L} = 2\beta \text{Tr} \left\{ \left( \mathbf{1}_M + BB^\dagger \right)^{-1} \partial_\alpha B^\dagger \left( \mathbf{1}_M + BB^\dagger \right)^{-1} \partial_\alpha B \right\} . \tag{4.1.27}\]
The following global symmetry of the Lagrangian
\[
B \to UBU , \tag{4.1.28}\]
can be used to generate the twisted mass potential with
\[
U = e^{iM\vartheta} , \tag{4.1.29}\]
\( M^\dagger = M \) being an \( M \)-by-\( M \) Hermitian mass matrix, \( \vartheta \in \mathbb{R} \) and \( U \) is manifestly unitary. As above, we expand the field in modes and keep just the lowest mode,
\[
B(t, z, \vartheta) = e^{iM\vartheta}B_0(t, z)e^{iM\vartheta} . \tag{4.1.30}\]
Upon inserting this field in the Lagrangian (4.1.27) and dropping the suffix, we obtain the following mass-deformed sigma model

\[
\mathcal{L} = 2 \beta \text{Tr} \left\{ \left( \mathbf{1}_M + B^\dagger B \right)^{-1} \partial_\alpha B^\dagger \left( \mathbf{1}_M + BB^\dagger \right)^{-1} \partial_\alpha B \\
+ \left( \mathbf{1}_M + B^\dagger B \right)^{-1} \{ M, B^\dagger \} \left( \mathbf{1}_M + BB^\dagger \right)^{-1} \{ M, B \} \right\}. 
\] (4.1.31)

As the mass matrix is Hermitian, the vacuum equation reads

\[
\{ M, B \} = 0, 
\] (4.1.32)

which in general can only be satisfied for \( B = 0 \). Since \( B \) is an inhomogeneous coordinate on the target space of the sigma model, we can conclude that there exist a vacuum for each patch, giving

\[
n_{\text{vacua}}^{USp(2M)} = 2^M, 
\] (4.1.33)

in the case of \( USp(2M) \) while it decomposes for topological reasons, as explained in Ref. [50], into

\[
n_{\text{vacua}}^{SO(2M)} = 2^{M-1} + 2^{M-1}, 
\] (4.1.34)

where the two sectors have different topological \( \mathbb{Z}_2 \) charge.

### 4.2 The \( \mathbb{C}P^{N-1} \) sigma model

The Ansatz for the orientational moduli of the coincident \( k \)-winding vortex in the \( U(N) \) model is given in chapter 3. The mass matrix is defined as

\[
M = \begin{pmatrix} m_1 & \mathbf{1}_{N-1} \\
& \mathbf{1}_{N-1} \end{pmatrix},
\] (4.2.1)

in which \( \mathbf{1}_{N-1} \) is a diagonal \( N - 1 \) by \( N - 1 \) matrix.

Working in the singular gauge will save a lot of time for calculating, the squark field \( q \) and gauge field \( A_i \) transform as

\[
q \rightarrow V q, \quad A_i \rightarrow VA_i V^\dagger + i\partial_i V V^\dagger.
\] (4.2.2)

where the \( V \) is written as

\[
V = \begin{pmatrix} e^{-i\theta} & \mathbf{1}_{N-1} \\
& \mathbf{1}_{N-1} \end{pmatrix}. 
\] (4.2.3)
4.2 The $\mathbb{C}P^{N-1}$ sigma model

Using the reducing matrix $U$ to act on squark $q$ and gauge field $A_i$, the Ansatz of $q$ and $A_i$ with the orientational zero-modes are written as

\[
q = \frac{1}{2}(\phi_1 + \phi_2)1_N + \frac{1}{2}(\phi_1 - \phi_2)UTU^\dagger, \tag{4.2.4}
\]

\[
A_i = -\epsilon_{ij}\frac{x_j}{r^2} \left[f \frac{1_N}{N} + \frac{1}{2}f_{NA}(UTU^\dagger + \frac{N - 2}{N}1_N)\right]. \tag{4.2.5}
\]

The Ansatz for the adjoint scalar field $\hat{\phi}$ is universal as given in Eq. (4.1.2), but $T$ is replaced to be $T$ in (3.1.36).

The potential term becomes

\[
D_i \hat{\phi} = -\frac{x_i}{2r^2}\partial_r b \left(M - UTU^\dagger M1_TU^\dagger\right) + \frac{i}{2}\epsilon_{ij}\frac{x_j}{r^2}f_{NA} b [UTU^\dagger, M], \tag{4.2.6}
\]

\[
\hat{\phi} q + q M = \frac{\phi_1 - \phi_2}{2} \left[UTU^\dagger, M\right] + \frac{\phi_1 + \phi_2}{2} \left[M - UTU^\dagger M1_TU^\dagger\right]. \tag{4.2.7}
\]

The first and the second matrix terms in Eq. (4.2.6) are orthogonal,

\[
\text{Tr} \left\{ [UTU^\dagger, M] (M - UTU^\dagger M1_TU^\dagger) \right\} = 0, \tag{4.2.8}
\]

but the trace of their squares are equal and agree with Ref. [66]

\[
\text{Tr} |[UTU^\dagger, M]|^2 = \text{Tr} |M - UTU^\dagger M1_TU^\dagger|^2 = 8 \text{Tr} [(M_\perp)^2], \tag{4.2.9}
\]

where $\text{Tr} [(M_\perp)^2]$ is defined as

\[
\text{Tr} [(M_\perp)^2] = \text{Tr} \left[\frac{m_1^2 + B^\dagger M_{N-1}B}{1 + B^\dagger B} \right] - \frac{m_1^2 + (B^\dagger M_{N-1}B)^2}{(1 + B^\dagger B)^2} \right]. \tag{4.2.10}
\]

in which $B$ is defined in (3.1.39).

The following steps are mainly the same with $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models. The effective action is calculated to be

\[
S_{\text{adj}} = 2\beta_2 \int d^2x \ 2\text{Tr} [(M_\perp)^2], \tag{4.2.11}
\]

where $\beta_2$ have the same expression given in Eq. (4.1.7). However, the BPS equations of $\mathbb{C}P^{N-1}$ are distinguished with that of $SO$ and $USp$ case, the solution of $b(r)$ need to be considered more carefully. Observing that the profile of non-Abelian gauge field $f_{NA}$ is the coincident with $f_{NA}$ in Eq. (3.1.8), and only this expression is necessary to solve $\beta_2$ here. Now the answer is that

\[
b = \frac{\phi_1}{\phi_2}. \tag{4.2.12}
\]
The $\theta$ term is calculated to be
\[
\frac{i\theta}{16\pi^2} \text{Tr} \left[ \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right] = -\frac{\theta}{2\pi} T \int d^2x \, e^{\alpha\beta} \text{Tr} \left[ \partial_\alpha (X^{-\frac{1}{2}} B^\dagger) \partial_\beta (Y^{-\frac{1}{2}} B) \right].
\] (4.2.13)

where $T = 1$ for the minimal winding case. It is also can be expressed in terms of $n$, which read
\[
e^{\alpha\beta} \text{Tr} \left[ \partial_\alpha (X^{-\frac{1}{2}} B^\dagger) \partial_\beta (Y^{-\frac{1}{2}} B) \right] = e^{\alpha\beta} \text{Tr} (\partial_\alpha n^\dagger \partial_\beta n) . \]
(4.2.14)

The recipe can also be extended to the higher winding case, as discussed in Sec. 3.2.2, now the only difference is that
\[
I_\theta = k .
\] (4.2.15)

### 4.2.1 SS dimension reduction and Vacua

Now we use SS dimension method to verify. We have the following Kähler potential for $\mathbb{C}P^{N-1}$,
\[
K = 2 \beta \log \left( 1 + B^\dagger B \right) ,
\] (4.2.16)

where $B$ is an vector of $N-1$ components. This Kähler potential leads to the Lagrangian [63, 66]
\[
\mathcal{L} = 2 \beta \left[ \frac{\partial_\alpha B^\dagger \partial_\alpha B}{1 + B^\dagger B} - \frac{(B^\dagger \partial_\alpha B) (\partial_\alpha B^\dagger B)}{(1 + B^\dagger B)^2} \right] .
\] (4.2.17)

As above we use the following global symmetry of the Lagrangian
\[
B \rightarrow UB ,
\] (4.2.18)

with $U$ unitary to generate the twisted mass potential. Keeping just the lowest mode
\[
B(t, z, \vartheta) = e^{\tilde{M} \vartheta} B(t, z) ,
\] (4.2.19)

where $\tilde{M}$ is an $(N-1) \times (N-1)$ matrix and is Hermitian, i.e., $\tilde{M}^\dagger = \tilde{M}$. Insertion of this field into the Lagrangian gives us the deformed sigma model
\[
\mathcal{L} = 2 \beta \left[ \frac{\partial_\alpha B^\dagger \partial_\alpha B + B^\dagger \tilde{M}^2 B}{1 + B^\dagger B} - \frac{(B^\dagger \partial_\alpha B) (\partial_\alpha B^\dagger B) + (B^\dagger \tilde{M} B)^2}{(1 + B^\dagger B)^2} \right] .
\] (4.2.20)
4.3 The quadratic surface $Q^{2M-2}$ sigma model

The mass term of Eq.(4.2.20) are the same with mass term calculated by using the Ansatz method if we draw on the relation

$$\tilde{M} = M_{N-1} - m_1 1_{N-1}. \quad (4.2.21)$$

The vacuum equations are

$$B^\dagger \tilde{M}^2 B = 0, \quad B^\dagger \tilde{M} B = 0, \quad (4.2.22)$$

which in general are only satisfied by $B = 0$. Hence, there will be $N$ vacua in accord with Witten’s result

$$n_{\text{vacua}}^{SU(N)} = N, \quad (4.2.23)$$

where $N$ is the number of patches needed to describe $\mathbb{C}P^{N-1}$.

4.3 The quadratic surface $Q^{2M-2}$ sigma model

As explained in detail in Ref. [56], the non-Abelian vortex in $U(1) \times SO(2M)$ has an irreducible orbit in the higher winding $k = 2$ case, which has as an effective low-energy theory on the world-sheet sigma model on the Hermitian symmetric space $Q^{2M-2} = \frac{SO(2M)}{SO(2) \times SO(2M-2)}$.

The mass matrix for the $Q^{2M-2}$ sigma model is written as

$$M = \begin{pmatrix} M_n & 0 \\ 0 & M_2 \end{pmatrix}, \quad (4.3.1)$$

where $M_n$ is Hermitian and anti-symmetric $(2M-2) \times (2M-2)$ matrix, $M_2$ is chosen to be $M_2 = \text{diag}(m, -m)$. The expression of $M$ here depends on the basis of the algebra $J$ [56], and belongs to the Cartan-subalgebra of the invariant subgroup $SO(2) \times SO(2M-2)$.

The Ansatz for squark $q$, gauge field $A_i$ of this vortex has already been given in last chapter and in [56], we follow the same notation here. The Ansatz for adjoint scalar has the form Eq.(4.1.2) with $M$ defined as in (4.3.1).

A lengthy but happy calculation shows the contribution as follow

$$\text{Tr}|D_i \hat{\phi}|^2 = \left[(b')^2 + \frac{f_{NA}b^2}{r^2}\right] \cdot 2\text{Tr}(M_1^2), \quad (4.3.2)$$

$$2\text{Tr} \hat{\phi} q + qM|^2 = \left[\frac{1}{2}(1 + b^2)(\phi_1^2 + \phi_2^2 + 2\phi_0^2) - 2b\phi_0(\phi_1 + \phi_2)\right] \cdot 2\text{Tr}(M_2^2), \quad (4.3.3)$$
where \( M_{\perp}^2 \) is defined as

\[
\text{Tr}(M_{\perp}^2) \equiv \frac{4}{D^2} \varphi^\dagger M_n \varphi - \frac{8}{D^4} (\varphi^\dagger M_n \varphi)^2 + \frac{4}{D^2} m^2 \varphi^\dagger \varphi - \frac{8}{D^4} m^2 (|\varphi^\dagger \varphi|^2 - |\varphi^T \varphi|^2)
\]

\[- \frac{8}{D^4} m \varphi^\dagger M_n \varphi (1 - |\varphi^T \varphi|^2) \]  

(4.3.4)

The partial 4 dimensional action will become

\[
\int d^4x \text{Tr} \left[ \frac{2}{g^2} |D_i \hat{\varphi}|^2 + 2 |q \varphi + q M|^2 \right] = 2 \beta_2 \int dx^2 4 \text{Tr}(M_{\perp}^2),
\]

(4.3.5)

where \( \beta_2 \) is written as

\[
\beta_2 \equiv \frac{2 \pi}{g^2} \int dr r \left\{ (\partial_r b)^2 + \frac{1}{g^2} f_{nA}^2 b^2 + \frac{g^2}{4} [(1 - b)^2 (\phi_1^2 + \phi_2^2 + 2 \phi_0^2)]
\]

\[+ \frac{g^2}{2} b [(\phi_1 - \phi_0)^2 + (\phi_2 - \phi_0)^2] \right\}.
\]

(4.3.6)

The solution for \( b(r) \) is easy to obtain, we obtain

\[
b = 1 - \rho = \frac{\phi_0}{\phi_2} = \frac{\phi_1}{\phi_0}.
\]

(4.3.7)

4.3.1 SS dimensional reduction

Now the SS dimensional reduction is used to check the result. The \( Q^{2M-2} \) sigma model has the following Kähler potential

\[
K = 2 \beta \log \left( 1 + 2 \varphi^\dagger \varphi + |\varphi^T \varphi|^2 \right),
\]

(4.3.8)

giving rise to the Lagrangian [56]

\[
\mathcal{L} = 8 \beta \left\{ \frac{\partial_\alpha \varphi^\dagger \partial_\alpha \varphi + 2 |\varphi^T \partial_\alpha \varphi|^2}{1 + 2 \varphi^\dagger \varphi + |\varphi^T \varphi|^2} - \frac{2 |\varphi^\dagger \partial_\alpha \varphi + (\varphi^\dagger \varphi) (\varphi^T \partial_\alpha \varphi)|^2}{[1 + 2 \varphi^\dagger \varphi + |\varphi^T \varphi|^2]^2} \right\},
\]

(4.3.9)

where \( \phi \) is a complex \( 2M - 2 \) component vector. The Lagrangian is symmetric under the following transformation

\[
\varphi \rightarrow U \varphi,
\]

(4.3.10)

where \( U^\dagger U = 1 \). Choosing

\[
U = e^{i \tilde{M} \phi},
\]

(4.3.11)
where $\tilde{M}$ is the Mass matrix for $\varphi$, and both the $SO(2M - 2)$ and $U(1)$ are contained in $\tilde{M}$, i.e.

$$\tilde{M} = M_n - m_1 n.$$  \hfill (4.3.12)

it is clear that $M_n$ has to be Hermitian and anti-symmetric and hence purely imaginary.

Now keeping only the zero modes upon compactification, we get

$$\varphi(t, z, \vartheta) = e^{i\tilde{M}\vartheta}\varphi_0(t, z).$$  \hfill (4.3.13)

Inserting this field into the Lagrangian (4.3.9) we obtain

$$L = 8\beta\left[\frac{\partial_\alpha\varphi^\dagger\partial_\alpha\varphi + 2|\varphi^T\partial_\alpha\varphi|^2}{1 + 2\varphi^\dagger\varphi + |\varphi^T\varphi|^2} - \frac{2|\varphi^\dagger\varphi + (\varphi^\dagger\varphi^\dagger\varphi)\varphi^T\partial_\alpha\varphi|}{1 + 2\varphi^\dagger\varphi + |\varphi^T\varphi|^2} + \frac{\varphi^\dagger M_n^2 \varphi + m^2 \varphi^\dagger \varphi}{1 + 2\varphi^\dagger\varphi + |\varphi^T\varphi|^2}\right].$$  \hfill (4.3.14)

where we have used that $\varphi^T M_n \varphi = 0$ due to the anti-symmetry of the mass matrix. The vacuum equation reads

$$\varphi^\dagger M_2 \varphi = 0, \quad \varphi^\dagger \tilde{M} \varphi = 0,$$  \hfill (4.3.15)

which for a generic choice of the mass matrix yields the only solution $\varphi = 0$. Hence, we find the number of vacua to be

$$n_{\text{vacua}}^{SO(2M), k=2} = M + M,$$  \hfill (4.3.16)

where one set of $M$ vacua are in the topological sector with positive $\mathbb{Z}_2$-charge while the other $M$ vacua have a negative one. This result is indeed expected as this irreducible orbit of the corresponding vortex should transform as a vector.

We have now considered a few sigma models which are all low-energy effective descriptions of non-Abelian vortex systems. The number of vacua in the classical regime is expected to remain the same in the quantum regime.

### 4.4 Conclusion

There is a concrete quantitative correspondence between supersymmetry theories in two and four dimensions. The BPS spectrum of the mass-deformed two-dimensional $\mathcal{N} = (2, 2)$
The $\mathbb{C}P^{N-1}$ sigma model coincides with the BPS spectrum of the four dimensional $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theories [77, 78]. This reason lies that the two-dimensional sigma models are effective low-energy theories describing orientational moduli on the world sheet of non-Abelian confining strings [66, 67]. In this chapter, appropriate Ansatz for the adjoint scalars will be given to realize the correspondence exactly. By the advantage of this Ansatz, we integrate two terms related to adjoint fields in the bulk four-dimensional theory, and find the result to be a mass potential term for the corresponding sigma models. The integration is analytically solvable due to the BPS equations. Three concrete examples are discussed, the $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models, the $\mathbb{C}P^{N-1}$ sigma models, and the quadratic surface $Q^{2M-2}$ sigma models. Some high winding cases in the last chapter can also be applied, for instance, the completely anti-symmetric $k$ winding vortices. We also check the result from Scherk-Schwarz dimensional reduction [57, 58], which calculates the mass term from kinetic term directly. We found that the integration and SS dimension reduction produce the same result.
Chapter 5

Group theory of non-Abelian vortices

5.1 Moduli space of non-Abelian $U(N)$ vortices

The moduli space of the non-Abelian $U(N)$ vortices governed by the BPS Eq. (2.2.4) was first studied by Hanany-Tong \[63\]. There the dimension of the moduli space $\mathcal{M}_k$ of $k$ vortices has been shown by using an index theorem calculation to be\(^1\)

$$\dim \mathcal{C} \mathcal{M}_k = kN, \quad (5.1.1)$$

with $k$ being the topological winding number. Moreover, they found a D-brane configuration and derived a Kähler quotient construction for $\mathcal{M}_k$. It is sometimes called a half-ADHM construction by analogy with the moduli space of instantons. In the D-brane configuration, the $k$ vortices are $k$ D2-branes suspended between $N$ D4-branes and an NS5-brane. The low-energy effective field theory on the $k$ D2-branes is described by a $U(k)$ gauge theory coupled with a $k$-by-$k$ matrix $Z$ in adjoint representation and a $k$-by-$N$ matrix $\psi$ in the fundamental representation $\mathbf{k}$ of the $U(k)$ gauge symmetry, given by D2–D2 strings and D2–D4 strings, respectively. The $U(k)$ gauge symmetry on the D2-branes acts on $Z$ and $\psi$ as

$$(Z, \psi) \rightarrow (gZg^{-1}, g\psi), \quad g \in U(k). \quad (5.1.2)$$

\(^1\)The general result of Ref. [63] in $U(N)$ theory for $N_f \geq N$ flavors is $\dim \mathcal{C} \mathcal{M}_k = kN_f$. However, we restrict our attention to the case $N_f = N$ and hence local vortices in this paper.
The moduli space $\mathcal{M}_k$ can be read off as the Higgs branch of vacua in the $U(k)$ gauge theory on the $k$ D2-branes, which is the Kähler quotient of the $U(k)$ action (5.1.2)\(^2\)

$$\mathcal{M}_k \cong \mathcal{M}_k^{\text{HT}} \equiv \{(Z, \psi) \mid \mu_D = r\mathbf{1}_k\} / U(k),$$

(5.1.3)

$$\mu_D \equiv [Z, Z^\dagger] + \psi \psi^\dagger.$$  

(5.1.4)

This Kähler quotient gives a natural metric on $\mathcal{M}_k$ provided that $(Z, \psi)$ has a flat metric on $\mathbb{C}^{k(k+N)}$. Unfortunately, the geodesics of such a metric do not describe the correct dynamics of vortices [63]. The 2d FI parameter $r$ is related to 4d gauge coupling constant by

$$r = \frac{4\pi}{g^2},$$

(5.1.5)

which holds under the RG flow if the 4d theory has $\mathcal{N} = 2$ supersymmetry and the 2d theory has $\mathcal{N} = (2, 2)$ supersymmetry [67, 41].

According to Ref. [79] the Kähler quotient (5.1.3) can be rewritten as a complex symplectic quotient as

$$\mathcal{M}_k \cong \{(Z, \psi)\} \sslash GL(k, \mathbb{C}),$$

(5.1.6)

where instead of having the $D$-term condition $\mu_D = r\mathbf{1}_k$, the pair of the matrices $(Z, \psi)$ are divided by the complexified non-compact group $U(k)^\mathbb{C} = GL(k, \mathbb{C})$ which acts in the same way as Eq. (5.1.2). Here the quotient denoted by the double slash “$\sslash$” means that points at which the $GL(k, \mathbb{C})$ action is not free should be removed so that the group action is free at any point. This quotient is also understood as the algebra-geometric quotient, so that the quotient space is parameterized by a set of $GL(k, \mathbb{C})$ holomorphic invariants with suitable constraints, see e.g. Ref. [80].

The starting point of our analysis, Eq. (5.1.6), can also be obtained directly from a purely field-theoretic point of view, based on the BPS equation (2.2.4). It has been shown by using the moduli-matrix approach [81, 46, 82] that all the moduli parameters of the $k$-vortex solutions are summarized exactly as in Eq. (5.1.6). The 4d field theory also provides the correct metric on $\mathcal{M}_k$ describing the dynamics of vortices as a geodesic motion on the moduli space. Although a general formula for the metric and its Kähler potential has

\(^2\)Here the normalization of the scalar fields $Z, \psi$ is chosen so that they have canonical kinetic terms in the two-dimensional effective gauge theory on the D2 branes. In this convention the eigenvalues of $Z$ (i.e. vortex positions) are dimensionless parameters.
been derived [53], the explicit form of the metric is however difficult to obtain since no analytic solutions are known to the BPS equation. Nevertheless, the asymptotic metric for well-separated vortices has recently been found in Ref. [51].

5.2 \( GL(k, \mathbb{C}) \) invariants

In what follows, we analyze the moduli space Eq. (5.1.6) without assuming any metric \textit{a priori}. Our prime concern is how the exact global \( SU(N) \) symmetry acts on the vortex moduli space \( \mathcal{M}_k \). The matrix \( Z \) is a singlet while \( \psi \) belongs to the fundamental representation \( N \). Namely, the \( SU(N) \) acts on \( Z \) and \( \psi \) as

\[
Z \rightarrow Z, \quad \psi \rightarrow \psi U, \quad U \in SU(N).
\]

As will be seen this action induces a natural \( SU(N) \) action on the moduli space of vortices. We will also discuss the metrics on the symmetry orbits on which the \( SU(N) \) acts isometrically. To this end, we use the algebra-geometric construction [80] of the moduli space by using the \( GL(k, \mathbb{C}) \) invariants which provide a set of coordinates of the moduli space.

Clearly, the coefficients \( \sigma_i \) \((i = 1, \ldots, k)\) of the characteristic polynomial of \( Z \) are invariants of \( GL(k, \mathbb{C}) \) action

\[
\det (\lambda 1_k - Z) = \lambda^k + \sum_{i=1}^{k} (-1)^i \sigma_i \lambda^{k-i}.
\]

Since the vortex positions \( z_I \) \((I = 1, \ldots, k)\) are defined as the eigenvalues of \( Z \) (roots of the characteristic polynomial)

\[
\det (\lambda 1_k - Z) = \prod_{I=1}^{k} (\lambda - z_I),
\]

the parameters \( \sigma_i \) and \( z_I \) are related by

\[
\sigma_i = P_i(z_1, \ldots, z_k),
\]

where \( P_i \) \((i = 1, \ldots, k)\) are the elementary symmetric polynomials defined by

\[
P_i(z_1, \ldots, z_k) \equiv \sum_{1 \leq I_1 < \cdots < I_i \leq k} z_{I_1} z_{I_2} \cdots z_{I_i}.
\]

\(^3\)See Ref. [52] for an alternative formula for vortices on Riemann surfaces.
Note that vortex positions $z_I$ are not fully invariant under $GL(k, \mathbb{C})$ transformations since they can be exchanged by the Weyl group $\mathfrak{S}_k$.

Other invariants can be constructed as follows. Let $Q^{(n)}$ ($n = 0, 1, \ldots$) be the following $(k, N)$ matrices of $SL(k, \mathbb{C}) \times SU(N)$ (Eqs. (5.1.2) and (5.2.1)):

$$Q^{(0)} \equiv \psi, \quad Q^{(1)} \equiv Z\psi, \quad \ldots, \quad Q^{(n)} \equiv Z^n\psi, \quad \ldots.$$  \hfill (5.2.6)

One can construct $SL(k, \mathbb{C}) \subset GL(k, \mathbb{C})$ invariants from $Q^{(n)}$ by using the totally anti-symmetric tensor $\varepsilon_{i_1i_2\cdots i_k}$ as

$$B_{r_1r_2\cdots r_k}^{n_1n_2\cdots n_k} \equiv e^{i_1i_2\cdots i_k}Q_{1r_1}^{(n_1)}Q_{2r_2}^{(n_2)}\cdots Q_{kr_k}^{(n_k)}.$$ \hfill (5.2.7)

We call these the “baryonic invariants” or sometimes simply “the baryons” below, relying on a certain analogy to the baryon states in the quark model (or in quantum chromodynamics).

**Remark:** although obviously they have no physical relation to the real-world baryons (the proton, neutron, etc.), no attentive reader should be led astray by such a short-hand notation.

Note that the baryons (5.2.7) are invariant under $SL(k, \mathbb{C})$ and transform under the remaining $U(1)^C \cong \mathbb{C}^*$ as

$$B_{r_1r_2\cdots r_k}^{n_1n_2\cdots n_k} \rightarrow e^{\lambda}B_{r_1r_2\cdots r_k}^{n_1n_2\cdots n_k}.$$ \hfill (5.2.8)

with a suitable weight $\lambda$.

The vortex positions $\{z_I\} \cong \mathbb{C}^k/\mathfrak{S}_k \cong \mathbb{C}^k$ are parameterized by the moduli parameters $\{\sigma_i\} \cong \mathbb{C}^k$. In addition to these parameters, there are baryons

$$\{B_{r_1r_2\cdots r_k}^{n_1n_2\cdots n_k}\} \cong V,$$

as moduli parameters, where $V$ denotes an infinite-dimensional complex linear space spanned by the baryons. The problem is that not all of these invariants are independent of each other; the baryons $B_{r_1r_2\cdots r_k}^{n_1n_2\cdots n_k}$ and $\sigma_i$ satisfy certain constraints by construction. Therefore, the vortex moduli space Eq. (5.1.6) can be rewritten as

$$\mathcal{M}_k \cong \{\mathbb{C}^k \times V \mid \text{constraints}\} \parallel \mathbb{C}^*.$$ \hfill (5.2.9)

Since the baryonic invariants transform under $SU(N)$, there exists a linear action of $SU(N)$ on $V$: this induces an $SU(N)$ action on the moduli space.
Consider now the constraints on the parameters \( \sigma_i \) and the baryons \( B_{r_1 r_2 \ldots r_k}^{n_1 n_2 \ldots n_k} \) in more detail. For this purpose it turns out to be convenient to introduce an auxiliary set of \( k \) linear harmonic oscillator states, each of which carrying an \( SU(N) \) label, and make a map from the vector space \( V \) to the Fock space of such oscillators. Let us introduce a “vortex state vector” \( |B\rangle \in V \) by

\[
|B\rangle \equiv \frac{1}{(n_1!n_2! \cdots n_k!)^2} B_{r_1 r_2 \ldots r_k}^{n_1 n_2 \ldots n_k} |n_1, r_1 \rangle \otimes |n_2, r_2 \rangle \otimes \cdots \otimes |n_k, r_k \rangle ,
\]

(5.2.10)

with \( n_i \in \mathbb{Z}_{\geq 0} \), \( 1 \leq r_i \leq N \); the associated annihilation and creation operators \( \hat{a}_i, \hat{a}_i^\dagger \) (\( i = 1, \ldots, k \))

\[
\hat{a}_i (\cdots \otimes |n_i, r_i \rangle \otimes \cdots) = \sqrt{n_i} (\cdots \otimes |n_i - 1, r_i \rangle \otimes \cdots),
\]

(5.2.11)

\[
\hat{a}_i^\dagger (\cdots \otimes |n_i, r_i \rangle \otimes \cdots) = \sqrt{n_i + 1} (\cdots \otimes |n_i + 1, r_i \rangle \otimes \cdots)
\]

(5.2.12)

satisfy the standard commutation relations

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 .
\]

(5.2.13)

Note that once \( |B\rangle \) is given, the baryonic invariants can be read off from the following relation

\[
B_{r_1 r_2 \ldots r_k}^{n_1 n_2 \ldots n_k} = \langle 0, r_1 ; \cdots ; 0, r_k | (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2} \cdots (\hat{a}_k)^{n_k} | B \rangle ,
\]

(5.2.14)

where \( |0, r_1 ; \cdots ; 0, r_k \rangle \equiv |0, r_1 \rangle \otimes \cdots \otimes |0, r_k \rangle \) are the ground states. Now there are three types of constraints to be taken into account (see Appendix D for more details):

1. From definition (5.2.7) one can see that the baryons satisfy the anti-symmetry property

\[
B^{A_1 \ldots A_i \ldots A_k} = -B^{A_1 \ldots A_j \ldots A_i \ldots A_k},
\]

(5.2.15)

where \( A_i \) stands for the pair of indices \( (n_i, r_i) \). This constraint can be rewritten as

\[
\hat{\rho} |B\rangle = \text{sign}(\rho) |B\rangle ,
\]

(5.2.16)

where \( \hat{\rho} \) denotes an element of the symmetric group \( \mathfrak{S}_k \). For an element \( \hat{\rho} \in \mathfrak{S}_k \)

\[
\rho = \begin{pmatrix}
1 & 2 & \cdots & k \\
I_1 & I_2 & \cdots & I_k
\end{pmatrix},
\]

(5.2.17)

\[\text{We hasten to add that no relation between the notion of vortex “state vectors” here and any quantum dynamics is implied by such a construction.}\]
the action on the state is defined by
\[
\hat{\rho} |n_1, r_1\rangle \otimes |n_2, r_2\rangle \otimes \cdots \otimes |n_k, r_k\rangle = |n_{I_1}, r_{I_1}\rangle \otimes |n_{I_2}, r_{I_2}\rangle \otimes \cdots \otimes |n_{I_k}, r_{I_k}\rangle
\] (5.2.18)

2. The second condition is a consequence of the relation \(Q^{(n+m)} = Z^m Q^{(n)}\). It follows that
\[
P_i(\hat{a}_1, \ldots, \hat{a}_k) |B\rangle = \sigma_i |B\rangle, \quad (i = 1, \ldots, k),
\] (5.2.19)
where \(P_i(\hat{a}_1, \ldots, \hat{a}_k)\) are the elementary symmetric polynomials made of \(\hat{a}_i\) (cfr. Eq. (5.2.5)).

3. The last type of constraints are the quadratic equations for the baryons, which follow from Eq. (5.2.7):
\[
B^{A_1 A_2 \cdots A_{k-1}} [A_k B^{B_1 B_2 \cdots B_k}] = 0,
\] (5.2.20)
where \(A_i\) stands for a pair of indices \((n_i, r_i)\). This constraint is a generalization of the Plücker relations for the Grassmannian.

Eqs. (5.2.16) and (5.2.19) can be viewed as linear constraints for baryons with \(\sigma_i\)-dependent coefficients. Therefore, for a given set of values \(\{\sigma_i\}\), they define a linear subspace \(W(\sigma_i) \subset V\) to which the vortex state vector \(|B\rangle\) belongs. We will see that the representation of the \(SU(N)\) action on \(W(\sigma_i)\) is independent of \(\sigma_i\) and isomorphic to \(k\) copies of the fundamental representation \(N\)
\[
W(\sigma_i) \cong \mathbb{C}^{N^k} \cong \bigotimes_{i=1}^{k} N_i.
\] (5.2.21)

Note that not all vectors in this “state space” \(W(\sigma_i)\) represent vortex state vectors since they must still satisfy Eq. (5.2.20). Namely, the vortex moduli space is defined by the constraints (5.2.20), which are quadratic homogeneous polynomials of the coordinates of \(W(\sigma_i)\) with \(\sigma_i\)-dependent coefficients.

### 5.2.1 The moduli space of \(k\) separated vortices

Let us first consider the case of winding-number \(k\) vortices with distinct centers, \(z_I \neq z_J\) (for all \(I \neq J\)). It follows from Eq. (5.2.19) that for \(i = 1, 2, \ldots, k\)
\[
\prod_{I=1}^{k} (\hat{a}_i - z_I) |B\rangle = \left((\hat{a}_i)^k + \sum_{n=1}^{k} (-1)^n \sigma_n (\hat{a}_i)^{k-n}\right) |B\rangle = \prod_{j=1}^{k} (\hat{a}_i - \hat{a}_j) |B\rangle = 0.
\] (5.2.22)
Thus, in the case of $z_I \neq z_J$, there exists an $I_i$ ($1 \leq I_i \leq k$) for each $i$ such that:

$$\hat{a}_i \ket{B} = z_{I_i} \ket{B}$$.  

Namely, the most generic form of the solution to the constraint (5.2.22) is

$$\ket{B} = \sum_{I_1, I_2, \ldots, I_k} \hat{B}_{I_1 I_2 \cdots I_k} |z_{I_1}, r_1\rangle \otimes |z_{I_2}, r_2\rangle \otimes \cdots \otimes |z_{I_k}, r_k\rangle$$,  

(5.2.24)

where $|z_{I_i}, r_i\rangle$ are the coherent states defined by

$$|z_{I_i}, r_i\rangle \equiv \exp \left( z_{I_i} \hat{a}_i^\dagger \right) |0, r_i\rangle$$.  

(5.2.25)

Recall that the coherent states are eigenstates of the annihilation operators

$$\hat{a}_i |z_{I_i}, r_i\rangle = z_{I_i} |z_{I_i}, r_i\rangle$$.

(5.2.26)

Then the constraint (5.2.19) reads

$$P_i(z_{I_1}, z_{I_2}, \cdots, z_{I_k}) \ket{B} = \sigma_i \ket{B} \quad \left( = P_i(z_1, z_2, \cdots, z_k) \ket{B} \right)$$.  

(5.2.27)

This means that $\{z_{I_1}, z_{I_2}, \cdots, z_{I_k}\}$ is a permutation of $\{z_1, z_2, \cdots, z_k\}$. Taking into account the anti-symmetry condition (5.2.16), the solution of the constraints (5.2.16) and (5.2.19) is given by

$$\ket{B} = \sum_{r_1, r_2, \cdots, r_k} \hat{B}_{r_1 r_2 \cdots r_k} \hat{\mathcal{A}} \left( |z_1, r_1\rangle \otimes |z_2, r_2\rangle \otimes \cdots \otimes |z_k, r_k\rangle \right)$$,  

(5.2.28)

where $\hat{\mathcal{A}}$ denotes the anti-symmetrization of the states

$$\hat{\mathcal{A}} \equiv \text{sign}({\mathbf{\rho}}) \hat{\rho}$$.

(5.2.29)

For a given set $\{z_1, z_2, \cdots, z_k\}$, the solutions (5.2.28) span an $N^k$-dimensional vector space $W(\sigma_i)$ and the redefined baryons $\hat{B}_{r_1 r_2 \cdots r_k}$ are the coordinates of $W(\sigma_i)$. As stated in Eq. (5.2.21), $\hat{B}_{r_1 r_2 \cdots r_k}$ is in the direct product representation $\bigotimes_{i=1}^{k} \mathbf{N}$. They can be expressed in terms of the original baryons $B_{r_1 r_2 \cdots r_k}^{n_1 n_2 \cdots n_k}$ by using the relation

$$\hat{B}_{r_1 r_2 \cdots r_k} = \langle 0, r_1; \cdots; 0, r_k | e_1(\hat{a}_1) \cdots e_k(\hat{a}_k) | B \rangle$$.

(5.2.30)

---

Note that this relation does not necessarily hold for coincident vortices. For example, if $z_I = z_J = z_0$ ($I \neq J$), the constraint (5.2.22) can also be satisfied by a state vector $|B\rangle$ such that

$$(\hat{a}_i - z_0)^2 |B\rangle = 0, \quad \hat{a}_i |B\rangle \neq z_0 |B\rangle$$.  

where $|0, r_1; \cdots; 0, r_k\rangle \equiv |0, r_1\rangle \otimes \cdots \otimes |0, r_k\rangle$ are the ground states and $e_I$ ($I = 1, \ldots, k$) are the polynomials defined as

$$e_I(\lambda) \equiv \prod_{J \neq I}^{\infty} \frac{\lambda - z_J}{z_I - z_J}, \quad (e_I(z_J) = \delta_{IJ}). \quad (5.2.31)$$

Since this polynomial is ill-defined for coincident vortices $z_I = z_J$ (for $I \neq J$), the coherent state representation (5.2.28) is valid only for separated vortices. As we will see later, there exist well-defined coordinates of $W(\sigma_i)$ for arbitrary values of $\sigma_i$. They can be obtained from $B_{r_1r_2\cdots r_k}$ by linear coordinate transformations with $z_I$-dependent coefficients. Hence the result that the linear space $W(\sigma_i)$ has the representation $\otimes_{i=1}^{k} N$ holds for arbitrary values of $\sigma_i$, including the coincident cases ($z_I = z_J$), as well.

So far we have specified the state space $W(\sigma_i)$ to which the vortex state vectors belong. Now let us examine which vectors in $W(\sigma_i)$ can be actually allowed as vortex state vectors. The remaining constraint is the Plücker relation (5.2.20) which reads

$$\tilde{B}_{r_1\cdots r_i\cdots r_k} \tilde{B}_{s_1\cdots s_i\cdots s_k} = \tilde{B}_{r_1\cdots s_i\cdots r_k} \tilde{B}_{s_1\cdots r_i\cdots s_k}, \quad (5.2.32)$$

for each $i = 1, 2, \ldots, k$. This is solved by

$$\tilde{B}_{r_1r_2\cdots r_k} = \phi_{r_1}^{1} \phi_{r_2}^{2} \cdots \phi_{r_k}^{k}, \quad (5.2.33)$$

Since the baryons are divided by $U(1)^{C} \subset GL(k, \mathbb{C})$, the multiplication of a non-zero complex constant on each of $\tilde{\phi}^i \in \mathbb{C}^N$ ($I = 1, \ldots, k$) is unphysical. Therefore, each $N$-vector $\tilde{\phi}^i = (\phi_1^i, \cdots, \phi_N^i)$ parameterizes $\mathbb{C}P^{N-1}$.

We thus see that for separated vortices the baryon given in Eq. (5.2.28) can be written as an anti-symmetric product of “single vortex states”

$$|B\rangle = \hat{A} \left[ \left( \sum_{r_1=1}^{N} \phi_{r_1}^{1} |z_1, r_1\rangle \right) \otimes \left( \sum_{r_2=1}^{N} \phi_{r_2}^{2} |z_2, r_2\rangle \right) \otimes \cdots \otimes \left( \sum_{r_k=1}^{N} \phi_{r_k}^{k} |z_k, r_k\rangle \right) \right]. \quad (5.2.34)$$

This means that the moduli space of the separated vortices is just a $k$-symmetric product of $\mathbb{C} \times \mathbb{C}P^{N-1}$ parameterized by the position of the vortices $z_I$ and the orientation $\tilde{\phi}^i$ [46]

$$\mathcal{M}^{k\text{-separated}} \simeq \left( \mathbb{C} \times \mathbb{C}P^{N-1} \right)^{k} / \mathfrak{S}_k, \quad (5.2.35)$$

where $\mathfrak{S}_k$ stands for the symmetric group. Note that the space of vortex states Eq. (5.2.34), which are just generic (anti-symmetrized) factorized states. It spans far fewer dimensions
than might na"ively be expected for the product-states made of \( k \) vectors, which would have a dimension of the order of \( 2^Nk \), ignoring the position moduli.

As is clear – hopefully – from our construction, the use of the vortex “state vector” notion is for convenience only here, made for exhibiting the group-theoretic properties of the non-Abelian vortices. In other words we do not attribute to |\( B \rangle \) any direct physical significance. Accordingly, we need not discuss the question of their normalization (metric on the vector space \( V \)) here. Note that two of the constraints (Eq. (5.2.16) and Eq. (5.2.19)) are indeed linear; the third, quadratic constraint (Eq. (5.2.20)) does not affect their normalization either.

It is tempting, on the other hand, to note that any choice of a metric in \( V \) would induce a metric on the vortex moduli space, which is of physical interest. As discussed briefly in Appendix E, however, a simple-minded choice of the metric for |\( B \rangle \) does not lead to fully correct behavior of the vortex interactions.

### 5.2.2 Highest-weight coincident vortices and \( SU(N) \) irreducible orbits

Let us next consider \( k \) vortices on top of each other, all centered at the origin. Namely we focus our attention on the subspace of the moduli space specified by the condition

\[
\sigma_i = 0 \quad \text{for all } i .
\]  

(5.2.36)

Since the coherent states of Eq. (5.2.24) are not the general solution to the constraint (5.2.22), the situation is now more complicated. To understand the structure of this subspace in detail, it is important to know how the \( SU(N)_{C+T} \) acts on it. As we have seen, the moduli space of vortices can be described in terms of the vortex state vector endowed with a linear representation of the \( SU(N) \) action. We will denote the \( SU(N) \) orbits of highest-weight vectors (to be defined below) the “irreducible \( SU(N) \) orbits” since the vectors belong to irreducible representations on those orbits. In this subsection we classify irreducible \( SU(N) \) orbits by Young tableaux.

The “highest-weight vectors” will be defined as the special configurations of \( \psi \) and \( Z \) satisfying the following conditions:

- Any \( U(1)^{N-1} \) transformation in the Cartan subgroup of \( SU(N) \) can be absorbed by
a $GL(k, \mathbb{C})$ transformation. Namely, for an arbitrary diagonal matrix $D \in U(1)^{N-1}$, there exists an element $g \in GL(k, \mathbb{C})$ such that

$$
\psi D = g \psi, \quad Z = gZg^{-1}.
$$

(5.2.37)

- Any infinitesimal $SU(N)$ transformation with a raising operator $\hat{E}_\alpha$ can be absorbed by an infinitesimal $SL(k, \mathbb{C})$ transformation. Namely, for an arbitrary lower triangular matrix $L$ whose diagonal entries are all 1, there exists an element $\tilde{g} \in SL(k, \mathbb{C})$ such that

$$
\psi L = \tilde{g} \psi, \quad Z = \tilde{g}Z\tilde{g}^{-1}.
$$

(5.2.38)

Such configurations are classified by a non-increasing sequence of integers $\{l_1, l_2, \cdots, l_{k_1}\}$ satisfying

$$
N \geq l_1 \geq l_2 \geq \cdots \geq l_{k_1} \geq 0, \quad l_1 + l_2 + \cdots + l_{k_1} = k.
$$

(5.2.39)

In other words, they are specified by Young tableaux (diagrams)$^6$ with $k$ boxes

\[
\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
2 & 2 & \cdots & 2 \\
\vdots & \vdots & & \\
\vdots & & & \\
l_1 & & & \\
\end{array}
\]

(5.2.40)

where the height of the $i$-th column is $l_i$ and the width of the $i$-th row is $k_i$. The total number of boxes is equal to the vortex winding number $k$. An example of a pair of matrices $(\psi, Z)$ corresponding the highest-weight state is given in Fig. 5.1. For such a pair of matrices $(\psi, Z)$, one can check the existence of $g$ and $\tilde{g}$ satisfying Eq. (5.2.37) and Eq. (5.2.38), given by

$$
g = \begin{pmatrix}
D_{l_1} & & \\
& \ddots & \\
& & D_{l_{k_1}}
\end{pmatrix}, \quad
\tilde{g} = \begin{pmatrix}
L_{l_1} & & \\
& \ddots & \\
& & L_{l_{k_1}}
\end{pmatrix},
$$

(5.2.41)

$^6$In the following, the term “Young tableaux” is used to denote diagrams without numbers in the boxes (Young diagrams), unless otherwise stated.
5.2 \( GL(k, \mathbb{C}) \) invariants

Fig. 5.1: An example of a \( k \)-by-\( (N + k) \) matrix \((\psi, Z)\) with \( k_1 = 4 \). The painted square boxes stand for unit matrices while the blank spaces imply that all their elements are zero.

where \( k_1 \) is the number of boxes in the first row of the Young tableau, and \( D_{l_i} \) and \( L_{l_i} \) are the upper-left \( l_i \)-by-\( l_i \) minor matrices of \( D \) and \( L \), respectively.\(^7\)

The baryons corresponding to \((\psi, Z)\) are given by

\[
|B\rangle = \hat{A} \left( |l_1\rangle \otimes |l_2\rangle \otimes \cdots \otimes |l_{k_1}\rangle \right), \quad |l_{n+1}\rangle \equiv |n, 1\rangle \otimes |n, 2\rangle \otimes \cdots \otimes |n, l_{n+1}\rangle \tag{5.2.42}
\]

We claim that this state is the highest-weight vector of the irreducible representation of \( SU(N) \) specified by the Young tableau. This can be verified as follows. Since \((\psi, Z)\) satisfy the condition Eq. (5.2.37), the baryons transform under the \( U(1)^{N-1} \) transformation according to

\[
|B\rangle \rightarrow \det g |B\rangle = \exp \left( \sum_{i=1}^{l_1} k_i \theta_i \right) |B\rangle, \quad \sum_{i=1}^{N} \theta_i = 0, \tag{5.2.43}
\]

where \( k_i \) is the number of boxes in the \( i \)-th row of the Young tableau. The weights of the \( U(1)^{N-1} \) action can be read off in terms of \( k_i \) as

\[
m_i = k_i - k_{i+1}, \tag{5.2.44}
\]

where the integers \([m_1, m_2, \ldots, m_{N-1}]\) are the Dynkin labels. On the other hand, since \((\psi, Z)\) satisfy the condition Eq. (5.2.38), the \( SL(k, \mathbb{C}) \) invariants \( B_{r_1 r_2 \cdots r_k}^{m_1 m_2 \cdots m_k} \) are annihilated by the raising operators

\[
\hat{E}_\alpha |B\rangle = 0. \tag{5.2.45}
\]

\(^7\)For example, \( D_{l_i} = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{l_i}}) \) for \( D = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}) \).
Fig. 5.2: An example $\mathcal{M}_{\text{orbit}} = \frac{SU(7)}{SU(2) \times SU(2) \times SU(3) \times U(1)}$, with $N = 7$, $m = [0, 1, 0, 1, 0, 0]$ and $k = 13$. The black nodes in the Dynkin diagram denote the removed nodes [2].

We have thus proved that (5.2.42) represents the highest-weight state of the representation (5.2.40) in the usual sense.

We define an “irreducible SU(N) orbit for the set of Dynkin labels: $[m_1, m_2, \ldots, m_{N-1}]$ ” as an SU(N) orbit of the corresponding highest-weight state. Note that this definition is obviously independent of the choice of $U(1)^{N-1} \in SU(N)$ in Eq. (5.2.37). It is known that such an orbit is a generalized flag manifold of the form $SU(N)/H$ with $H$ being a subgroup of $SU(N)$ which acting on the highest-weight state as

$$\hat{h} |B\rangle = e^{i\theta(h)} |B\rangle \sim |B\rangle, \quad \forall \hat{h} \in H. \quad (5.2.46)$$

The subgroup $H$ can be specified by removing the nodes in the Dynkin diagram which correspond to non-zero Dynkin labels $m_i \neq 0$, i.e. it is specified by a painted Dynkin diagram [2]. Therefore, the irreducible orbits can be written as generalized flag manifolds

$$\mathcal{M}_{\text{orbit}} = \frac{SU(N)}{SU(q_1 + 1) \times \cdots \times SU(q_{p+1} + 1) \times U(1)^p}, \quad (5.2.47)$$

where $p (1 \leq p \leq N - 1)$ is the number of removed nodes and $q_i (i = 1, \ldots, p + 1)$ is the number of nodes in the connected component between $(i - 1)$-th and $i$-th removed nodes (see Fig. 5.2). The number $p$ is denoted the rank of the Kähler coset space (5.2.47). One can also verify that an $H$-transformation on $(\psi, Z)$ can indeed be absorbed by $GL(k, \mathbb{C})$ transformations.

It will now be shown that the irreducible orbits are the fixed-point set of the spatial rotation

$$(\psi, Z) \rightarrow (\psi, e^{i\theta} Z). \quad (5.2.48)$$

---

8These orbits were studied in a non-systematic way in Ref. [83].
To see this, it is sufficient to check that the highest-weight state is invariant under the rotation (5.2.48), since the $SU(N)$ transformations commute with the spatial rotation. One way to show the invariance of the highest-weight state is to find a $GL(k, \mathbb{C})$ transformation which cancels the transformation (5.2.48) on the matrix of Fig. 5.1. A different, but easier, way is to check the invariance of the highest-weight state (5.2.39) under the action of the spatial rotations explicitly. Since the generator of the spatial rotation $\hat{J}$ acts on the ground state $|0\rangle$ and the operators $\hat{a}, \hat{a}^\dagger$ as ($J$ is just a number operator)

$$\hat{J}|0\rangle = 0, \quad [\hat{J}, \hat{a}_i] = \hat{a}_i, \quad [\hat{J}, \hat{a}_i^\dagger] = -\hat{a}_i^\dagger,$$

the highest-weight state (5.2.39) is an eigenstate of $\hat{J}$, hence the state transforms as

$$|B\rangle \rightarrow \exp (i\theta \hat{J}) |B\rangle = \exp \left(-i \sum_{n=0}^{k_1-1} n l_{n+1} \theta \right) |B\rangle.$$  

Since the phase of the state vector is unphysical, Eq. (5.2.50) shows that the highest-weight state is invariant under the spatial rotation. Therefore, the irreducible orbits are in the fixed-point set of the spatial rotation. The inverse also turns out to be true: we can show by using the moduli-matrix formalism that any fixed points of the spatial rotation are contained in one of the irreducible orbits. Therefore, the fixed-point set is precisely the disjoint union of the irreducible orbits.

All this can be seen more explicitly in terms of the original fields. The solution $(q, A_\mu)$ to the BPS equation (2.2.4) corresponding to the irreducible orbits can be determined from the fact that they are invariant under the spatial rotation

$$q(z, \bar{z}) \rightarrow q(e^{-i\theta} z, e^{i\theta} \bar{z}), \quad A_{\bar{z}}(z, \bar{z}) \rightarrow e^{i\theta} A_{\bar{z}}(e^{-i\theta} z, e^{i\theta} \bar{z}),$$

where $A_{\bar{z}} = A_1 + iA_2$. Let $(q^{(k)}, A^{(k)}_\mu)$ be the solution of $k$ ANO vortices situated at the origin $z = 0$. They transform under the rotation as

$$q^{(k)}(e^{-i\theta} z, e^{i\theta} \bar{z}) = e^{-i k \theta} q^{(k)}(z, \bar{z}), \quad A^{(k)}_{\bar{z}}(e^{-i\theta} z, e^{i\theta} \bar{z}) = A^{(k)}_{\bar{z}}(z, \bar{z}).$$

The solution on the irreducible orbits can be obtained by embedding the ANO solutions into diagonal components

$$q = U^\dagger \diag (q^{(k_1)}, q^{(k_2)}, \ldots, q^{(k_N)}) U, \quad A_{\bar{z}} = U^\dagger \left( A^{(k_1)}_{\bar{z}}, A^{(k_2)}_{\bar{z}}, \ldots, A^{(k_N)}_{\bar{z}} \right) U,$$

where $U \in SU(N)_{C+F}$. Note that the sequence of the numbers $\{k_1, k_2, \ldots, k_N\}$ can always be reordered as $k_1 \geq k_2 \geq \cdots \geq k_N \geq 0$ by using the Weyl group $S_N \subset SU(N)_{C+F}$. This
solution is invariant under the rotation since the phase factors of the Higgs fields can be absorbed by the following gauge transformation

\[ q \rightarrow gq, \quad A_z \rightarrow gA_zg^\dagger, \quad g = U^\dagger \text{diag} \left( e^{-ik_1 \theta}, e^{-ik_2 \theta}, \ldots, e^{-ik_N \theta} \right) U \in U(N)_C. \] (5.2.54)

We can also see that the solution (5.2.53) is invariant under the same subgroup of \( SU(N) \) as the state on the irreducible orbit specified by the Young tableau with \( k_i \) boxes in the \( i \)-th row. Therefore, the irreducible orbit with the set of Dynkin labels \( [m_1, \ldots, m_{N-1}] \) (\( m_i = k_i - k_{i+1} \)) corresponds to the BPS solutions of the form of Eq. (5.2.53).

In the next section, we will show that a vortex state at a generic point on the moduli space is given by a linear superposition of vectors corresponding to various irreducible representations. Furthermore, in Section 5.4, metrics for all irreducible \( SU(N) \) orbits will be obtained by assuming that the metrics are Kähler and isometric under the \( SU(N) \) action.

### 5.3 \( SU(N) \) decomposition of general \( k \) vortex states

In this section we solve the constraints (5.2.20) and (5.2.19) in order to find out the \( SU(N) \) property of a general \( k \)-winding vortex. The cases of \( k = 1, 2 \) and 3 are solved concretely; a general recipe for solution will be given, valid for any \( N \) and for any winding number \( k \). A particular attention will be paid to the vortices with coincident centers. The results of these analysis provide the \( SU(N) \) decomposition rule for a generic vortex state of a given winding number.

#### 5.3.1 \( k = 1 \) vortices

\( k = 1 \) is a trivial example. In this case, we have

\[ \sigma_1 = z_1, \quad |B\rangle = \sum_{r=1}^{N} \phi_r |z_1, r\rangle. \] (5.3.1)

There is no non-trivial constraint, so that the moduli space is

\[ \mathcal{M}^{k=1} = \mathbb{C} \times \mathbb{C} P^{N-1} \cong \mathbb{C} \times \frac{SU(N)}{SU(N-1) \times U(1)}. \] (5.3.2)

As \( |B\rangle \) is in the fundamental representation of \( SU(N) \), the orientational moduli space is given by the orbit of a vector in the fundamental representation.
5.3 $SU(N)$ decomposition of general $k$ vortex states

5.3.2 Solution of the constraints for $k = 2$

This is the first case with non-trivial constraints.

$k = 2$ $U(N)$ vortices

With coordinates $\sigma_1 = z_1 + z_2 \in \mathbb{C}$ and $\sigma_2 = z_1 z_2 \in \mathbb{C}$, the linear constraints (5.2.19) in this case are given by

$$ (\hat{a}_1 + \hat{a}_2) |B\rangle = \sigma_1 |B\rangle, \quad \hat{a}_1 \hat{a}_2 |B\rangle = \sigma_2 |B\rangle, \quad (5.3.3) $$

which are equivalent to the following equations for the baryonic invariants

$$ B_r^{n+1m} + B_r^{nm+1} s = \sigma_1 B_r^{nm}, \quad B_r^{n+1m+1} s = \sigma_2 B_r^{nm}. \quad (5.3.4) $$

In Section 5.2.1, we have seen that the solution can be expressed by the coherent states for separated vortices. Let us see what happens to the coherent states in the coincident limit. In the case of $k = 2$, the coherent state representation of the solution is given by

$$ |B\rangle = \frac{1}{2} \tilde{B}_{r_1 r_2} \left( |z_1, r_1\rangle \otimes |z_2, r_2\rangle - |z_2, r_2\rangle \otimes |z_1, r_1\rangle \right). \quad (5.3.5) $$

It is convenient to decompose $\tilde{B}_{r_1 r_2}$ into the irreducible representations of $SU(N)$

$$ \tilde{A}_{r_1 r_2} \equiv \frac{\tilde{B}_{r_1 r_2} - \tilde{B}_{r_2 r_1}}{2}, \quad \tilde{S}_{r_1 r_2} \equiv \frac{\tilde{B}_{r_1 r_2} + \tilde{B}_{r_2 r_1}}{2}. \quad (5.3.6) $$

Then, the solution can be rewritten as

$$ |B\rangle = \left[ \tilde{A}_{r_1 r_2} \cosh \left( \frac{z_1 - z_2}{2}(\hat{a}_1^\dagger - \hat{a}_2^\dagger) \right) + \tilde{S}_{r_1 r_2} \sinh \left( \frac{z_1 - z_2}{2}(\hat{a}_1^\dagger - \hat{a}_2^\dagger) \right) \right] \left| \sigma_1 \frac{r_1}{2}, r_1 \right\rangle \otimes \left| \sigma_1 \frac{r_2}{2}, r_2 \right\rangle, $$

where $\sigma_1 = z_1 + z_2$. If we naively take the coincident limit $z_2 \to z_1$, the symmetric part drops out

$$ |B\rangle \to \tilde{A}_{r_1 r_2} \left| \sigma_1 \frac{r_1}{2}, r_1 \right\rangle \otimes \left| \sigma_1 \frac{r_2}{2}, r_2 \right\rangle. \quad (5.3.7) $$

Although this state satisfies the constraint (5.3.3), this is not the most general solution of the coincident case. To obtain the correct expression for the most general solution, let us redefine

$$ A_{r_1 r_2} \equiv \tilde{A}_{r_1 r_2}, \quad S_{r_1 r_2} \equiv \frac{z_1 - z_2}{2} \tilde{S}_{r_1 r_2}. \quad (5.3.8) $$
Then, the solution (5.3.5) can be rewritten as

\[ |B\rangle = \sum_{n=0}^{\infty} \frac{1}{(2n)!} w^n (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^{2n} \left[ A_{r_1r_2} + \frac{1}{2n+1} S_{r_1r_2} (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^{2n+1} \right] |\frac{\sigma_1}{2}, r_1\rangle \otimes |\frac{\sigma_1}{2}, r_2\rangle , \tag{5.3.9} \]

where we have introduced a square of the relative position as

\[ w \equiv \frac{\sigma_1^2}{4} - \sigma_2 = \frac{(z_1 - z_2)^2}{4} . \tag{5.3.10} \]

In this expression, it is obvious that the symmetric part also survives in the coincident limit \( w \to 0 \)

\[ |B\rangle \to \left[ A_{r_1r_2} + S_{r_1r_2} (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^{2n} \right] |\frac{\sigma_1}{2}, r_1\rangle \otimes |\frac{\sigma_1}{2}, r_2\rangle . \tag{5.3.11} \]

Therefore, Eq. (5.3.9) is the most general form of the solution which is valid also in the coincident limit. The symmetric and anti-symmetric tensors \( S_{rs} \) and \( A_{rs} \) in Eq. (5.3.9) are the well-defined coordinate of the vector space \( W(\sigma) \) for arbitrary values of \( \sigma \). Clearly these correspond to the decomposition of the tensor product \( \mathbf{N} \otimes \mathbf{N} \) into irreducible representations of \( SU(N) \). A generic point on the moduli space is described by a superposition of the states belonging to different irreducible representations.

In terms of \( A_{rs} \) and \( S_{rs} \), the baryonic invariants can be read off from the solution using (5.2.14)

\[ B_{rs}^{00} = A_{rs} , \quad B_{rs}^{10} = S_{rs} + \frac{\sigma_1}{2} A_{rs} , \quad B_{rs}^{11} = \sigma_2 A_{rs} , \quad \cdots , \tag{5.3.12} \]

and hence, the Plücker conditions (5.2.20), which are the remaining constraints, can be rewritten as

\[ A_{pq} A_{rs} + A_{pr} A_{sq} + A_{ps} A_{qr} = 0 , \tag{5.3.13} \]

\[ A_{pq} S_{rs} + A_{rp} S_{qs} + S_{ps} A_{qr} = 0 , \tag{5.3.14} \]

\[ w A_{pq} A_{rs} + S_{pr} S_{qs} - S_{ps} S_{qr} = 0 . \tag{5.3.15} \]

By these constraints, the moduli space of two vortices is embedded into \( \mathbb{C}^2 \times \mathbb{C} P^{N^2-1} \) which is parameterized by independent coordinates \( \{\sigma_1, \sigma_2, A_{rs}, S_{rs}\} \).

Now, let us look into two different subspaces corresponding to the irreducible \( SU(N) \) orbits. They are obtained by setting 1) \( S_{rs} = 0, A_{rs} \neq 0 \) and 2) \( A_{rs} = 0, S_{rs} \neq 0 \).

1) Consider first the subspace with \( S_{rs} = 0 \). Eq. (5.3.15) allows \( S_{rs} = 0 \) only in the coincident case \( w = 0 \). Note that Eq. (5.3.14) is automatically satisfied by \( S_{rs} = 0 \), and that
5.3 $SU(N)$ decomposition of general $k$ vortex states

Eq. (5.3.13) gives the ordinary Plücker conditions which embed the complex Grassmannian $Gr_{N,2}$ into a complex projective space $\mathbb{C}P^{N(N-1)/2-1} \simeq \{A_{pq}\}/\mathbb{C}^*$. We find therefore that the subspace $S_{rs} = 0$ is:

$$\mathcal{M}^\parallel \cong \mathbb{C} \times Gr_{2,N} \cong \mathbb{C} \times \frac{SU(N)}{SU(2) \times SU(N-2) \times U(1)}. \quad (5.3.16)$$

According to the results in the previous section, this is the irreducible $SU(N)$ orbit for $\parallel$.

2) In the other subspace characterized by $A_{rs} = 0$, we have a non-trivial constraint $S_{pr}S_{qs} = S_{ps}S_{qr}$. The general solution is

$$S_{rs} = \phi_r \phi_s, \quad \phi_r \in \mathbb{C}^N. \quad (5.3.17)$$

Here $\phi_r$ is nothing but the orientation vector given in Eq. (5.3.1), so $S_{rs} = \phi_r \phi_s$ corresponds to the $k = 2$ vortices with parallel orientations. The corresponding moduli subspace is given by

$$\mathcal{M}^\Box \cong \mathbb{C}^2 \times \mathbb{C}P^{N-1} \cong \mathbb{C}^2 \times \frac{SU(N)}{SU(N-1) \times U(1)}, \quad (5.3.18)$$

which is indeed the other irreducible orbit, extended for generic $w$. We have thus identified the two moduli subspaces, the irreducible $SU(N)$ orbits of anti-symmetric and symmetric representations, respectively. They correspond to the vortex states in Eq. (5.3.9) without the second or the first term, respectively. The generic vortex state (5.3.9) is a linear superposition of these two states.

Note that in some cases the orbits of different representations are described by the same coset manifold. For example, both $\Box$ and $\Box \Box$ are given by $\mathbb{C}P^{N-1}$, see Eqs. (5.3.2) and (5.3.18). As we shall see in Section 5.4, however, the Kähler class completely specifies the representations and distinguishes the orbits belonging to different representations $^9$.

More on $k = 2$ coincident $U(2)$ vortices

Let us study $k = 2$ vortices in the $U(2)$ case in some more detail by looking at another slice of the moduli space. This case in particular has been studied in the Refs. [84, 85, 47, 45, 35, 76]. In this case, there exist only a singlet $A_{12}$ and a triplet $\{S_{11}, S_{12}, S_{22}\}$ of $SU(2)$.

$^9$Except for the cases of pairs of conjugate representations. They are found to be described by the same Kähler metric, i.e., by the same low-energy effective action. See Subsection 5.4.2 below.
Among the constraints (5.3.13)–(5.3.15), the only non-trivial one is
\[ w(A_{12})^2 + S_{11}S_{22} - (S_{12})^2 = 0. \] (5.3.19)

Let us consider the moduli space of coincident vortices which corresponds to the subspace \( w = 0 \). In this case, the above constraint is solved by \( S_{rs} = \phi_r\phi_s \) again. Now, the moduli subspace is parameterized by the center of mass position \( z_0 = \frac{a_1}{2} \) and \( \{ \eta, \phi_1, \phi_2 \} \) with \( \eta \equiv A_{12} \). Thus, the vortex state is given by, without constraints
\[ |B\rangle_{w=0} = \eta |z_0\rangle_1 + \sum_{r,s=1}^2 \phi_r\phi_s |z_0; r, s\rangle_3, \] (5.3.20)

where the singlet \( |z_0\rangle_1 \) and the triplet \( |z_0; r, s\rangle_3 \) are given by
\[ |z_0\rangle_1 \equiv |z_0, 1\rangle \otimes |z_0, 2\rangle - |z_0, 2\rangle \otimes |z_0, 1\rangle, \] (5.3.21)
\[ |z_0; r, s\rangle_3 \equiv (a_1^\dagger - a_2^\dagger) (|z_0, r\rangle \otimes |z_0, s\rangle + |z_0, s\rangle \otimes |z_0, r\rangle). \] (5.3.22)

Note that the \( \mathbb{C}^* \subset GL(k, \mathbb{C}) \) acts as
\[ \{ \eta, \phi_1, \phi_2 \} \sim \{ \lambda^2 \eta, \lambda \phi_1, \lambda \phi_2 \}, \quad \lambda \in \mathbb{C}^*. \] (5.3.23)

Hence the moduli subspace for the two coincident vortices is found to be the two dimensional weighted projective space with the weights \( (2, 1, 1) \)
\[ \mathcal{M}_{k=2}^{\text{coincident}} \cong \mathbb{C} \times \mathbb{C}P^2_{(2,1,1)} \cong \mathbb{C} \times \mathbb{C}P^2_{\mathbb{Z}_2}. \] (5.3.24)

This is exactly the result obtained previously [85, 47]. Although this might be seen as just a reproduction of an old result, there is a somewhat new perspective on the irreducible representation of \( SU(2) \). Here we would like to stress again that \( A_{12} = \eta \) is the singlet while \( S_{rs} = \phi_r\phi_s \) is the triplet. Together they form the coordinate of \( \mathbb{C}P^2_{(2,1,1)} \). In Fig. 5.3, we show the space \( \mathbb{C}P^2_{(2,1,1)} \) in the \( |\phi_1|^2 - |\phi_2|^2 \) plane with a natural metric given by \( 2|\eta|^2 + |\phi_1|^2 + |\phi_2|^2 = 1 \). The states \( 3 \) and \( 1 \) live on the boundaries of \( \mathbb{C}P^2_{(2,1,1)} \); the points in the bulk of \( \mathbb{C}P^2_{(2,1,1)} \) are described by the superposition \( 1 \oplus 3 \).

In Appendix F we discuss possible metrics on \( \mathbb{C}P^2_{(2,1,1)} \) and show that independently of the choice of the metric, they indeed yield at the diagonal edge of Fig. 5.3 the Fubini-Study metric with the same Kähler class on \( \mathbb{C}P^1 \).
5.3 $SU(N)$ decomposition of general $k$ vortex states

Fig. 5.3: $WCP_{(2,1,1)}^{2}$ in the gauge $2|\eta|^{2} + |\phi_{1}|^{2} + |\phi_{2}|^{2} = 1$. The diagonal edge corresponds to the triplet state $\mathbf{3}$ and the origin to the singlet state $\mathbf{1}$. The bulk is a non-trivial superposition of $\mathbf{1}$ and $\mathbf{3}$. The diagonal edge and the origin are the only irreducible orbits in this system.

5.3.3 Solution for the $k = 3$ coincident vortices

In this section, we consider $k = 3$ vortices sitting all at the origin, $\sigma_{1} = \sigma_{2} = \sigma_{3} = 0$ ($z_{1} = z_{2} = z_{3} = 0$). (The $k = 3$ vortex solutions of more general types – with generic center positions – will be discussed in Appendix G.) The constraint (5.2.19) reduces to

$$\hat{a}_{1}\hat{a}_{2}\hat{a}_{3}|B\rangle = 0, \quad (\hat{a}_{1}\hat{a}_{2} + \hat{a}_{2}\hat{a}_{3} + \hat{a}_{3}\hat{a}_{1})|B\rangle = 0, \quad (\hat{a}_{1} + \hat{a}_{2} + \hat{a}_{3})|B\rangle = 0 \quad (5.3.25)$$

which lead to $(\hat{a}_{i})^{3}|B\rangle = 0$ for $i = 1, 2, 3$. Taking into account the anti-symmetry condition (5.2.15), we obtain the following solution to the constraints (see Appendix G)

$$|B\rangle = \left[ A_{r_{1}r_{2}r_{3}} + \left( X_{r_{1}r_{2}r_{3}}^{1} \hat{a}_{1}^{\dagger} + X_{r_{1}r_{2}r_{3}}^{2} \hat{a}_{2}^{\dagger} + X_{r_{1}r_{2}r_{3}}^{3} \hat{a}_{3}^{\dagger} \right) \right.$$

$$\left. -\frac{1}{2} \left( Y_{r_{1}r_{2}r_{3}}^{1} (\hat{a}_{1}^{\dagger} - \hat{a}_{3}^{\dagger})^{2} + Y_{r_{1}r_{2}r_{3}}^{2} (\hat{a}_{1}^{\dagger} - \hat{a}_{3}^{\dagger})^{2} + Y_{r_{1}r_{2}r_{3}}^{3} (\hat{a}_{1}^{\dagger} - \hat{a}_{2}^{\dagger})^{2} \right) \right.$$

$$\left. -\frac{1}{2} S_{r_{1}r_{2}r_{3}} (\hat{a}_{1}^{\dagger} - \hat{a}_{2}^{\dagger})(\hat{a}_{2}^{\dagger} - \hat{a}_{3}^{\dagger})(\hat{a}_{3}^{\dagger} - \hat{a}_{1}^{\dagger}) \right] |0, r_{1}\rangle \otimes |0, r_{2}\rangle \otimes |0, r_{3}\rangle, \quad (5.3.26)$$

where $Y_{r_{1}r_{2}r_{3}}^{i}$ ($i = 1, 2, 3$) and $X_{r_{1}r_{2}r_{3}}^{i}$ ($i = 1, 2, 3$) are tensors satisfying

$$Y_{r_{1}r_{2}r_{3}}^{1} + Y_{r_{1}r_{2}r_{3}}^{2} + Y_{r_{1}r_{2}r_{3}}^{3} = 0, \quad X_{r_{1}r_{2}r_{3}}^{1} + X_{r_{1}r_{2}r_{3}}^{2} + X_{r_{1}r_{2}r_{3}}^{3} = 0. \quad (5.3.27)$$
The tensors $S, Y, X, A$ have the following index structures

\[
S_{r_1r_2r_3} = S_{r_\rho(1)r_\rho(2)r_\rho(3)},
\]

(5.3.28)

\[
Y^i_{r_1r_2r_3} = \text{sign}(\rho)Y_{r_\rho(1)r_\rho(2)r_\rho(3)}^i,
\]

(5.3.29)

\[
X^i_{r_1r_2r_3} = \text{sign}(\rho)X_{r_\rho(1)r_\rho(2)r_\rho(3)}^i,
\]

(5.3.30)

\[
A_{r_1r_2r_3} = \text{sign}(\rho)A_{r_\rho(1)r_\rho(2)r_\rho(3)},
\]

(5.3.31)

where $\rho$ denotes elements of the symmetric group $S_3$. The first and last equation show that $S_{r_1r_2r_3}$ and $A_{r_1r_2r_3}$ are totally symmetric and anti-symmetric, respectively. The second (third) equation indicates that only one of $Y^1, Y^2, Y^3$ ($X^1, X^2, X^3$) is independent. Hence we arrive at a natural correspondence between the baryons and the Young tableaux as

\[
A_{r_1r_2r_3} : \quad X^i_{r_1r_2r_3} : \quad Y^i_{r_1r_2r_3} : \quad S_{r_1r_2r_3} : \quad .
\]

(5.3.32)

This looks perfectly consistent with the standard decomposition of $\square \otimes \square \otimes \square$.

Actually this is not quite straightforward, and this example nicely illustrates the subtlety alluded in the Introduction. As we have seen in the previous section, there is a one-to-one correspondence between the highest-weight states of the baryons $|B\rangle$ and the Young tableaux with $k$ boxes of a definite type. This means that there is only one vortex state of highest weight, corresponding to the mixed-symmetry Young tableau\(^{10}\). However, we seem to have $Y$ and $X$ in (5.3.32), both of which correspond to the same Young tableau. This apparent puzzle is solved by looking at the following Plücker relation rewritten in terms of $S, X, Y, A$

\[
(Y^1_{rst})^2 = -S_{rst}X^1_{rst} - X^1_{srts}S_{rts} + X^1_{srts}S_{rst},
\]

(5.3.33)

(no sum over $r, s, t$),

which shows that the tensor $Y$ is determined in terms of the others up to a sign. This implies that no solution to Eq. (5.3.33) of “pure $Y$” type, i.e., with $Y \neq 0$, $A = S = X = 0$, exists. Hence we have verified the one-to-one correspondence between the highest-weight baryon states $|B\rangle$ and the Young tableaux, as in Figure 5.4.

By setting two among $S, X$ or $A$ to be zero, we obtain the corresponding $SU(N)$ irreducible orbits, which can be immediately read off from the Young tableaux as (for

\(^{10}\text{In contrast to the standard composition-decomposition rule for three distinguishable objects in the } N \text{ representation, two inequivalent highest weight states in the same irreducible representation, described by the same mixed-type Young tableau, will appear. This is not so for our } k \text{ vortices.}\)
5.3 SU(N) decomposition of general k vortex states

\[ m = [3, 0, 0, \cdots, 0] \quad m = [1, 1, 0, \cdots, 0] \quad m = [0, 0, 1, 0, \cdots, 0] \]

\[ \begin{array}{ccc}
\bullet & \cdots & \bullet \\
\text{SU}(N-1) & & \text{SU}(N-2)
\end{array} \]

\[ S : \begin{array}{c}
\text{Fig. 5.4: The irreducible orbits in the moduli space of } k = 3 \text{ vortices.}
\end{array} \]

\[ N \geq k = 3 \]

\[ \mathcal{M}^S \cong \frac{SU(N)}{SU(N-1) \times U(1)} \cong \mathbb{C}P^{N-1}, \quad (5.3.34) \]

\[ \mathcal{M}^X \cong \frac{SU(N)}{SU(N-2) \times U(1)^2}, \quad (5.3.35) \]

\[ \mathcal{M}^A \cong \frac{SU(N)}{SU(3) \times SU(N-3) \times U(1)} \cong Gr_{N,3}. \quad (5.3.36) \]

Due to the existence of \( Y \), the whole subspace with \( \sigma_i = 0 \) is more complicated than the \( k = 2 \) case. The simplest non-trivial case \( N = 2 \) (SU(2) global symmetry) somewhat enlightens our understanding. In that case, \( A \) is identically zero and the following parameterization using the coordinates \( \{ \eta, \xi^1, \xi^2, \phi_1, \phi_2 \} \in \mathbb{C}^5 \)

\[ X^1_{r12} = \epsilon_{rs} \xi^s, \quad Y^1_{r12} = \eta \phi_r, \quad S_{rst} = \phi_r \phi_s \phi_t, \quad r, s, t = 1, 2 \quad (5.3.37) \]

solves all of the Plücker relations except for

\[ \eta^2 = \xi^r \phi_r. \quad (5.3.38) \]

Therefore, \( \eta \) is a locally dependent coordinate. Since the equivalence relation is

\[ \{ \xi^r, \eta, \phi_r \} \simeq \{ \lambda^3 \xi^r, \lambda^2 \eta, \lambda \phi_r \}, \quad (5.3.39) \]

the moduli space in this case is a hypersurface in \( WCP^4_{(3,3,2,1,1)} \simeq \mathbb{C}P^4 / \mathbb{Z}_3 \). The irreducible orbits corresponding to \( S \) and \( X \) are the subspaces obtained by setting \( \xi^r = 0 \) or \( \phi_r = 0 \), respectively. Both of them are isomorphic to

\[ \mathcal{M}^{[2,1,0]} \cong \mathcal{M}^{[1,0,0]} \cong \frac{SU(2)}{U(1)} \cong \mathbb{C}P^1. \quad (5.3.40) \]

According to the results of the next section, however, they are characterized by the different Kähler classes while their Kähler potentials are given by

\[ K \simeq \begin{cases} 
3r \log |\phi_r|^2 & \text{as } |\xi^r|^2 \to 0 \\
r \log |\xi^r|^2 & \text{as } |\phi_r|^2 \to 0
\end{cases} \quad . \quad (5.3.41) \]
5.3.4 Generalization to arbitrary winding number

In this section, we comment on a generalization to the case of an arbitrary winding number $k$. As we have seen in the $k = 2, 3$ cases, the coherent states (5.2.24) become insufficient to describe the general solution to the constraint (5.2.19) when two or more vortex centers coincide. The procedure to obtain the general solution for $k = 3$ vortices can be generalized to the case of arbitrary $k$ as follows. Let $|S; r_1, \ldots, r_k; \{z_i\}\rangle$ be the following linear combination of the coherent states

$$|S; r_1, \ldots, r_k; \{z_i\}\rangle \equiv \frac{1}{k!} \Delta \sum_{\rho \in S_k} \text{sign} (\hat{\rho}) \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1\rangle \otimes \cdots \otimes |0, r_k\rangle, \quad (5.3.42)$$

where the polynomial $\Delta$ and the operators $\hat{v}$ are defined by

$$\Delta \equiv \prod_{I > J} (z_I - z_J), \quad \hat{v} \equiv \exp \left( \sum_{i=1}^{k} z_i \hat{a}_i^\dagger \right); \quad (5.3.43)$$

$\hat{\rho} \hat{v} \hat{\rho}^{-1}$ then reads

$$\hat{\rho} \hat{v} \hat{\rho}^{-1} = \exp \left( z_1 \hat{a}_{\rho(1)}^\dagger + z_2 \hat{a}_{\rho(2)}^\dagger + \cdots + z_k \hat{a}_{\rho(k)}^\dagger \right). \quad (5.3.44)$$

This state vector (5.3.42) is a solution of the constraint (5.2.19) which is well-defined even in the coincident limit $z_I \rightarrow z_J$:

$$|S; r_1, \ldots, r_k; \{z_i\}\rangle \rightarrow \Delta (\hat{a}_{\rho(1)}^\dagger, \ldots, \hat{a}_{\rho(k)}^\dagger) |0, r_1\rangle \otimes \cdots \otimes |0, r_k\rangle. \quad (5.3.45)$$

Other well-defined solutions can be obtained by acting with polynomials of annihilation operators $\hat{a}_i$ on $|S; r_1, \ldots, r_k; \{z_i\}\rangle$. The linearly independent solutions are generated by the polynomials $h_i(\hat{a}_1, \ldots, \hat{a}_k)$ satisfying the following property\(^{11}\) for arbitrary symmetric polynomials $P$:

$$\langle 0 | h_i(\hat{a}_1, \ldots, \hat{a}_k) P(\hat{a}_1^\dagger, \ldots, \hat{a}_k^\dagger) = 0, \quad (5.3.46)$$

where $\langle 0 | \equiv \langle 0, r_1 | \otimes \cdots \otimes \langle 0, r_k |$. Such polynomials $h_i(\hat{a}_1, \ldots, \hat{a}_k)$ span a $k!$-dimensional vector space $H$ on which the symmetric group $S_k$ acts linearly\(^{12}\)

$$\hat{\rho} h_i(\hat{a}_1, \ldots, \hat{a}_k) \hat{\rho}^{-1} = h_i(\hat{a}_{\rho(1)}, \ldots, \hat{a}_{\rho(k)}) = g_i^j(\rho) h_j(\hat{a}_1, \ldots, \hat{a}_k), \quad (5.3.47)$$

\(^{11}\)The conditions (5.3.46) can be written in an alternative, equivalent form $P(\partial_1, \ldots, \partial_k) h_i(\eta_1, \ldots, \eta_k) = 0$, where $\partial_i \equiv \partial / \partial \eta_i$.

\(^{12}\)The representation of $H$ is isomorphic to the regular representation of $S_k$. 
where $g_{ij}(\rho)$ is a matrix corresponding to the transformation $\rho \in \mathcal{G}_k$. By using a linearly independent basis $\{h_i\}$, the general solution to Eq. (5.2.19) can be written as a superposition of $h_i |S; r_1, \cdots, r_k)$

$$|B\rangle = \sum_{r_1, \cdots, r_k} \sum_{i=1}^{k!} X^i_{r_1 \cdots r_k} h_i(\hat{a}_1, \cdots, \hat{a}_k) |S; r_1, \cdots, r_k; \{z_i\}\rangle.$$  

(5.3.48)

Since $|S; r_1, \cdots, r_k; \{z_i\}\rangle$ is well-defined for arbitrary vortex positions, this expression of the general solution is valid even in the coincident limit. Taking into account the constraint Eq. (5.2.16), we find that $X^i_{r_1 \cdots r_k}$ should have the following index structure

$$X^i_{r_1 \cdots r_k} = X^i_{\rho^{-1}(1) \cdots \rho^{-1}(k)} g_{ij}(\rho), \quad \text{for all } \rho \in \mathcal{G}_k.$$  

(5.3.49)

This condition reduces the number of degrees of freedom to $N^k = \dim W(\sigma_i)$. Since Eq. (5.2.24) and Eq. (5.3.48) are related by the change of basis from coherent states to $h_i |S; r_1, \cdots, r_k; \{z_i\}\rangle$, the coordinates $X^i_{r_1 \cdots r_k}$ can be obtained from $\tilde{B}_{r_1 \cdots r_k}$ by a linear coordinate transformation with $z_I$-dependent coefficients. Therefore, it is obvious that $X^i_{r_1 \cdots r_k}$ transforms under $SU(N)$ as a multiplet in the direct product representation $\otimes_{i=1}^k \mathbb{N}$.

We can also confirm this fact by decomposing the $k!$-dimensional vector space $H$ into the irreducible representations of the symmetric group $\mathcal{G}_k$. They are classified by the standard Young tableaux with $k$ boxes (Young tableaux with increasing numbers in each row and column) and correspondingly, the set of the coefficients $\{X^i_{r_1 \cdots r_k}\}$ can also be decomposed into subsets classified by the standard Young tableaux. Eq. (5.3.49) then tells us that the subset of $X^i_{r_1 \cdots r_k}$ for each irreducible representation of $\mathcal{G}_k$ forms a multiplet in the irreducible representation of $SU(N)$ specified by the corresponding Young tableau.

Finally, the remaining constraint (5.2.20) can be rewritten by using the relation (5.2.14) to quadratic constraints for $X^i_{r_1 \cdots r_k}$, which give the vortex moduli space as a subspace in $\mathbb{C}^k \times \mathbb{C}P^{N^k}$.

## 5.4 Kähler potential on irreducible $SU(N)$ orbits

In this section we will obtain the metric on each of the irreducible orbits inside the vortex moduli space $\mathcal{M}_k$ by use of a symmetry argument. We only use the fact that the metric of the whole vortex moduli space is Kähler and has an $SU(N)$ isometry.
One of the most important characteristics of non-Abelian vortices is that it possesses internal orientational moduli. They arise when the vortex configuration breaks the $SU(N)_{C+F}$ symmetry to its subgroup $H \subset SU(N)$. For a single vortex, it is broken to $SU(N-1) \times U(1)$ and the moduli space is homogeneous. On the other hand, the moduli space for multiple vortices, i.e. $k > 1$, is not homogeneous and has some anisotropic directions (even if we restrict ourselves to consider the subspace of coincident vortices). Consequently, the shape of the metric at generic points cannot be determined from the symmetry alone. The metric is not isometric along such a direction, and the isotropic subgroup $H$ (and the orbit $SU(N)/H$) can change as we move along such a direction in $M_k$.\(^{13}\) The moduli space $M_k$ contains all irreducible $SU(N)$ orbits associated with all possible Young tableaux having $k$ boxes, as its subspaces which are invariant under the action of the spatial rotation. In the following, we uniquely determine the metrics for all irreducible $SU(N)$ orbits. The irreducible orbits are all Kähler manifolds although generic $SU(N)$ orbits are not.\(^{14}\) We shall derive the Kähler potentials instead of the metrics directly.

The pair of matrices $(\psi, Z)$ corresponding to generic points on an orbit is obtained by acting with $SU(N)$ on a specific configuration $(\psi_0, Z_0)$. Let us decompose any element $U \in SU(N)$ as

$$U = LDU,$$

where $D$ is a diagonal matrix of determinant one and $L (U)$ is a lower (upper) triangular matrix whose diagonal elements are all 1. This is called the LDU decomposition.\(^{15}\) In this case, the matrix $U$ is a unitary matrix $UU^\dagger = 1$, and hence the matrices $L$, $D$ and $U$ are related by

$$UU^\dagger = (LD)^{-1}(LD)^{\dagger-1}.$$\(^{15}\)

Therefore, once the matrix $U$ is given, the lower triangular matrix $LD$ is uniquely determined up to multiplication of diagonal unitary matrices $u$ as $LD \to uLD$. That is, entries

\(^{13}\)This usually occurs in supersymmetric theories with spontaneously broken global symmetries and is called the supersymmetric vacuum alignment \([86]\). This phenomenon was discussed for non-Abelian vortices in Ref. \([83]\) and for domain walls in Ref. \([87]\). For non-Abelian $SO$, $USp$ vortices see Ref. \([50]\).

\(^{14}\)All irreducible $SU(N)$ orbits, which are the set of zeros of the holomorphic Killing vector for the spatial rotation, can be obtained as subspaces in $M_k$ with holomorphic conditions $f(\sigma, B) = 0$. Therefore the Kähler metrics are induced by these constraints from the Kähler metric on $M_k$. It is an interesting question if a Kählerian coset space in $M_k$ always corresponds to an irreducible orbit.

\(^{15}\)An invertible matrix admits an LDU decomposition if and only if all its principal minors are non-zero.
of $U$ are complex coordinates of the flag manifold $SU(N)/U(1)^{N-1}$.

Let $\psi_0$ and $Z_0$ be matrices of the form given in Fig. 5.1 and $m = [m_1, m_2, \ldots, m_{N-1}]$ be the set of Dynkin labels of the corresponding highest-weight state. Since the matrices $\psi_0$ and $Z_0$ satisfy the conditions (5.2.37) and (5.2.38), $LD$ can be always absorbed by $g \in GL(k, \mathbb{C})$ and $\tilde{g} \in SL(k, \mathbb{C})$ given in Eq. (5.2.41)

\[ \psi_0 U = (\tilde{g} g) \psi_0, \quad Z_0 = (\tilde{g} g) Z_0 (\tilde{g} g)^{-1}. \quad (5.4.3) \]

This implies that a pair $(\psi, Z)$ parameterizing the irreducible $SU(N)$ orbit is given by

\[ \psi_{\text{orbit}} = \psi_0 U, \quad Z_{\text{orbit}} = Z_0, \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1,N} \\ 1 & u_{23} & \cdots & u_{2,N} \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & \cdots & \cdots & u_{N-1,N} \\ & \cdots & \cdots & \cdots & 1 \end{pmatrix}, \quad u_{ij} \in \mathbb{C}. \quad (5.4.4) \]

The vortex state constructed by the latter is obtained as

\[ |B_{\text{orbit}}\rangle \equiv |B(\psi_{\text{orbit}}, Z_{\text{orbit}})\rangle = \hat{U} |B(\psi_0, Z_0)\rangle = \det g^{-1} \hat{U} |B(\psi_0, Z_0)\rangle, \quad (5.4.5) \]

with operators $\hat{U}$ and $\hat{U}$ corresponding to $U$ and $\hat{U}$ respectively.

In supersymmetric theories, $\psi$ and $Z$ can be regarded as chiral superfields. The complex parameters contained in $U$ are also lifted to chiral superfields and can be regarded as Nambu-Goldstone zero-modes of $SU(N)/U(1)^{N-1}$. If $m_i \neq 0$ for all $i = 1, \ldots, N-1$, then $SU(N)$ is broken to the maximal Abelian subgroup (the maximal torus) $U(1)^{N-1}$ and all the parameters $u_{ij}$ are physical zero modes. One can easily check that the dimension of the flag manifold $SU(N)/U(1)^{N-1}$ counts the degrees of freedom in $U$. On the other hand, if $m_i = 0$ for some $i$'s, then the unbroken group $H$ is enlarged from the maximal torus $U(1)^{N-1}$ to $SU(N)/H$ being generalized flag manifolds, from which we can further eliminate some of $u_{ij}$ by using $GL(k, \mathbb{C})$.

Since the vortex moduli space $\mathcal{M}_k$ has an $SU(N)$ isometry, the Kähler potential for $\mathcal{M}_k$, which is a real function of $\sigma_i$ and $B$, should be invariant under the $SU(N)$ transformation

\[ K(|B\rangle) = K(\hat{U} |B\rangle), \quad (5.4.6) \]

16The generic Kähler potential on $SU(N)/U(1)^{N-1}$, which contains $N - 1$ free parameters (Kähler classes), can be obtained from the method of supersymmetric non-linear realizations [88]. When all chiral superfields contain two Nambu-Goldstone scalars as in our case, they are called the pure realizations.
where \(|B\rangle\) is the vortex state vector satisfying all the constraints (5.2.16), (5.2.19) and (5.2.20). Furthermore, the \(\mathbb{C}^*\) transformations on the Kähler potential should be absorbed by the Kähler transformations

\[
K(e^\lambda |B_0\rangle) = K(|B_0\rangle) + f(\lambda) + \overline{f(\lambda)},
\]

(5.4.7)
since the \(\mathbb{C}^*\) action on \(|B_0\rangle\) gives a physically equivalent state \(e^\lambda |B_0\rangle \sim |B_0\rangle\). Note that this transformation can be absorbed only when \(\lambda\) is holomorphic in the moduli parameters.

We can easily show that the function \(f(\lambda)\) has the following properties

\[
f(2\pi i) + \overline{f(2\pi i)} = f(0) + \overline{f(0)},
\]

(5.4.8)

\[
f(\lambda_1 + \lambda_2) + \overline{f(\lambda_1 + \lambda_2)} = f(\lambda_1) + \overline{f(\lambda_1)} + f(\lambda_2) + \overline{f(\lambda_2)}.
\]

(5.4.9)

From these relations the form of the function \(f\) can be determined as

\[
f(\lambda) + \overline{f(\lambda)} = r(\lambda + \bar{\lambda}), \quad r \in \mathbb{R}.
\]

(5.4.10)

Now we are ready to derive the Kähler potential for the irreducible \(SU(N)\) orbits. With the above assumptions, the Kähler potential for the \(SU(N)\) orbit can be calculated as

\[
K(u_{ij}, \bar{u}_{ij}) \equiv K(|B_{\text{orbit}}\rangle) = K(\det g^{-1} \hat{U} |B\rangle)
\]

\[
= K(|B\rangle) - r \log |\det g|^2,
\]

(5.4.11)

where \(B = B(\psi_0, Z_0)\). Since the first term of Eq. (5.4.11) is a constant, it can be eliminated by the Kähler transformation. It follows from Eqs. (5.4.2) and (5.2.41) that

\[
K(u_{ij}, \bar{u}_{ij}) = - r \log |\det g|^2 = r \sum_{l=1}^{N-1} m_l \log \det(U_l U_l^\dagger),
\]

(5.4.12)

where \(U_l\) are \(l\)-by-\(N\) minor matrices of \(U\) given by

\[
U_l = \begin{pmatrix}
1 & u_{12} & \cdots & u_{1,l} & u_{1,l+1} & \cdots & u_{1,N} \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
& & u_{l-1,l} & \ddots & \ddots & \ddots & \\
& & & 1 & u_{l,l+1} & \cdots & u_{l,N}
\end{pmatrix}.
\]

(5.4.13)

Note that if \(m_l = 0\) for some \(l\)’s, the dimension of the manifold decreases in a way that is consistent with the enhancement of the symmetry \(H\).
The coefficients $r m_l$ of the terms in the Kähler potential (5.4.12) determine the Kähler class of the manifold. As noted in the footnote 16 the generic Kähler potential contains $N - 1$ free parameters, which is now determined from the set of Dynkin labels $[m_1, m_2, \cdots, m_{N-1}]$. We see that the Kähler classes are quantized to be integers multiplied by $r$ which implies that these Kähler manifolds are Hodge. This can be expected from the Kodaira theorem stating that Hodge manifolds are all algebraic varieties, i.e. they can be embedded into some projective space $\mathbb{CP}^n$ by holomorphic constraints.

The overall constant $r$ of the Kähler potential cannot be determined by the above argument based on the symmetry. It can however be obtained by a concrete computation, for instance, $k = 1$ vortex ($m = [1, 0, \cdots, 0]$) results in Refs. [67, 41, 68, 66]

$$r = \frac{4\pi}{g^2},$$

(5.4.14)

which matches the result (5.1.5) based on the D-brane picture [63]. It can be also determined from the charge of instantons trapped inside a vortex [68].

Recently, some of us constructed [56] the world-sheet action and computed the metrics explicitly from the first principles for the vortices in $SO$, $USp$ and $SU$ theories, generalizing the work of Refs. [41, 66]. The systems considered include the cases of some higher-winding vortices in $U(N)$ and $SO$ theories: the results found there agree with the general discussion given here.

### 5.4.1 Examples

In this subsection we provide two examples with $N = 2$ and $N = 3$ to illustrate the determination of the Kähler potentials. In all cases, the results obtained here agree with what follows from the explicit calculations of low-energy effective action discussed in Chapter 3, providing us with nontrivial check of consistency for both of our approaches.

$N = 2$

To be concrete, let us take some simple examples for $N = 2$. For simplicity, we first consider the $k = 2$ case. There are two highest-weight states: triplet and singlet, for which
\( \psi_0 \) and \( Z_0 \) take the form, see Fig. 5.1,

\[
(\psi, Z)_{\mathfrak{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (\psi, Z)_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\] (5.4.15)

In the former case \( SU(2) \) is broken to \( U(1) \) and the orbit is \( SU(2)/U(1) \cong \mathbb{C}P^1 \). Applying Eq. (5.4.12), we obtain the Kähler potential for the Fubini-Study metric on \( \mathbb{C}P^1 \)

\[
K_{N=2} = 2r \log(1 + |a|^2), \quad U = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\] (5.4.16)

On the other hand, \( SU(2) \) is unbroken in the singlet case. Indeed \( \psi_0 \) is just the unit matrix, so that an arbitrary \( SU(2) \) transformation can indeed be canceled by \( GL(2, \mathbb{C}) \). This can be easily extended to the generic case with \( k > 2 \). In the case of \( k_1 > k_2 \), \( SU(2) \) is broken to \( U(1) \) while if \( k_1 = k_2 \), \( SU(2) \) is unbroken. From Eq. (5.4.12), we find the Kähler potential for the Fubini-Study metric on \( \mathbb{C}P^1 \) for \( k_1 > k_2 \):

\[
K_{N=2} = r m_1 \log(1 + |a|^2), \quad m_1 = k_1 - k_2,
\] (5.4.17)

while the orbits are always \( \mathbb{C}P^1 \) for arbitrary \( k_1 \) and \( k_2 \) (\( k_1 > k_2 \)), one can distinguish them by looking at the Kähler class \( rm_1 = r(k_1 - k_2) \). For instance, one can distinguish two \( \mathbb{C}P^1 \)’s in Eqs. (5.3.2) and (5.3.18) for one and two vortices, respectively.

\( N = 3 \)

Next, let us study the \( N = 3 \) case. There are four different types according to the Young tableaux and the unbroken groups \( H \), see Table 5.1. We parameterize the matrix \( U \) as

\[
U = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.
\] (5.4.18)

The complex parameters \( a, b, c \) are (would-be) Nambu-Goldstone zero-modes associated with \( SU(3) \to H \). Applying Eq. (5.4.12), we find

\[
K_{N=3} = r m_1 \log \left( 1 + |a|^2 + |b|^2 \right) + r m_2 \log \left( 1 + |c|^2 + |b - ac|^2 \right),
\] (5.4.19)

with \( m_1 = k_1 - k_2 \) and \( m_2 = k_2 - k_3 \). When \( m_1 > 0 \) and \( m_2 > 0 \) (\( k_1 > k_2 > k_3 \)), this represents the Kähler potential for the Kähler manifold \( SU(3)/U(1)^2 \) with a particular
5.4 Kähler potential on irreducible $SU(N)$ orbits

<table>
<thead>
<tr>
<th>$k_1 &gt; k_2 &gt; k_3$</th>
<th>$k_1 &gt; k_2 = k_3$</th>
<th>$k_1 = k_2 &gt; k_3$</th>
<th>$k_1 = k_2 = k_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>$U(1)^2$</td>
<td>$U(1) \times SU(2)$</td>
<td>$U(1) \times SU(2)$</td>
</tr>
<tr>
<td>H</td>
<td>$U(1)^2$</td>
<td>$U(1) \times SU(2)$</td>
<td>$U(1) \times SU(2)$</td>
</tr>
</tbody>
</table>

Table 5.1: Four different types of $N=3$ coincident vortices.

choice of the complex structure [89]. When $m_1 > 0$ and $m_2 = 0$ ($k_1 > k_2 = k_3$), the parameter $c$ disappears from the Kähler potential and hence it reduces to

$$K_{m_2=0} = r m_1 \log \left( 1 + |a|^2 + |b|^2 \right),$$

which is nothing but the Kähler potential of $\mathbb{CP}^2 \simeq SU(3)/[U(1) \times SU(2)]$. When $m_1 = m_2 = 0$ ($k_1 = k_2 = k_3$), $SU(3)$ is unbroken, so that the orbit is just a point (with a vanishing Kähler potential).

5.4.2 Conjugate orbits

Note that in the $SU(3)$ example discussed in the last subsection the replacement

$$a \rightarrow -c, \quad b \rightarrow ac - b, \quad c \rightarrow -a$$

(5.4.21)

together with the exchange $m_1 \leftrightarrow m_2$, leaves invariant the Kähler potential (5.4.19). In other words, irreducible orbits for $m = [m_1, m_2]$ and $m = [m_2, m_1]$ are identical. In fact, this is a special case of duality between two $SU(N)$ conjugate representations, relating the irreducible orbits for $[m_1, m_2, \cdots, m_{N-1}]$ to the one with $[m_{N-1}, m_{N-2}, \cdots, m_1]$. As we are interested here in the motion of the orientational moduli parameters only, it is very reasonable that we find the same Kähler metric for a vortex in $r$ representation and another in $r^*$ representation.

Generalization to arbitrary $(N, k)$ of the mapping (5.4.21) leaving the Kähler potential invariant is give by

$$[m_1, m_2, \cdots, m_{N-1}] \leftrightarrow [m_{N-1}, m_{N-2}, \cdots, m_1],$$

$$U \leftrightarrow E (U^T)^{-1} E,$$

(5.4.22)

where $(E)_{ij} = \delta_{i,n-j+1}$. 
Coming back to the concrete $SU(3)$ examples in Subsection 5.4.1, the case with $(k_1, k_2, k_3) = (2, 1, 0)$ corresponds to 8 of $SU(3)$ which of course is self-dual. A pair of $(k_1, k_2, k_3) = (3, 3, 0)$ and $(4, 1, 1)$ provides the first nontrivial example of duality between two different irreducible orbits: they correspond to $10^*$ and 10, respectively. Finally, the orbits $(k_1, k_2, k_3) = (5, 4, 0)$ and $(k_1, k_2, k_3) = (6, 2, 1)$ belong to the pair of irreducible representations, $35^*$ and 35.

5.5 Conclusion

In this chapter, we investigated the structure of the moduli space of multiple BPS non-Abelian vortices in $U(N)$ gauge theory with $N$ fundamental Higgs fields, focusing our attention on the action of the exact global (color-flavor diagonal) $SU(N)$ symmetry on it. The moduli space of a single non-Abelian vortex, $\mathbb{C}P^{N-1}$, is spanned by a vector in the fundamental representation of the global $SU(N)$ symmetry. The moduli space of winding-number $k$ vortices is instead spanned by vectors in the direct-product representation: they decompose into the sum of irreducible representations each of which is associated with a Young tableau made of $k$ boxes, in a way somewhat similar to the standard group composition rule of $SU(N)$ multiplets. The Kähler potential is exactly determined in each moduli subspace, corresponding to an irreducible $SU(N)$ orbit of the highest-weight configuration.
Chapter 6

Conclusion and Outlook

In this thesis, we have constructed the low-energy effective action describing the fluctuations of the non-Abelian orientational zero-modes on the vortex world-sheet in a certain class of models. We also constructed the massive world sheet actions in several concrete examples. Group-theoretic and dynamical properties of higher winding vortices in the \( U(N) \) model are investigated, taking full advantage of the Kähler quotient construction \[60\].

In chapter 2, we studied the GNOW dual property of non-Abelian vortices. In the cases of minimal vortices in \( SO(2M) \times U(1) \) and \( USp(2M) \times U(1) \) theories, their moduli and transformation laws have been found to be isomorphic to spinor orbits in the GNOW duals, \( Spin(2M) \) and \( Spin(2M+1) \). The GNOW duality property is tested in the \( SO(2M+1) \) case as well. This could possibly be important in view of the general vortex-monopole connection, implied in a hierarchical symmetry breaking scenario, in which our vortex systems appear as a low-energy approximation \[35, 37, 48\].

In chapter 3, we generalized the \( \mathbb{C}P^{N-1} \) world-sheet action found some time ago in the \( U(N) \) model to a wider class of gauge groups. In the cases of the minimal vortices in \( U(1) \times SO(2M) \) and \( U(1) \times USp(2M) \) theories, they are given by two-dimensional sigma models in Hermitian symmetric spaces \( SO(2M)/U(M) \) and \( USp(2M)/U(M) \), respectively. Not much has appeared yet in the literature about the study of orientational moduli and their fluctuation properties in the case of higher-winding vortices \[35, 47, 76\]. We have also found the effective action for some higher-winding vortices in \( U(1) \times SO(2M) \) as well as in the \( U(N) \) theory.

Our vortex effective actions define the way the vortex orientational modes fluctuate
just below the typical mass scales characterizing the vortex solutions, and are somewhat analogous to the bare Lagrangian defining a given four-dimensional (4D) gauge-matter system, at some ultraviolet scale.

On the other hand, the effective vortex sigma models obtained here are, either in the non-supersymmetric version [70] or in a supersymmetric extension [71], all known to be asymptotically free. They become strongly coupled at the mass scale much lower than the typical vortex mass scale. The vortex effective action does not tell immediately what happens at such long distances, just as the form of the bare (ultraviolet) Lagrangian of an asymptotically-free 4D system does not immediately teach us about the infrared behavior of the system (Quantum Chromodynamics being a famous example). Let us note that the infrared behavior of our vortex fluctuations depends on whether or not the system is supersymmetric, or more generally, which other bosonic or fermionic matter fields are present, even though they do not appear explicitly (i.e. these fields are set to zero) in the classical vortex solutions.

In chapter 4, the mass-deformed effective action are constructed. There is a concrete quantitative correspondence between supersymmetry theories in two and four dimensions. The BPS spectrum of the mass-deformed two-dimensional $\mathcal{N} = (2, 2)$ $\mathbb{CP}^{N-1}$ sigma model coincides with the BPS spectrum of the four dimensional $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theories [77, 78]. The reason is that the two-dimensional sigma models are effective low-energy theories describing orientational moduli on the world sheet of non-Abelian confining strings [66, 67]. The appropriate Ansatz for the adjoint scalars has been given to realize the correspondence exactly. By the advantage of this Ansatz, we integrate two terms related to adjoint fields in the bulk four-dimensional theory, and find the result to be a mass potential term for the corresponding sigma models. The integration is analytically solvable due to the BPS equations. We generalized the result of $\mathbb{CP}^{N-1}$ sigma model to the $SO(2M)/U(M)$ and $USp(2M)/U(M)$ sigma models, and also the higher winding case of the quadratic surface $Q^{2M-2}$ sigma models. We also check the result from Scherk-Schwarz dimensional reduction [57, 58], which calculates the mass term from kinetic term directly. We found that the integration and SS dimension reduction produce the same result.

In chapter 5, we have investigated the moduli spaces of higher-winding BPS non-Abelian vortices in $U(N)$ theory by using the Kähler-quotient construction, for the purpose of clarifying the transformation properties of the points in the moduli under the exact
global $SU(N)$ symmetry group. In the case of vortices with distinct centers, the moduli space is basically just the symmetrized direct product of those of individual vortices, $(\mathbb{C} \times \mathbb{C}P^{N-1})^k / \mathfrak{S}_k$. It turns out to be a rather nontrivial problem to exhibit the group-theoretic properties of the points in the sub-moduli, in the cases of the vortex solutions with a common center. The results found show that they do behave as a superposition of various “vortex states” corresponding to the irreducible representations, appearing in the standard $SU(N)$ decomposition of the products of $k$ objects in the fundamental representations (Young tableaux).

In particular, various “irreducible $SU(N)$ orbits” have been identified: they correspond to fixed-point sets invariant under the spatial rotation group. These solutions are axially symmetric and they transform according to various irreducible representations appearing in the decomposition of the direct product.

Although some of our results might be naturally expected on general grounds, a very suggestive and non-trivial aspect of our findings is the fact that the points of the vortex moduli, describing the degenerate set of classical extended field configurations, are formally mapped to oscillator “quantum-state” vectors, endowed with simple $SU(N)$ transformation properties. Also, the way the irreducible orbits are embedded in the full moduli space appears to be quite non-trivial, and exhibits special features of our vortex systems. For instance, an irreducible orbit associated with a definite type of Young tableaux appears only once, unlike in the usual decomposition of $k$ distinguishable objects in $\mathbf{N}$.

We have determined the Kähler potential on each of these irreducible orbits. Since we have used symmetry only, our Kähler potential cannot receive any quantum corrections except for the overall constant $r$. However, the Kähler potential can receive quantum corrections in non-supersymmetric theories\(^1\).

The results found here agree, in all cases the comparison has been made, with the explicit calculations of low-energy effective actions discussed in Chapter 3, providing us with highly nontrivial checks of our analyses. Let us note that the explicit construction of the effective world-sheet action in Chapter 3 and the Kähler quotient construction of vortices in Chapter 5 are in many senses complementary. The first approach is more physical, by using the standard field equations and transformation properties of concrete

\(^1\)The renormalization group flow for $r$ in the case of $k = 1$ vortex in $\mathcal{N} = 2$ $U(N)$ supersymmetric theories was found in Refs. [67, 41].
soliton vortex solutions under the exact symmetry groups. The effective world-sheet action represents low-energy excitations of Nambu-Goldstone like modes, propagating along the vortex length and in time, which might be directly correlated with the fluctuation of non-Abelian monopoles, once the whole system is embedded in a hierarchically broken larger, gauge system, as discussed in the Introduction. The second approach, on the other hand, seems to be more powerful in exhibiting the group-theoretic features of the BPS non-Abelian vortices. In particular, we find it most remarkable that the points of the vortex moduli space, through Eq. (5.2.10), can be precisely mapped to multi-component quantum-mechanical harmonic oscillator states, which are in general superpositions of various terms, each belonging to definite irreducible representation of the symmetry group.

Non-Abelian vortices open a new chapter of solitons. The research on this promises in an important manner to contribute to the study of supersymmetric gauge field theories, string theory, high temperature superconductivity, quark confinement, and possibly other problems in physics.

Many issues can be studied based on the investigations of non-Abelian vortices. Extension of our considerations to more general situations in $U(N)$ theories (question of non-irreducible, general orbits in the vortex moduli space considered here, or the metric in the case of semi-local vortices, which occur when the number of flavors exceeds the number of colors [90, 91]) remains an open issue. A particularly interesting extension would however be the study of a more general class of gauge theories, such as $SO, USp$ or exceptional groups, as the group-theoretic features of our findings would manifest themselves better in such wider testing grounds, extending the results obtained in this thesis.

The BPS vortices of a fixed winding number possess a large degeneracy. However, several authors showed that when non-BPS corrections are considered, the degeneracy is eliminated. The vortices with higher tension will decay into vortices with lower tension. so it is interesting to ask which kind of vortices is most stable. By varying fine-tuned parameters, many possible phases can be discussed. There can be a “phase transition” from Abelian to non-Abelian for vortices, when some masses of the squarks become the same from the generic values. The confined monopoles can also transform from the Abelian monopoles to non-Abelian ones. Semi-classically such transformation can be studied in detail, the answer may change when quantum corrections are considered. The unbroken group $H$ can break down to an Abelian subgroup dynamically, which result in an approxi-
mately degeneracy set of monopoles. The quantization of non-Abelian monopoles remains an open question. The study of renormalization group flow will help us to understand this question.

In a recent work [42, 92], vortices and the moduli space of singular monopoles are proved to be isomorphic on a Riemann surface with an interval. One question is that given a vortex solution, is there a corresponding monopole solution satisfying certain boundary condition? It is suggested that the relation between symplectic quotients and Nahm’s equations can help to answer these questions. In $d = 1 + 1$ dimensions, vortices are finite action solutions in the Euclidean equations of motion. In other words, they play the role of instantons in the theory. In string theory, the vortices are related to the worldsheet instantons wrapping the 2-cycles of the Calabi-Yau Higgs branch. The moduli space of vortices is exactly half the fields of the ADHM construction, which means the vortex moduli space is half of instanton moduli space. The investigation of moduli space of vortices could well turn out to be a helpful tool to count instantons in two- and four-dimensions.
Appendix

A Supersymmetric gauge field theory

In this section, we want to introduce the $\mathcal{N} = 2$ supersymmetry and supersymmetric field theory. After many years investigation, the materials in this section now become standard [93, 94, 95, 96]. Some special conventions are taken in order to arrive at our final purpose: the bosonic truncation of the $\mathcal{N} = 2$ supersymmetric Lagrangian. The superfields are introduced, the Lagrangian of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theory with matter are constructed.

A1 Superfields and $\mathcal{N} = 1$ supersymmetry

The conventions are accordance with Lykken’s review [94]. The metric is

$$g^{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

which is different with the well-know Wess-Bagger conventions. The super-covariant derivatives acting on functions of $(x, \theta, \bar{\theta})$ are defined as

$$\bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\sigma_\alpha^\mu \theta^\alpha \partial_\mu.$$  \hspace{2cm} (A2)

The covariant derivative acting on functions of $(y, \theta, \bar{\theta})$ are expressed as

$$D_{\alpha} = \partial_\alpha + 2i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad D_{\bar{\alpha}} = \partial_{\bar{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}}.$$  \hspace{2cm} (A3)

A superfield is a function on the superspace. Superfield formalism is an elegant way to construct supersymmetric models. The most general scalar superfield $\Phi(x, \theta, \bar{\theta})$ is a scalar field in $\mathcal{N} = 1$ rigid superspace which can always be expanded as

$$\Phi(x, \theta, \bar{\theta}) = f(x) + \theta \phi(x) + \bar{\theta} \bar{\chi}(x) + \theta \theta m(x) + \bar{\theta} \bar{\theta} n(x)$$
$$+ \theta \sigma^\mu \bar{\theta} A_\mu(x) + \theta \theta \bar{\theta} \bar{\lambda}(x) + \bar{\theta} \bar{\theta} \theta \psi(x) + \theta \theta \bar{\theta} \bar{\theta} d(x).$$  \hspace{2cm} (A4)

An $\mathcal{N} = 1$ chiral superfield is obtained by one constraints

$$\bar{D}_{\dot{\alpha}} \Phi = 0,$$  \hspace{2cm} (A5)
where $\tilde{D}_\alpha$ is the super-covariant derivative of superfields. Define a bosonic coordinate $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$, we have

$$\tilde{D}_\alpha y^\mu = 0, \quad \tilde{D}_\alpha\theta^\beta = 0.$$ (A6)

Here the definition of the super-covariant derivative is given in (A3). So, any chiral superfield can be expanded as

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y).$$ (A7)

Similarly, an anti-chiral superfield is defined by $D_\alpha\Phi^\dagger(y^\dagger, \bar{\theta}) = 0$, which can be expanded as

$$\Phi^\dagger(y^\dagger, \bar{\theta}) = \phi^\dagger(y) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\theta F^\dagger(y^\dagger).$$ (A8)

Generically, any arbitrary function of chiral superfields is a chiral superfield, we can define a superpotential as

$$W(\Phi_i) = W(\phi_i) + \frac{\delta W}{\delta\phi_i} \sqrt{2}\theta\psi_i + \theta\theta \left(\frac{\delta W}{\delta\phi_i} F_i + \frac{1}{2} \frac{\delta^2 W}{\delta\phi_i\delta\phi_j} \psi_i\psi_j\right).$$ (A9)

In terms of the coordinates $(x^\mu, \theta, \bar{\theta})$, $\Phi$ and $\Phi^\dagger$ can be expanded as

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4} \theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) + \sqrt{2}\theta\psi(x)$$ (A10)

$$- \frac{i}{\sqrt{2}} \theta\theta\partial_\mu\psi^\dagger = \theta\theta F(x),$$ (A11)

$$\Phi^\dagger(x, \theta, \bar{\theta}) = \phi^\dagger(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger(x) - \frac{1}{4} \theta\theta\bar{\theta}\bar{\theta}\Box\phi^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x)$$ (A12)

$$+ \frac{i}{\sqrt{2}} \bar{\theta}\theta\theta\sigma^\mu\partial_\mu\bar{\psi} + \bar{\theta}\theta F^\dagger(x).$$ (A13)

Vector superfields $V(x, \theta, \bar{\theta})$ are defined from the general scalar superfields by imposing a reality constraint:

$$V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta}).$$ (A14)

The general expansion of the vector fields takes the form

$$V(x, \theta, \bar{\theta}) = C + i\theta\chi - i\bar{\theta}\bar{\chi} + \frac{i}{2} \theta\theta(M + iN) - \frac{i}{2} \bar{\theta}\bar{\theta}(M - iN) - \theta\sigma^\mu\bar{\theta}A_\mu$$

$$+ i\theta\bar{\theta}(\lambda + \frac{i}{2} \sigma^\mu\partial_\mu\chi) - i\theta\theta\bar{\theta}(\lambda + \frac{i}{2} \sigma^\mu\partial_\mu\bar{\chi}) + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta}(D - \frac{1}{2} \Box C).$$ (A15)

An infinitesimal Abelian transformation can be applied $V \rightarrow V + \Phi^\dagger + \Phi$, in components

$$A_\mu \rightarrow A_\mu + i\partial_\mu\phi^\dagger - i\partial_\mu\phi.$$ (A16)
A Supersymmetric gauge field theory

So, many components field can be gauged away by the Wess-Zumino gauge, the vector superfield can be written as

\[ V(x, \theta, \bar{\theta})_{WZ} = -\theta \sigma^\mu \bar{\theta} A_\mu + i \theta \partial \theta \bar{\lambda} - i \partial \bar{\theta} \theta \bar{\lambda} + \frac{1}{2} \theta \partial \bar{\theta} \bar{D}. \]  

(A17)

The Wess-Zumino gauge breaks supersymmetry, but not the Abelian gauge symmetry, and also does not fix the Abelian gauge. The Abelian superfield strengths are defined by

\[ W_\alpha \equiv -\frac{1}{4} (D \bar{D}) D_\alpha V_{WZ}(x, \theta, \bar{\theta}), \quad \bar{W}_\dot{\alpha} \equiv -\frac{1}{4} (D \bar{D}) \bar{D}_\dot{\alpha} V_{WZ}(x, \theta, \bar{\theta}). \]  

(A18)

\( W_\alpha \) and \( \bar{W}_\dot{\alpha} \) are chiral left-handed superfields and anti-chiral right-handed superfields respectively. Working on the variables \((y, \theta, \bar{\theta})\), notice that \( x^\mu = y^\mu - i \theta \sigma^\mu \bar{\theta} \), the \( W_\alpha \) can be written as

\[ W_\alpha = -i \lambda_\alpha(y) + \theta_\alpha D(y) - i (\sigma^\mu \theta)_\alpha F_{\mu \nu}(y) + \theta \theta (\sigma^\mu \partial_\mu \bar{\lambda}(y))_\alpha. \]  

(A19)

In the non-Abelian case, \( V \) belongs to the adjoint representation of the gauge group: \( V = V^a T^a \), where the generator \( T^a \) satisfy that

\[ [T^a, T^b] = if^{abc} T^c, \quad \text{Tr}(T^a T^b) = k \delta^{ab}, \]  

(A20)

\( k \) is taken to be 1 in the calculation of component fields of the \( \mathcal{N} = 2 \) supersymmetry gauge theory Lagrangian. When we consider the non-Abelian vortices, \( k \) is taken to be \( \frac{1}{2} \).

The non-Abelian superfield strength is defined as

\[ W_\alpha \equiv -\frac{1}{8} (D \bar{D}) e^{-2V_{WZ}} D_\alpha e^{2V_{WZ}} \]

\[ = -i \lambda_\alpha(y) + \theta_\alpha D(y) - i (\sigma^\mu \theta)_\alpha F_{\mu \nu}(y) + \theta \theta (\sigma^\mu \partial_\mu \bar{\lambda}(y))_\alpha, \]  

(A21)

in which

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu], \quad D_\mu \bar{\lambda} = \partial_\mu \bar{\lambda} + i [A_\mu, \bar{\lambda}]. \]  

(A22)

The super Yang-Mills Lagrangian with a \( \theta \) term can be constructed as

\[ \mathcal{L} = \frac{1}{8\pi} \text{Im} \left( \tau \text{Tr} \int d^2 \theta W^\alpha W_\alpha \right) \]

\[ = \text{Tr} \left[ -\frac{1}{4g^2} F_{\mu \nu} F^{\mu \nu} + \frac{\theta}{32\pi^2} F_{\mu \nu} \tilde{F}^{\mu \nu} + \frac{1}{g^2} \left( \frac{1}{2} D^2 - i \lambda \sigma^\mu D_\mu \bar{\lambda} \right) \right], \]  

(A23)

where \( \tau \) is the complex coupling constant, written as

\[ \tau \equiv \frac{\theta}{2\pi} + \frac{4\pi}{g^2}. \]  

(A24)
This is the pure $\mathcal{N} = 1$ gauge theory. Now we add chiral (matter) multiplets $\Phi$ which transforms in a representation of the gauge group,

$$\Phi \rightarrow e^{-2i\Lambda} \Phi, \quad \Phi^\dagger \rightarrow e^{2i\Lambda^\dagger} \Phi^\dagger,$$

(A25)

the gauge invariant kinetic term reads as $\Phi^\dagger e^{2V} \Phi$. The full $\mathcal{N} = 1$ supersymmetric Lagrangian can be written as

$$L = \frac{1}{8\pi} \text{Im} \left( \tau \text{Tr} \int d^2\theta W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{2V} \Phi + \int d^2\theta \mathcal{W} + \int d^2\bar{\theta} \bar{\mathcal{W}}$$

(A26)

The normalization factors of the Yang-Mills part and the matter part are not fixed by the $\mathcal{N} = 1$ supersymmetry. There is still a gauge and supersymmetric invariant term that can be added: the Fayet-Iliopoulos D-term. Suppose the gauge group is simply $U(1)$, or contains some $U(1)$ factors. Let $V_A$ denote the vector superfield in the Abelian case, or the component of $V_A$ corresponding to an Abelian factor. The Abelian gauge transformation is already given in Eq.(A16), while its supersymmetry transformation is only a total derivative term. It can be written as

$$L_{FI} = \sum_A \xi^A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi^A D^A,$$

(A27)

in which $A$ belongs to some Abelian factors. The FI D-term is important to induce the condensation of the squark fields when we consider vortex solutions.

**A2 $\mathcal{N}=2$ Supersymmetry**

There are several ways to extend the $\mathcal{N} = 1$ supersymmetry to the $\mathcal{N} = 2$ case. One way is to extend the superspace variables to $(x, \theta, \bar{\theta})$ to $(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$. Then, redefine the superfields, the $\mathcal{N} = 2$ super Yang-Mills can be written as the prepotential form [95, 94]

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \text{Tr} \int d^2\theta d^2\bar{\theta} \mathcal{F}(\Psi).$$

(A28)

However we can also construct the $\mathcal{N} = 2$ Lagrangian with the following argument. $\mathcal{N} = 2$ theory has a global $SU(2)_R$ symmetry which acts on the two chiral supercharges $Q^i_\alpha$ and
A Supersymmetric gauge field theory

$Q^2\alpha$. In superfields language, the $\mathcal{N} = 1$ scalar multiplet $(\phi, \psi)$ and vector multiplet $(A_\mu, \lambda)$ are in the same field content as the $\mathcal{N} = 2$ vector multiplet. Since $A_\mu$ and $\lambda$ belong to the adjoint representation, $\phi$ and $\psi$ must also belong to adjoint representation. Because the two supercharges are a doublet of the $SU(2)_R$ symmetry, the two fermions $\psi$ and $\lambda$ also form a doublet of $SU(2)_R$. A non-trivial superpotential would give $\psi$ interactions which are absent for that of $\lambda$, so the superpotential must be set to zero. $SU(2)_R$ symmetry also determines the normalization factors before the scalar part and the vector part. The $\mathcal{N} = 2$ super Yang-Mills Lagrangian can be written as

$$L_{SYM} = \frac{1}{8\pi} \text{Im} \text{Tr} \left[ \tau \left( \int d^2\theta W^\alpha W_\alpha + 2 \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{3V} \Phi \right) \right]$$

$$= \frac{1}{g^2} \text{Tr} \left[ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \bar{\lambda} + (D^\mu \phi)^\dagger D_\mu \phi - i \bar{\psi} \sigma^\mu D_\mu \psi + g^2 \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right.$$  

$$+ \frac{1}{2} D^2 + F^\dagger F + D[\phi, \phi^\dagger] + i \sqrt{2} \phi^\dagger \{\lambda, \psi\} - i \sqrt{2} \{\bar{\psi}, \bar{\lambda}\} \phi].$$  

(A29)

The commutators and anti-commutators arise since in the adjoint representation there is such formula

$$f^{abc} = -i \text{Tr} (T^a [T^b, T^c]).$$  

(A30)

The potential for supersymmetric systems has the universal form

$$V = -\text{Tr} \left( \frac{1}{2} D^2 + F^\dagger F \right).$$  

(A31)

There are no kinetic term for $D$ and $F$, so they are called auxiliary fields. By virtue of the equations of motion, they can be solved by the equations of motion. In the Lagrangian (A29), we have such solutions

$$D = [\phi^\dagger, \phi], \quad F = F^\dagger = 0.$$  

(A32)

This Lagrangian is also the supersymmetric generalization of the Yang-Mills-Higgs system. The vacuum is determined by Eq.(A32), so the vacuum expectation value(VEV) of $\phi$ lies in the Cartan sub-algebra of the gauge group. The spontaneously symmetry breaking of the system indicates the 't Hooft Polyakov monopole solutions and dyonic solutions exist.

$\mathcal{N} = 2$ matter fields are called the hypermultiplets. The component fields of the hypermultiplets can be arranged in two $\mathcal{N} = 1$ superfields. One is chiral superfield $Q$ and another is anti-chiral superfield $\bar{Q}$, transforming under the fundamental and anti-
fundamental representation of the gauge group respectively, which read as

\[ Q_i = q_i + \sqrt{2} \theta \psi_i + \theta \theta f, \]
\[ \tilde{Q}_i = \tilde{q}_i + \sqrt{2} \theta \tilde{\psi}_i + \theta \theta \tilde{f}. \]  

(A33)

Adding \( N_F \) hypermultiplets, which interact with one \( \mathcal{N} = 2 \) vector multiplet, the Lagrangian can be written as

\[ L_{\text{hyper}} = \sum_{i=1}^{N} \left[ \int d^2 \theta d^2 \bar{\theta} \left( Q_i^\dagger e^{2V} Q_i + \tilde{Q}_i e^{-2V} \tilde{Q}_i^\dagger \right) + \int d^2 \theta \left( \sqrt{2} \tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i \right) ight. 
\]
\[ + \int d^2 \bar{\theta} \left( \sqrt{2} Q_i^\dagger \Phi^\dagger \tilde{Q}_i^\dagger + m_i Q_i^\dagger \tilde{Q}_i^\dagger \right) \].

(A34)

The first term is Kähler potential which gives the kinetic term of matter fields. The second and third terms are referred to as the superpotential term, which describes the interaction and the mass of the system.

A3 Fayet-Iliopoulos D-term and F-term

In a \( \mathcal{N} = 2 \) supersymmetric theory, a small mass term for the adjoint field can be added by the superpotential

\[ \mathcal{A} = \mu \text{Tr} \Phi^2. \]  

(A35)

\( \Phi \) is a chiral superfield, which is the superpartner of the \( \mathcal{N} = 2 \) Abelian gauge field, defined as

\[ \Phi \equiv \phi + \sqrt{2} \theta \lambda + \theta \theta F, \]  

(A36)

The superpotential \( \mathcal{A} \) term will softly break the \( \mathcal{N} = 2 \) supersymmetry down to \( \mathcal{N} = 1 \). However, if we expand the superpotential around the vacuum expectation value (VEV) of the adjoint scalars, and keep only the linear term in \( \Phi \), the \( \mathcal{N} = 2 \) supersymmetry is conserved. The truncated superpotential is called Fayet-Iliopoulos F-term. Now we add both the FI D-term and FI F-term to the \( \mathcal{N} = 2 \) Lagrangian Eq. (A34), the superpotential can be written as

\[ V = \sum_{i=1}^{N_f} 2 \epsilon^2 \text{Tr} |\tilde{q}_i t^0 q_i| + \mathcal{A}'(m) |^2 + \sum_{i=1}^{N_f} \frac{\epsilon^2}{2} \left[ \text{Tr}(q_i^\dagger t^0 q_i) - \text{Tr}(\tilde{q}_i t^0 \tilde{q}_i) - \xi \right]^2, \]  

(A37)
where the color index is suppressed, and the coupling constant is changed to $e$ for the convention to discuss color-flavor locking. $\mathcal{A}(m)$ means the adjoint field $\phi$ is expanded around its VEV which is equal to some $m$ of squarks. This potential has the $\mathcal{N} = 2$ supersymmetry, which can be expressed in a manifest $SU(2)_R$ invariant form. Constructing the doublet of $SU(2)_R$ symmetry, $q^r = (q, \tilde{q}^\dagger)$, the superpotential can be rewritten as

$$V = \sum_{r,s} e^2 \left[ \text{Tr}(q^r t^0 q_s) - \frac{1}{2} \delta^{rs} \text{Tr}(q^{r'} t^0 q_{r'}) - \xi^a (\sigma^a)^{rs} \right]^2.$$  \hspace{1cm} (A38)

in which $r, s = 1, 2$ are the $SU(2)_R$ indices, also $a = 1, 2, 3$. $\sigma$ is the Pauli matrices, and $\xi^a$ are defined as

$$-\xi^1 + i \xi^2 = A(m), \hspace{0.5cm} \xi^3 = \frac{1}{2} \xi.$$  \hspace{1cm} (A39)

The superpotential can be rotated by $SU(2)_R$, which will have the same form as the $V_D$ with FI D-term, written as

$$V = \sum_i N_f e^2 \left[ \text{Tr}(q_i t^0 q_i) - \text{Tr}(\tilde{q}_i t^0 \tilde{q}_i^\dagger) - \xi' \right]^2,$$  \hspace{1cm} (A40)

where $\xi'$ is defined as

$$\xi' = \sqrt{\mathcal{A}'(m)^2 + \xi^2}.$$  \hspace{1cm} (A41)

Both the FI D- and F-terms can induce the SSB. However, they are not the same object. First we discuss $\mathcal{N} = 2$ supersymmetry. If both of them are in the triplet of $SU(2)_R$, and also the higher order terms of $\mathcal{A}$ are excluded, $\mathcal{N} = 2$ supersymmetry is not broken. when we drop the D-term, F-term softly breaks $\mathcal{N} = 2$ supersymmetry, i.e., the lowest order term does not break and higher order terms explicitly break. Otherwise, if we have only D-term, $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = 1$ explicitly. Secondly for $\mathcal{N} = 1$ supersymmetry, F-term breaks supersymmetry completely in the vortex core, while theory with D-term breaks supersymmetry partially [97]. When gauge groups are taken into account, F-term tends to represent models with non-Abelian gauge group, and D-term represents Abelian ones. F-term superpotential can be shifted by a perturbation of $\phi$, the color-flavor are not locked but only marginally locked [98]. Meanwhile, the FI D-term truly lock the color-flavor vacuum.
B Spinor representation of $SO(2M + 1)$

The spinor generators of the $SO(2M + 1)$ group $(a, b = 1, 2, \ldots, 2M + 1)$

$$[\Sigma_{ab}, \Sigma_{cd}] = -i (\delta_{bc}\Sigma_{ad} - \delta_{ac}\Sigma_{bd} - \delta_{bd}\Sigma_{ac} + \delta_{ad}\Sigma_{bc}) ,$$

(B1)
can be constructed as [69]

$$\Sigma_{2j-1,2M+1} = \frac{1}{2} j^{-1} \tau_3 \otimes \tau_j \otimes \mathbf{1} ,$$
$$\Sigma_{2j,2M+1} = \frac{1}{2} j^{-1} \tau_3 \otimes \tau_j \otimes \mathbf{1} ,$$

(B2)
in which $j = 1, 2, \ldots, M$, acting on the $M$-dimensional spin-$\frac{1}{2}$ system

$$|s_1\rangle \otimes |s_2\rangle \otimes \cdots |s_M\rangle ,$$

(B3)
with the sub-algebra $SO(2M)$ generated by:

$$\Sigma_{\alpha\beta} = -i \left[ \Sigma_{\alpha,2M+1}, \Sigma_{\beta,2M+1} \right] , \quad \alpha, \beta = 1, 2, \ldots, 2M .$$

(B4)
The annihilation and creation operators are defined by

$$a_k = \frac{1}{\sqrt{2}} (\Sigma_{2k-1,2M+1} - i \Sigma_{2k,2M+1}) = \frac{1}{2} k^{-1} \otimes \tau_3 \otimes \tau_1 \otimes \mathbf{1} ,$$
$$a_k^\dagger = \frac{1}{\sqrt{2}} (\Sigma_{2k-1,2M+1} + i \Sigma_{2k,2M+1}) = \frac{1}{2} k^{-1} \otimes \tau_3 \otimes \tau_2 \otimes \mathbf{1} ,$$

(B5)
where

$$\tau_\pm = \frac{\tau_1 \pm i\tau_2}{\sqrt{2}} .$$

(B6)

By expressing the generators $\Sigma_{ab}$ in terms of $a_j, a_j^\dagger$ and using \{a_j, a_k^\dagger\} = $\delta_{jk}/2$, we find that the spinors transform as follows:

$$S = e^{i\omega_{\alpha\beta}\Sigma_{\alpha\beta} + i\omega_{\gamma,2M+1}\Sigma_{\gamma,2M+1}}$$
$$= 1 + \alpha_{ij} a_i^\dagger a_j + \beta_{ij} a_i^\dagger a_j^\dagger + \beta_{ij} a_i a_j + d_i a_i^\dagger - d_i^\dagger a_i + i\omega_{2i,2i+1} + O (\omega^2) ,$$

(B7)
where

$$\alpha_{jk} \equiv 2 (\omega_{2j,2k} + \omega_{2j-1,2k-1} + i\omega_{2j-1,2k} - i\omega_{2j,2k-1}) ,$$
$$\beta_{jk} \equiv - (\omega_{2j,2k} - \omega_{2j-1,2k-1} + i\omega_{2j-1,2k} + i\omega_{2j,2k-1}) ,$$
$$d_j \equiv \frac{1}{\sqrt{2}} (\omega_{2j,2M+1} + i\omega_{2j-1,2M+1}) ,$$

(B8)
in terms of the original real rotation parameters $\omega_{ij}$. $\alpha_{jk}$ represent the parameters of $U(M) \subset SO(2M + 1)$ which leaves invariant the origin Eq. (2.3.11), whereas $\beta_{jk}$ and $d_j$ parameterize the coset, $SO(2M + 1)/U(M)$. The imaginary constants in Eq. (B7) contribute simply to the complex phase of $S$. $\beta_{jk}$ are antisymmetric complex matrices and $d_j$ is a complex $M$-component vector.

### C Transformation properties of $SO(2M + 1)$ vortices

The $SO(2M + 1)_C^{+F}$ generator is written as

$$\delta U = \begin{pmatrix} \alpha_{11} & \alpha_{12}^T & 0 & \beta_{12}^T & \xi^* \\ \alpha_{12}^* & \alpha_{22} & -\beta_{12} & \beta_{22} & \xi^* \\ 0 & -\beta_{12}^* & -\alpha_{11} & -\alpha_{12}^T & \xi \\ \beta_{12}^* & -\beta_{22}^* & -\alpha_{12} & -\alpha_{22}^T & \xi^* \\ -\xi & -\xi^T & -\xi^* & -\xi^T & 0 \end{pmatrix}, \quad \text{(C1)}$$

where the $\alpha$ block is anti-Hermitian, the $\beta$ block is antisymmetric. This form also satisfies the unitary condition. Acting with $\delta U$ on $q_0(z)$ from the right and then using a $\delta V$ to pull it back. The $\delta V$ is defined as

$$\delta V = \begin{pmatrix} V_{11} & \tilde{V}_{12}^T & W_{11} & \tilde{W}_{12}^T & \zeta_1^* \\ \tilde{V}_{12}^* & V_{22} & W_{21} & W_{22} & \tilde{\zeta}_1^* \\ \mu & \mu^T & \rho_{11} & \rho_{12}^T & \zeta_2 \\ \tilde{\mu}_{11} & \tilde{\mu}_{22} & \tilde{\rho}_{21} & \tilde{\rho}_{22} & \tilde{\zeta}_2 \\ \eta_1 & \tilde{\eta}_1^T & \eta_2 & \tilde{\eta}_2^T & \varepsilon \end{pmatrix}. \quad \text{(C2)}$$

It indeed satisfies the relation

$$\delta V^T \cdot J + J \cdot \delta V = 0, \quad J = \begin{pmatrix} 0 & 1_M & 0 \\ 1_M & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(C3)}$$

Here we set different parameters to get the most general solutions. Solving the matrix equations

$$\delta q_0(z) = q_0(z) \cdot \delta U + \delta V \cdot q_0(z), \quad \text{(C4)}$$
we get the solution of $\delta V$

\[
V_{11} = -\alpha_{11} - \alpha_{12}^T \cdot \tilde{C}_2 - \beta_{12}^T \cdot \tilde{C}_1 - C_2 \cdot \xi^* , \quad V_{12}^T = -z\alpha_{12}^T, \\
W_{11} = 0, \quad W_{12}^T = -z\beta_{12}^T, \\
\zeta_1^* = -z\xi^*, \\
V_{22} = \tilde{C}_2 \cdot \tilde{\alpha}_{12}^T - \alpha_{22} - \tilde{\beta}_{12} \cdot \tilde{C}_1^T, \\
W_{22} = \tilde{C}_2 \cdot \tilde{\beta}_{12}^T - \beta_{22} - \tilde{\beta}_{12} \cdot \tilde{C}_2^T, \\
\mu_{11} = 0, \quad \mu_{12}^T = 0, \\
\rho_{11} = \tilde{C}_1 \cdot \tilde{\beta}_{12} + \alpha_{11} + \tilde{C}_2 \cdot \tilde{\alpha}_{12} + C_2 \xi^*, \\
\zeta_2 = 0, \\
\mu_{22} = \beta^T - \tilde{\alpha}_{12} \cdot \tilde{C}_1^T + \tilde{C}_1 \cdot \tilde{\alpha}_{12}^T, \\
\eta_{11} = 0, \\
\eta_{22} = z\xi^*, \\
\varepsilon = 0, \\
\end{align*}

We also get the variation of $\tilde{C}_1$, $\tilde{C}_2$ and $C_2$ as

\begin{align*}
\delta \tilde{C}_1 &= \beta_{22}^T \cdot \tilde{C}_2 + A\alpha_{12} + \tilde{C}_2 \cdot \tilde{\alpha}_{12}^T \tilde{C}_1 + \tilde{C}_1 \cdot \tilde{\beta}_{12}^T \tilde{C}_1 \\
&+ \alpha_{22}^T \cdot \tilde{C}_1 + C_2 \xi^* \tilde{C}_1 - C_2 \xi + \alpha_{11} \tilde{C}_1 + \tilde{\beta}_{12}, \\
\delta \tilde{C}_2 &= -\beta_{22} \cdot \tilde{C}_1 + A\beta_{12} + \tilde{C}_2 \cdot \tilde{\beta}_{12} \tilde{C}_1 + \tilde{C}_2 \cdot \tilde{\alpha}_{21} \tilde{C}_2 \\
&- \alpha_{22} \tilde{C}_2 + \alpha_{11} \tilde{C}_2 - C_2 \xi^* \tilde{C}_2 - \tilde{\alpha}_{12}^T , \\
\delta C_2 &= A\xi^* + \tilde{C}_1^T \cdot \xi^* + \tilde{C}_1^T \cdot \xi + C_2 \beta_{12}^T \cdot \tilde{C}_1 \\
&+ C_2 \alpha_{11} + C_2 \tilde{C}_2^T \cdot \tilde{\alpha}_{12} + C_2 \xi^* + \xi. \\
\end{align*}

At the origin, setting

\[
\tilde{C}_1 = 0, \tilde{C}_2 = 0, C_2 = 0, \\
\] we obtain that

\begin{align*}
\delta \tilde{C}_1 &= \tilde{\beta}_{12}^T , \quad \delta \tilde{C}_2 = -\tilde{\alpha}_{12}, \quad \delta C_2 = \xi , \\
\end{align*}

The local neighborhood is indeed a vector representation. The dimension of this neighborhood is $4M - 2$. 

In this appendix, we derive the constraints (5.2.16), (5.2.19) and (5.2.20) from the definition of the baryons

\[ B_{t_1 t_2 \ldots t_k}^{n_1 n_2 \ldots n_k} = \epsilon^{i_1 i_2 \ldots i_k} Q_{i_1 t_1}^{(n_1)} Q_{i_2 t_2}^{(n_2)} \ldots Q_{i_k t_k}^{(n_k)}, \quad (Q^{(n)} \equiv Z^n \psi), \]  

(D1)

and the vortex state vector

\[ |B\rangle = \sum_{n_1, n_2, \ldots, n_k} \frac{1}{n_1! n_2! \ldots n_k!} B_{t_1 t_2 \ldots t_k}^{n_1 n_2 \ldots n_k} |n_1, r_1 \rangle \otimes |n_2, r_2 \rangle \otimes \cdots \otimes |n_k, r_k \rangle. \]  

(D2)

1. Eq. (5.2.16) implies that the baryon is anti-symmetric under the exchange of any pair of indices \((n, r)\). This can easily be seen from the definition of the baryons

\[ B_{t_1 t_2 \ldots t_k}^{n_1 n_2 \ldots n_k} = \epsilon^{i_1 i_2 \ldots i_k} Q_{i_1 t_1}^{(n_1)} \ldots Q_{i_j t_j}^{(n_j)} \ldots Q_{i_k t_k}^{(n_k)} = -\epsilon^{j_1 j_2 \ldots j_k} Q_{j_1 t_1}^{(n_1)} \ldots Q_{j_j t_j}^{(n_j)} \ldots Q_{j_k t_k}^{(n_k)} = -B_{t_1 t_2 \ldots t_k}^{n_1 n_2 \ldots n_k}. \]  

(D3)

2. The annihilation operator \(\hat{a}_I\) acts on the state as

\[ \hat{a}_I |B\rangle = \sum \frac{1}{(n_1! \ldots (n_I - 1)! \ldots n_k!)} B_{t_1 t_2 \ldots t_k}^{n_1 n_2 \ldots n_k} |n_1, r_1 \rangle \cdots |n_I - 1, r_I \rangle \cdots |n_k, r_k \rangle = \sum \frac{1}{(n_1! \ldots n_I! \ldots n_k!)} B_{t_1 t_2 \ldots t_k}^{n_1 \ldots n_I + \ldots n_k} |n_1, r_1 \rangle \cdots |n_I, r_I \rangle \cdots |n_k, r_k \rangle. \]  

(D4)

This means that the baryon is mapped by the operator \(\hat{a}_I\) as

\[ B_{t_1 t_2 \ldots t_k}^{n_1 \ldots n_k} \mapsto B_{t_1 \ldots t_k}^{n_1 \ldots n_I + 1 \ldots n_k} = \epsilon^{i_1 \ldots i_k} Z_{j_1 t_1}^{(n_1)} \ldots Q_{j_I t_I}^{(n_I)} \ldots Q_{i_k t_k}^{(n_k)}. \]  

(D5)

Therefore, we find that the operator \(\prod_{I=1}^{k} (\lambda - \hat{a}_I)\) acts on the baryons as

\[ B_{t_1 \ldots t_k}^{n_1 \ldots n_k} \mapsto \epsilon^{j_1 j_2 \ldots j_k} (\lambda \mathbf{1}_k - Z)_{j_1 t_1}^{(n_1)} \cdots (\lambda \mathbf{1}_k - Z)_{j_k t_k}^{(n_k)} Q_{i_1 t_1}^{(n_1)} \ldots Q_{i_k t_k}^{(n_k)} = \det(\lambda \mathbf{1}_k - Z) B_{t_1 \ldots t_k}^{n_1 \ldots n_k}. \]  

(D6)

Namely, the vortex state should be an eigenstate of the operator \(\prod_{I=1}^{k} (\lambda - \hat{a}_I)\)

\[ \prod_{I=1}^{k} (\lambda - \hat{a}_I) |B\rangle = \det(\lambda \mathbf{1}_k - Z) |B\rangle. \]  

(D7)

Comparing the coefficient of \(\lambda^i\) on both sides, we obtain the constraint (5.2.19).
3. The left hand side of Eq. (5.2.20) is

$$B^{A_1 \cdots A_k B_1 \cdots B_k} = \sum_{i_1, \cdots, i_k} \sum_{j_1, \cdots, j_k} \epsilon^{i_1 \cdots i_k} \epsilon^{j_1 \cdots j_k} Q^{A_1}_{i_1} \cdots Q^{A_k}_{i_k} Q^{B_1}_{j_1} \cdots Q^{B_k}_{j_k},$$  \hspace{1cm} (D8)

where $A_i$ and $B_i$ each denote a pair of indices $(n, r)$. Let us focus on the following part

$$\sum_{j_1, \cdots, j_k} \epsilon^{j_1 \cdots j_k} Q^{A_k}_{i_k} Q^{B_1}_{j_1} \cdots Q^{B_k}_{j_k}.$$  \hspace{1cm} (D9)

Since the indices $j_1, \cdots, j_k$ are contracted with $\epsilon^{j_1 \cdots j_k}$, there exist a number $I$ ($1 \leq I \leq k$) such that $i_k = j_I$ for each term in the sum. Therefore, all the terms in Eq. (D9) vanish since the indices $A_k$ and $B_1, \cdots, B_k$ are anti-symmetrized. This fact leads to the constraint Eq. (5.2.20).

### E A toy metric on the vector space spanned by $|B\rangle$

We have not considered in the main text the metric for the vector space spanned by $|B\rangle$, introduced in Subsection 5.2, for reasons explained at the end of Subsection 5.2.1. Such a metric would however induce a natural metric on the vortex moduli space, which is of physical interest. For instance, one could simply assume the standard inner product $\langle B|B\rangle$; it would induce a metric specified by the following Kähler potential

$$K_{\text{toy}} = r \log \langle B|B\rangle.$$  \hspace{1cm} (E1)

Note that the equivalence relation (5.2.8) is realized as Kähler transformations. Namely, the moduli space is embedded into the projective space with suitable constraints (5.2.20).

In the case of well-separated vortices $|z_I - z_J| \gg m^{-1}$, we find that the Kähler potential (E1) takes the form

$$K_{\text{toy}} = r \sum_{I=1}^{k} \left(|z_I|^2 + \log |\vec{\phi}_I|^2\right) - r \sum_{I, J(\neq I)} \frac{|\vec{\phi}_I^T \cdot \vec{\phi}_J^T|^2}{|\vec{\phi}_I^T|^2|\vec{\phi}_J^T|^2} e^{-|z_I - z_J|^2} + \cdots.$$  \hspace{1cm} (E2)

The first term correctly describes free motions of $k$ vortices while the second term describes interactions between the vortices.

Unfortunately, the interaction terms do not have the correct form; terms which behave as $1/|z_I - z_J|^2$ or $K_0(m|z_I - z_J|)$ must be present if massless or massive modes propagate.
between vortices, respectively. The former is the case of the Hanany-Tong metric [63] (which still does not describe the correct interactions), while the latter is the case of the correct asymptotic form obtained from the BPS equations [51].

**F Metrics on \( \mathbb{C}P^2 \) for \( k = 2 \) and \( N = 2 \)**

In this Appendix we will study some metrics on the intrinsic subspace \( \mathbb{C}P^2 \) for \( k = 2 \) coincident vortices in the \( U(2) \) gauge theory \( (N = 2) \). We show that two different metrics on \( \mathbb{C}P^2 \) contain the Fubini-Study metric with the same Kähler class on \( \mathbb{C}P^1 \) at the diagonal edge of Fig. 5.3.

For any choice of metric on the moduli space, a subspace specified by a holomorphic constraint should also be a Kähler manifold. Its Kähler potential must be invariant under the global \( SU(2) \) and the transformation (5.3.23) as

\[
K_{\mathbb{C}P^2} = r f(X) \sim r \tilde{f}(X) + \text{const.} \times \log |\phi_i|^2, \quad X \equiv \frac{|\phi_i|^4}{r|\eta|^2} \tag{F1}
\]

with an arbitrary function \( f \). For the Hanany-Tong model, \( f(X) \) can be written as [76]²,

\[
f(X) = w^2 - \log(1 - w^4), \quad w^2 = \frac{2X}{1 + X + \sqrt{1 + 6X + X^2}} \tag{F2}
\]

For the toy model (E1) in Appendix E, \( f(X) \) can be written as

\[
f(X) = r \log (1 + rX). \tag{F3}
\]

These two models have the same behavior

\[
f(X) \sim \begin{cases} 
\log X + \text{const.}, & X \gg 1, \\
\text{const.} \times X, & X \ll 1.
\end{cases} \tag{F4}
\]

Since \( \log X \simeq 2 \log |\phi_i|^2 \), they give the usual Fubini-Study metric on \( \mathbb{C}P^1 \) with the same Kähler class, \( 2r \), for \( \eta = 0 \), and they have a conical singularity at \( \phi_i = 0 \). These features are not accidental but are guaranteed for any choice of the moduli space metric, as we show in Section 5.4.

²Here \( w \) is identical to that of Eq. (32) of the paper presented by Auzzi-Bolognesi-Shiftman [76]. Actually, we can reproduce the metric Eq. (34) in their work from the above potential.
G General solution of the linear constraints for $k = 3$

In this section, we consider the general solution of the linear constraints (5.2.16) and (5.2.19) for the $k = 3$ case. We have seen in Section 5.2.1 that the solution can be expressed by the coherent states

$$|B\rangle = \sum_{r_1,r_2,r_3} \tilde{B}_{r_1 r_2 r_3} \hat{A}( |z_1, r_1 \rangle \otimes |z_2, r_2 \rangle \otimes |z_3, r_3 \rangle ).$$  \hfill (G1)

However, this expression is not valid globally on the moduli space since the coherent states become linearly dependent when some vortices coincide $z_i = z_j$. In order to derive a globally well-defined expression for the general solution, let us rewrite the coherent state of Eq. (G1) as

$$|B\rangle = \frac{1}{3!} \sum_{r_1,r_2,r_3} \sum_{\rho \in S_3} \text{sign}({\rho}) \tilde{B}_{r_1 r_2 r_3} \hat{\rho} |z_1, r_1 \rangle \otimes |z_2, r_2 \rangle \otimes |z_3, r_3 \rangle \hat{\rho}^{-1},$$  \hfill (G2)

where $\hat{\rho}$ is an element of the symmetric group $S_3$. Defining an operator $\hat{v}$ by

$$\hat{v} \equiv \exp \left( z_1 \hat{a}_1^\dagger + z_2 \hat{a}_2^\dagger + z_3 \hat{a}_3^\dagger \right),$$  \hfill (G3)

and the action of the symmetric group

$$\hat{\rho} \hat{v} \hat{\rho}^{-1} \equiv \exp \left( z_{\rho^{-1}(1)} \hat{a}_{\rho^{-1}(1)}^\dagger + z_{\rho^{-1}(2)} \hat{a}_{\rho^{-1}(2)}^\dagger + z_{\rho^{-1}(3)} \hat{a}_{\rho^{-1}(3)}^\dagger \right),$$  \hfill (G4)

we can rewrite the state $|B\rangle$ as

$$|B\rangle = \frac{1}{3!} \sum_{r_1,r_2,r_3} \sum_{\rho \in S_3} \text{sign}({\rho}) \tilde{B}_{r_1 r_2 r_3} \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1 \rangle \otimes |0, r_2 \rangle \otimes |0, r_3 \rangle.$$  \hfill (G5)

This means that the solution $|B\rangle$ is a linear combination of $3! = 6$ states $\hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1 \rangle \otimes |0, r_2 \rangle \otimes |0, r_3 \rangle$, which form a basis of the vector space of states satisfying the constraint

$$P(\hat{a}_1, \hat{a}_2, \hat{a}_3) |B\rangle = P(z_1, z_2, z_3) |B\rangle,$$  \hfill (G6)

for all symmetric polynomials $P$. However this basis is well-defined only for separated vortices since the states become degenerate when some vortices coincide. A globally well-defined basis can however be constructed as follows. Let $|S; r_1, r_2, r_3; \{z_i\}\rangle$ be the state defined by

$$|S; r_1, r_2, r_3; \{z_i\}\rangle \equiv \frac{1}{3!} \sum_{\rho \in S_3} \text{sign}({\rho}) \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1 \rangle \otimes |0, r_2 \rangle \otimes |0, r_3 \rangle,$$  \hfill (G7)
where $\Delta$ is the Vandermonde polynomial

$$\Delta(z_1, z_2, z_3) \equiv (z_1 - z_2)(z_2 - z_3)(z_3 - z_1).$$

This state is a solution of the constraint (G6) and well-defined even when the vortex centers coincide

$$|S; r_1, r_2, r_3; \{z_i\} \rightarrow \Delta(\hat{a}_1^1, \hat{a}_2^1, \hat{a}_3^1) \ |0, r_1 \rangle \otimes \ |0, r_2 \rangle \otimes \ |0, r_3 \rangle.$$  

(G9)

The other globally well-defined solutions can be constructed by acting with polynomials of $\hat{a}_i$ on $|S; r_1, r_2, r_3; \{z_i\} \rangle$. Note that any polynomial can be decomposed as

$$f(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \sum_i g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3),$$

(G10)

where $g_i$’s are symmetric polynomials and $h_i$ are polynomials satisfying

$$\langle 0, r_1 | \otimes \langle 0, r_2 | \otimes \langle 0, r_3 | h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) P(\hat{a}_1^1, \hat{a}_2^1, \hat{a}_3^1) = 0.$$  

(G11)

for all symmetric polynomials $P$ (without the constant term). Since the state $|S; r_1, r_2, r_3; \{z_i\} \rangle$ satisfies

$$g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) \ |S; r_1, r_2, r_3; \{z_i\} \rangle = g_i(z_1, z_2, z_3) \ |S; r_1, r_2, r_3; \{z_i\} \rangle,$$

(G12)

a symmetric polynomial $g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ does not create a new state. Therefore, it is sufficient to consider the polynomials $h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ satisfying Eq. (G11). The space of such polynomials $H$ is a $3! = 6$-dimensional vector space which can be decomposed as

$$H^{(0)} \ni S,$$

$$H^{(1)} \ni \tilde{Y}^1 \hat{a}_1 + \tilde{Y}^2 \hat{a}_2 + \tilde{Y}^3 \hat{a}_3,$$

$$H^{(2)} \ni \tilde{X}^1(\hat{a}_2 - \hat{a}_3)^2 + \tilde{X}^2(\hat{a}_3 - \hat{a}_1)^2 + \tilde{X}^3(\hat{a}_1 - \hat{a}_2)^2,$$

$$H^{(3)} \ni A(\hat{a}_1 - \hat{a}_2)(\hat{a}_2 - \hat{a}_3)(\hat{a}_3 - \hat{a}_1),$$

(G13)

(G14)

(G15)

(G16)

where $S, \tilde{Y}^i, \tilde{X}^i, A$ are complex numbers satisfying

$$\tilde{Y}^1 + \tilde{Y}^2 + \tilde{Y}^3 = 0, \quad \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3 = 0.$$  

(G17)

The spaces $H^{(i)}$ are closed under the action of the symmetric group and the decomposition $H = \bigoplus_i H^{(i)}$ corresponds to the decomposition of the regular representation of $\mathfrak{S}_3$. Acting
with the elements of $H^{(i)}$ on $|S\rangle$, we obtain the following basis

$$|S\rangle \equiv \sum_{r_1, r_2, r_3} S_{r_1 r_2 r_3} |S; r_1, r_2, r_3; \{ z_i \}\rangle,$$

$$|Y\rangle \equiv \sum_{r_1, r_2, r_3} (\tilde{Y}^1_{r_1 r_2 r_3} \hat{a}_1 + \tilde{Y}^2_{r_1 r_2 r_3} \hat{a}_2 + \tilde{Y}^3_{r_1 r_2 r_3} \hat{a}_3) |S; r_1, r_2, r_3; \{ z_i \}\rangle,$$

$$|X\rangle \equiv \sum_{r_1, r_2, r_3} \left( \tilde{X}^1_{r_1 r_2 r_3} (\hat{a}_2 - \hat{a}_3)^2 + \tilde{X}^2_{r_1 r_2 r_3} (\hat{a}_3 - \hat{a}_1)^2 + \tilde{X}^3_{r_1 r_2 r_3} (\hat{a}_1 - \hat{a}_2)^2 \right) |S; r_1, r_2, r_3; \{ z_i \}\rangle,$$

$$|A\rangle \equiv \sum_{r_1, r_2, r_3} A_{r_1 r_2 r_3} (\hat{a}_1 - \hat{a}_2)(\hat{a}_2 - \hat{a}_3)(\hat{a}_3 - \hat{a}_1) |S; r_1, r_2, r_3; \{ z_i \}\rangle,$$

From the anti-symmetry condition $\tilde{\rho} |B\rangle = \text{sign}(\rho) |B\rangle$, we find that for all $\rho \in \mathcal{S}_3$

$$S_{r_1 r_2 r_3} = S_{\rho(1) r_2 \rho(3)}; \quad (G18)$$

$$\tilde{Y}^i_{r_1 r_2 r_3} = \text{sign}(\rho) \tilde{Y}^{\rho(i)}_{\rho(1) r_2 \rho(3)}; \quad (G19)$$

$$\tilde{X}^i_{r_1 r_2 r_3} = \text{sign}(\rho) \tilde{X}^{\rho(i)}_{\rho(1) r_2 \rho(3)}; \quad (G20)$$

$$A_{r_1 r_2 r_3} = \text{sign}(\rho) A_{\rho(3) r_2 \rho(3)}; \quad (G21)$$

These relations imply that the tensors are in the irreducible representations of $SU(N)$. Note that in the coincident limit $z_1 = z_2 = z_3$, these states reduce to

$$|S\rangle \rightarrow \sum_{r_1, r_2, r_3} S_{r_1 r_2 r_3} (\hat{a}_1^\dagger - \hat{a}_2^\dagger) (\hat{a}_2^\dagger - \hat{a}_3^\dagger) (\hat{a}_3^\dagger - \hat{a}_1^\dagger) |0, r_1, r_2, r_3\rangle,$$

$$|Y\rangle \rightarrow \sum_{r_1, r_2, r_3} (Y^1_{r_1 r_2 r_3} (\hat{a}_2 - \hat{a}_3)^2 + Y^2_{r_1 r_2 r_3} (\hat{a}_3 - \hat{a}_1)^2 + Y^3_{r_1 r_2 r_3} (\hat{a}_1 - \hat{a}_2)^2) |0, r_1, r_2, r_3\rangle,$$

$$|X\rangle \rightarrow \sum_{r_1, r_2, r_3} (X^1_{r_1 r_2 r_3} \hat{a}_1^\dagger + X^2_{r_1 r_2 r_3} \hat{a}_2^\dagger + X^3_{r_1 r_2 r_3} \hat{a}_3^\dagger) |0, r_1, r_2, r_3\rangle,$$

$$|A\rangle \rightarrow \sum_{r_1, r_2, r_3} A_{r_1 r_2 r_3} |0, r_1, r_2, r_3\rangle; \quad (G22)$$

where $|0, r_1, r_2, r_3\rangle = |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle$, and

$$Y^1 \equiv \tilde{Y}^2 - \tilde{Y}^3, \quad Y^2 \equiv \tilde{Y}^3 - \tilde{Y}^1, \quad Y^3 \equiv \tilde{Y}^1 - \tilde{Y}^2;$$

$$X^1 \equiv -6 (\tilde{X}^2 - \tilde{X}^3), \quad X^2 \equiv -6 (\tilde{X}^3 - \tilde{X}^1), \quad X^3 \equiv -6 (\tilde{X}^1 - \tilde{X}^2). \quad (G23)$$

By rewriting the solution (G5) as a linear combination of these states, we obtain the globally well-defined general solution to the linear constraints.
Bibliography


