Elastic Continua as seen from Cosmology

Lagrangian Relativistic Methods applied to problems of Theory of Elasticity

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Introduction

Maybe, only few other subjects in physics are as intriguing as cosmology. This is probably because the study of cosmology involves other fields of human knowledge which are rather philosophical, until the point where they border on existential issues. This is why it should fascinate everybody, since it entails our deeper beliefs to be continuously questioned without providing any reassuring and firm answer. Uncertainty reigns when we try to give a copacetic description of the cosmos: the way man has been coping with this issue for centuries is to invent two entities, namely *space* and *time*, that govern the laws of the world we live in. However, they are not trivial notions at all: if we tried to give a clear definition of them (of time especially), we would find it impossible to provide one withstandng attacks from many critics. And this is what really happened in history. Philosophers and physicists have debated a lot, and no winner has emerged yet. For instance, modern physics has revolutionized the nature of space and time, but only reformulating the landmark concepts introduced by Newton, not effectively solving the problem.

In my treatise I tackle cosmology in simple terms, in order to prepare the background for the so-called Strained State Cosmology which is a newly proposed cosmological theory (A. Tartaglia) to solve the aforementioned everlasting and mystifying riddle. The Strained State Theory pretends to apply the fully geometrical description of three-dimensional solids to the four-dimensional space-time, elaborating a correspondence between the general notions of theory of elasticity and the general properties of space-time. Finally, my results prove that this theory also encompasses the deformation occurring in three-dimensional material continua, a sort of *classical limit* in practice. The cosmological model is in fact applied to problems which were encountered before in the ordinary theory of elasticity, and yields satisfactory responses, even though it might not be easy to find them at first sight - we need to be aware not only of what we are looking for, but also of the smartest way to get to
In conclusion, the structure of my work reflects the need for a thorough and fully justified thesis. Before arriving at cosmology, I give a proper account of tensor calculus, fundamental since it constitutes the new mathematical language used throughout the treatise; theory of elasticity, to describe continuum mechanics using tensors and solving simple problems; Lagrangian mechanics and the basics of general relativity, important to talk about cosmology. The Strained State Theory is then illustrated and in the last chapter it is applied to the problems illustrated within the classical theory of elasticity, with a full analysis of the results.
Chapter 1

Tensor Calculus

The language I will use in my treatise is that of tensors. In this chapter I will try to give a general overview of tensor calculus, not claiming to give a thorough explanation of all the remarkable topics, since it would be too long and complicated, indeed. It is however important to bear in mind some of these definitions and concepts, because they will be useful, if not fundamental, in the following chapters. Even though the subject might seem a little harsh, I have chosen to deal with this matter in a more intuitive (and less rigorous than it could be done in a course in differential geometry) way.

Tensor calculus was firstly developed about 1890 by Gregorio Ricci-Curbastro under the name of absolute differential calculus. With another famous mathematician, and also his pupil, Tullio Levi-Civita, he published the classic text *Méthodes de calcul différentiel absolu et leurs applications* (Methods of absolute differential calculus and their applications). At the beginning engineers and physicists were not much attracted by this new mathematical language. In the 1900s the absolute calculus became known as tensor analysis, achieving broader acceptance with Einstein’s theory of general relativity (1915), which is formulated completely in the language of tensors. With difficulty had Einstein learned them, with the help from the geometer Marcel Grossmann; Einstein was then supported by Levi-Civita, so as to correct his mistakes in manipulating tensor analysis. Moreover, tensors were also found to be useful in other fields: mechanics of continua is currently described in terms of tensors for instance. Other well-known tensors in differential geometry are the metric tensor and the Riemann curvature tensor. These are among the most
important tensors that I will define and use hereafter.

What I shall try to express in mathematical terms is that tensors must be independent of a particular choice of coordinate system. The aim of tensor calculus is to study those expressions which are invariant to any admissible change of coordinates.

1.1 The Einstein Summation Convention

A convention which simplifies the notation for sums is the so-called *Einstein summation convention*. It allows to suppress the use of $\Sigma$ for sums, because the repetition of an index twice in the same expression designates itself an implied summation. Even though the meaning of the position of the indices will be clarified later, the summation convention holds if and only if the two repeated indices appear one as a subscript and the other as a superscript. For example

$$x^i y_i$$

where $1 \leq i \leq n$ stands for

$$\sum_{i=1}^{n} x^i y_i = x^1 y_1 + x^2 y_2 + \ldots + x^n y_n$$

Consider indices $\mu$ and $\nu$, both of which range from 1 to $n$. In the expression

$$c_{\mu\nu} x^\mu = c_{1\nu} x^1 + c_{2\nu} x^2 + \ldots + c_{n\nu} x^n$$

there are two sort of indices. The index of summation $\mu$ runs from 1 to $n$; the use of the character $\mu$ is inessential, since any symbol or letter would mean the same thing (e.g. $c_{\mu\nu} x^\mu = c_{\lambda\nu} x^\lambda = c_{\rho\nu} x^\rho$). So $\mu$ is called *dummy* index. The index $\nu$ is not repeated, thus it does not imply any summation. It can take on any value in its range independently: it is called *free* index.

◊ **Remark:** any index cannot be repeated more than twice in an expression. For example $c_{ij} x_i x_i$ is meaningless, but if written $c_{ij}(x_i)^2$, it becomes meaningful. However, $a_i (x^i + y^i)$ makes sense, since it is obtained by composing the well-defined expressions $a_i z^i$ and $x^i + y^i = z^i$. 

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More sums can appear in a single expression, e.g.

\[ a_{ij}^{k} b_{k} c^{ij} \]

\( \Diamond \) **Remark**: the following expressions should be carefully noted

1. \( a_{ij}(x^i + y^i) \neq a_{ij}x^i + a_{ij}y^i \)
2. \( a_{ij}(x^i + y^i) = a_{ij}x^i + a_{ij}y^i \)
3. \( a_{ij}x^iy^j \neq a_{ij}y^ix^j \)
4. \( a_{ij}x^iy^j = a_{ij}y^ix^j \)
5. \( (a_{ij} + a_{ji})x^iy^j \neq 2a_{ij}x^iy^j \)
6. \( (a_{ij} + a_{ji})x^ix^j = 2a_{ij}x^ix^j \)

### 1.2 Kronecker Tensor

A very simple and useful symbol is the **Kronecker delta**. Note that for now, the position of the indices is unimportant, i.e. the tensor is always the same independently of the position of the indices. In section 1.4, the difference will be explained.

**Definition 1** (Kronecker Delta).

\[
\delta_{ij} = \delta_{i}^{j} = \delta^{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \tag{1.1}
\]

is a symmetric tensor (\( \delta_{ij} = \delta_{ji}, \delta^{ij} = \delta^{ji} \), and so on) and corresponds to the \( n \times n \) identity matrix

\[
I = \begin{pmatrix} 
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]

For example, in a Euclidean space \( \mathbb{R}^n \) the length of the line element \( dl^2 \) is expressed as

\[
dl^2 = \delta_{\alpha\beta} dx^\alpha dx^\beta
\]

If \( n = 3 \), then \( dl^2 = dx^2 + dy^2 + dz^2 \).
1.3 Admissible Changes of Coordinates

Definition 2. Suppose $x^\mu = x^\mu (\xi^1, \ldots, \xi^n)$ expresses a change of coordinate system (from the $\xi$ to the $x$ coordinates) in $\mathbb{R}^n$. It is an admissible change of coordinates iff the map $x^\mu = x^\mu (\xi^1, \ldots, \xi^n)$ is $C^1$, i.e. differentiable and with continuous differential, and the Jacobian of the transformation

$$|J [x^\mu (\xi^1, \ldots, \xi^n)]| = \left| \frac{\partial x^\mu}{\partial \xi^\nu} \right| \neq 0 \quad (1.2)$$

Thus, the application $x^\mu = x^\mu (\xi^1, \ldots, \xi^n)$ is invertible, and $JJ^{-1} = I$ or

$$\frac{\partial x^\alpha}{\partial \xi^\rho} \frac{\partial \xi^\rho}{\partial x^\beta} = \delta^\alpha_\beta$$

using Einstein summation convention. Conversely, $J^{-1}J = I$ or

$$\frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial x^\rho}{\partial \xi^\beta} = \delta^\alpha_\beta$$

♦ Remark: recall that the Jacobian is the determinant of the Jacobian matrix. I indicate it as $\text{det}(J) = |J|$.

1.4 Contravariance and Covariance

Contravariance and covariance are used to describe how physical quantities change with a change of coordinates. Albeit in some particular situations, e.g. when the coordinate change is just a rotation, this distinction is invisible, as far as more general coordinate systems are concerned, such as curvilinear coordinates, the distinction becomes critically important. The terms covariant and contravariant were introduced by J.J. Sylvester in 1853 in order to study an algebraic invariant theory.

Recall that tensors must constitute a reference for physical laws. We define as manifold the support we use to place our points, which can be physically existing points, in a three-dimensional space for instance, or just abstract, like points in the phase space, in which all possible states of a system are represented. There exist various different ways to indicate these points or quantities: for example there are scalars and vectors. A vector is an operator which, in an $n$-dimensional space, needs
n numbers to be fully described: those numbers are just scalars, that are associated with the n reference vectors, which together form a basis for our space.

Consider a vector \( \mathbf{a} \): given a basis \((\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)\) the vector \( \mathbf{a} \) is represented as

\[
\mathbf{a} = a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + \ldots + a^n \mathbf{e}_n = a^\mu \mathbf{e}_\mu
\]

where \( \mathbf{e}_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu \).

The different position of the index \( \mu \) gives information on how the quantities vary with a change of coordinates. In fact, changing the coordinates from \( x \) to \( \xi \), where the basis is \((\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)\), we have that

\[
\mathbf{a} = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 + \ldots + \alpha^n \mathbf{e}_n = \alpha^\alpha \mathbf{e}_\alpha
\]

Here

\[
\alpha^\alpha = a^\mu \frac{\partial \xi^\alpha}{\partial x^\mu} \tag{1.3}
\]

expresses the contravariant transformation law, in which the new quantity is a function of the old quantity and of the derivatives of the new coordinates with respect to the old coordinates. Conversely, the covariant law is expressed by the subscript and

\[
\mathbf{e}_\alpha = e_\mu \frac{\partial x^\alpha}{\partial \xi^\mu} \tag{1.4}
\]

so the new quantity is a function of the old quantity and of the derivatives of the old coordinates with respect to the new coordinates.

It is now trivial to prove that under any admissible coordinate change the vector \( \mathbf{a} \) is an invariant. That is, the components \( a^\mu \) must vary in the opposite way as the change of basis. Transformations 1.3 and 1.4 thus compensate each other, and the result is invariant. From the behaviour of its components we say that \( \mathbf{a} \) is a contravariant vector, or simply vector. The most important type of contravariant vector is the tangent vector. A vector \( \mathbf{u} \) is tangent to a manifold, for instance to a curve \( x^\mu = x^\mu(\lambda) \) where \( \lambda \) is the affine parameter describing the curve (it could be time), if its components have the form

\[
\mathbf{u}^\mu = \frac{dx^\mu}{d\lambda}
\]

Now, the components inevitably contra-vary with the change of basis:

\[
\alpha^\alpha = \frac{d\xi^\alpha}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial \xi^\alpha}{\partial x^\mu} = u^\mu \frac{\partial \xi^\alpha}{\partial x^\mu}
\]
Examples of (contravariant) vectors include the radius vector, or any derivative of it with respect to time, including velocity and acceleration.

On the other hand, in a so-called dual vector, or covector, the components of the vector co-vary with a change of basis to maintain the same meaning. A relevant example of a covariant vector is the gradient: given a scalar regular function $\phi : X \rightarrow \mathbb{R}$, where $X \in \mathbb{R}^n$ stands for the manifold, the components of the gradient vector are

$$\nabla_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$$

(thus we obtain the gradient effectively dividing the function $\phi$ by a vector $x^\mu$).

Its components, following the rules of differential calculus, vary as

$$\nabla_\alpha \phi = \frac{\partial \phi}{\partial \xi^\alpha} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \xi^\alpha} = \nabla_\mu \phi \frac{\partial x^\mu}{\partial \xi^\alpha}$$

which manifestly represents a covariant law.

In conclusion, any vector $v^\mu$ is associated with a dual vector $v_\mu$: more in general to any vector space $V$ corresponds a dual vector space $V^*$, whose elements are called covectors. The existence of a scalar product induces an isomorphism between $V$ and $V^*$, i.e. between vectors and covectors. If the scalar product is Euclidean, then with a natural choice of basis in $V$, as the orthonormal Cartesian basis ($i, j, k$), a vector and the corresponding covector have coinciding components. This is the reason why in classical physics, where we are used to describing the ordinary space $\mathbb{R}^3$ in rectangular coordinates, we do not perceive the difference.

### 1.5 Vector Contraction

To start off, I suggest we work on vectors, that are only a particular type of tensors. However, vectors are more familiar to us than the ‘abstract’ idea of tensor: so, it is easier to introduce new important concepts setting momentarily aside the general definition of tensor. Then, in the following sections, all these topics will be then generalized.

A vector contraction is the operation that arises from the natural pairing of a vector space and its dual.

Consider a contravariant vector and a covector (a differential in this case):

$$b = b^\mu \partial_\mu \quad \chi = \chi_\mu \, dx^\mu$$
Using them as operators, let us apply one to the other, or vice-versa:

\[ b^\mu \partial_\mu \chi_\alpha \, dx^\alpha \]

Recall that \( \partial_\mu \) and \( dx^\alpha \) are the true operators and so they cannot be interchanged, whereas the remaining terms are just numbers. Rearranging we get

\[ b^\mu \chi_\alpha \frac{\partial x^\alpha}{\partial x_\mu} = b^\mu \chi_\alpha \delta^\alpha_\mu \]

where the last expression may be worked out to find that it is zero if \( \alpha \neq \mu \): the only choice is setting \( \alpha = \mu \), thus obtaining

\[ b^\mu \chi_\mu \]

This expression is a scalar and is again an invariant, since its components vary with complementary laws. The operation of associating a scalar to a vector and a covector (a sort of scalar product) is called *contraction* and is indicated by

\[ \langle b, \chi \rangle = b^\mu \chi_\mu \quad (1.5) \]

### 1.6 Definition of Tensor and Properties

After a brief introduction on tensors, in which some of the main ideas were exposed, I will try to give a formal definition of tensor. It is true that it could be done in a much more rigorous way; anyway, here I am following the point of view of the founders of tensor calculus, such as in Levi-Civita, [11].

For the rest of the section consider two coordinate systems \( x^\mu \) and \( \xi^\mu \) (where in the latter all the quantities are barred, for example \( u \) becomes \( \bar{u} \)), with \( \mu = 1, \ldots, n \). Vectors are *first-order* tensors, since there is only one free index indicating the components: then, the components are as many as the dimension of the space (note also that a scalar is a tensor of order zero: it has no free indices, and for this it is obviously an invariant. For example the temperature measured in a particular point in space remains the same in any coordinate system). Recall that a tensor of *rank* or order one has two parts, which under any admissible coordinate change vary in two opposite ways compensating each other. When we have a vector \( \mathbf{v} = v^\mu \mathbf{e}_\mu \) for example, we say that this tensor (vector) is contravariant because we consider the
law with which \( v^\mu \) changes, implying of course the existence of the basis \( e_\mu \) which is covariant. Thus, there exist two types of first-order tensors \( T \), which, recalling from before, are the contravariant vectors

\[
T^\lambda = T_\gamma \frac{\partial \xi^\lambda}{\partial x^\gamma}
\]

and the covariant vectors

\[
\overline{T}_\lambda = T_\gamma \frac{\partial x^\gamma}{\partial \xi^\lambda}
\]

with \( \gamma \) and \( \lambda \) varying between 1 and \( n \).

Before generalizing to higher-order tensors, let me illustrate what happens when the free indices are two. We have second-order tensors. Now \( \mathbf{V} = (V^{\lambda\rho}) \) is a matrix field, being \( (V^{\lambda\rho}) \) an \( n \times n \) square matrix of scalar fields (the components of the tensor) \( V^{\lambda\rho}(x^\mu) \), all defined over the same manifold \( X \in \mathbb{R}^n \). As before assume that the components are \( (T^{\lambda\rho}) \) in the coordinates \( x^\lambda \) and \( (\overline{T}^{\lambda\rho}) \) in the \( \xi^\lambda \). Then three cases are possible:

1. **contravariant tensor** if the transformation obeys the law

\[
\overline{T}^{\lambda\rho} = T_\gamma \frac{\partial \xi^\lambda}{\partial x^\gamma} \frac{\partial \xi^\rho}{\partial x^\kappa}
\] (1.6)

2. **covariant tensor** if

\[
\overline{T}_{\lambda\rho} = T_\gamma \frac{\partial x^\gamma}{\partial \xi^\lambda} \frac{\partial x^\rho}{\partial \xi^\kappa}
\] (1.7)

3. **mixed tensor** if the tensor is contravariant of order one and covariant of order one

\[
\overline{T}^{\rho}_\lambda = T_\gamma \frac{\partial x^\gamma}{\partial \xi^\lambda} \frac{\partial \xi^\rho}{\partial x^\kappa}
\] (1.8)

Consider also the following theorem which will be essential in the future. It is clear that if we wish to use arrays to represent tensors, a second-order tensor is obviously a matrix.

**Theorem 1.** Suppose that \( T^{\lambda\rho} \) is a covariant tensor of rank two \((1 \leq \lambda \leq n, 1 \leq \rho \leq n)\). If the matrix \( (T^{\lambda\rho}) \) is invertible on the manifold \( X \in \mathbb{R}^n \), with inverse matrix \( (T^{\lambda\rho})^{-1} \), one has that

\[
(T^{\lambda\rho})^{-1} = (T^{\lambda\rho})
\]

where \( T^{\lambda\rho} \) is a contravariant tensor of rank two. Conversely,

\[
(T^{\lambda\rho})^{-1} = (T^{\lambda\rho})
\]
For tensors of arbitrary order the necessity of a generalized vector field, which is commonly called tensor field, arises. Denote this tensor field by $V = \left( V^{\lambda_1 \lambda_2 \ldots \lambda_p}_{\rho_1 \rho_2 \ldots \rho_q} \right)$: it is an array of $n^{p+q}$ scalar components defined over $X$.

**Definition 3 (Tensor).** The components of a general tensor of order $p + q$, contravariant of order $p$ and covariant of order $q$ (also called a tensor of type $(q, p)$), $(T^{\lambda_1 \lambda_2 \ldots \lambda_p}_{\rho_1 \rho_2 \ldots \rho_q})$ in the $x^\lambda$ coordinates, and $(\overline{T}^{\lambda_1 \lambda_2 \ldots \lambda_p}_{\rho_1 \rho_2 \ldots \rho_q})$ in the $\xi^\lambda$ coordinates, transform as

$$
T^{\lambda_1 \lambda_2 \ldots \lambda_p}_{\rho_1 \rho_2 \ldots \rho_q} = T^{\gamma_1 \gamma_2 \ldots \gamma_p}_{\sigma_1 \sigma_2 \ldots \sigma_q} \frac{\partial x^{\lambda_1}}{\partial x^{\gamma_1}} \frac{\partial x^{\lambda_2}}{\partial x^{\gamma_2}} \ldots \frac{\partial x^{\lambda_p}}{\partial x^{\gamma_p}} \frac{\partial x^{\sigma_1}}{\partial \xi^{\rho_1}} \frac{\partial x^{\sigma_2}}{\partial \xi^{\rho_2}} \ldots \frac{\partial x^{\sigma_q}}{\partial \xi^{\rho_q}}
$$

(1.9)

where the Einstein convention for sums has been used, and all indices have the obvious range.

Hence, the term tensor is a wide concept encompassing both scalars and vectors. The following example might help clarify a bit, as well as it introduces the tensor product $\otimes$. Consider two tangent vectors $a = a^\mu e_\mu$ and $b = b^\nu e_\nu$. Applying the tensor product to them we get

$$a \otimes b = a^\mu e_\mu \otimes b^\nu e_\nu = a^\mu b^\nu e_\mu \otimes e_\nu$$

In this way we have built a new tensor, whose rank has been increased by one. That is, the two bases $e_\mu$ and $e_\nu$, that constitute two operators, are paired and recombined to yield a new operator, namely $e_\mu \otimes e_\nu$, indicating a matrix basis. On the other hand, $a^\mu b^\nu$ is the numerical part of the tensor, and it is easy to understand that it is well represented by an array of numbers in the form of a matrix. Then,

$$a \otimes b = \begin{pmatrix}
    a^1 b^1 & a^1 b^2 & \ldots & a^1 b^n \\
    \vdots & \vdots & \ddots & \vdots \\
    a^\mu b^1 & a^\mu b^2 & \ldots & a^\mu b^n \\
    \vdots & \vdots & \ddots & \vdots \\
    a^n b^1 & a^n b^2 & \ldots & a^n b^n
\end{pmatrix}$$

which is also named bivector. I shall now prove that $a \otimes b$ is a tensor, starting from the assumption that $a$ and $b$ are both tensors. In other words, I want to prove that even if we change coordinates (from $x^\mu$ to $\xi^\mu$)

$$a \otimes b = a^\mu b^\nu e_\mu \otimes e_\nu = \overline{a}^\alpha \overline{b}^\beta \overline{e}_\alpha \otimes \overline{e}_\beta$$
applying laws 1.3 and 1.4. Working out the change of basis:

\[ a^\nu b^\mu e_\mu \otimes e_\nu = a^\alpha \partial x^\mu / \partial \xi^\alpha \otimes b^\beta \partial x^\nu / \partial \xi^\beta \]

Thus proving that \( a \otimes b \) is a tensor.

As it can be understood from the previous example, our aim is now forging new invariants (and so tensors) with one or more existing tensors.

I will not prove it here, but it is true that if \( T_1, T_2, \ldots, T_\mu \) are tensors of the same type and order, then any linear combination

\[ c_1 T_1 + c_2 T_2 + \ldots + c_\mu T_\mu \quad c_i \in \mathbb{R} \]

is again a tensor of the same type and order.

Another operation generating a tensor from two tensors is the **inner product**. It consists of equating a superscript (i.e. a contravariant index) of one tensor to a subscript (i.e. a covariant index) of another tensor, thus summing products of components over the repeated index. It is similar to the operation 1.5, \( \langle b, \chi \rangle = b^\mu \chi_\mu \), defined for a contravariant and a covariant vector respectively. Here is an example:

\[ C^\delta_{\gamma \sigma} = A^\delta_{\gamma \nu} B^\nu_\sigma = A^\delta_{\gamma 1} B^1_\sigma + A^\delta_{\gamma 2} B^2_\sigma + \ldots + A^\delta_{\gamma n} B^n_\sigma \]

the inner product over the index \( \nu \) yields a tensor where \( \nu \) is not figuring any more, even though it was appearing in the two tensors \( A^\delta_{\gamma \nu} \) and \( B^\nu_\sigma \) separately.

The inner product must not be mistaken for the **outer product**, where we would write

\[ C^\delta_{\gamma \nu \sigma} = A^\delta_{\gamma \nu} B^\mu_\sigma \]

Here the two tensors are just placed side by side: then, indices of the same type sum up. The order of the tensor increases, whereas the inner product decreases the total number of indices.

Another order-reducing operation, is obtained by applying the inner product to a single tensor, i.e. equating two different indices of the same tensor, also defined as **contraction**. For example take the tensor \( T^{\gamma_1 \gamma_2 \ldots \gamma_n}_{\sigma_1 \sigma_2 \ldots \sigma_n} \) and contract it over the indices
\( \gamma_1 \) and \( \sigma_2 \). This means setting \( \gamma_1 = \alpha \) and \( \sigma_2 = \alpha \) in the above expression to obtain a new tensor of type \( (q - 1, p - 1) \):

\[
T_{\alpha \gamma_2 \ldots \gamma_p}^{\alpha \gamma_2 \ldots \gamma_p} = T_{\sigma_1 \alpha \ldots \sigma_q}^{\gamma_1 \gamma_3 \ldots \gamma_q}
\]

Consider now a tensor whose rank is two, expressed in mixed form, like \( T_\mu^\nu \). If we contract this tensor over its indices, we obtain the scalar

\[
T_\alpha^\alpha = T_1^1 + T_2^2 + \ldots + T_n^n
\]

Since \( T_\mu^\nu \) represents a matrix, \( T_\alpha^\alpha \) is simply its trace. Then, the trace of mixed tensors is an invariant.

Eventually, I am showing that not all the operations on tensors yield other tensors. To do this, consider the symbol \( \varepsilon_{ijk} \), called Levi-Civita symbol, or also permutation symbol. It is defined as follows

\[
\varepsilon_{ijk} = \begin{cases} 
+1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1), \\
-1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3), \\
0 & \text{if } i = j \text{ or } j = k \text{ or } k = i 
\end{cases}
\]

and it can be used to write the formula for the determinant of a second-order tensor \( B = B^{ij} \) in three dimensions \( (n = 3) \). Namely

\[
\det(B^{ij}) = \varepsilon_{ijk} B_1^i B_2^j B_3^k
\]

Generalizing in \( n \) dimensions we have that

\[
\varepsilon_{\mu\nu\lambda\ldots} = \begin{cases} 
+1 & \text{if } (\mu,\nu,\lambda,\ldots) \text{ is an even permutation of } (1,2,3,4,\ldots) \\
-1 & \text{if } (\mu,\nu,\lambda,\ldots) \text{ is an odd permutation of } (1,2,3,4,\ldots) \\
0 & \text{otherwise}
\end{cases}
\]

and so

\[
\det(C^{\mu\nu}) = \varepsilon_{\mu\nu\lambda\ldots} C_1^{\mu} C_2^{\nu} C_3^{\lambda} C_4^{\ldots}
\]

If the coordinates are changed, we might apply the transformation laws seen before and:

\[
\det(C^{\mu\nu}) = \varepsilon_{\mu\nu\lambda\ldots} \left( \frac{\partial x^\mu}{\partial \xi^\alpha} \right) \left( \frac{\partial x^\nu}{\partial \xi^3} \right) \left( \frac{\partial x^\lambda}{\partial \xi^5} \right) \left( \frac{\partial x^\mu}{\partial \xi^\gamma} \right) \cdots = \det(C^{\mu\nu}) |J|^2
\]
where \( J = \left( \frac{\partial x^\mu}{\partial \xi^\nu} \right) \) is the Jacobian matrix of the transformation. Thus the determinant of a tensor is not a tensor!

### 1.7 Distance: the Metric Tensor

This section illustrates the way in which we can measure the length of a tangent vector or of a curve on a given manifold. The notion of distance, also called metric, is paramount in mathematics, as well as in everyday life. Tensor calculus provides a general formulation of distance, indistinctly for Euclidean and non-Euclidean manifolds. It accounts also for the form assumed by the Euclidean metric with a particular choice of coordinates, i.e. other than Cartesian.

Theorem 2. The metric tensor \( g_{\mu\nu} \) is a completely covariant tensor of rank two.

The distance \( dl \) between two nearby points, adopting the \( x^\mu \) coordinates is expressed by

\[
 dl^2 = g_{\mu\nu} dx^\mu dx^\nu
\]

(1.15)

For example, the arc-length formula for a Euclidean three-dimensional space in the rectangular (or Cartesian) coordinates \((x, y, z)\) is

\[
 dl^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2
\]

this being the famous Pythagoras’ theorem in three dimensions. \( \delta_{ij} \) is the well-known Kronecker delta 1.1 and it may be written out explicitly in matrix form

\[
 \delta_{ij} = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{pmatrix}
\]

\[ \diamond \textbf{Remark}: \text{so far I have used both Latin and Greek indices without explaining the distinction between them. The convention I am adopting here, unless stated differently, is to use Latin indices for three-dimensional manifolds, and Greek for general n-dimensional spaces.} \]

Continuing with the previous example, I wish to show that the Kronecker tensor is not the only possibility for a flat metric (i.e. a Euclidean space). More properly, \( \delta_{\alpha\beta} \)
is the metric for a flat manifold in rectangular coordinates. Hence, I will now change
the coordinate system (leaving the flatness of the manifold unvaried) according to
the transformation law 1.7, to obtain the metric \( g_{\mu \nu} \) from the metric \( \delta_{\mu \nu} \), or

\[
g_{\mu \nu} = \delta_{\alpha \beta} \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu}
\]  

(1.16)

For instance, I choose to switch to spherical polar coordinates \((r, \theta, \varphi)\). From appendix A.2 we have

![Spherical coordinate system](image)

Figure 1.1. Spherical coordinate system \((r, \theta, \varphi)\)

\[
x = r \sin \theta \cos \varphi
\]

\[
y = r \sin \theta \sin \varphi
\]

\[
z = r \cos \theta
\]

Then, all the possible derivatives \( \frac{\partial x^a}{\partial \xi^i} \) are to be worked out, i.e.

\[
\frac{\partial x}{\partial r} = \sin \theta \cos \varphi \quad \frac{\partial x}{\partial \theta} = r \cos \theta \cos \varphi \quad \frac{\partial x}{\partial \varphi} = -r \sin \theta \sin \varphi
\]

\[
\frac{\partial y}{\partial r} = \sin \theta \sin \varphi \quad \frac{\partial y}{\partial \theta} = r \cos \theta \sin \varphi \quad \frac{\partial y}{\partial \varphi} = r \sin \theta \cos \varphi
\]

\[
\frac{\partial z}{\partial r} = \cos \theta \quad \frac{\partial z}{\partial \theta} = -r \sin \theta \quad \frac{\partial z}{\partial \varphi} = 0
\]

Replacing all these results into 1.16 we obtain

\[
g_{ij} = \begin{pmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]  

(1.17)
In conclusion, we might write that
\[ dl^2 = \delta_{ij} dx^i dx^j = g_{ab} d\xi^a d\xi^b \]
or explicitly
\[ dl^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \]

In the rest of the treatise the metric tensor will be widely used, and the metrics of non-Euclidean manifolds will be carefully illustrated. This is also because the essence of my thesis lies in the various forms that the metric tensor may attain; further on, all these ideas will be analysed thoroughly.

1.7.1 Properties

The metric tensor is one of the tenets of tensor calculus. It is also named fundamental tensor. It does not only express the length of a line element, but it has many other fundamental functions. Here I will introduce its main properties, thus allowing to talk about more complex mathematical structures in the following pages.

- \( g_{\mu \nu} \) is of class \( C^2 \), i.e. all its second-order derivatives exist and are continuous;
- symmetry, or \( g_{\mu \nu} = g_{\nu \mu} \);
- \( g_{\mu \nu} \) is non-singular, i.e. \( |g_{\mu \nu}| \neq 0 \);
- \( g_{\mu \nu} \) is positive definite, which means that \( g_{\mu \nu} v^\mu v^\nu > 0 \) for all non-zero vectors \( v^\mu \). Also \( |g_{\mu \nu}| > 0 \) and \( g_{11}, g_{22}, \ldots, g_{nn} \) are all positive;
- since the determinant of the matrix \((g_{\mu \nu})\) is positive (but to this purpose it is only important that it be non-zero) the metric is invertible. Hence, there exists the inverse matrix \((g^{\mu \nu})\) such that \( g_{\mu \nu} g^{\nu \rho} = \delta^\rho_\mu \). Moreover, \( g^{\mu \nu} \) is itself a tensor, namely a totally contravariant tensor, obeying the law of transformation 1.6:

\[ g^{\mu \nu} = g^{\alpha \beta} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} \]

**Definition 4.** The inverse of the fundamental matrix (tensor) field \( g_{\mu \nu} \) is the conjugate metric tensor
\[ g_{\mu \nu}^{-1} = g^{\mu \nu} \]
The same properties listed above apply also to $g^\mu\nu$.

Using these definitions, we can now build new tensors from a given tensor, by simply applying the inner product with the metric tensor, or with its inverse. This operation is defined as raising or lowering indices in a tensor.

Given a contravariant tensor $T^\mu$ we may combine it with the metric to obtain a covariant tensor $U_\lambda$ as follows

$$U_\lambda = g_{\mu\lambda} T^\mu$$

If $g_{\mu\lambda}$ is the metric tensor, it proves useful in many cases to consider $U_\lambda$ and $T^\lambda$ as the covariant and contravariant version of the same tensor: so we can set $U_\lambda = T^\lambda$.

The same procedure holds with the inverse metric: given a covariant tensor $T_\mu$, its contravariant version is easily understood to be $T^\mu$, i.e.

$$U^\lambda = g^{\mu\lambda} T_\mu = T^\lambda$$

**Definition 5** (Lowering and raising indices). Given the metric tensor $g_{\mu\nu}$ and its conjugate $g^{\mu\nu}$, the operation of lowering a contravariant index of the tensor $T_{\gamma_1 \gamma_2 ... \gamma_i ... \gamma_p}$, $p$ times contravariant and $q$ times covariant, consists of the following expression

$$g_{\gamma_i \mu} T_{\sigma_1 \sigma_2 ... \sigma_j ... \sigma_q}^{\gamma_1 \gamma_2 ... \gamma_i ... \gamma_p} = T_{\mu \sigma_1 \sigma_2 ... \sigma_j ... \sigma_q}^{\gamma_1 \gamma_2 ... \gamma_i ... \gamma_p}$$

which is $p-1$ times contravariant and $q+1$ times covariant.

The operation of raising a covariant index of the tensor $T_{\gamma_1 \gamma_2 ... \gamma_i ... \gamma_p}$, $p$ times contravariant and $q$ times covariant, consists of

$$g^{\sigma_j \mu} T^{\gamma_1 \gamma_2 ... \gamma_i ... \gamma_p}_{\sigma_1 \sigma_2 ... \sigma_j ... \sigma_q} = T^{\mu \gamma_1 \gamma_2 ... \gamma_i ... \gamma_p}_{\sigma_1 \sigma_2 ... \sigma_q}$$

which is $p+1$ times contravariant and $q-1$ times covariant.

The determinant of the metric in not an invariant, since it transforms as we saw at the end of the previous section:

$$|g| = |g| |J|^2 \quad (1.18)$$

The determinant of the metric, $|g|$, is involved in the expression of the volume element in any integral. As a start, consider a three-dimensional Euclidean space with rectangular coordinates. The volume element is the well-known

$$dV = dx \, dy \, dz$$
A change of coordinates from \((x, y, z)\) to \((\xi, \eta, \zeta)\) leads to the reformulation of the volume element as

\[
dV = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} \, d\xi \, d\eta \, d\zeta = |J^{-1}| \, dV
\]

In the most general case we have

**Definition 6.** The volume element over an \(N\)-dimensional manifold with metric \(g_{\mu\nu}\) is

\[
dV = \sqrt{|g|} \, d^N x
\]  

(1.19)

where \(d^N x = dx^1 \, dx^2 \ldots dx^N\).

The following result is then paramount

**Theorem 3.** The volume element \(dV\) over an \(N\)-dimensional manifold is invariant, i.e.

\[
dV = \sqrt{|g|} \, d^N x = \sqrt{|g|} \, d^N \xi = d\tilde{V}
\]

**Proof.** The proof of this theorem comes easily if we consider relation 1.18, or

\[
\sqrt{|g|} = \sqrt{|g|} \, |J^{-1}| = \sqrt{|g|} \, |J|^{-1}
\]

where the law for transformation of a covariant tensor 1.7 has been used. Conversely, \(d^N x\) contravaries, so that

\[
d^N x = d^N \xi \, |J|
\]

Replacing these expressions in 1.19 the theorem is proved:

\[
dV = \sqrt{|g|} \, |J|^{-1} \, |J| \, d^N \xi
\]

\(\Box\)

### 1.8 Tensor Derivatives

With this section we enter the heart of tensor calculus. The notions illustrated hereby are absolutely non-trivial, but in a way more natural, in the sense that they better reflect the geometric properties of the investigation we wish to pursue in the field of tensor analysis.

Suppose we want to differentiate a tensor field along a curve in the flat space \(\mathbb{R}^2\).
Applying the definition of a derivative, we first need to be able to compute the difference between two vectors: in a Euclidean manifold the difference of two vectors is carried out component-wise. Suppose now that $\Sigma$ is a curved (not flat any more) surface of $\mathbb{R}^3$, and $\mathbf{V}$ is the field of vectors tangent to a curve along $\Sigma$. If we intend to work out the difference between two of the tangent vectors, we are not allowed to operate component-wise like before, because the result would not be \textit{intrinsic} any more, i.e. it would not be possible to express the resulting derivative field in terms of the coordinates of the surface. The reason for this stands in the fact that $\Sigma$ is not Euclidean\footnote{We use to say that it is curved; in next section the notion of curved manifolds will be tackled} like $\mathbb{R}^2$, so it has to be treated differently. The notion of derivative consequently needs a generalization, that is it must be formulated in a fashion which holds in whatsoever manifold we might be considering. This is the goal of the present section.

Consider a contravariant vector $\mathbf{a} = a^\mu \mathbf{e}_\mu$, of which we are to compute the directional derivative along the path $\lambda$ connecting two near points $P$ and $P'$ as in figure 1.2.

Figure 1.2. Path $\lambda$ between two points $P$ and $P'$
This derivative will be indicated as
\[ \frac{D a}{d\lambda} \]
instead of \( \frac{da}{d\lambda} \), because \( D \) signifies that there are two contributions in the computation of the derivative. One is due to the vector field \( a^\mu \) itself; the other, which must not be forgotten, depends on the manifold over which we move, i.e. on the variation of the orientation of the basis vectors \( e_\mu \). Applying the simple rule for differentiating a product, we have
\[
\frac{D a}{d\lambda} = \frac{da^\mu}{d\lambda} e_\mu + a^\mu \frac{D e_\mu}{d\lambda}
\]
The second term of the sum is the derivative of a vector. So, let me write
\[
\frac{D e_\mu}{d\lambda} = \frac{D e_\mu}{dx^\alpha} \frac{dx^\alpha}{d\lambda} = u^\alpha \left( \Gamma^\beta_{\mu \alpha} e_\beta \right)
\]
where \( u^\alpha = \frac{dx^\alpha}{d\lambda} \) is simply the vector tangent to \( \lambda \). The remaining part expresses the change of basis while moving over the manifold: its complete form will be illustrated further on. For now, let me say that \( \Gamma^\alpha_{\mu \nu} \) is called connection, symmetric in the covariant part. It has to be noted that the connection is not a tensor, but it will serve our purposes anyway\(^2\). Then, manipulating a bit the above expressions, i.e. making the tangent vector \( u^\alpha \) appear explicitly, and working on the indices, we end up with
\[
\frac{D a}{d\lambda} = u^\nu \left( \frac{\partial a^\mu}{\partial x^\nu} + a^\rho \Gamma^\mu_{\nu \rho} \right) e_\mu = u^\nu \frac{D a^\mu}{\partial x^\nu} e_\mu
\]
where

**Definition 7.** The covariant derivative of a contravariant vector \( a^\mu \) is given by the expression
\[
\frac{D a^\mu}{\partial x^\nu} = \frac{\partial a^\mu}{\partial x^\nu} + a^\rho \Gamma^\mu_{\nu \rho}
\]  

or, using the notation for which
\[
\frac{D a^\mu}{\partial x^\nu} = a^\mu_{\nu} \quad \text{and} \quad \frac{\partial a^\mu}{\partial x^\nu} = a^\mu_{\nu}
\]

\(^2\)The connection depends on the choice of the manifold and of the coordinates. For instance, changing the coordinate system from \( x^\mu \) to \( \xi^\mu \) it is easily seen that the connection it is not a tensor:
\[
\Gamma^\alpha_{\beta \delta} = \Gamma^\lambda_{\mu \nu} \frac{\partial \xi^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \xi^\delta} \frac{\partial x^\nu}{\partial \xi^\delta} + \frac{\partial^2 x^\lambda}{\partial \xi^\beta \partial \xi^\delta} \frac{\partial \xi^\alpha}{\partial x^\lambda}
it is shortly written as

\[ a_{\mu}^{\nu} = a_{\nu}^{\mu} + \alpha^{\rho} \Gamma_{\nu \rho}^{\mu} \]  

(1.21)

In a flat space with Cartesian coordinates \( \Gamma_{\nu \rho}^{\mu} = 0 \) always (this will be clear as soon as we provide an expression for \( \Gamma_{\nu \rho}^{\mu} \)). This explains why

\[ a_{\nu}^{\mu} = a_{\mu}^{\nu} \]

and so why in ordinary space the definition of partial derivative we use is correct and complete.

I now wish to show the formula to compute the covariant derivative of a covector, namely of \( \chi = \chi_{\alpha} \omega^{\alpha} \). Reintroducing the notion of connection, we guess, in analogy with 1.21, that

\[ \chi_{\mu;\nu} = \chi_{\mu,\nu} + \Psi_{\mu \nu}^{\alpha} \chi_{\alpha} \]

where \( \Psi_{\mu \nu}^{\alpha} \) is another type of connection. I shall now investigate around the relationship coexisting between \( \Psi_{\mu \nu}^{\alpha} \) and \( \Gamma_{\mu \nu}^{\alpha} \). Consider the scalar \( \phi \), defined as the contraction 1.5 of vectors \( a \) and \( \chi \):

\[ \phi = a^{\mu} \chi_{\mu} \]

Being a scalar, it has to be that \( \phi_{,\alpha} = \phi_{,\alpha} \) or

\[ a_{,\alpha}^{\mu} \chi_{\mu} + a^{\mu} \chi_{\mu,\alpha} = a_{,\alpha}^{\mu} \chi_{\mu} + a^{\mu} \chi_{\mu;\alpha} \]

\[ a_{,\alpha}^{\mu} \chi_{\mu} + a^{\mu} \chi_{\mu,\alpha} = a_{,\alpha}^{\mu} \chi_{\mu} + a^{\mu} \chi_{\mu;\alpha} + \Gamma_{\mu \nu}^{\delta} a^{\mu} \chi_{\delta} + \Psi_{\mu \nu}^{\delta} a^{\mu} \chi_{\delta} \]

from which it clearly is

\[ \Psi_{\mu \nu}^{\delta} = -\Gamma_{\mu \nu}^{\delta} \]

**Definition 8.** The covariant derivative of a covariant vector \( \chi^{\alpha} \) is given by the expression

\[ \chi_{\alpha;\nu} = \chi_{\alpha,\nu} - \chi_{\delta} \Gamma_{\nu \alpha}^{\delta} \]

(1.22)

The above expressions for differentiating covariant and contravariant objects may be generalized to higher order tensors. Although I am not writing the general formula here, I am illustrating the case of the covariant derivative of the three possible types of second-order tensors.
• Completely contravariant tensor $T = T^{\mu\nu} e_\mu \otimes e_\nu$

\[
T^{\mu\nu;\alpha} = T^{\mu\nu}_{,\alpha} + \Gamma^{\mu}_{\alpha\delta} T^{\delta\nu} + \Gamma^{\nu}_{\alpha\delta} T^{\mu\delta}
\]

• Completely covariant tensor $T = T_{\mu\nu} \omega^\mu \otimes \omega^\nu$

\[
T_{\mu\nu;\alpha} = T_{\mu\nu,\alpha} - \Gamma^\delta_{\mu\alpha} T_{\delta\nu} - \Gamma^\delta_{\alpha\nu} T_{\mu\delta}
\]

• Mixed Tensor $T = T^\mu_{\nu} e_\mu \otimes \omega^\nu$

\[
T^\mu_{\nu;\alpha} = T^\mu_{\nu,\alpha} + \Gamma^\mu_{\alpha\delta} T^\delta_{\nu} - \Gamma^\delta_{\alpha\nu} T^\mu_{\delta}
\]

Therefore, for the metric tensor we have

\[
g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^\delta_{\mu\alpha} g_{\delta\nu} - \Gamma^\delta_{\alpha\nu} g_{\mu\delta}
\]

The following result is fundamental.

**Theorem 4.** The covariant derivative of the metric tensor is always zero.

\[
g_{\mu\nu;\alpha} = 0 \tag{1.23}
\]

**Proof.** Consider the expression

\[
v^\mu w_\mu = g_{\mu\nu} v^\nu w^\nu
\]

Computing the covariant derivative of both members we must obtain the same result. The first member becomes

\[
(v^\mu w_\mu)_{,\alpha} = v^\mu_{,\alpha} w_\mu + v^\mu w_{\mu;\alpha}
\]

and the second member, lowering the superscripts when possible,

\[
(g_{\mu\nu} v^\nu w^\nu)_{,\alpha} = g_{\mu\nu;\alpha} v^\nu w^\nu + v^\mu_{,\alpha} w_\mu + v^\mu w_{\mu;\alpha}
\]

For the equality of the two members to hold, $g_{\mu\nu;\alpha}$ must vanish. \qed

I shall now explore the relation between the connection and the metric.
Theorem 5 (Christoffel symbols). Suppose the metric is non-singular, i.e. \(|g| \neq 0\). Then, there exists a relation between the connection \(\Gamma^\alpha_{\beta\delta}\) and the fundamental tensor \(g_{\mu\nu}\), giving rise to the so-called Christoffel symbols

\[
\Gamma^\alpha_{\beta\delta} = \frac{1}{2} g^{\alpha\epsilon} \left( \frac{\partial g_{\beta\epsilon}}{\partial x^\delta} + \frac{\partial g_{\delta\epsilon}}{\partial x^\beta} - \frac{\partial g_{\beta\delta}}{\partial x^\epsilon} \right) \tag{1.24}
\]

Proof. From theorem 4 it follows that

\[
g_{\mu\nu,\alpha} - \Gamma^\delta_{\mu\alpha} g_{\delta\nu} - \Gamma^\delta_{\alpha\nu} g_{\mu\delta} = 0
\]

Now, computing all the possible permutations and using the inverse metric, the Christoffel symbols are obtained.

Finally, consider the following examples in two dimensions.

First, suppose we have a flat two-dimensional manifold. The metric in rectangular coordinates \(x, y\) is indeed

\[
\delta_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Being constant it is clear that for all values of \(\alpha, \beta, \delta\), \(\Gamma^\alpha_{\beta\delta} = 0\) as we expected.

In the polar coordinates \(r, \theta\) the metric is not constant any more, since we have

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
\]

The inverse metric is easily found:

\[
g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}
\]

With all the possible permutations of the indices \(\alpha, \beta, \delta\), the only non-zero Christoffel are

\[
\Gamma^1_{22} = \Gamma^r_{\theta\theta} = -r \\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}
\]

Then, in polar coordinates not all the covariant derivatives equal the partial derivatives, namely:

\[a^r_{;\theta} = \frac{\partial a^r}{\partial \theta} - r a^\theta \]
\[a^\theta_{;\theta} = \frac{\partial a^\theta}{\partial \theta} + \frac{a^r}{r} \]
Consider now a non-Euclidean surface, for example that of a sphere. Two angular coordinates are needed to specify points on the surface: the longitude $\varphi$ and the colatitude $\vartheta$. The metric on a sphere of radius $a$ is then

$$dl^2 = a^2 d\vartheta^2 + a^2 \sin^2 \vartheta d\varphi^2$$

or

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix}$$

Recall that locally the metric can always be reduced to a Cartesian metric, since with each point is associated a tangent plane.

The non-zero Christoffels are

$$\Gamma^\vartheta_{\varphi\varphi} = \frac{\cos \vartheta}{\sin \vartheta}$$

$$\Gamma^{\varphi}_{\vartheta \varphi} = -\sin \vartheta \cos \vartheta$$

Furthermore, the surface element in this manifold is

$$dS = a^2 \sin \vartheta d\vartheta d\varphi$$

### 1.9 Curvature

Curvature is the origin of the fact that when we displace a tangent vector on a curved manifold it undergoes an increment, which we took into account when defining the covariant derivative of a vector field. If the space is flat, this increment vanishes, as the Christoffel symbols also vanish. So, the presence of curvature, i.e. of distortion, is indicated by the existence of non-vanishing Christoffel symbols. By the way, the Christoffel symbols are not tensors, so they are not good candidates to represent the curvature of a manifold. In any case, we need to look for a tensor that contains the Christoffels, and maybe also the metric.

Consider the tensor of rank three completely covariant

$$T_{\mu;\alpha\beta} - T_{\mu;\beta\alpha}$$
Note that it is not zero in general, because the result of a second order derivative is usually dependent of the order of differentiation (but in a Cartesian metric it is independent). \( T_\mu \) is an arbitrary covariant tensor. Recalling the rules for differentiating a covariant vector 1.22 and tensor, we have:

\[
T_{\mu;\alpha\beta} = T_{\mu,\alpha\beta} - \Gamma^\rho_{\mu\alpha\beta} T_\rho - \Gamma^\rho_{\mu\alpha} T_{\rho,\beta} - \Gamma^\rho_{\mu\beta} T_{\sigma,\alpha} + \Gamma^\sigma_{\mu\beta} \Gamma^\rho_{\sigma\alpha} T_\rho - \Gamma^\delta_{\alpha\beta} T_{\mu,\delta} + \Gamma^\delta_{\alpha\beta} \Gamma^\rho_{\mu\delta} T_\rho
\]

Analogously,

\[
T_{\mu;\beta\alpha} = T_{\mu,\beta\alpha} - \Gamma^\rho_{\mu\beta\alpha} T_\rho - \Gamma^\rho_{\mu\beta} T_{\rho,\alpha} - \Gamma^\rho_{\mu\alpha} T_{\sigma,\beta} + \Gamma^\sigma_{\mu\alpha} \Gamma^\rho_{\sigma\beta} T_\rho - \Gamma^\delta_{\beta\alpha} T_{\mu,\delta} + \Gamma^\delta_{\beta\alpha} \Gamma^\rho_{\mu\delta} T_\rho
\]

Working out the difference of the two expressions we get

\[
T_{\mu;\alpha\beta} - T_{\mu;\beta\alpha} = (\Gamma^\rho_{\mu,\alpha\beta} - \Gamma^\rho_{\mu,\beta\alpha} + \Gamma^\sigma_{\mu,\beta} \Gamma^\rho_{\sigma\alpha} - \Gamma^\sigma_{\mu,\alpha} \Gamma^\rho_{\sigma\beta}) T_\rho
\]

**Definition 9 (Riemann).** The tensor of rank four, one time contravariant and three times covariant

\[
R^{\rho}_{\mu\alpha\beta} = \Gamma^\rho_{\mu,\alpha\beta} - \Gamma^\rho_{\mu,\beta\alpha} + \Gamma^\sigma_{\mu,\beta} \Gamma^\rho_{\sigma\alpha} - \Gamma^\sigma_{\mu,\alpha} \Gamma^\rho_{\sigma\beta}
\]  

(1.26)

is called Riemann tensor, and it is the most general expression of the curvature of a manifold.

Making use of the metric, the totally covariant form is obtained, or

\[
R_{\nu\mu\alpha\beta} = g_{\rho\nu} R^{\rho}_{\mu\alpha\beta}
\]  

(1.27)

The Riemann curvature tensor 1.27 is antisymmetric if we interchange \( \nu \) with \( \mu \) or \( \alpha \) with \( \beta \). For example:

\[
R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \quad R_{\nu\mu\beta\alpha} = -R_{\nu\mu\alpha\beta} \quad R_{\mu\nu\beta\alpha} = R_{\nu\mu\alpha\beta}
\]

It is symmetric if we switch \( \nu \mu \) with \( \alpha\beta \):

\[
R_{\nu\mu\alpha\beta} = R_{\alpha\beta\nu\mu}
\]

Finally the Bianchi identity holds for 1.27:

\[
R_{\nu\mu\alpha\beta} + R_{\nu\mu\beta\alpha} + R_{\nu\alpha\beta\mu} = 0
\]  

(1.28)

where the last three indices have been permuted three times.

With no difficult computations other two tensors are consequently defined.
**Definition 10** (Ricci). *Contracting the Riemann curvature tensor 1.26 as follows, the so-called Ricci tensor is*

\[ R^\lambda_{\mu \alpha \lambda} = R^\mu_{\mu \alpha} \quad (1.29) \]

**Definition 11** (Scalar curvature). *The trace of the Ricci tensor in mixed form \( R^\rho_{\mu \rho} \) is equal to the invariant known as scalar of curvature \( R \), or simply scalar curvature:*

\[ R^\alpha_{\alpha} = R \quad (1.30) \]

Pay attention to the fact that the only condition assuring that a space has no curvature is

\[ R^\rho_{\rho \alpha \beta} = 0 \]

Consequently also \( R_{\lambda \sigma} = 0 \) and \( R = 0 \). Then, if \( R = 0 \) we cannot say that the manifold is flat since we have to look at Riemann tensor 1.26. However, if \( R \neq 0 \), it suffices the scalar of curvature to imply the distortion of the manifold. \( R = 0 \) is only a necessary condition, not sufficient to infer the curvature of a space.

As a last example, recall the case of the spherical manifold

\[ dl^2 = a^2 d\vartheta^2 + a^2 \sin^2 \vartheta \, d\phi^2 \]

of which I intend to compute the Riemann tensor, the Ricci tensor, and the scalar curvature to show that it is not Euclidean. The Christoffels, already computed, are

\[ \Gamma^\varphi_{\vartheta \varphi \varphi} = \frac{\cos \vartheta}{\sin \vartheta} \quad \Gamma^\varphi_{\varphi \varphi} = -\sin \vartheta \cos \vartheta \]

After some computations, the non zero elements are the following

\[ R^\varphi_{\vartheta \varphi \vartheta} = -1 \quad R^3_{\varphi \varphi \vartheta} = \sin^2 \vartheta \]

\[ R_{\vartheta \vartheta} = 1 \quad R_{\varphi \varphi} = \sin^2 \vartheta \]

\[ R^\vartheta_{\vartheta} = \frac{1}{a^2} \quad R^\varphi_{\varphi} = \frac{1}{a^2} \]

\[ R = \frac{2}{a^2} \]
Chapter 2

Theory of Elasticity: an Overview

Under the action of applied forces solids deform to some extent, i.e. they exhibit changes both in volume and in shape. In continuum mechanics deformation is the transformation of a body from a reference to a natural configuration, which is the real configuration currently assumed by the solid body. Moreover, deformation happens to be caused by external loads, body forces (such as gravity or electromagnetic forces), or temperature changes within the solid. In other words a deformation field exists as the result of a stress field induced either by applied forces or by variations in the temperature field of the body.

Deformations that completely disappear when the stress field ceases, i.e. the continuum recovers its original configuration, are called elastic. Otherwise, when some deformation remains after the stresses have been removed, the deformation is defined as plastic.

In this chapter I discuss the basics of the theory of elasticity. I will deduce the strain and stress tensors, and, through thermodynamics, the relation between them, known as Hooke’s law; study homogeneous deformations and the equations of equilibrium for isotropic bodies; illustrate particular axially symmetric problems, namely spherical or cylindrical shaped cavities subjected to a uniform axial load. Throughout the treatise temperature changes will not be taken into account; also, the effects of gravitational and electromagnetic fields will be neglected. Thus external forces are the only cause contributing to the creation of a stress field.

Unlike one usually does in theory of elasticity, here I will distinguish among covariant, contravariant, and mixed tensors. This choice would be unjustified if it
were not preparatory to the topics which I will treat further on. In fact, in theory of elasticity only Cartesian tensors are used, and making a distinction regarding the position of the indices could seem a little redundant. Nonetheless, in general relativity, which is where I am heading for, this distinction is of great importance.

### 2.1 Strain Tensor

In this section I investigate the way in which deformation may be expressed mathematically\(^1\). Any point in the body is defined by a radius vector \(\mathbf{r}\): suppose we are in three dimensions, so the components of \(\mathbf{r}\) are \((x^1, x^2, x^3)\) in some coordinates. If the body undergoes a certain amount of deformation it implies that in general its points are displaced. Therefore the radius vector changes from \(\mathbf{r}\) to \(\mathbf{r}'\), with components \((x'^1, x'^2, x'^3)\). Then, the relation

\[
u^i = x'^i - x^i
\]

defines the _displacement vector_. Since the coordinates \(x'^i\) are functions of the coordinates \(x^i\) of the undeformed state, also \(u^i\) is a function of \(x^i\). If \(\mathbf{u}\) is given as a function of \(x^i\), then the deformation of the solid body is entirely determined.

Deformation is also known as _strain_. Let me consider only small deformations: the theory I will refer to thus goes under the name of _infinitesimal strain theory_, also known as _small strain theory_, _small deformation theory_, _small displacement theory_, or _small displacement-gradient theory_. In fact, except for some special cases (thin rods or thin plates), for a small deformation the displacement is itself small. It must be underlined that strain is a description of deformation in terms of _relative_ displacement of points in the body: this idea is well conveyed by the so-called _Cauchy strain_ or _engineering strain_ which is expressed as the ratio of total deformation to the initial dimension of the material body. For example the _engineering normal strain_ \(e\) equals the change in length \(\Delta L\) per unit of the original length \(L\):\(^4\)

\[
e = \frac{\Delta L}{L}
\]

---

\(^1\)See [8] for deeper details.
\(^2\)in the rest of the treatise I will prefer the use of tensor or component notation, i.e. \(x^i\) instead of \(\mathbf{r}\).
\(^3\)\(x'^i\).
\(^4\)It is not my intention to fully illustrate each aspect of the theory of elasticity, so I have quoted the Cauchy strain only to explain the concept of relativity in strain.
Consider two points whose distance is small, say $dx^i$ in the $e_i$ direction: after deformation occurs, their distance has changed to $dx'^i$. The total length of the segment joining the two points is indeed

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \delta_{ij} dx^i dx^j \quad (2.2)$$

before deformation and

$$dl'^2 = (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2 = \delta_{ij} dx'^i dx'^j \quad (2.3)$$

after it. From relation 2.1 one has

$$dx'^i = dx^i + du^i = dx^i + \frac{\partial u^i}{\partial x^k} dx^k \quad (2.4)$$

replacing this identity in 2.3 gives

$$dl'^2 = \delta_{ij} dx'^i dx'^j = \delta_{ij} (dx^i + du^i) (dx^j + du^j) =$$

$$= \delta_{ij} dx^i dx^j + 2 \delta_{ij} \frac{\partial u^j}{\partial x^k} dx^i dx^k + \delta_{ij} \frac{\partial u^i}{\partial x^k} dx^k dx^j \quad (2.5)$$

We first lower the index of the displacement vector with the Kronecker tensor (which corresponds to the metric here), i.e. $\delta_{ij} \partial u^i = \partial u_j$, and then assume that the partial derivative in the second term is symmetric, i.e.

$$\frac{\partial u_i}{\partial x^k} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} \right)$$

Eventually, we interchange indices $l$ and $i$ in the third term to yield

$$dl'^2 = dl^2 + 2 \varepsilon_{ik} dx^i dx^k \quad (2.6)$$

where $\varepsilon_{ik}$ is the strain tensor

$$\varepsilon_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} + \frac{\partial u^l}{\partial x^k} \frac{\partial u_l}{\partial x^i} \right) \quad (2.7)$$

The strains occurring in a solid are almost always small. Since also the displacements are assumed to be small, compared to the dimensions of the solid at least, it is possible to neglect second order terms in 2.7 to get the strain tensor:

$$\varepsilon_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} \right) = \frac{1}{2} (u_{i,k} + u_{k,i}) \quad (2.8)$$
Consider now a small volume \( dV \) of dimensions \( dx_1, dx_2, dx_3 \), so that \( dV = dx_1dx_2dx_3 \). After deformation in the normal direction of each face the volume has changed to \( dV' = dx'_1dx'_2dx'_3 \). Making use of 2.4 the new volume becomes

\[
dV' = dV + \left( \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} \right) dV = dV (1 + \varepsilon)
\]

where \( \varepsilon \) is equal to the trace of the strain tensor in mixed form, namely \( \varepsilon^k_k \), which is invariant under coordinate changes\(^5\). The change in volume is then \( \delta(dV) = \varepsilon dV \) or

\[
\frac{\delta(dV)}{dV} = \varepsilon
\]

So, the trace of \( \varepsilon^j_j \) has a clear physical meaning: it measures the volume change.

### 2.1.1 Strain Transformation Rules

Choosing an orthonormal basis of the coordinate system \((e_1, e_2, e_3)\), the strain tensor is a tensor of rank 2 and is indeed represented by a matrix

\[
\varepsilon = \varepsilon^{ij} e_i \otimes e_j
\]

\[
\varepsilon = \begin{pmatrix}
\varepsilon^{11} & \varepsilon^{12} & \varepsilon^{13} \\
\varepsilon^{12} & \varepsilon^{22} & \varepsilon^{23} \\
\varepsilon^{13} & \varepsilon^{23} & \varepsilon^{33}
\end{pmatrix}
\]

Using another orthonormal coordinate system \((e^*_1, e^*_2, e^*_3)\) we have

\[
\varepsilon = \varepsilon^{*ij} e^*_i \otimes e^*_j
\]

\(^5\)If rectangular coordinates are used, no distinction exists between contravariance and covariance. Then, the trace of the strain tensor is the same for the totally contravariant, totally covariant, and mixed forms. It is one of the three invariants of the strain tensor \( \varepsilon \) (as in section 2.1.1, \( \varepsilon \) indicates the matrix corresponding to the strain tensor):

\[
I_1 = \text{tr}(\varepsilon)
\]

\[
I_2 = \frac{1}{2} \{\text{tr}(\varepsilon^2) - [\text{tr}(\varepsilon)]^2\}
\]

\[
I_3 = |\varepsilon|
\]
2 – Theory of Elasticity: an Overview

\( \varepsilon^* = \begin{pmatrix} \varepsilon^{*11} & \varepsilon^{*12} & \varepsilon^{*13} \\ \varepsilon^{*12} & \varepsilon^{*22} & \varepsilon^{*23} \\ \varepsilon^{*13} & \varepsilon^{*23} & \varepsilon^{*33} \end{pmatrix} \)

The relation between the components of the strain tensor (for instance in mixed form) in the two systems is

\[ \varepsilon^{*j}_i = a_{in} a^{jm} \varepsilon^n_m \]  (2.13)

where \( a_{ij} = e_i^* \cdot e_j \) and in matrix form

\[ \varepsilon^* = A \varepsilon A^T \]

\[
\begin{pmatrix}
\varepsilon^{*1}_1 & \varepsilon^{*1}_2 & \varepsilon^{*1}_3 \\
\varepsilon^{*2}_1 & \varepsilon^{*2}_2 & \varepsilon^{*2}_3 \\
\varepsilon^{*3}_1 & \varepsilon^{*3}_2 & \varepsilon^{*3}_3
\end{pmatrix} =
\begin{pmatrix}
a^{11} & a^{12} & a^{13} \\
a^{12} & a^{22} & a^{23} \\
a^{13} & a^{23} & a^{33}
\end{pmatrix}
\begin{pmatrix}
\varepsilon^1_1 & \varepsilon^1_2 & \varepsilon^1_3 \\
\varepsilon^2_1 & \varepsilon^2_2 & \varepsilon^2_3 \\
\varepsilon^3_1 & \varepsilon^3_2 & \varepsilon^3_3
\end{pmatrix}^T
\]

### 2.1.2 Strain Tensor in Spherical Coordinates

It is often useful to refer to the components of the strain tensor in terms of spherical coordinates \((r, \theta, \phi)\):

\[
\mathbf{u} = u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta + u^\phi \mathbf{e}_\phi
\]

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} \]  (2.14)

\[ \varepsilon_{r\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) \]  (2.15)

\[ \varepsilon_{r\phi} = \frac{1}{r \sin \theta} \left( \frac{\partial u_\phi}{\partial \phi} + u_r \sin \theta + u_\theta \cos \theta \right) \]  (2.16)

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \]  (2.17)

\[ \varepsilon_{\theta\phi} = \frac{1}{2r} \left( \frac{1}{\sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - u_\phi \cot \theta \right) \]  (2.18)

\[ \varepsilon_{\phi\phi} = \frac{1}{r \sin \theta} \left( \frac{1}{r} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) \]  (2.19)
2.1.3 Strain Tensor in Cylindrical Coordinates

Using the coordinates \((r, \theta, z)\):

\[
\mathbf{u} = u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta + u^z \mathbf{e}_z
\]

\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}
\] (2.20)

\[
\varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right)
\] (2.21)

\[
\varepsilon_{zz} = \frac{\partial u_z}{\partial z}
\] (2.22)

\[
\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)
\] (2.23)

\[
\varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)
\] (2.24)

\[
\varepsilon_{zr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)
\] (2.25)
2.2 Stress Tensor

In this section I will analyse the cause of strain, i.e. stress. By stress is meant the intensity of the internal contact forces acting across imaginary internal surfaces. The internal stresses, which arise when the body is deformed, are due to molecular forces, whose range of action is very short compared to the distances considered in theory of elasticity (that is a macroscopic theory). Consider figure 2.1: the force \( \mathbf{dF} \) acts on the cube represented in the drawing. In terms of stress a general force \( \mathbf{dF} \) may be expressed as

\[
\mathbf{dF} = dF^j \mathbf{e}_j = \sigma^{ij} dS_i \mathbf{e}_j = (\sigma^{11} dS_1 + \sigma^{21} dS_2 + \sigma^{31} dS_3) \mathbf{e}_1 + (\sigma^{12} dS_1 + \sigma^{22} dS_2 + \sigma^{32} dS_3) \mathbf{e}_2 + (\sigma^{13} dS_1 + \sigma^{23} dS_2 + \sigma^{33} dS_3) \mathbf{e}_3
\]  

(2.26)

Each component force has been written in terms of stresses. In particular, in picture 2.1 the force \( \mathbf{dF} \) acts only on the face whose surface is \( dS_2 \): then \( \mathbf{dF} = (\sigma^{21} \mathbf{e}_1 + \sigma^{22} \mathbf{e}_2 + \sigma^{23} \mathbf{e}_3) dS_2 \).

\( \sigma^{ij} \) indicates a new tensor, called the stress tensor, which is fully specified in all of its components in contravariant form in 2.27. The first index indicates the face on
which the stress is acting, where the second index indicates its direction. According to Cauchy, the stress at any point in a continuum body is completely defined by the nine components of the stress tensor $\sigma^{ij}$.

$$\sigma^{ij} = \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

(2.27)

As shown in figure 2.2 and referring to 2.27, the components on the diagonal indicate normal stresses (red), whereas the others indicate shear (green).

Figure 2.2. Graphical representation of the stress tensor in a small volume

The total force acting on a portion of a body is equal to the sum of all the forces.
acting on all its volume elements.

$$\int_V \mathbf{F} \, dV$$

Furthermore, the internal forces cancel out because of Newton’s third law: the total force can then be interpreted as the resultant of all the forces exerted on a given portion of a body by the portions surrounding it. The latter idea is well expressed by figure 2.2 where the cube can be seen as a portion extracted from a body. So, let me integrate the stresses over the whole surface of the element $V$:

$$\int_V F^i dV = \oint_{S=\partial V} \sigma^{ij} n_j \, dS = \int_V \sigma^{ij} \, dV \quad (2.28)$$

In 2.28 Gauss’ divergence theorem for vector fields$^6$ has been used. At equilibrium the total force must cancel out, thus we conclude that the equilibrium equations for a deformed solid element are:

$$\sigma^{ij}_{,j} = 0 \quad (2.30)$$

If some external force $\mathbf{H}$ (it can be a body force, like gravity force $\mathbf{H} = m \mathbf{g}$) acts on the volume, then it must balance internal forces: equation 2.30 becomes

$$\sigma^{ij}_{,j} + h^i = 0 \quad (2.31)$$

where

$$h^i = \frac{dH^i}{dV}$$

is the force per unit volume ($[N/m^3]$).

### 2.2.1 Symmetry of the Stress Tensor

The topic of equilibrium continues here, because when a body is at equilibrium, not only must the sum of the forces in each direction vanish, but also the summation

$^6$It states that given a vector field $\mathbf{A}$ (with components $a^\mu$) defined on a compact region $V$, whose boundary is $S$,

$$\int_{\partial V = S} a^\mu n_\mu \, dS = \int_V a^\mu \, dV \quad (2.29)$$

It is a special case of Stokes’ generalized theorem, which I will not state here. Also remember that

$$\sigma^{ij} = \frac{\partial \sigma^{ij}}{\partial x^j} = \nabla \cdot \sigma^{ij}$$
of torques with respect to any arbitrary point must be identically zero. Consequently the stress tensor $\sigma^{ij}$ takes a simplified form; in this section, a mathematical proof of the symmetry of the stress tensor is given.

**Theorem 6** (Symmetry of the stress tensor). The stress tensor $\sigma^{ij}$, as defined in contravariant form in 2.27, is symmetric, i.e. $\sigma^{ij} = \sigma^{ji}$, thus reducing the number of independent parameters from 9 to 6.

**Proof.** Consider the torques of the forces acting on the system with respect to a general point $O$ (figure 2.3): it must equal zero.

![Figure 2.3.](image)

$$\tau_O = \int_S \mathbf{r} \wedge \mathbf{T} \, dS$$

$r = x^j \mathbf{e}_j$ and $\mathbf{T} = \sigma^{ij} n_i \mathbf{e}_j$ is the stress vector. In component form we have:

$$\int_S \varepsilon_{ijk} x^j \sigma^{mi} n_m \, dS = 0 \quad (2.32)$$

applying again the divergence theorem as in 2.28, the left hand side of 2.32 becomes

$$\int_S \varepsilon_{ijk} x^j \sigma^{mi} n_m \, dS = \int_V (\varepsilon_{ijk} x^j \sigma^{mi})_{,m} \, dV = \int_V \varepsilon_{ijk} x^j \sigma^{mi} \, dV + \int_V \varepsilon_{ijk} x^j \sigma^{mi} \, dV \quad (2.33)$$
Using the equilibrium equations in absence of external forces 2.30, the second integral in 2.33 vanishes. Moreover, considering that $x^i_j = \delta^i_j$, 2.33 combined with 2.32 yields

$$\int_V \varepsilon_{ijk} \sigma^{ij} dV = 0 \implies \varepsilon_{ijk} \sigma^{ij} = 0 \implies \sigma^{ij} = \sigma^{ji}$$

(2.34)

according to the meaning of the permutation symbol.

This proves that the stress tensor is symmetric.

\[ \square \]

### 2.3 Thermodynamics and Energy of Deformation

At this point I propose to compute the work done by the internal stresses during the deformation process. I will suppose, as usual, that all the strains are small, as well as the displacements. The work $\delta \mathcal{L}$ per unit volume is easily obtained by multiplying the force $F^i = \sigma^{ij}_j$ by the small change in the displacement vector $\delta u_i$:

$$\delta \mathcal{L} = F^i \delta u_i = \sigma^{ij}_j \delta u_i$$

(2.35)

integrating over the volume of the solid body and applying again Gauss’ theorem \(^7\) we get

$$\int_V \delta \mathcal{L} dV = \oint_S \sigma^{ij} \delta u_i n_j dS - \int_V \sigma^{ij} (\delta u_i)_j dV$$

(2.36)

Since at infinity the medium, which is itself infinite, is not deformed, the first integral of 2.36 tends to zero (because the surface $S$ of integration tends to infinity, where $\sigma^{ij} = 0$). Then, using the property of symmetry of the stress tensor 2.33 the rest of 2.36 may be rewritten as

$$\int_V \delta \mathcal{L} dV = -\frac{1}{2} \int_V \sigma^{ij} \left[ (\delta u_i)_j + (\delta u_j)_i \right] dV$$

(2.37)

By comparison with 2.8:

$$\left[ (\delta u_i)_j + (\delta u_j)_i \right] = \delta \left( u_{i,j} + u_{j,i} \right) = 2 \delta \varepsilon_{ij}$$

and we get

$$\delta \mathcal{L} = -\sigma^{ij} \delta \varepsilon_{ij}$$

(2.38)

\(^7\)In fact

$$\int_V (\sigma^{ij} \delta u_i)_j dV = \oint_S \sigma^{ij} \delta u_i n_j dS$$
Another important aspect is about the rate at which the deformation occurs. I assume it to be slow enough to consider this process reversible: this means that at each moment the thermodynamic equilibrium is established in the body. Then we can write the infinitesimal change of internal energy \( d\mathcal{E} = TdS - d\mathcal{L} \), i.e. the heat exchanged reversibly plus the deformation energy \( W = -\mathcal{L} \), or

\[
d\mathcal{E} = TdS + \sigma^{ij} d\varepsilon_{ij} \quad (2.39)
\]

Introducing the Helmholtz free energy \( \mathcal{F} = \mathcal{E} - TS \), this becomes

\[
d\mathcal{F} = -SdT + \sigma^{ij} d\varepsilon_{ij} \quad (2.40)
\]

In conclusion, knowing either the internal energy \( \mathcal{E} \) or the free energy \( \mathcal{F} \) in terms of the strain tensor, it is easy to obtain the components of the stress tensor:

\[
\sigma^{ij} = \left( \frac{\partial \mathcal{E}}{\partial \varepsilon_{ij}} \right)_{S=\text{const}} = \left( \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}} \right)_{T=\text{const}} \quad (2.41)
\]

Trying to give an expression for \( \mathcal{F} \) or \( W \) (they actually coincide when the temperature is held constant) we could use Euler’s theorem (see Appendix C theorem 8), which states that

\[
\varepsilon_{ij} \frac{\partial W}{\partial \varepsilon_{ij}} = 2W
\]

Combining this relation with 2.41:

\[
W = \frac{1}{2} \sigma^{ij} \varepsilon_{ij} \quad (2.42)
\]

Eventually, in hydrostatic compression the stress tensor equals \( \sigma^{ij} = -p \delta^{ij} \). Then in 2.39 we have

\[
\sigma^{ij} d\varepsilon_{ij} = -p \delta^{ij} d\varepsilon_{ij} = -pd\varepsilon^{i}_i
\]

By comparison with what is written at page 30, \( d\varepsilon^{i}_i = d\varepsilon = dV \), i.e. 2.39 takes the well known form

\[
d\mathcal{E} = TdS - pdV
\]

### 2.4 Hooke’s Law

Now, following a thermodynamic approach like the one I illustrated in the previous section, I will find the relation between stress and strain. In particular, I will
treat only isotropic bodies at the beginning, and I will give some information about the general case at the end of the section.

I start off considering that the Helmholtz free energy $\mathcal{F}$ is in practice a potential energy. It is then possible to write it in a series expansion where it must be noted that the linear term will not figure in the expression. This is simply because the linear term represents a straight line which does not possess any extremal points, thus it does not satisfy the request for a potential function. This fact can be otherwise proven because in an undeformed state $\varepsilon_{ij} = 0$ implies that also $\sigma_{ij} = 0$: being $\sigma_{ij} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}}$ (at constant temperature), it would not be possible to have the internal stresses vanish if there were a linear term in $\mathcal{F}$. Since the free energy is a scalar, each term of its expansion must be a scalar too.

A suitable expansion up to the second order is then

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \lambda \varepsilon^2 + \mu \varepsilon_{lm} \varepsilon^{lm}$$

(2.43)

Two independent scalars of second degree have been formed, $\varepsilon^2 = (\varepsilon^k_k)^2$, which is the square of the trace of the strain matrix, and $\varepsilon_{lm} \varepsilon^{lm}$, i.e. the trace of the matrix multiplied by itself. $\lambda$ and $\mu$ are the so called Lamé coefficients: $\lambda$ is also called the Lamé ‘s first parameter, $\mu$ is the Lamé ‘s second parameter, or shear modulus. They are internal factors, since they depend on the choice of the material only. Actually, they are not the only choice possible: other parameters exist, such as $E$, $\nu$ and $K$, but here I will not go through them because they give nothing but an equivalent description (in the Appendix the conversion formulas for them are reported, though).

Whereas $\lambda$ and $\mu$ are internal parameters, $\mathcal{F}_0$ depends upon external parameters. However, the constant term cancels out in derivation, so it is not in my interest to determine its form. From 2.42 the free energy also corresponds to

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \sigma^{ij} \varepsilon_{ij}$$

It is now clear that we are aiming at finding a minimum for $\mathcal{F}$. In this fashion let me apply formula 2.41 to the expansion of $\mathcal{F}$ proposed in 2.43, remembering that $\varepsilon = \varepsilon^k_k = \delta^{ij} \varepsilon_{ij}$ ($T$ is constant). After some algebra:

$$\sigma^{ij} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}} = \lambda \delta^{ij} \varepsilon + 2 \mu \varepsilon^{ij}$$

(2.44)
This is known as the Hooke’s law for isotropic bodies, which are bodies characterized by properties which are independent of direction in space.

Instead, considering a general body, Hooke’s Law takes the following form, also known as generalized Hooke’s law

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl}
\]  

(2.45)

where \( C_{ijkl} \), called the stiffness tensor or elasticity tensor, is a fourth order tensor containing 81 elastic coefficients. However, the 81 independent parameters easily decrease thanks to symmetries and other properties: isotropic bodies are one of these cases, when we observe that the number of independent elastic constants has shrunk to only two, \( \lambda \) and \( \mu \), or \( E \) and \( \nu \) for example.

### 2.5 Equilibrium Equations for an Isotropic Body

The field of the theory of elasticity I am trying to tackle in these paragraphs can also be defined as elastostatics, which involves the study of linear elasticity under the conditions of equilibrium, where all forces and displacements are independent of time. The governing equations are 2.30 in absence of external forces, or 2.31 more in general. The latter, with the stress tensor in mixed form, is:

\[
\sigma_{i,j}^j + h_i = 0
\]

What I am about to derive is the displacement formulation of the equilibrium equations. Strains and stresses are now replaced, leaving the displacements as the only unknowns to be solved for. First, the strain-displacement relations 2.8 are substituted into the constitutive equations 2.44, i.e. Hooke’s law for isotropic bodies. For this purpose let me write

\[
\varepsilon = \varepsilon^i_i = u^k_k
\]

and

\[
\varepsilon^j_i = \delta^{jm} \varepsilon_{im} = \frac{1}{2} \delta^{jm} (u_i,m + u_m,i)
\]

Differentiating 2.44 in mixed form after the above substitutions

\[
\sigma_{i,j}^j = \lambda \delta^j_k u_{k,j} + \mu \delta^{im} (u_{i,mj} + u_{m,ij})
\]
Performing the contractions in the formula above and replacing it in the general equilibrium equations we get:

$$\mu u_{i,j}^j + (\lambda + \mu) u_{j|i}^j + h_i = 0 \quad (2.46)$$

The last equation is named *Navier-Cauchy* equation, and it is simply the displacement formulation of the equilibrium equations. Note that $u_{i,j}^j = (g^{jk}u_{i,j})_k = g_{j,k}^k u_{i,j} + g^{jk}u_{i,jk}$ is the Laplacian of $u_i$, where $g^{jk}$ is the inverse metric, that in the case of a flat Cartesian manifold equals $\delta^{jk}$. In vector form it is

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu \nabla^2 u + h = 0 \quad (2.47)$$

From the vector identity

$$\nabla(\nabla \cdot u) = \nabla^2 u + \nabla \times \nabla \times u \quad (2.48)$$

equation 2.47 may be also written as

$$(\lambda + 2\mu)\nabla(\nabla \cdot u) - \mu \nabla \times \nabla \times u + h = 0 \quad (2.49)$$
2.6 Particular Symmetric Solutions

In this last section I present two examples, which will be important in the following chapters. Both involve symmetry properties which will make the job of computing the stress and strain tensors much easier.

2.6.1 Spherical Symmetry

I first consider the case of a hollow sphere subjected to a spherically symmetric loading. To describe the problem, I shall use a set of spherical polar coordinates \((r, \theta, \varphi)\) along with a basis \((e_r, e_\theta, e_\varphi)\) (see figure 2.4). This particular case yields very simple results in terms of the stress and strain tensors. In fact, taking into account the position vector \( \mathbf{x} \) and the displacement vector \( \mathbf{u} \), they are functions of the only variable \( r \) and are represented as follows:

\[
\mathbf{x} = r \mathbf{e}_r
\]

\[
\mathbf{u} = u(r) \mathbf{e}_r
\]

Then, considering the strain-displacement relations in spherical coordinates (page 42).
31), the strain tensor $\varepsilon^i_j$ becomes:

$$\varepsilon^i_j = \begin{pmatrix} \varepsilon^r_r & 0 & 0 \\ 0 & \varepsilon^\vartheta_\vartheta & 0 \\ 0 & 0 & \varepsilon^\varphi_\varphi \end{pmatrix} = \begin{pmatrix} \frac{du}{dr} & 0 & 0 \\ 0 & \frac{u}{r} & 0 \\ 0 & 0 & \frac{u}{r} \end{pmatrix}$$

(2.51)

Due to the symmetry of both the solid and its loading, $\varepsilon^\vartheta_\vartheta = \varepsilon^\varphi_\varphi$. The same symmetry is found in the stress tensor, and again two of the diagonal elements happen to be the same, i.e. $\sigma^\vartheta_\vartheta = \sigma^\varphi_\varphi$. We might therefore write down the stress tensor $\sigma^i_j$:

$$\sigma^i_j = \begin{pmatrix} \sigma^r_r & 0 & 0 \\ 0 & \sigma^\vartheta_\vartheta & 0 \\ 0 & 0 & \sigma^\varphi_\varphi = \sigma^\vartheta_\vartheta \end{pmatrix}$$

(2.52)

From the stress-strain relations for an isotropic, homogeneous and linear elastic material 2.44 we can easily deduce that

$$\sigma^r_r = (\lambda + 2\mu) \varepsilon^r_r + 2\lambda \varepsilon^\vartheta_\vartheta$$  

(2.53)

$$\sigma^\vartheta_\vartheta = \sigma^\varphi_\varphi = 2(\lambda + \mu) \varepsilon^\vartheta_\vartheta + \lambda \varepsilon^r_r$$  

(2.54)

Now, let me treat the problem of a hollow sphere of internal radius $R$ and infinite external radius. This is the case of a spherical cavity whose walls are so thick to be considered infinite. In figure 2.5 a section through the center of the sphere is represented. The cavity is loaded by a uniform hydrostatic pressure $p$ from the inside; there is no pressure from the outside. According to 2.50 the displacement is a function of $r$ only. This implies

$$\nabla \times \mathbf{u} = 0$$

Since in this case I do not consider any body force, $\mathbf{h} = 0$ too. With these observations the Navier-Cauchy equilibrium equation 2.49 becomes:

$$\nabla (\nabla \cdot \mathbf{u}) = 0$$

or

$$\nabla \cdot \mathbf{u} = \text{constant}$$

Expressing the divergence in spherical coordinates and setting the constant equal to $3a$:

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 u \right) = 3a$$
Integrating we find the form which the displacement, the stress and the strain tensors assume

$$u = ar + \frac{b}{r^2}$$

$$\varepsilon_r = a - \frac{2b}{r^3}$$

$$\varepsilon_\theta = \varepsilon_\varphi = a + \frac{b}{r^3}$$

$$\sigma_r = (3\lambda + 2\mu) a - 4\mu \frac{b}{r^3}$$

$$\sigma_\theta = \sigma_\varphi = (3\lambda + 2\mu) a + 2\mu \frac{b}{r^3}$$

The hydrostatic pressure, and in general any external force, figures in the boundary conditions of the problem. In this case they are

$$\sigma_r = \begin{cases} 0 & \text{if } r \rightarrow \infty \\ p & \text{if } r = R \end{cases}$$

(2.55)

applying conditions 2.55 the constants of integration must assume the following values

$$a = 0 \quad b = -\frac{pR^3}{4\mu}$$
thus yielding the results for this problem

\[ u = -\frac{pR^3}{4\mu} \frac{1}{r^2} \]  
\[ \varepsilon_r = \frac{pR^3}{2\mu} \frac{1}{r^3} \]  
\[ \varepsilon_\theta = \varepsilon_\varphi = \frac{-pR^3}{4\mu} \frac{1}{r^3} \]  
\[ \sigma_r = \frac{pR^3}{r^3} \]  
\[ \sigma_\theta = \sigma_\varphi = -\frac{pR^3}{2r^3} \]  

2.6.2 Cylindrical Symmetry

The problem of the cylindrical pipe is very similar to that of the spherical cavity. The only great difference lies in the fact that this pipe is not bounded, i.e. it is a cylinder of infinite length (also the walls have infinite thickness as with the sphere). This gives rise to a particular case of deformation called plane deformation, in which the displacement along the axis of the cylinder, which I assume to be the \( z \)-axis, is zero throughout the body, and the other components of displacement do not depend on \( z \). As a consequence \( \varepsilon_i^z = 0 \) \( 8 \) for all values of \( i \); also \( \sigma_i^z = 0 \) for all \( i \neq z \): \( \sigma_z^z \) is in fact necessarily different from zero in order to keep the length of the body constant in the \( z \) direction. Nonetheless, in this problem we are unable to perceive any displacement along the revolution axis \( z \), on the grounds that while the displacement is finite, the longitudinal dimension of the body is not. Then, I propose that the pipe be in a state of generalized plane strain, and I will not be investigating upon the longitudinal deformation any further.

I shall use a set of cylindrical coordinates \( (r,\theta,z) \) along with a basis \( (e_r,e_\theta,e_z) \) (see figure 2.6). The position vector \( \mathbf{x} \) and the displacement vector \( \mathbf{u} \) are functions of \( r \) only, so as to write again 2.50

\[ \mathbf{x} = r \mathbf{e}_r \]
\[ \mathbf{u} = u(r) \mathbf{e}_r \]

\(^8\) Also \( \varepsilon_i^z = \varepsilon_i z = 0 \)
Using the strain-displacement relations in cylindrical coordinates (page 32), the strain tensor $\varepsilon^i_j$ becomes:

$$
\varepsilon^i_j = \begin{pmatrix}
\varepsilon^r_r & 0 & 0 \\
0 & \varepsilon^\theta_\theta & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{du}{dr} & 0 & 0 \\
0 & \frac{u}{r} & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

(2.61)

The stress tensor $\sigma^i_j$ is simply just the diagonal matrix

$$
\sigma^i_j = \begin{pmatrix}
\sigma^r_r & 0 & 0 \\
0 & \sigma^\theta_\theta & 0 \\
0 & 0 & \sigma^z_z
\end{pmatrix}
$$

(2.62)

From the stress-strain relations for an isotropic, homogeneous and linear elastic material 2.44 we can easily deduce that

$$
\sigma^r_r = (\lambda + 2\mu) \varepsilon^r_r + \lambda \varepsilon^\theta_\theta
$$

(2.63)

$$
\sigma^\theta_\theta = (\lambda + 2\mu) \varepsilon^\theta_\theta + \lambda \varepsilon^r_r
$$

(2.64)

The problem consists of determining the deformation of a cylindrical pipe having an infinite external radius, and an internal radius $R$, with a pressure $p$ inside and no
pressure outside. The procedure is the same as that used in the previous problem. Setting $h = 0$ we have:

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{d(ru)}{dr} \equiv 2a$$

The displacement, the stress and the strain tensors are

$$u = ar + \frac{b}{r}$$

$$\varepsilon^r_r = a - \frac{b}{r^2}$$

$$\varepsilon^\varphi_\varphi = a + \frac{b}{r^2}$$

$$\sigma^r_r = 2(\lambda + \mu) a - 2\mu \frac{b}{r^2}$$

$$\sigma^\varphi_\varphi = 2(\lambda + \mu) a + 2\mu \frac{b}{r^2}$$

The boundary conditions are the same as 2.55

$$\sigma^r_r = \begin{cases} 0 & \text{if } r \to \infty \\ p & \text{if } r = R \end{cases}$$

from which we find

$$a = 0 \quad b = -\frac{pR^2}{2\mu}$$

thus yielding the results for this problem

$$u = -\frac{pR^2}{2\mu} \frac{1}{r}$$

$$\varepsilon^r_r = \frac{pR^2}{2\mu} \frac{1}{r^2}$$

$$\varepsilon^\varphi_\varphi = -\frac{pR^2}{2\mu} \frac{1}{r^2}$$

$$\sigma^r_r = \frac{pR^2}{r^2}$$

$$\sigma^\varphi_\varphi = -\frac{pR^2}{r^2}$$
Chapter 3

Lagrangian Mechanics and Variational Principles in Classical Physics and in General Relativity

The purpose of this chapter is to derive the general equations of motion for a particle or a system of particles. This represents a fundamental point in my thesis, since the topics and methods that I will introduce here will be paramount in the last part of the treatise.

First, I will obtain the so-called Euler-Lagrange equations that describe a system of particles: the path I am following to get to these equations passes through the famous variational principle known as the principle of least action or Hamilton’s principle. In this process the system is described by a function, the Lagrangian: I will find and define it as it is usually done in classical mechanics. Thereupon I have to cope with the problem of jumping to relativistic mechanics using the same principles. In this attempt, I am showing the form that the Lagrangian shall take in general relativity, as well as define the Lagrangian density in field theory. Eventually, before moving on to a new step of the treatise, I will prove that Einstein’s field equation may be obtained by applying variational methods to the relativistic Lagrangian previously defined.
Consider the most general system of $N$ particles\textsuperscript{1}. The position of each particle is defined by its radius vector $\mathbf{r}$, which is expressed in terms of the Cartesian coordinates $x,y,z$.\textsuperscript{2} The velocity and acceleration of any particle are then written as $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, respectively. In the ordinary three-dimensional space describing the position of a system of $N$ particles involves $N$ radius vectors, i.e. $3N$ coordinates. $3N$ also indicates the degrees of freedom, which are the number of independent parameters needed to uniquely describe the system. Notwithstanding the definition of radius vector I hinted at above, it must be clear that the coordinates need not be the Cartesian coordinates. On the contrary, depending on the problem we are willing to solve, there might be some choices of coordinates that are more convenient.

Hence, any $n$ variables $q^1, q^2, ..., q^n$ which completely and unambiguously determine a system with $n$ degrees of freedom are called \textit{generalized} or \textit{Lagrangian coordinates}, and the time derivatives $\dot{q}^h$ are called \textit{generalized velocities}. When all the $q^h$ and the $\dot{q}^h$, $h = 1,2,...,n$, are given at some instant, we say that the state of a system is completely determined at that instant, i.e. the accelerations $\ddot{q}^h$ are uniquely defined. Therefore, we are able, at least in principle, to describe the motion of the particles of the system.

Nonetheless, the equations of motion arising from the method of Lagrange, that I am going to describe, can be non-linear. Unfortunately, this is the unpleasant case which will render rather complicated computations: in the following chapters I am deriving highly non-linear equations, whose search for solutions is going to be extremely difficult indeed.

\textsuperscript{1}As far as the part on classical mechanics is concerned I followed the approach used in the first chapter of [9].

\textsuperscript{2}$x^1,x^2,x^3$ more in general.
3.1 Hamilton’s Principle

For any mechanical system there exists a definite function that characterises it. This function is the Lagrangian $L$:

$$L = L(q, \dot{q}, t)$$

where $q = \left( q^1 \ q^2 \ q^3 \ \ldots \ q^n \right)^T$ and $\dot{q} = \left( \dot{q}^1 \ \dot{q}^2 \ \dot{q}^3 \ \ldots \ \dot{q}^n \right)^T$. Now, fixing two instants of time $t_A$ and $t_B$, at which the system assumes the configurations $Q_A = (q_A, \dot{q}_A)$ and $Q_B = (q_B, \dot{q}_B)$ respectively, the principle of least action or Hamilton’s principle states that:

The system evolves from $Q_A$ to $Q_B$ following a path (see figure 3.1) such that the integral $S$, called action,

$$S = \int_{t_A}^{t_B} L(q, \dot{q}, t) \, dt$$

is stationary. The action integral 3.2 must have an extremum, but not necessarily a minimum, for an entire path.

From the principle stated above, I will now derive the differential equations of the motion of a system. Let $q = q(t)$ be the position for which $S$ is an extremum, say a minimum. $\delta q(t)$ is called a variation of the function $q(t)$: it is small everywhere except at the endpoints of the interval where it is zero:

$$\delta q(t_A) = 0 \text{ and } \delta q(t_B) = 0$$

$$50$$
When \( q(t) \) is increased by \( \delta q(t) \), i.e. it becomes \( q(t) + \delta q(t) \), also the value of \( S \) is increased (simply because I supposed that \( q(t) \) minimizes \( S \)), namely

\[
\delta S = \int_{t_A}^{t_B} \mathcal{L} (q + \delta q(t), \dot{q} + \delta \dot{q}(t), t) \, dt - \int_{t_A}^{t_B} \mathcal{L} (q, \dot{q}, t) \, dt
\]

Hence, Hamilton’s principle may be restated as follows

\[
\delta S = \delta \left( \int_{t_A}^{t_B} \mathcal{L} (q, \dot{q}, t) \, dt \right) = 0 \quad (3.4)
\]

This expression might be expanded in powers of \( \delta q \) and \( \delta \dot{q} \), and it is called the first variation or simply variation of the action. To have a minimum, or more in general an extremum, we require that this variation be zero.

\[
\int_{t_A}^{t_B} \left( \frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right) dt = 0
\]

Since \( \delta \dot{q} = \frac{d}{dt} (\delta q) \) we integrate the second term of the integrand by parts to get

\[
\left[ \frac{\partial \mathcal{L}}{\partial q} \delta q \right]_{t_A}^{t_B} + \int_{t_A}^{t_B} \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \, dt = 0
\]

Applying conditions 3.3 the first term between square brackets is zero. The remaining integral must vanish for all values of \( \delta q \). In this way we obtain the so called Lagrange equations.

**Theorem 7** (Lagrange equations). Suppose that a system of particles is described by the function 3.1. Then the differential equations of its subsequent motion are written in the following form

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad (3.5)
\]

### 3.2 Determination of the Lagrangian

Before finding the form which the Lagrangian takes on in classical mechanics, I want to spend some words on the following remark:

**Proposition 1.** Let \( \mathcal{L} = \mathcal{L} (q, \dot{q}, t) \) and \( \mathcal{L}' = \mathcal{L}' (q, \dot{q}, t) \) be two functions differing only by the total time derivative of some function \( \gamma (q, t) \) of coordinates and time. Then, being \( S \) and \( S' \) the action integrals defined in 3.2,

\[
\delta S = \delta S'
\]
Proof. Assume that

\[ \mathcal{L}'(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{d}{dt}\gamma(\mathbf{q}, t) \]  

(3.7)

The actions 3.2 associated with the two Lagrangians are related:

\[ S' = \int_{t_A}^{t_B} \mathcal{L}'(\mathbf{q}, \dot{\mathbf{q}}, t) \, dt = \int_{t_A}^{t_B} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \, dt + \int_{t_A}^{t_B} \frac{d\gamma(\mathbf{q}, t)}{dt} \, dt = S + \gamma(\mathbf{q}_B, t_B) - \gamma(\mathbf{q}_A, t_A) \]

Effecting the variation for the last two terms and taking into account 3.3 we have that

\[ \delta \left( \gamma(\mathbf{q}_A, t_A) \right) = \gamma(\mathbf{q}_A + \delta \mathbf{q}_A, t_A) - \gamma(\mathbf{q}_A, t_A) = 0 \]

\[ \delta \left( \gamma(\mathbf{q}_B, t_B) \right) = \gamma(\mathbf{q}_B + \delta \mathbf{q}_B, t_B) - \gamma(\mathbf{q}_B, t_B) = 0 \]

and so, not only \( \delta S = 0 \), but also \( \delta S' = 0 \), thus proving the proposition. \( \square \)

Hence, the Lagrangian is defined up to an additive total time derivative of whatsoever function of coordinates \( \mathbf{q} \) and time \( t \).

Eventually, let me investigate on the form of the Lagrangian. The simplest case is that of a free particle moving with velocity \( \mathbf{v} \) with respect to an inertial frame of reference. Here the Lagrangian depends only on the square of the velocity of the particle, i.e. \( \mathcal{L} = \mathcal{L}(v^2) \). Then, if two frames \( Oxyz \) and \( O'x'y'z' \) move with a velocity \( \epsilon \) one with respect to the other, in the \( O' \) reference system the particle has velocity \( \mathbf{v}' = \mathbf{v} + \epsilon \). The two Lagragians computed in the two frames must differ only by the total time derivative of a function of coordinates and time, as stated in proposition 1. According to 3.7 we have

\[ \mathcal{L}'(v^2) = \mathcal{L}(v^2) + \frac{d}{dt}\gamma(\mathbf{q}, t) \]

Moreover

\[ \mathcal{L}'(v^2) = \mathcal{L}(v'^2) = \mathcal{L}(v^2 + 2\mathbf{v} \cdot \epsilon + \epsilon^2) \]

Expanding the above expression in terms of \( \epsilon \) up to the first order we get

\[ \mathcal{L}'(v^2) = \mathcal{L}(v^2) + \frac{\partial \mathcal{L}}{\partial v^2} 2\mathbf{v} \cdot \epsilon \]

The equations of motion must have the same form in every frame. This means that

\[ \mathcal{L}(v^2) = \mathcal{L}'(v^2) \]
For this to be viable the second term of the expansion \( \frac{\partial \mathcal{L}}{\partial v^2} 2v \cdot \epsilon \) needs to be linear in the velocity \( v \), implying that \( \frac{\partial \mathcal{L}}{\partial v^2} \) does not depend on \( v \). In conclusion the Lagrangian for a particle has the form

\[
\mathcal{L} = \frac{1}{2} mv^2
\]  

(3.8)

where \( m \) is the mass of the particle, and 3.8 is otherwise called kinetic energy.

Furthermore, for a closed system, i.e. a system of \( N \) particles interacting with one another but not with other bodies, the Lagrangian found above is not enough. In classical mechanics, the missing term needs to describe the interaction among the particles: it is function of the position \( q \) and its form depends on the nature of interaction\(^3\). This function is called potential \( V = V(q) \). Note that \( V = -U \), where \( U \) denotes the potential energy. Hence, the Lagrangian is made up of the sum of the kinetic energies 3.8 of all the particles and the potential:

\[
\mathcal{L} (q, \dot{q}, t) = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 + V(q)
\]  

(3.9)

Two remarks can be made. Firstly, the fact that the potential depends on the positions of the particles only means that a change in the position of any particle immediately affects the other particles of the system. Secondly, 3.8 and 3.9 show that time is regarded as both homogeneous and isotropic, i.e. the Lagrangian does not change, and so do not the equations of motion, if the direction of time is reversed, from \( t \) to \( -t \).

\(^3\)I am not illustrating the precise form of this function in the treatise. However, it can be said that it depends on the presence of a conservative field, like a gravitational, electric or elastic field. Only in the latter case I am giving an expression: if a spring has elastic constant \( k \), rest length \( l_0 \), and it is elongated of \( \Delta l = l - l_0 \), its potential energy is

\[
U = \frac{1}{2} k \Delta l^2
\]
3.3 The Lagrangian of an Elastic Field

From this section on, I turn my attention to the concept of field. All the problems that I will analyse involve field theory (in relativity it is paramount). I firstly have to show the form which the Lagrangian functional attains within this theory. In the most general case, suppose we are in a space of \( N \) dimensions. I shall now define the density of Lagrangian \( L \). This is a key concept. In fact, in classical physics the action \( S \) is the integral of \( L \) with respect to time (see 3.2). In field theory one integrates the quantity \( L \) over the ‘volume’ \( \Omega^{(N)} \), with \( d\Omega^{(N)} = \sqrt{|g|} d^N x \), where \( N \) is the dimension of the manifold. No matter what \( N \) is, the action is an integral over the whole set of coordinates that describe the manifold, i.e.

\[
S = \int_{\Omega^{(N)}} \mathcal{L} \sqrt{|g|} d^N x = \int_{\Omega^{(N)}} L d^N x \tag{3.10}
\]

**Definition 12** (Lagrangian density). In field theory the quantity \( L \), where

\[
L = \mathcal{L} \sqrt{|g|} \tag{3.11}
\]

such that the action may be written as in 3.10, is named Lagrangian density.

\( \diamond \) **Remark**: the quantity \( \mathcal{L} \) is not a Lagrangian now, even though this name can be found, also in this treatise. The proper concept to use in field theory is in fact that of a Lagrangian density. However, writing in general \( d^N x = d^{N-1} x \, dt = dV \, dt \), and setting \( V = \Omega^{(N-1)} \) action becomes

\[
S = \int_{\Omega^{(N)}} L d^N x = \int_{t_a}^{t_B} \int_V L dV \, dt
\]

where the real Lagrangian is the following

\[
L^* = \int_V L dV = \int_V \mathcal{L} \sqrt{|g|} dV
\]

The Lagrangian I have talked about until this point describes either a single body or a system of points. Both cases are examples of discrete physical models. Since the aim of my thesis is to apply relativistic methods to continuum mechanics, both of which involve fields, I suppose that as a first step it would be appropriate to give a Lagrangian formulation of a continuous elastic medium. (For a richer treatise on the Lagrangian formulation in continuum mechanics see [5], chapter 16).
Consider a homogeneous elastic rod as in figure 3.2. It can be represented by a discrete set of \( N \) material points. I will now try to perform the passage to a continuous rod, imposing the limit \( N \to \infty \). Let \( \rho \) and \( m = \rho Al \) be the density and the mass of the rod respectively, where \( A \) is its section and \( l \) its length. Since the rod is divided in \( N \) equal parts, each of them has mass \( \frac{m}{N} \) and length \( \varsigma = \frac{l}{N} \). Let me indicate with \( u_i \) the displacement of the point \( P_i \), and assume that every two successive points are connected by means of a spring with elastic constant \( k \) (all the springs are equal). Then, the elongation of the spring attached to the points \( P_i \) and \( P_{i+1} \) is indeed \( u_{i+1} - u_i \); this gives a potential energy

\[
U = \frac{1}{2} k \sum_{i=1}^{N-1} (u_{i+1} - u_i)^2
\]

as well as a kinetic energy

\[
T = \frac{1}{2} m \sum_{i=1}^{N} \dot{u}_i^2
\]

with \( \dot{u}_i \) indicating the velocity of the point \( P_i \). The Lagrangian of this discrete system is then

\[
\mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{2} \left\{ \frac{m}{N} \sum_{i=1}^{N} \dot{u}_i^2 - k \sum_{i=1}^{N-1} (u_{i+1} - u_i)^2 \right\}
\]

Formula 3.12 might be rewritten as

\[
\mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{2} \left\{ \rho \sum_{i=1}^{N} \dot{u}_i^2 - E \sum_{i=1}^{N-1} \left( \frac{u_{i+1} - u_i}{\varsigma} \right)^2 \right\} A \varsigma
\]

where \( E \) is the Young’s modulus and here the assumption has been made that \( E = k \varsigma \). Performing the passage to the limit, \( N \to \infty \), and considering that \( A \) is
equivalent to a double integral of the form $\int_A dS$, the Lagrangian becomes

$$\mathcal{L} = \int \frac{1}{2} \left[ \rho \left( \frac{\partial u}{\partial t} \right)^2 - E \left( \frac{\partial u}{\partial x} \right)^2 \right] dx dS \tag{3.13}$$

It is now clear that the key in describing a continuum medium in a field theory is that of Lagrangian density (note that the Lagrangian depends not only on its density, but also on the region of integration). In this problem

$$L \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \right) = \frac{1}{2} \left[ \rho \left( \frac{\partial u}{\partial t} \right)^2 - E \left( \frac{\partial u}{\partial x} \right)^2 \right]$$

Moreover, in this example, the meaning of field should have been clarified a little. It involves the existence of a function of the coordinates of the chosen manifold, a field function, describing the whole continuous system: in this case the field was described by the displacement function, representing the displacement of all the points of the system from an equilibrium configuration.
3.4 Lagrangian Methods in General Relativity

In a fashion similar to that of the previous sections (the computations are less trivial, though), I will show how variational methods are powerful also in relativity. In the following chapters this material is widely used as a hinge to argue the thesis of my treatise. Of course, I will adopt only the Lagrangian formulation. In 1915, David Hilbert was the first mathematician to propose this view and propose the action which is now known as Hilbert or Einstein-Hilbert action.

In relativity the quadri-volume $d\Omega^{(4)} = \sqrt{|g|} d^4x$ is also called space-time and, because of the signature\(^4\), $|g| = -g$. Hence, the action becomes

$$S = \int_{\Omega^{(4)}} \mathcal{L} \sqrt{-g} d^4x = \int_{\Omega^{(4)}} L d^4x$$

(3.14)

and, following definition 3.11, the Lagrangian density in relativity takes the form

$$L = \mathcal{L} \sqrt{-g}$$

(3.15)

such that the action may be written as in 3.14.

It is now time to determine the Lagrangian density for the vacuum Einstein equation, i.e. for the gravitational field: the following derivation is carried out following the theory of fields as described in [10]. I begin considering that the equation of the field must contain at most second order derivatives of the coordinates. This requires that $L$ contain at most first order derivatives of the metric $g_{\mu\nu}$, i.e.

---

\(^4\)In relativity a Lorentzian manifold is used. It is the most important example of a pseudo-Riemaniann manifold. In a few words, a pseudo-Riemaniann manifold has a metric tensor not necessarily positive definite (in fact a Riemaniann manifold always has a positive definite metric). A Lorentzian manifold in relativity possesses a signature, called Lorentz signature, which is either $(+, -, -, -)$ or $(-, +, +, +)$, and this implies that the determinant to be $g < 0$. For example the Galilean metric $g^{(0)}_{\mu\nu}$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

---
that $L$ be built with only $g_{\mu\nu}$ and $\Gamma_{\nu\rho}^{\mu}$: in addition $L$ must be a scalar. It is practically impossible to construct a scalar invariant with only $g_{\mu\nu}$ and $\Gamma_{\nu\rho}^{\mu}$: with a proper set of coordinates it would be easy to make all the $\Gamma_{\nu\rho}^{\mu}$ vanish at a given point. Whilst the scalar curvature $R$, expressing the average curvature of any point of space-time, might be a suitable candidate, even though it contains second derivatives of the metric $g_{\mu\nu}$.

However, getting rid of the second order derivatives is not a tough job. $R$ is in fact linear in the second derivatives, thus allowing us to write integral 3.10 with no second derivatives. The trick consists of integrating the second order derivatives by parts, so as to obtain a boundary condition which vanishes when the variation is performed. For instance suppose that $R(g'', g', g) = ag'' + f(g', g)$, which means that $R$ is equal to the second derivatives of $g_{\mu\nu}$ (a is a constant) plus another function containing first order derivatives of $g'_{\mu\nu}$, and $g_{\mu\nu}$ itself. Then

$$S = \int_{\Omega^{(4)}} R(g'', g', g) \sqrt{-g} \, d^4x = \int_{\Omega^{(4)}} a g'' \sqrt{-g} \, d^4x + \int_{\Omega^{(4)}} f(g', g) \sqrt{-g} \, d^4x$$

Integrating the first term of the right hand side by parts

$$a \int_{\partial\Omega^{(4)}} g' \sqrt{-g} \, d^3x - a \int_{\Omega^{(4)}} g' (\sqrt{-g})' \, d^4x + \int_{\Omega^{(4)}} f(g', g) \sqrt{-g} \, d^4x$$

The first term is an integral over a hypersurface $S$ enclosing the volume $\Omega^{(4)}$: computing the first variation of the action, this term goes identically to zero, because the variation is obviously zero on the boundary (this condition was already encountered in 3.3). Hence, the remaining terms surely do not contain second derivatives of the metric. The following result is then clear

$$\delta S = \delta \left( \int_{\Omega^{(4)}} R \sqrt{-g} \, d^4x \right) = \delta \left( \int_{\Omega^{(4)}} L \, d^4x \right)$$

$$L = R \sqrt{-g}$$

(3.16)

In this way, where Hamilton’s principle is applied to derive the equations of motion, and so the variation of the gravitational action is taken into account, the Lagrangian happens to equal the scalar curvature $R$ computed by contracting the Ricci’s tensor. The gravitational action in 3.16 is the integral over time of an energy, obtained by integrating with respect to a three-dimensional space the curvature which geometrically expresses an energy density due to the gravitational field.

For further details on the Lagrangian formulation of general relativity follow Appendix E.1 of [16].
3.4.1 Einstein’s Field Equations

As a consistent example in literature, I will now prove that the Lagrangian formulation discussed above brings to the famous Einstein’s field equations (conversely, for a proof of the fact that these equations involve the stationarity of the gravitational action see [6], chapter 12). First, let me give a useful definition. With the (+, −, −, −) signature, the functional known as Hilbert action is

\[ S_H = -\frac{1}{2\kappa} \int \Omega \sqrt{-g} R d^4x \]  
(3.17)

where \( \kappa = \frac{8\pi G}{c^4} \) where \( G \) is the Newton’s gravitational constant, \( 6.673 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \) and \( c \) is the speed of light in vacuum, \( 299792458 \text{ ms}^{-1} \).

To determine the equations we need to add to the Lagrangian for the vacuum, a term \( L_M \) representing the Lagrangian for the matter to yield

\[ S = -\int \Omega \left( \frac{1}{2\kappa} R + L_M \right) \sqrt{-g} d^4x \]

Imposing \( \delta S = 0 \):

\[ \int \Omega \left[ \frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x = 0 \]

The above relation must hold for any \( \delta g^{\mu\nu} \), and so we can remove the integral sign to get:

\[ \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \]

The right hand side is by definition \( \kappa T^{\mu\nu} \), i.e. it is proportional to what we define as the stress-energy tensor \( ^5 T^{\mu\nu} \).

\(^5\)‘Geometry tells matter how to move, and matter tells geometry how to curve’ by John Archibald Wheeler (see [13], chapter 5, for further information). The stress-energy tensor is a tool determining how much mass-energy is in a unit volume. In other words, much information is provided by this tensor, i.e. knowledge of the energy density, momentum density, and of the stresses. Explicitly

\[ T^{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \]

it is symmetric, where

\[ \bullet T_{00} \text{ is the energy density;} \]
To compute the variation of the scalar of curvature $\delta R$ I shall begin with the Riemann tensor:

$$R^\rho_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$

Its variation is

$$\delta R^\rho_{\sigma\mu\nu} = \delta \Gamma^\rho_{\nu\sigma,\mu} - \delta \Gamma^\rho_{\mu\sigma,\nu} + \delta \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$

The covariant derivative of the variation of the connection is written as

$$\delta \Gamma^\rho_{\nu\mu;\lambda} = \delta \Gamma^\rho_{\nu\mu,\lambda} + \Gamma^\rho_{\sigma\lambda} \delta \Gamma^\sigma_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\rho_{\sigma\mu} - \Gamma^\rho_{\sigma\mu} \delta \Gamma^\rho_{\nu\lambda}$$

- the column $(T_{10}, T_{20}, T_{30})^T$ is the momentum density;
- the row $(T_{01}, T_{02}, T_{03})$ is the energy flux;
- the remaining $3 \times 3$ submatrix indicates the momentum flux

$$\begin{pmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{pmatrix}$$

where the elements on the diagonal express pressure and the others shear stresses.

**Stress-energy tensor for a perfect fluid**

A perfect fluid is a fluid (liquid or gaseous) which moves in space-time with a 4-velocity vector $v$, and possesses, in the rest frame of each fluid element, a density of mass-energy $\rho c^2$ and an isotropic pressure $p$. In order to be perfect there must not be any anisotropic pressures, viscosity, and shear stresses. Then, the stress-energy tensor is simply built using the metric, the velocity, the at-rest density and pressure of the fluid

$$T_{\mu\nu} = (p + \rho c^2) v_\mu v_\nu + p g_{\mu\nu}$$

(3.18)

**Conservation Law for Energy-Momentum**

Considering that energy-momentum is another way of calling the stress-energy tensor, it is possible to write a conservation law for this quantity, too. In physics, conservation laws have a similar form. For instance, taking the quantity mass $m$, a conservation law may be written. Given the density function $\rho(x,y,z,t) = \frac{dm}{dV}$, and the current density $j = \rho v$, this law, in differential form, is written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

In a four-dimensional space the current density is defined as $J = (\epsilon \rho, \rho \dot{x}, \rho \dot{y}, \rho \dot{z})$, and the conservation law states

$$J_\alpha,\alpha = 0$$

Then, generalizing the latter expression for pseudo-Riemannian manifolds, we need to use covariant differentiation, yielding

$$T^\gamma_{\delta,\gamma} = 0$$

(3.19)

which represents of course a conservation law.
Thus, thanks to this observation, the variation of the Riemaniann curvature can be easily turned into

$$\delta R^\rho_{\sigma\mu\nu} = \delta \Gamma^\rho_{\nu\sigma\mu} - \delta \Gamma^\rho_{\mu\sigma\nu}$$

Contracting two indices of the Riemann tensor we obtain the Ricci tensor

$$\delta R_{\mu\nu} = \delta R^\rho_{\mu\rho\nu} = \delta \Gamma^\rho_{\nu\mu\rho} - \delta \Gamma^\rho_{\rho\mu\nu}$$

Hence, the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ gives on variation

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} = R_{\mu\nu} \delta g^{\mu\nu} + (g^{\mu\nu} \delta \Gamma^\sigma_{\nu\mu} - g^{\mu\sigma} \delta \Gamma^\rho_{\rho\mu})_{,\sigma}$$

where the metric tensor has been taken inside the brackets of the covariant differentiation thanks to the well known rule $g^{\alpha\beta}_{,\gamma} = 0$. Besides, the second term $(g^{\mu\nu} \delta \Gamma^\sigma_{\nu\mu} - g^{\mu\sigma} \delta \Gamma^\rho_{\rho\mu})_{,\sigma}$ becomes a boundary term when integrated, thanks to Stokes’ theorem, and it vanishes on variation. Finally, we get:

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \quad (3.20)$$

Following Jacobi’s formula, the rule for differentiating a determinant, or $\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}$, the variation of the inverse metric is

$$\delta \left( \frac{1}{g} \right) = -\frac{\delta g}{g^2} = \frac{1}{g} g_{\mu\nu} \delta g^{\mu\nu}$$

From this formulas, and considering that $\delta \sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} :$

$$\frac{1}{\sqrt{-g}} \delta \sqrt{-g} = -\frac{1}{2} g_{\mu\nu}$$

Finally, gathering together all the variations we have computed, Einstein’s field equations arise:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$
The left hand side is commonly named *Einstein Tensor*, thus producing 

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]  

(3.22)

---

6An important property of the Einstein tensor comes from the conservation law 3.19 written in the footnote at page 60. From equation 3.22 easily follows that

\[ G_{\delta\gamma} = 0 \]  

(3.21)

7In geometrized units c and G are set equal to unity (see [16], Appendix F), and 3.22 simplifies to

\[ G_{\mu\nu} = 8\pi T_{\mu\nu} \]
Chapter 4

A ‘Strained Space-Time’ Theory

So far, I have introduced many different classical concepts in the field of physics. By the word classical I refer to the physical knowledge which is already well established on the scientific scenario, and clearly not to the branch of physics officially beginning in the 1600s and whose end is fixed at the dawn of the twentieth century. Yet, in those chapters I tried to merge many different results coming from the whole physics, no matter the field which they belong to. This meant that I did not follow any chronological pattern in presenting them, abruptly jumping from one century to another, putting together several different topics, some of which are also not trivial at all. On the other hand, I hope the treatise was clear enough to prepare rather solid tenets for the following chapters.

A second intention is about the language, scientific and not, which I used to present this preparatory knowledge. Not only wished I to present a background like the one that is easily found in any physics textbook, but also I strove to reformulate it in a way such that the reader could get acquainted with the notation which I will use throughout the treatise. It all comes down to the fact that, sometimes, the complexity of a subject can be avoided by adopting a different approach: in this case the approach I am talking about is rather mathematical. In fact, albeit there is no need to describe elasticity with a tensor notation typical of general relativity, I highly believe that my ‘reformulation’ of it in chapter 2 was not a waste of time. This belief should be supported by the variety of topics put together in chapter 3, where I began with classical mechanics and ended up talking about relativity. In this sense, chapter 3 represents a transition from classical (this time I mean the real
classical physics) to modern (actually, only relativistic) physics, where we cannot do without the notation which seemed (and is) indeed superfluous in the theory of elasticity.

Here, I move on by another step: continuing with general relativity stuff, I will talk about *Cosmology*. This chapter presents, more than others, what could be defined as the state of the art. Right after, in the following chapter, I will finally get to the point, performing the computations which will prove an aspect of the new cosmological theory exposed hereafter. More in details, in this chapter I will start off by illustrating what the standard cosmology looks like (Robertson-Walker Universe and Schwarzschild metric) and then I will switch to the recently proposed Strained State Cosmology (SSC) theory. The bibliography I was inspired by mainly comes from [1], [13] and [16] for the standard cosmology and from [3], [14] and [15] for the SSC.

![Figure 4.1. Graphical representation of the universe expansion. NASA/WMAP Science Team](image)
4.1 Homogeneous and Isotropic Universe

In the previous chapter I accurately derived Einstein’s field equation through variational principles. The essence of general relativity is that space-time corresponds to a four-dimensional manifold with metric $g_{\mu\nu}$ of Lorentz signature. Einstein’s equation 3.22 establishes the relation between the metric and the matter-energy distribution in space-time. Now the question is to find which solution of equation 3.22 describes the universe we observe. The answer must be formulated only with sufficient observational data, and of course with some \textit{a priori} assumptions about the nature of our universe. Here, I will only be investigating the structure of the universe starting from the \textit{ansatz} that it is homogeneous and isotropic.

Since Copernicus, it has been clear to us that we are not in a privileged position in our universe: if we were located in another region of it, our surroundings would almost surely look alike. Locally our universe appears anisotropic and heterogeneous, but at large scales, greater than 100 Mpc\textsuperscript{1}, it seems isotropic and homogeneous, indeed. This means that on sufficiently large scales, observations should yield results not depending on the direction towards which we are looking. Furthermore, homogeneity means that any event in the four-dimensional space-time should look like any other event. An example of strong evidence for the homogeneity and isotropy, is represented by thermal radiation at about 3 K filling our universe. So far, the models developed from the assumption of isotropy and homogeneity have been successful in describing properties of the universe.

One can prove that the requirements of homogeneity and isotropy separately imply the constancy of the curvature of the manifold. The parameter $k$, called curvature parameter, gives information on which type of spatial slice we are living in. Only three values of $k$ are allowed: +1 for positively curved, 0 for flat and −1 for negatively curved spatial sections.

- $k = 0$ is that of a flat three-dimensional manifold, whose metric is

$$ds^2 = dx^2 + dy^2 + dz^2$$

\textsuperscript{1}A parsec is the distance from the Sun (or the Earth) to an ideal star which has a parallax angle of one arcsecond. Or, considering picture 4.2, it is the distance at which the mean distance between the Earth and the Sun (1.496 × 10\textsuperscript{11} m also called Astronomical Unit, AU) subtends a second of arc. Dividing an AU by tan(1\textdegree) we find what a parsec is, i.e. 3.086 × 10\textsuperscript{16} m.
in Cartesian coordinates and

\[ ds^2 = d\chi^2 + \chi^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \]

in spherical coordinates. Pre-relativity physics only recognised this metric. However, even under the restrictive assumptions of homogeneity and isotropy, general relativity admits two other possibilities.

- \( k = +1 \) is attained by the 3-spheres, i.e. surfaces in four-dimensional flat Euclidean space \( \mathbb{R}^4 \) whose Cartesian coordinates satisfy

\[ x^2 + y^2 + z^2 + w^2 = R^2 \]

In spherical coordinates the 3-spheres have metric

\[ ds^2 = d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \]

This spatial geometry is a compact manifold, consequently describing a finite universe, but with no boundary. Such a universe is named \textit{closed}. 
• $k = -1$ is the curvature of three-dimensional hyperboloids, i.e. surfaces in four-dimensional flat Lorentzian manifold whose coordinates satisfy

$$t^2 - x^2 - y^2 - z^2 = R^2$$

In hyperbolic coordinates the typical metric is

$$ds^2 = d\chi^2 + \sinh^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)$$

This geometry, along with the flat metric, constitutes a case of open universe.

In conclusion, the space-time metric may be expressed as follows

$$ds^2 = d\tau^2 - a^2(\tau) \begin{cases} 
    d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \\
    d\chi^2 + \chi^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \\
    d\chi^2 + \sinh^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)
\end{cases}$$

Equation 4.1 is called a Robertson-Walker cosmological model. However, this is not the way in which this metric is usually written: there exists a more compact one. To arrive at this compact formula recall the following derivatives:

$$\frac{d}{dx} \left( \sin^{-1} x \right) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx} \left( \sinh^{-1} x \right) = \frac{1}{\sqrt{1+x^2}}$$

In the case of a flat manifold, just assume that $\chi = r$, and so $d\chi = dr$. When $k = +1$ instead set $\sin \chi = r$ so that

$$d\chi^2 = \left( \frac{d \left( \sin^{-1} r \right)}{dr} \right)^2 dr^2 = \frac{dr^2}{1-r^2}$$

Repeating the same procedure when $k = -1$ ($r = \sinh \chi$), the Robertson-Walker metric 4.1 becomes

$$ds^2 = d\tau^2 - a^2(\tau) \left[ \frac{dr^2}{1-kr^2} + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right]$$

It can be observed that spatial homogeneity and isotropy lower the number of unknown quantities from ten in the most general case to only one, the scale factor $a(\tau)$, function of the cosmic time $^2\tau$.

\[ ^2 \text{The cosmic time is the time coordinate used to define the evolution of our universe. It originates} \]
4.2 Schwarzschild Metric

The reason why I decided to talk about this metric is symmetry: in fact in the next chapter I will treat only symmetric three-dimensional solids. The Schwarzschild solution (1916) describes the gravitational field outside a spherical, uncharged, non-rotating distribution of mass. This metric is paramount in general relativity.

Consider a set of spherical coordinates, defined by the transformation (see Appendix A.2)

\[ x = r \sin \vartheta \cos \varphi \]
\[ y = r \sin \vartheta \sin \varphi \]
\[ z = r \cos \vartheta \]

The line element in spherical coordinates in the flat \( \mathbb{R}^3 \) Euclidean manifold is

\[ ds_0^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \]  
(4.3)

i.e. the metric tensor is

\[ \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \vartheta \end{pmatrix} \]  
(4.4)

Placing a spherical distribution of mass \( M \) in the origin \( O \), the space-time is deformed, and its metric is not Euclidean any more. However, we can infer the form that the new metric tensor \( g_{\mu\nu} \) attains: since it describes a static field with central symmetry it will look like

\[ g_{\mu\nu} = \begin{pmatrix} a(r) & 0 & 0 & 0 \\ 0 & -b(r) & 0 & 0 \\ 0 & 0 & -r^2 c(r) & 0 \\ 0 & 0 & 0 & -r^2 d(r) \sin^2 \vartheta \end{pmatrix} \]

from the initial singularity (compare with the Cosmic Defect in section 4.3.4).

The cosmic time is astonishingly a length, like the other three space dimensions. We do not perceive time in terms of metres, but in relativity we ought to get acquainted with metres and drop seconds. Hence, the letter \( \tau \) indicates the cosmic time expressed as a length. If we prefer to express time \( t \) in seconds, then we have to multiply it by the speed of light, so that

\[ d\tau^2 = c^2 dt^2 \]
4 – A ‘Strained Space-Time’ Theory

or

\[ ds^2 = a(r) \, d\tau^2 - b(r) \, dv^2 - r^2 \, c(r) \, d\vartheta^2 - r^2 \, d(r) \, \sin^2 \vartheta \, d\varphi^2 \]

\[ a(r), b(r), c(r), d(r) \text{ are to be determined. They are functions of } r \text{ only, in order to preserve the central symmetry. Since the field is static we must have } g_{0\lambda} = 0 \text{ for all } \lambda \neq 0. \]

Furthermore, at infinity the metric must be Euclidean again so that the spatial components of \( g_{\mu\nu} \) are equal to \(-\eta_{\mu\nu}\) of formula 4.4. \( r \) is not the same radius vector as in classical physics. Nonetheless, I impose that any circle centred in \( O \) have length \( s = 2\pi r \). I shall then set \( \vartheta, r, \) and \( \tau \) constant: in particular, with no loss of generality, set \( \vartheta = \frac{\pi}{2} \). Then, getting rid of the minus sign,

\[ ds = r \sqrt{d(r)} \, d\varphi \]

integrated and imposing the length of the circle as said above

\[ r \sqrt{d(r)} \, 2\pi = 2\pi r \]

thus \( d(r) = 1 \). The same procedure may be repeated for \( \varphi \) constant, letting \( \vartheta \) vary, to gain \( c(r) = 1 \). Hence,

\[ g_{\mu\nu} = \begin{pmatrix} a(r) & 0 & 0 & 0 \\ 0 & -b(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \vartheta \end{pmatrix} \]

From this metric, the Christoffel symbols are (computed with Maxima)

\[ \Gamma^\tau_{\tau r} = \Gamma^r_{\tau \tau} = \frac{a'}{2a}, \quad \Gamma^r_{rr} = \frac{b'}{2b}, \quad \Gamma^r_{tt} = \frac{a'}{2b} \]

\[ \Gamma^\varphi_{\varphi \vartheta} = \Gamma^\varphi_{\vartheta \varphi} = \frac{\cos \vartheta}{\sin \vartheta}, \quad \Gamma^r_{\vartheta \vartheta} = -\frac{r}{b}, \quad \Gamma^r_{\varphi \varphi} = -\frac{r \sin^2 \vartheta}{b} \]

\[ \Gamma^\vartheta_{r \vartheta} = \Gamma^\vartheta_{\vartheta r} = \Gamma^\varphi_{r \varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^\vartheta_{\varphi \varphi} = -\sin \vartheta \cos \vartheta \]

where ‘ indicates differentiation with respect to \( r \). The non-zero components of the Ricci’s tensor are therefore

\[ R_{00} = \frac{a''}{2b} - \frac{a'}{4b} (\log(ab))' + \frac{a'}{rb}, \quad R_{11} = -\frac{a''}{2a} + \frac{a'^2}{4a^2} + \frac{a'b'}{4ab} + \frac{b'}{rb} \]

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\[
R_{22} = 1 - \frac{1}{b} + \frac{r}{2b} \left( \log \left( \frac{b}{a} \right) \right)'
\]
\[
R_{33} = \sin^2 \vartheta \left( 1 - \frac{1}{b} + \frac{r}{2b} \left( \log \left( \frac{b}{a} \right) \right) \right)'
\]

In \textit{vacuo} curvature is zero, i.e. \( R_{\mu\nu} = 0 \), thus generating the following non-linear system of ordinary differential equations:

\[
\begin{cases}
\frac{a''}{2a} - \frac{a'}{2b} \left( \log(ab) \right)' + \frac{a'}{2b} &= 0 \\
-\frac{a''}{2a} + \frac{a'^2}{4a^2} + \frac{a'b'}{4ab} + \frac{b'}{2b} &= 0 \\
1 - \frac{1}{b} + \frac{r}{2b} \left( \log \left( \frac{b}{a} \right) \right)' &= 0 \\
\sin^2 \vartheta \left( 1 - \frac{1}{b} + \frac{r}{2b} \left( \log \left( \frac{b}{a} \right) \right) \right)' &= 0
\end{cases}
\]

Note that the last two equations are the same. We are looking for solutions of the form

\[ a = 1 + \frac{C}{r} \]

because for big values of \( r \) the metric becomes Galilean, indeed. Casting the proposed \( a \) into the first equation we obtain that

\[ b = \frac{D}{1 + \frac{C}{r}} \]

with \( C \) and \( D \) constants. \( D = 1 \) is easily obtained by putting \( a \) and \( b \) into the third equation.

Only the constant \( C \) needs to be determined now. To do this, I must introduce some physical considerations along with the mathematical treatise of this section.

**Classical Limit**

Starting from Einstein’s equation 3.22 I shall now illustrate what happens to the metric of a manifold in the case of a feeble gravitational field. Recall the Lagrangian 3.9 for a particle in the classical case is

\[ \mathcal{L} = \frac{1}{2}mv^2 - mU \]

where \( U = \frac{\dot{U}}{m} \). For simplicity, following proposition 1, let me rewrite this Lagrangian like

\[ \mathcal{L} = \frac{1}{2}mv^2 - mU - mc^2 \] (4.5)
In fact $-mc^2$ is only a constant and is of course the total time derivative of a function of only coordinates and time. The classical action is given by 3.2

$$S_C = \int_{t_A}^{t_B} \mathcal{L} \, dt$$

The action in special relativity, between two events $A$ and $B$ is

$$S_{SP} = -mc \int_{AB} ds$$

(4.6)

A weak gravitational field implies of course small velocities with respect to $c$. Then the two actions, 3.2 and 4.6, must look alike. Casting the Lagrangian 4.5 into the classical action 3.2:

$$S_C = -mc \int_{t_A}^{t_B} \left( c - \frac{v^2}{2c} + \frac{U}{c} \right) \, dt = S_{SP}$$

Then, the classical limit consists of

$$ds = \left( c - \frac{v^2}{2c} + \frac{U}{c} \right) \, dt$$

Squaring both terms in this equality, rearranging, and getting rid of higher order infinitesimals (where $c^2$ appears at the denominator)

$$ds^2 = \left( c - \frac{v^2}{2c} + \frac{U}{c} \right)^2 \, dt^2 = \left( 1 + \frac{2U}{c^2} \right) d\tau^2 - v^2 \, dt^2$$

and the explicit metric tensor of the classical limit is

$$g_{\alpha\beta} = \begin{pmatrix}
1 + \frac{2U}{c^2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

(4.7)

The potential energy may be expressed as the gravitational Newtonian potential energy

$$U = -G\frac{M}{r}$$

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Now that we are equipped with the necessary information, we are able to determine the value of $C$:

$$a(r) = 1 + \frac{C}{r} = 1 - \frac{2GM}{rc^2} = g_{00}$$

yielding

$$C = -\frac{2GM}{c^2} = -r_g$$

$r_g$ is named gravitational radius or Schwarzschild radius. Therefore the Schwarzschild metric is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) d\tau^2 - \frac{1}{1 - \frac{r_g}{r}} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2$$

$$g_{\mu\nu} = \begin{pmatrix}
1 - \frac{r_g}{r} & 0 & 0 & 0 \\
0 & -\frac{1}{1 - \frac{r_g}{r}} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2 \sin^2 \vartheta \\
\end{pmatrix} \quad (4.8)$$

♦ Remark: to establish a relationship between the mathematics and physics of the problem for a second time, note that the metric 4.8 has physical meaning if and only if $g_{00} > 0$, i.e. the existence condition for the Schwarzschild metric is

$$r > r_g$$

For smaller values of $r$ the metric becomes singular. The physical significance of this singularity, and whether this singularity could ever occur in nature, was debated for many decades until the second half of the twentieth century. Now it is widely accepted by the scientific world that an object smaller than its Schwarzschild radius\(^3\) is a so-called black hole. The surface of this body at $r_g$ behaves as an event horizon, from which neither light nor particles can escape.

\(^3\)It is easily computed that the Sun has a Schwarzschild radius of approximately 3.0 km while the Earth’s is only about 9.0 mm, the size of a peanut.
4.3 The Strained State Cosmology

In chapter 2, about the theory of elasticity, I gave a quite accurate description of the three-dimensional material continua. Tensors, even though at the beginning they were not very well seen by both physicists and engineers, are now widely used for this description. As already mentioned, I chose to adopt this mathematical tool in a fashion typical of general relativity (so, I always distinguish between covariant and contravariant form) throughout the treatise.

Einstein’s equation 3.22 links together space-time (on the left side) and matter/energy (on the right side). In cosmology there are few but important facts that do not fit in the classical general relativity. On the other hand, a relevant number of theories have been proposed to solve this problem. For instance dark energy and dark matter have been hypothesised, but they are not the only ones: by now, one of the most successful theories is the so-called lambda-cold dark matter theory. Most of the times, in cosmology, new theories are forged by manipulating the Lagrangian or adding fields on heuristic bases. In any case it is difficult, or almost impossible, to devise experiments (like that of Michelson and Morley which proved the non-existence of ether) whose aim would be to discriminate among these theories.

Besides, another way which could be chosen to investigate this tangled situation is to stick to physical knowledge we already possess, instead of continuously building new scenarios to explain new problems. This principle corresponds to the well known Occam’s razor which states. frustra fit per plura quod fieri potest per pauciora, i.e. it is useless to do with more what can be done with less, or: Entia non sunt multiplicanda praeter necessitatem, which means that entities are not to be multiplied unnecessarily.

This is the idea underneath the theory proposed by Angelo Tartaglia, with the support of Ninfa Radicella and Mauro Sereno, which goes under the name of Strained State Cosmology. It is a theory based on physical intuition, extending our knowledge of continuum mechanics to a four-dimensional Lorentzian manifold. The presence of matter/energy or the existence of texture defects like those in 3D crystals create deformations in space-time. In other words, they induce strain, and the core of this theory is the presence in space-time of a strain energy analogous to the elastic potential energy. The strain energy of space-time is eventually associated with the
 curvature of the space-time itself \(^4\); strain affects the metric tensor and is produced by some cause. As already mentioned, this cause might be either external, meaning matter/energy, or internal, referring to the structural defects in the texture of the manifold, well known in elasticity and plasticity theories.

### 4.3.1 Metric Properties of Elastic Continua

Consider an \(N + n\) dimensional space, with \(N\) and \(n\) \(\in\) \(\mathbb{N}\). This space will be our embedding manifold which is assumed to be flat, i.e. with Euclidean geometry, with coordinates \(X^a, \ a = 1, 2, \ldots, N + n\). Inside this space are embedded two \(N\) dimensional manifolds. The first is called reference manifold and it is assumed to be flat; the second is the natural manifold and corresponds to the reference manifold after some action has been applied to it: in this way the natural manifold appears deformed, and in general it has acquired an intrinsic curvature. Each of these two manifolds has its own coordinates, \(\xi^\mu\) and \(x^\mu\) for the reference and the natural manifold respectively \((\mu \in \mathbb{N}, \mu = 1, 2, \ldots, N)\).

In the embedding space the natural frame is indicated by \(n\) non-linear constraints of the form

\[
H_i \left(X^1, \ldots, X^{N+n}\right) = \text{constant} \quad (4.9)
\]

The reference frame, being flat, is associated with \(n\) linear constraints of the type

\[
F_i \left(X^1, \ldots, X^{N+n}\right) = \text{constant} \quad (4.10)
\]

\(i \in \mathbb{N}, i = 1, 2, \ldots, n\)

Suppose that the natural manifold is sufficiently regular, and all functional dependencies are smooth and differentiable as many times as needed. Moreover, it is possible to put the points of each embedded manifold in one-to-one correspondence (this operation can be thought as a gauge transformation - see later, section 5.1.1). This is readily done by creating a displacement vector field \(u\) in the \(N + n\) dimensional space that unambiguously and unequivocally connects points of the natural manifold to points of the reference manifold, both of which are \(N\) dimensional subspaces. This procedure, apart from being rigorously mathematical, has the meaning

\(^4\)The idea of assigning a sort of rigidity to space-time is actually old (Sakharov, 1968), but it was formulated in terms of quantum physics, where another problem usually arises, i.e. the mismatch between the values obtained from quantum computations and those needed to be in accordance with cosmological phenomena.
of connecting each point after deformation to its reciprocal in the unstrained state, i.e. to the position it occupied before deformation occurred.

To give a pictorial view of this procedure, consider $n = 1$, and $N = 2$. The situation is then visualized in picture 4.3.

![Diagram](image)

Figure 4.3. The embedding three-$(N+n)$-dimensional space contains two surfaces. The flat one is the reference manifold, whereas the distorted one is the natural manifold: the displacement vector field $u$ is indicated by thick dashed arrows. (Picture issued from A. Tartaglia, [14])

In general we assume

$$\xi^\mu = \xi^\mu (X^1, ..., X^{N+n}) \quad \text{and} \quad x^\mu = x^\mu (X^1, ..., X^{N+n}) \quad (4.11)$$

Now, the job consists of taking on the two subspaces corresponding pairs of nearby points and compare the length of the line elements. On the reference manifold we
have

\[ dl^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu \]  \hspace{1cm} (4.12)

On the natural manifold the metric becomes non-Euclidean and

\[ dl'^2 = g_{\mu\nu} dx^\mu dx^\nu \]  \hspace{1cm} (4.13)

I shall remark that only the reference manifold is flat, thus the following relation holds \textit{only} for the metric \( \eta_{\mu\nu} \) (\( y \) indicates Cartesian coordinates):

\[ \eta_{\mu\nu} = \delta_{\alpha\beta} \frac{\partial y^\alpha}{\partial \xi^\mu} \frac{\partial y^\beta}{\partial \xi^\nu} \]  \hspace{1cm} (4.14)

and \textit{not} for the metric \( g_{\mu\nu} \). On the other hand, both line elements can be expressed in the flat embedding space as, in general,

\[ dl^2 = \delta_{ab} dX^a dX^b \]  \hspace{1cm} (4.15)

which has been written in Cartesian coordinates for simplicity. Note that I use Latin indices for the embedding \( N + n \) dimensional space, and Greek indices for the two submanifolds. Anyway, if all functions are differentiable and invertible

\[ x^\mu = f^\mu (\xi^1, \xi^2, ..., \xi^N) \]  \hspace{1cm} (4.16)

holds, as well as its inverse function \( f^{-1} \), thus allowing us to use the same coordinates in the two different submanifolds. It is possible to go from 4.15 to 4.12 or to 4.13 applying constraints 4.10 and 4.9, and noticing that, observing figure 4.3

\[ r'(X) = r(X) + u(X) \]  \hspace{1cm} (4.17)

where \( r' \) or \( r_n \) is the radius vector joining the origin of the embedding coordinate system to a point on the natural manifold, and the same is true for \( r \) or \( r_r \) concerning the reference manifold. If we are considering four-dimensional reference and natural manifolds, the vectors \( u \) are in five dimensions.

Equipped with these definitions, we may now proceed to compare the line elements. This only makes sense if the comparison is done in one single manifold, and here I choose the embedding space, for the moment. Using the flat metric \( \delta_{ab} \) as in 4.15, unprimed coordinates for the reference and primed for the natural manifold, 4.12 and 4.13 become

\[ dl^2 = \delta_{ab} dX^a dX^b \]
\[ dl^2 = \delta_{ab} \, dX^a dX^b \]

As described before, applying the constraints the first of the equations just written becomes

\[ dl^2 = \delta_{ab} \frac{\partial X^a}{\partial \xi^\mu} \frac{\partial X^b}{\partial \xi^\nu} \, d\xi^\mu d\xi^\nu = \eta_{\alpha \beta} \, d\xi^\alpha d\xi^\beta = \eta_{\mu \nu} \, dx^\mu dx^\nu \]

This is the flat metric tensor of the reference manifold (Euclidean or Minkowskian).

\[ \diamond \text{ Remark: } \text{the last equality of the equation above, involving the factors} \]

\[ \eta_{\alpha \beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \, dx^\mu dx^\nu = \eta_{\mu \nu} \, dx^\mu dx^\nu \]

does not represent a ‘pure’ coordinate change, in the sense that the metric \( \eta_{\mu \nu} \) is not in general a metric tensor on the natural manifold.

The second equation turns into

\[ dl^2 = \delta_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu} \, dx^\mu dx^\nu = g_{\mu \nu} \, dx^\mu dx^\nu \]

In this case the constraint 4.9 leads to a distorted, curved metric with Lorentz signature. Recall that \( a, b \) run from 1 to \( N + n \), and \( \mu, \nu \) range from 1 to only \( N \).

Hence, now that we have expressed the two line elements 4.12 and 4.13 in the same coordinates, namely those of the natural manifold \( x^\mu \), let us compute their difference \( \delta (dl^2) \). To do this consider the assumption

\[ dx^\mu = d\xi^\mu + du^\mu = d\xi^\mu + \frac{\partial u^\mu}{\partial \xi^\alpha} d\xi^\alpha \]

so that

\[ dl^2 = g_{\mu \nu} \left( d\xi^\mu + \frac{\partial u^\mu}{\partial \xi^\alpha} d\xi^\alpha \right) \left( d\xi^\nu + \frac{\partial u^\nu}{\partial \xi^\beta} d\xi^\beta \right) \]

After some algebra we get

\[ dl^2 = dl^2 + \left( \eta_{\alpha \mu} \frac{\partial u^a}{\partial x^\nu} + \eta_{\nu b} \frac{\partial u^b}{\partial x^\mu} + \eta_{ab} \frac{\partial u^a}{\partial x^\mu} \frac{\partial u^b}{\partial x^\nu} \right) \, dx^\mu dx^\nu \]

Then, \( \delta (dl^2) = dl^2 - dl^2 \) is equal to twice the so-called strain tensor of the natural manifold

\[ \varepsilon_{\mu \nu} = \frac{1}{2} \left( \eta_{\alpha \mu} \frac{\partial u^a}{\partial x^\nu} + \eta_{\nu b} \frac{\partial u^b}{\partial x^\mu} + \eta_{ab} \frac{\partial u^a}{\partial x^\mu} \frac{\partial u^b}{\partial x^\nu} \right) \]

and

\[ g_{\mu \nu} = \eta_{\mu \nu} + 2\varepsilon_{\mu \nu} \]

(4.18)

(4.19)
The physical interpretation is that of a continuous deformation leading from the reference to the natural manifold. The displacement vector tells where any single point has moved, and the differential part does indicate the strain induced in the manifold.

4.3.2 Defects

If it were possible to find a transformation such that

\[ g_{\mu \nu} = \eta_{\alpha \beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \] (4.20)

the strained and unstrained manifolds would coincide, i.e. on the natural manifold there would not be any intrinsic curvature to perceive. In fact the integrability condition for 4.20 is

\[ R_{\alpha \lambda \mu \nu} = 0 \] (4.21)

This happens in the cases when the deformation is elastic and the stress has external origin: the strain goes to zero when external action on the body ceases. In other words, a smooth and continuous vector field \( u \) assures that 4.20 is integrable. On the other hand, intrinsic strains arise from singularities in the field \( u \) or in its derivatives. Another important notion which becomes important at this point is the one of defect: defects are paramount in the study of the properties of crystals or of solids in general. Volterra was one of the first scientists to account for a consistent description of them in 1907, in terms of dislocations and disclinations. Now, when I hinted at a singular displacement field before, I meant that the continuum medium must contain one or more texture defects. Consequently, the singularity in \( u \) affects the deformation field and in general the elementary deformation can be written like a non-integrable one form

\[ dr^\mu = \omega^\mu_\nu dr^\nu \]

and 4.13 like

\[ dl'^2 = \eta_{\alpha \beta} \omega_\mu^\alpha \omega_\nu^\beta d\xi^\mu d\xi^\nu \]

Since the SSC has its application in space-time, we might conclude that violation of condition 4.21 leads to non-trivial space-times, in the sense that defects do exist.

A defect may be represented as in figure 4.4. On the reference manifold there exists a bounded region \( C \) containing infinitely many points. The vector field mapping points between the two manifolds is in general regular, but it is singular in the
Figure 4.4. Representation of a defect. Not for all points the correspondence is one-to-one: a whole region $C$ of the reference manifold is mapped onto a point $O$ of the natural manifold. (Picture issued from A. Tartaglia, [14])

region $C$, i.e. the correspondence ceases to be one-to-one along and inside $C$. In fact, the entire region corresponds to a lower dimensional subset on the natural manifold, say a point or a line. Therefore, defects imply a natural manifold with non-zero strain; moreover, the strain is singular where the defects occur. Defects are able to induce peculiar symmetries: a pointlike defect generates a spherical symmetry, to which the Schwarzschild solution might be applied; linear defects induce cylindrical symmetry. As a matter of fact Volterra’s work on texture defects was subsequently extended to space-time by Puntingam and Soleng, in 1997.
4.3.3 Modified Lagrangian

As I already mentioned, the intrinsic distortion of the natural manifold makes us wonder about the existence of a sort of deformation energy. In some way, this energy must be part of the Lagrangian as an additional potential (and in the equations it has to lead to a dynamical history of the universe as well). For now, recall what I reported in chapter 2, about the elastic continua. I talked about linear elasticity, thus assuming that the relation between stresses and strains is the simplest possible, i.e. linear. This led us to Hooke’s law 2.45

\[ \sigma_{\mu\nu} = C_{\kappa\lambda\mu\nu} \varepsilon^{\kappa\lambda} \]

where the elastic modulus tensor \( C_{\kappa\lambda\mu\nu} \) contains several independent parameters (81) in the most general case. By the way, assuming that our solid body is locally isotropic simplified the situation until the point where \( C_{\kappa\lambda\mu\nu} \) depends only on two parameters, \( \lambda \) and \( \mu \), the Lamé’s parameters. Hooke’s law 2.45 shrank to 2.44, which here I rewrite with respect to the flat metric of the reference manifold \( \eta_{\alpha\beta} \) (note that in general it is not described by Cartesian coordinates; however \( \eta_{\alpha\beta} \) also includes \( \delta_{\alpha\beta} \), which is usually found in the standard elasticity theory)

\[ \sigma_{\mu\nu} = \lambda \eta_{\mu\nu} \varepsilon + 2 \mu \varepsilon_{\mu\nu} \quad (4.22) \]

The elastic modulus thus becomes

\[ C_{\kappa\lambda\mu\nu} = \lambda \eta_{\kappa\lambda} \eta_{\mu\nu} + \mu \left( \eta_{\kappa\mu} \eta_{\lambda\nu} + \eta_{\kappa\nu} \eta_{\lambda\mu} \right) \quad (4.23) \]

Also, in chapter 2, I introduced the concept of deformation energy, which can be successfully recovered here. I started by assuming that it must equal the usual elastic potential energy 2.42

\[ W = \frac{1}{2} \sigma_{\mu\nu} \varepsilon^{\mu\nu} \]

Casting 4.22 inside this relation, we discover that the deformation energy might be written as in 2.43, or

\[ W = \frac{1}{2} \lambda \varepsilon^2 + \mu \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} \quad (4.24) \]

Note that equation 4.24 was previously obtained (see page 39) considering the lowest significant terms of the Helmholtz free energy written in terms of strain.
It is now time to apply this knowledge on three-dimensional continua to space-time, i.e. to a four-dimensional manifold with Lorentz signature. Space-time will be treated as a physical continuum endowed with properties analogous to the ones of ordinary elastic solids. As a first step I reconsider the Einstein-Hilbert Lagrangian density 3.16,

$$L = R \sqrt{-g}$$

to which the deformation energy density must be added: the latter term is indeed, from 4.24, $W \sqrt{-g}$. Hence, the complete Lagrangian density is

$$L = (R + W + \mathcal{L}_M) \sqrt{-g} \quad (4.25)$$

so that the Hilbert action is changed into

$$S = \int_{\Omega^4} \left( R + \frac{1}{2} \lambda \varepsilon^2 + \mu \varepsilon_{\alpha \beta} \varepsilon^\alpha \beta + \mathcal{L}_M \right) \sqrt{-g} d^4x$$

The scalar curvature $R$ is the dynamical term, the analogous of the kinetic energy in practice, because it contains the derivatives of the metric. Now it is in general $R \neq 0$, because we assume the presence of a defect that generates intrinsic distortion. $\mathcal{L}_M$ is the Lagrangian density of the matter/energy, and is zero in vacuo. In the remaining chapters of the treatise this term will be usually left off, since I am not interested in studying exogenous causes.

From 4.25 new generalized Einstein’s equations can be written. The deformation energy $W$ contributes with a new additional stress-energy tensor, $T_{\varepsilon \mu \nu}$, which has to be read as an effective elastic stress-energy tensor. Hence, the final equations, the ‘Elastic’ Einstein’s equations, look like

$$G_{\mu \nu} = \kappa T_{\mu \nu} + T_{\varepsilon \mu \nu} \quad (4.26)$$

Since $T_{\varepsilon \mu \nu}$ is obtained varying a scalar with respect to a true tensor (the metric tensor $g_{\mu \nu}$), as well as $T_{\mu \nu}$, it is also a good tensor, with all the properties of tensors. This elastic tensor is partially built from the metric, but it also constitutes an additional source together with the ordinary matter/energy term. In vacuo the Bianchi identities bring the Einstein tensor to zero implying the conservation of $T_{\varepsilon \mu \nu}$, whereas in presence of matter/energy the whole quantity $\kappa T_{\mu \nu} + T_{\varepsilon \mu \nu}$ is conserved. The latter relation

$$\kappa T_{\mu \nu} + T_{\varepsilon \mu \nu} = \text{constant}$$
signifies the possibility of a transfer of energy between the matter and the strain, which is a well-known subject in classical physics. In explicit form

\[ T_{\varepsilon\mu\nu} = \lambda \varepsilon \varepsilon_{\mu\nu} + 2 \mu \varepsilon_{\mu\nu} \] (4.27)

One last remark about homogeneity and isotropy of space-time can be made. In fact, the unstrained manifold is obviously isotropic, since neither exogenous action nor intrinsic defects are present. When we pass to the natural strained manifold, the locally anisotropic strain can induce some anisotropy in the elastic parameters. Nonetheless, this anisotropy may be considered as of second order with respect to the strain, thus the theory we consider is linear and the conditions of homogeneity and isotropy hold. Note that this procedure, which might be regarded as a ‘linearization’, is typically used for ordinary three-dimensional continua.

### 4.3.4 The Strained State Theory (SST) and a Robertson-Walker Universe

At the beginning of this chapter I discussed the fact that it is commonly assumed that our universe is homogeneous and isotropic, i.e. it has a Robertson-Walker symmetry. This symmetry is an intrinsic property and it is independent of the presence of matter: matter only fits in the symmetry, reinforcing it. The primordial symmetry is explained by the presence of a space-like defect, which is commonly defined as a Cosmic Defect\(^5\). From this defect, which is commonly identified with the Big Bang in standard cosmology, the cosmic time has a start. The Cosmic Defect is also the origin of the signature of space-time.

The approach of the SSC is represented in picture 4.5, where \( O \) stands for the defect causing the Robertson-Walker metric to hold. As I thoroughly explained in the previous section, here the representation of the reference and natural manifolds is evident. The reference manifold is a plane with Euclidean or Minkowskian geometry, and the natural manifold is a distorted, i.e. curved, surface, with Lorentz signature. All the geodesics which originate from the defect are time-like, \( \tau \) being the cosmic time. Space is a generic intersection between the natural manifold and an open

\(^{5}\)It is obvious that one might ask why the defect should be there. A. Tartaglia, in [14], states that ‘we know that going back along the chain of ‘why?’s sooner or later we exit the domain of physics. We can only try and minimizing the number of independent assumptions and if possible look for physically consistent interpretations of their meaning’.

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Figure 4.5. Visualization of the approach of the SSC. To be as intuitive as possible the submanifolds are only in two dimensions instead of four, and the embedding space is in three instead of five dimensions - it would be impossible to depict something that has more than three dimensions, too! (Picture issued from A. Tartaglia, [14])

surface, like a hyperplane. In picture 4.5, the expansion of the universe may be deduced from the fact that we may perform such intersections for increasing values of the cosmic time.

It is important to find a correspondence between the unstrained and strained manifolds. In the manner used in the previous sections, I will now write down and compare the two line elements computed for each manifold. Recalling the Robertson-Walker metric 4.2, the line element on the natural manifold is

$$ds^2 = d\tau^2 - a^2(\tau)dt^2$$

(4.28)
where \( a(\tau) \) is the scale factor, and \( dl \) the space element. Now, consider the reference manifold: what is important to stress here is that the correspondence with the natural Robertson-Walker space-time may be carried out in infinitely different ways. Using the same coordinates as in 4.28, we have four free functions which express the correspondence, with the requirement that the reference manifold be flat. However, the symmetry of the manifold reduces the free functions by three: we remain with only one, that we call \( b(\tau) \). The line element is then, in the Euclidean reference subspace

\[
ds^2 = b^2(\tau) \, d\tau^2 + dl^2
\]

(4.29)

\( b(\tau) \) was defined gauge function (Radicella, 2011—see page 94 for clarification): this is not entirely correct, because this function does not indicate a degree of freedom. Applying relation 4.19 to the two metrics 4.28 and 4.29 computed above gives rise to the strain tensor for a Robertson-Walker space-time, whose non-zero elements are

\[
\begin{align*}
\varepsilon_{00} &= \frac{1 - b^2}{2} \\
\varepsilon_{\alpha\alpha} &= -\frac{1 + a^2}{2}
\end{align*}
\]

(4.30)

These elements enable us to compute the deformation energy 4.24 (in the next chapter attention will be drawn towards this computations). The metric used to raise and lower indices is indeed 4.28.

\[
W = \frac{\lambda}{8} \left( 1 - b^2 + 3 \frac{1 + a^2}{a^2} \right) + \frac{\mu}{4} \left[ (1 - b^2)^2 + 3 \left( \frac{1 + a^2}{a^4} \right)^2 \right]
\]

(4.31)

Moreover, the missing required terms to build the Lagrangian are

\[
R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)
\]

(4.32)

\[
\sqrt{-g} = a^3
\]

(4.33)

with \( \dot{a} = \frac{da}{d\tau}, \ddot{a} = \frac{d^2a}{d\tau^2} \).

In another step, we reduce the unknown functions from two to only one: the Lagrangian density is extremized with respect to \( b(\tau) \). Imposing \( \frac{dW}{db} = 0 \) we get two solutions for \( b^2 \):

\[
b_{1,2}^2 = 0, \quad \frac{2\lambda + \mu}{\lambda + 2\mu} + \frac{3}{a^2} \frac{\lambda}{\lambda + 2\mu}
\]

(4.34)
where the first one, i.e. $b = 0$, has no physical correspondence, so it has to be rejected.

One last specification needs to be made, i.e. the matter/energy is dust plus radiation, being the simplest choice possible. Finally, we apply Hamilton’s principle of least action and obtain the famous *Hubble parameter*:

$$
H = \frac{\dot{a}}{a} = c \sqrt{\frac{3}{16}} B \left(1 - \frac{(1 + z)^2}{a_0^2}\right)^2 + \frac{k}{6} (1 + z)^3 [\rho_{m0} + \rho_{r0}(1 + z)] \quad (4.35)
$$

$\rho_{m0}$ and $\rho_{r0}$ are the present values of the average matter and radiation densities respectively. The constant $k$ equals $\frac{16\pi G}{c^2}$. $B$ is the term containing information on the elastic behaviour of space-time, i.e. the Lamé’s parameters

$$
B = \frac{3}{2\mu} \frac{2\lambda + \mu}{\lambda + 2\mu}
$$

$z$ is the *redshift factor* which is formally defined as

$$
z = \frac{\lambda_o - \lambda_e}{\lambda_e} \quad (4.36)
$$

where $\lambda$ is the wavelength (measured in metres), and $z$ reads as the relative variation of the wavelength, i.e. between the value $\lambda_e$ at which the wave is emitted and $\lambda_o$ at which it is observed. Also

$$
1 + z = \frac{\lambda_o}{\lambda_e}
$$

However, in the Robertson-Walker Cosmology one usually expresses $z$ in terms of the scale factor $a$:

$$
1 + z = \frac{a_0}{a} \quad (4.37)
$$

where $a_0$ is the present value of the scale factor. As a consequence $a(1+z) = a_0 = \text{constant}$. The positive sign of 4.35 translates into an expanding universe. At very early
times \(z \gg 1\), when \(z \gg 1\), the strain contributes boosting the expansion and

\[
H_{z \gg 1} \approx cz^2 \sqrt{\frac{3B}{16a_0^2} + \frac{k}{6\rho_0}}
\] (4.38)

\[z = \frac{a_0}{a} - 1\]

At very early times the scale factor \(a\) attains particularly small values, so that

\(z \gg 1\)

At late times \(a\) becomes big, and it is called \(a_\infty\): according to the definition of \(z\) above

\(z \rightarrow -1\)
At late times, \( z \rightarrow -1 \), \( H \) becomes constant

\[
H_{z \rightarrow -1} \approx c \sqrt{\frac{3}{16} B} \tag{4.39}
\]

\[
a_\infty \approx \exp \left( c \sqrt{\frac{3}{16} B \tau} \right) \tag{4.40}
\]

In conclusion, the SST preserves the structure of general relativity, and gives a configuration of space-time accounting for both the initial inflation and for the late acceleration (see picture 4.1, page 64, for comparison). Namely, it is locally undistinguishable from general relativity, but predicts emerging effects at cosmic scales. I will not spend more words here, but it is important to highlight that the SST passes several consistency tests: for instance the theory satisfactorily fits the luminosity data of type Ia supernovae (for further details follow the bibliography: [14], by A. Tartaglia, and [15], by A. Tartaglia and N. Radicella).
Chapter 5

Strained State Three-Dimensional Continua

As I have piled up some knowledge necessary to understand the problem in deeper details, aiming at giving an in-depth study of the case, I may proceed to stick to the facts I intend to prove in order to bring this treatise to a sensible close. In this last chapter comes the conjunction of all the notions which I have illustrated so far. It is neither a farther step in a cosmological theory what I am going to prove, nor another manipulation of the Lagrangian functional, nor a newly baked method useful for treating elastic continua.

As the title of the thesis suggests, and as I already mentioned in the introduction, I will now show that the method outlined by the ‘Strained State’ cosmological theory might be successfully applied to three-dimensional material continua. Again, I shall start from an only ansatz: in cosmology we assume that the universe is homogeneous and isotropic, so as to end up labelling space-time with the so-called Robertson-Walker symmetry. Here the assumption is similar, since it is apparently the same, that of homogeneity and isotropy, but we leave one dimension out of the discussion: forgetting about time and space-time, we remain with a space in three dimensions with no Lorentz signature. This space need not be Euclidean of course, since the gist of the question is this one: solids might be seen as three-dimensional manifolds that turn from their unstrained Euclidean reference state to a curved non-Euclidean natural condition. This is indeed a non-conventional interpretation of the theory of elasticity, maybe more complicated, but it should satisfactorily lead to agreeing
results.

Hence, the ansatz here is about solid bodies with particularly symmetric shapes, as well as loaded symmetrically. As I hinted at when talking about the cosmic defects of space-time, we might have pointlike defects or straight linear defects. In the same spirit I will treat a case of central symmetry, i.e. a sphere, and a case of axial symmetry, or that of an infinitely long cylinder. The stresses operating in these solids originate from uniform and isotropic loadings which respect the symmetries of the body. Note that these two cases have already been treated in the ordinary way, i.e. following the classical theory of elasticity, in the last section of chapter two, where I provided the geometrical description of the solids, as well as the expressions for the displacement vectors, and for the stress and strain tensors.

Anyway, one remark shall be made. So far, I have never really talked about time: consider, in fact, that in the theory of relativity time resembles a spatial dimension, and it is indeed far from being associated with ‘the time’ we usually think about, which is absolute, being ‘an equally immutable regular flow setting the pace for all changes and movements’ (Newton, 1687). And, from now on, there is no reason at all to recover time. In three-dimensional space, time is not even present as a spatial dimension, and I am not willing to introduce it. Therefore, all the problems treated pertain to the field of statics. This partially justifies the copious digression on Lagrangian mechanics I pursued in chapter 3, when, with the aid of field theory, I developed the whole subject of Lagrange equations independently of the time variable.

In conclusion, the content of the following chapter is prompted by the ground ideas of the SSC exposed in the previous chapter. All the subjects developed in the treatise contribute to a full understanding and to the awareness of the topics treated hereafter.
All the exact results which I will use in the present chapter are issued from chapter 2, ‘Theory of Elasticity: an Overview’. In particular two examples, about the spherical and cylindrical cases, were completely worked out in section 2.6. I will first treat the spherical case, in analogy with the Schwarzschild solution in general relativity. Then, at the end the results will be provided also for the cylinder.

I now proceed to the construction of the Lagrangian density for the spherical case. In the fashion of the SST, the Lagrangian function is made up of two parts: the curvature and the deformation energy of the strained solid. These two ‘ingredients’ happen to be paramount. This chapter aims at thoroughly describing them at first. Then, the action integral will be deduced; eventually minimization of it will take us to the so-called Lagrange equations. To begin, I shall define the metric tensor of the various spaces and subspaces involved in the problem.

5.1 Metric Tensors for the Natural and Reference Manifolds

First of all, let me just remind what I am doing, since this method could still sound a little abstract to the reader for now. Following the definitions of embedding space, reference and natural manifolds given when illustrating the SST, here I recall that

- the natural manifold is a three-dimensional deformed material continuum, which in general is not flat, but it has assumed a curvature as a consequence of some action applied to it;

- the reference manifold is an ordinary three-dimensional flat space: with no loss of generality I assume it coincides with $\mathbb{R}^3$, described with a convenient set of coordinates, spherical or cylindrical for instance;

- the embedding space, therefore, has to be at least in four dimensions to contain the manifolds described above. Hence, I choose the simplest possible embedding space, i.e. the four dimensional Euclidean space $\mathbb{R}^4$.

Remark that there is no Lorentzian signature in any of the three spaces described above.
The situation is not difficult to visualise: have a look at picture 5.1. Suppose we have the two-dimensional flat embedding space $\mathbb{R}^2$ (on the left): lowering the dimension by one, we get a one-dimensional subspace, for instance a straight line (in black), or $\mathbb{R}$. In the ordinary $\mathbb{R}^3$ (on the right) we can analogously find a two-dimensional subspace, say $\mathbb{R}^2$, by selecting any arbitrary flat plane. And so repeating the same procedure in four dimensions, $\mathbb{R}^3$ is a flat ‘plane of $\mathbb{R}^4$', which is properly named *hyperplane*.

Nonetheless, when in $\mathbb{R}^2$, I could have chosen an arbitrary line, which needs not be straight. If the line were curved like the red one in the picture, I would have picked up a *non-*Euclidean one-dimensional space. The same holds in one or two more dimensions. So, the coloured surface contained in $\mathbb{R}^3$ is still two-dimensional, but not flat any more.

![Figure 5.1. Euclidean and non-Euclidean subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$](image)

After this clarification, I can write down the line element in the natural manifold. Recall that in the unstrained state the metric is the flat 4.3 in spherical polar coordinates

$$ds_0^2 = dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

As I also assumed in section 2.6, the loading respects the central symmetry of the sphere. This means that after deformation, the symmetry is preserved, even though dimensions are slightly scaled. Fortunately, the number of scale factors reduces, as I showed in the discussion on the Schwarzschild metric: here, only one is necessary.
I will call it \( f \), and remember that it is a function of \( r \) only, so

\[
f = f(r)
\]

There is more than a single choice for where to place the scale factor \( f \) in the metric. Here I choose to place it in front of \( dr^2 \), so as to obtain the metric

\[
ds^2 = f(r) \, dr^2 + r^2 \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right)
\]

(5.1)

\[
g_{\mu\nu} = \begin{pmatrix}
f(r) & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \vartheta
\end{pmatrix}
\]

Or, another plausible choice could be

\[
ds^2 = dr^2 + f^2(r) \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right)
\]

Now, turn to the reference manifold. The most important topological property is its flatness, and its physical meaning is that it represents the manifold before deformation. Expressing the length of the line element in this manifold involves knowing how the material was deformed. Picture 5.2 should give a clear idea of the problem. Point \( P \) on the natural strained manifold may correspond to infinitely many points on the reference manifold. For example, assuming that \( P \) lay on the blue circle before any stress was applied, the material shrank more than if it were on the orange circle, but less than if it were on the green one. The various ways to go from the strained solid backward to the reference flat state depend on the strategy of deformation adopted. This strategy is represented by the function

\[
w = w(r)
\]

which in practice establishes a one-to-one correspondence between points on the manifolds. The choice for the correspondence is totally arbitrary. Note an important property of \( w \): it avoids the need to step in the four-dimensional embedding space to define the vector field joining together pairs of points. The notion of this vector field is in fact implicit in the definition of \( w \). The coloured arrows in figure 5.2 are the effective vectors of the field \( u \): we may think of them like strings that pull points of the flat manifold to displace into the natural configuration. Therefore, the same line element on the natural manifold may have several corresponding line elements.
on the reference manifold, as picture 5.3 shows: of course we must fix the form that \( w \) assumes, eventually. In conclusion, the function \( w \) simply defines the line element on the flat reference space: given the orange \( ds \) on the section of the sphere in figure 5.3, \( w \), once arbitrarily fixed, associates it with the red, green or blue \( ds_r \), or with infinitely many others, depending on the choice of \( w \). In this fashion, the metric \( \eta_{\mu\nu} \) of the reference manifold is

\[
\begin{align*}
 ds_r^2 &= w'^2(r) \, dr^2 + w^2(r) \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \\
 \eta_{\mu\nu} &= \begin{pmatrix}
 w'^2(r) & 0 & 0 \\
 0 & w^2(r) & 0 \\
 0 & 0 & w^2(r) \sin^2 \vartheta 
\end{pmatrix}
\end{align*}
\]
5.1.1 Gauge Function

The function \( w \) has been defined (Radicella, 2011) gauge function. Before discussing whether this name is proper or not, I am trying to explain what a gauge function is (see also Landau, [10], chapter 3, 18). Gauge theories are field theories that, thanks to some particular functions, called gauge functions, describe how to transfer physical laws from one manifold to another granting the conservation of the symmetries. In practice, gauge theories establish the physical rules that make us observe the same phenomena in two different places. An example will certainly clarify the concept.

The well-known Maxwell equation\(^1\)

\[
\nabla \cdot \mathbf{B} = 0
\]

implies that the magnetic field \( \mathbf{B} \) is solenoidal, and it can be written as the curl of another vector field \( \mathbf{A} \), called vector potential:

\[
\mathbf{B} = \nabla \times \mathbf{A}
\]

Actually the correspondence is not one-to-one, since \( \mathbf{B} \) may determined by more than one vector field \( \mathbf{A} \). This fact becomes trivial when we recall the famous vector

\(^1\)I assume we are in three dimension, so that I unequivocally wrote \( \nabla \) instead of the general \( \nabla \).
identity $\vec{\nabla} \times \vec{\nabla} \phi = 0$. This allows us to write a vector potential determined up to an arbitrary constant vector $\vec{\nabla} \phi$, which happens to be our gauge function. In the end

$$A' = A + \vec{\nabla} \phi$$

In the same way, any scalar potential is always defined up to an arbitrary constant term. This term is another simpler example of gauge invariance. Hence, a gauge function should reflect the fact that it represents a degree of freedom. The non-uniqueness of potentials gives us the possibility to choose it as we prefer in order to fulfil some requirements chosen by us.

Getting back to our case, we see how the concept behind a gauge function is similar because it involves the transfer of laws (and metrics) between two manifolds. Even if $w$ is an arbitrary function, it is not free at all, since it needs to be determined, and this is what I will be doing here.

### 5.2 Lagrangian Density

After the metrics have been computed, I can build the Lagrangian density $L$, following of course the SST exposed in the previous chapter. The Lagrangian density is indeed similar 4.25 without the matter/energy term, and with the more general $\sqrt{|g|}$ instead of $\sqrt{-g}$:

$$L = (R + W) \sqrt{|g|}$$

The two ingredients $R$ and $W$ are ready to be built.

#### 5.2.1 Curvature $R$

The curvature of the reference manifold is obviously zero, whereas the curvature of the natural manifold is directly computed by operating on the metric tensor $g_{\mu\nu}$ of formula 5.1

$$ds^2 = f(r) dv^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

The whole procedure involves many differentiations and algebraic manipulations: it is described in chapter 1, where the theory on tensors is outlined. Here I use a proper software, i.e. Maxima, to compute the scalar curvature $R$ of the deformed
continuum material. The inputs to enter on the commend line and the output of the program is reported below.

Maxima 5.25.0 http://maxima.sourceforge.net
using Lisp Clozure Common Lisp Version 1.7-r14925M
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.

(%i1) load(ctensor);
(%o1) C:/Programmi/Maxima-5.25.0/share/maxima/5.25.0/share/tensor/ctensor.mac
(%i2) depends(f,r);
(%o2) \([f(r)]\)
(%i3) csetup();
Enter the dimension of the coordinate system:
3; Do you wish to change the coordinate names?
y; Enter a list containing the names of the coordinates in order
[r,theta,phi]; Do you want to
1. Enter a new metric?
2. Enter a metric from a file?
3. Approximate a metric with a Taylor series?
1; Is the matrix 1. Diagonal 2. Symmetric 3. Antisymmetric 4. General
Answer 1, 2, 3 or 4 :
1; Row 1 Column 1:
f; Row 2 Column 2:
r^2;
Row 3 Column 3:
r^2*(sin(theta))^2;
Matrix entered.
Enter functional dependencies with DEPENDS or 'N' if none.

Do you wish to see the metric?

Do you wish to see the metric inverse?

Do you wish to see the Ricci tensor?

Do you wish to see the Christoffel symbols?

Do you wish to see the Riemann tensor?

Do you wish to see the Ricci tensor?

\[
\begin{align*}
\text{uric}_{1,1} &= \frac{f}{2}\frac{r}{f} \\
\text{uric}_{2,1} &= \frac{1}{2}f\frac{r}{f} + \frac{2}{f} - \frac{2}{f} \\
\text{uric}_{2,2} &= \frac{2}{f} + \frac{2}{f} - \frac{2}{f} \\
\text{uric}_{2,2} &= \frac{2}{f} + \frac{2}{f} - \frac{2}{f}
\end{align*}
\]
\[ R = \frac{2rf' + 2f^2 - 2f}{r^2f^2} \]  

5.2.2 Strain Tensor \( \varepsilon_{\mu\nu}, \varepsilon^\nu_{\mu}, \varepsilon^{\mu\nu} \) and Deformation Energy \( W \)

From the well-known metrics \( g_{\mu\nu} (5.1) \) and \( \eta_{\mu\nu} (5.2) \), applying formula 4.19

\[ \varepsilon_{\mu\nu} = \frac{g_{\mu\nu} - \eta_{\mu\nu}}{2} \]
The non-zero components of the strain tensor in covariant form are
\[
\varepsilon_{rr} = \frac{f - w'^2}{2f} \quad (5.4)
\]
\[
\varepsilon_{\theta\theta} = \frac{r^2 - w^2}{2} \quad (5.5)
\]
\[
\varepsilon_{\varphi\varphi} = \frac{r^2 - w^2}{2r^2} \sin^2 \vartheta \quad (5.6)
\]
To raise one subscript of the strain tensor to obtain the mixed form \(\varepsilon^\nu_\mu\), I shall use the inverse metric \(g^{\mu\nu}\), which, being diagonal, is simply inverted into
\[
g^{\mu\nu} = \begin{pmatrix}
\frac{1}{f} & 0 & 0 \\
0 & \frac{1}{r^2} & 0 \\
0 & 0 & \frac{1}{r^2 \sin^2 \vartheta}
\end{pmatrix} \quad (5.7)
\]
and
\[
\varepsilon^r_r = \frac{f - w'^2}{2f} \quad (5.8)
\]
\[
\varepsilon^\theta_\theta = \frac{r^2 - w^2}{2r^2} = \varepsilon^\varphi_\varphi \quad (5.9)
\]
In this way, the scalar strain \(\varepsilon = \varepsilon^\alpha_\alpha = \varepsilon^r_r + \varepsilon^\theta_\theta + \varepsilon^\varphi_\varphi\) is immediately built:
\[
\varepsilon = \frac{f - w'^2}{2f} + \frac{r^2 - w^2}{r^2} \quad (5.10)
\]
Applying again the inverse metric \(g^{\mu\nu}\) to \(\varepsilon^\nu_\mu\), \(\varepsilon^{\mu\nu}\) is
\[
\varepsilon^{rr} = \frac{f - w'^2}{2f} \quad (5.11)
\]
\[
\varepsilon^{\theta\theta} = \frac{r^2 - w^2}{2r^2} \quad (5.12)
\]
\[
\varepsilon^{\varphi\varphi} = \frac{r^2 - w^2}{2r^4 \sin^2 \vartheta} \quad (5.13)
\]
and so we can compute the second degree scalar
\[
\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = \left(\frac{f - w'^2}{4f^2}\right)^2 + \frac{(r^2 - w^2)^2}{2r^4} \quad (5.14)
\]
The deformation energy is then obtained by casting these results into 4.24
\[
W = \frac{1}{2} \lambda e^2 + \mu \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}
\]
Eventually, the determinant of the metric yields

$$\sqrt{|g|} = \sqrt{fr^2 \sin \vartheta}$$  \hspace{1cm} (5.15)

The Lagrangian density is finally

$$\frac{L}{\sin \vartheta} = \left(\frac{2rf' + 2f^2 - 2f}{r^2f^2} + \frac{1}{2} \lambda \left(\frac{f - w'^2}{2f} + \frac{r^2 - w^2}{r^2}\right)^2 + \mu \left(\frac{(f - w'^2)^2}{4f^2} + \frac{(r^2 - w^2)^2}{2r^4}\right)\right) \sqrt{fr^2}$$  \hspace{1cm} (5.16)

### 5.3 Lagrange Equations

The same equations already derived in chapter 3, the so-called Lagrange equations 3.5, may be completely recovered here, with only one sensible change. The role of time is assigned to the only independent variable available, i.e. the radius $r$. Comparing this choice with what I stated when talking about Cosmology, where the cosmic time is a length, nothing prevents me from using a length in place of the ‘absolute’ time. The following is then a corollary of theorem 7 on the ordinary Lagrange equations.

**Corollary 1** (Lagrange equations). The differential equations of motion of the problem described above, where the Lagrangian density $L$ is expressed by 5.16 are

$$\frac{d}{dr} \frac{\partial L}{\partial f'} - \frac{\partial L}{\partial f} = 0$$

$$\frac{d}{dr} \frac{\partial L}{\partial w'} - \frac{\partial L}{\partial w} = 0$$  \hspace{1cm} (5.17)

where $'$ denotes derivation with respect to $r$.

Applying this corollary to the Lagrangian density 5.16, and after long computations, the Lagrange equations describing the displacement of the points of the deformed sphere (or also the strain field of the material), expressed in terms of the functions $f$ and $w$, are

$$\left(\frac{1}{2} \lambda + \mu\right) \frac{3}{8} r^2 (w')^4 + \left(w^2 \lambda - r^2 \mu - \frac{3}{2} r^2 \lambda\right) \frac{1}{4} f (w')^2 +$$

$$- (\lambda + \mu) \frac{1}{4} f^2 w^4 + \left(\frac{3}{2} \lambda + \mu\right) \frac{1}{2} f^2 w^2 - \left(\frac{9}{16} \lambda + \mu\right) r^2 + 1 \right) f^2 + f = 0$$  \hspace{1cm} (5.18)
and
\[
(w^2 \lambda - r^2 \mu - \frac{3}{2} r^2 \lambda + \frac{3}{2} f r^2 (w')^2 \lambda + \frac{3}{f} r^2 (w')^2 \mu) w''
\]
\[
+ \left( \lambda + 2 \mu - \frac{3}{4} f r (f') \lambda - \frac{3}{2} f r (f') \mu \right) r \frac{(w')^3}{f}
\]
\[
+w \lambda (w')^2 + \left( \frac{3}{4} f r^2 (f') \lambda - 2 r \mu - 3 r \lambda + \frac{1}{2} f r^2 (f') \mu - \frac{1}{2} f w^2 (f') \lambda \right) (w')
\]
\[- (\lambda + \mu) 2 \frac{f}{r^2} w^3 + (3 \lambda + 2 \mu) f w = 0 \quad (5.19)
\]

5.3.1 Linearization of the Lagrange Equations

As it is easily understood, the system of two equations 5.18 and 5.19 above is highly non-linear. In practice, this means that it will be very difficult (if not impossible!) to determine at least one solution, if any solution exists, of course. Consequently, as a first step of my discussion, I would rather show how 5.18 and 5.19 could be linearized using perturbation methods.

I will not give a deep insight into perturbation theory, but the following ideas are enough for our purpose. Recall that, for the spherical symmetry, the length of the line element on the reference manifold is 5.2
\[
ds^2_r = w^2(r) \, dr^2 + w^2(r) \left( d\theta^2 + \sin^2 \vartheta \, d\varphi^2 \right)
\]
However, when stresses are relaxed, the elastic material goes back to its unperturbed state, and the line element of the unstrained state obviously returns to the well-known flat 4.3
\[
ds^2_{r_0} = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \vartheta \, d\varphi^2
\]
or, in other words, \( w \) assumes a particular value, i.e.
\[
w(r) = r
\]
This solution is called of order ‘0’. It has a simple meaning: the solution of order ‘0’ is the value assumed by \( w \) when the manifold is undeformend, and it must identically satisfy the equations of motion 5.18 and 5.19 together with the order ‘0’ solution
\[
f = 1
\]
whose value is simply obtained by comparing 5.1 and 4.3.

It is plausible to suppose that the values of $f$ and $w$ that satisfy the strained solid equations are of the following form

$$f = 1 + o(r)$$

$$w = r \left(1 + o(r)\right)$$

where $o(r)$ contains terms of higher order, i.e. with $r$ appearing in any power, but negligible with respect to the solution of order ‘0’: we could think of terms with $r$ figuring at the denominator only. However, not all the negligible terms are to be neglected: in fact they represent the so-called perturbation, from which this method borrows its name.

Moreover, the negligible terms $o(r)$ might be collected into a set of arbitrary infinitesimal functions. For $w(r)$, let me denote this set by $\chi_i(r,\lambda,\mu)$, where $\lambda$ and $\mu$ are only arbitrary parameters. Then we can write

$$w = r \left(1 + o(r)\right) = r \left(1 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \varepsilon^3 \chi_3 + \ldots\right)$$

where the perturbation term $o(r)$ was reorganized into several (actually infinite) perturbations: $\varepsilon$ is simply a small number inserted into the expansion for simplicity, so as to indicate the order of each term in the solution. Analogously

$$f = 1 + o(r) = 1 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \ldots$$

Here I will stop at order ‘1’, which means replacing into equations 5.18 and 5.19 the proposed functions

$$f = 1 + \varepsilon \phi$$

$$f' = \varepsilon \phi'$$

$$w = ar (1 + \varepsilon \chi)$$

$$w' = a (1 + \varepsilon \chi) + a r \varepsilon \chi'$$

$$w'' = 2 \varepsilon a \chi' + a r \varepsilon \chi''$$

where the subscript of the functions $\phi$ and $\chi$ is omitted for obvious reasons, and $a$ is a real number which may be interpreted as a scale factor for $w$. Casting these values into the Lagrange equations of the problem, expanding and sorting out the
terms where $\varepsilon$ appears in powers greater than 1, the resulting equations in $\phi$ and $\chi$ are linear ($\varepsilon$ is set equal to 1 in the end).

\[
\begin{align*}
&\left(\frac{5}{2}a^2\lambda - \frac{3}{2}\lambda + 3a^2\mu\right) \frac{1}{2}a^2r^3 (\chi') + \left(\frac{3}{2}\lambda + \mu + \frac{3}{2}a^2\lambda + a^2\mu\right) \frac{1}{2}a^2r^2\chi \\
&\quad + \left[\left(\frac{3}{2}a^2\lambda - \frac{3}{4}\mu - \frac{9}{8}\lambda + a^2\mu - \frac{7}{8}a^4\lambda - \frac{5}{4}a^4\mu\right) \frac{1}{2}r^2 - 1\right] \phi \\
&\quad + \left(\frac{3}{2}a^2\lambda - \frac{3}{2}\mu - \frac{9}{4}\lambda + a^2\mu + \frac{3}{4}a^4\lambda + \frac{1}{2}a^4\mu\right) \frac{1}{4}r^2 = 0 \quad (5.22)
\end{align*}
\]

\[
\begin{align*}
r^2\chi'' + 4r\chi' - \frac{1}{2}r\phi' + \frac{\left(3a^2\lambda - 2\mu - 3\lambda + 4a^2\mu\right)}{\left(\frac{3}{2}\lambda - 3a^2\mu - \frac{5}{2}a^2\lambda + \mu\right)} \phi = 0 \quad (5.23)
\end{align*}
\]
5.4 Proof of the Theory: Extremization of the Lagrangian

Eventually, I will show that the SST accounts also for problems involving three-dimensional material continua, like those solved in section 2.6. Those are problems where the symmetry simplifies the computations a lot: the spherical case, for instance, involved a spherical cavity immersed in an infinite medium. Solutions were fully illustrated, and the geometry of the problem was completely described by the displacement function. Here the job becomes much more complicated: first of all, it is because the theory of elasticity is linear, whereas this is not. Other reasons will be illustrated and clarified in the following section. Here, instead, a proof for my thesis will be formulated.

First, two ideas were devised to try to prove the fact. Directly solving equations 5.18 and 5.19 would be impossible, so one could try to solve the linearized equations 5.22 and 5.23, and compare the results obtained with the exact solutions previously computed in the theory of elasticity (at page 45). Or, one could take the exact results, and cast them into equations 5.18 and 5.19 to see that they are identically satisfied. Unfortunately, both of these methods do not give satisfactory responses, so I have to abandon them; at least for the moment, since afterwards this failure in the proof will be accurately analysed.

The proof I will carry out makes use of the Lagrangian density only, and not of the equations: so, it is also simpler that I could have expected. Recall the theory on Lagrangian mechanics, in particular Hamilton’s principle 3.4. In the case of a field theory, using 3.10, it can be restated as

\[
\delta S = \delta \left( \int_{\Omega^{(3)}} \mathcal{L} \sqrt{|g|} \, d^3x \right) = \delta \left( \int_{\Omega^{(3)}} L \, d^3x \right) = 0 \tag{5.24}
\]

where the meaning of all symbols is clear.

Now, Hamilton’s principle 5.24 states that the action must be extremal (a minimum in general, but not always so), and from this, one usually derives the Lagrange equations 5.18 and 5.19. However, I wish to skip this part because unfruitful, as I have already mentioned, and adopt the following method: 5.24 involves that in order for \( \delta S \) to be zero, the Lagrangian density \( L \) must necessarily be extremal.
This is why Hamilton’s principle will be restated as

$$\delta L = 0 \quad (5.25)$$

which means that when we effect the variation of the Lagrangian density we shall find zero. One last point needs to be defined: when does (5.25) hold, i.e. for which values of $w$ and $f$? This question is tantamount to asking from where $w$ and $f$ are being derived, provided that I am not using the Lagrange equations. Well, the only choice I could successfully think of is to use the exact solutions yielded by the classical elasticity. In particular, (2.57) and (2.58) express the components $\varepsilon^r_r$, $\varepsilon^\vartheta_\vartheta$ and $\varepsilon^\varphi_\varphi$ of the strain tensor respectively

$$\varepsilon^r_r = \frac{pb^3}{2\mu} \frac{1}{r^3},$$

$$\varepsilon^\vartheta_\vartheta = \varepsilon^\varphi_\varphi = -\frac{pb^3}{4\mu} \frac{1}{r^3}.$$

(the radius of the cavity is indicated by $b$ instead of $R$ in order not to create confusion with the scalar curvature)

These components are to be expressed in covariant form. To do so we use the metric of the unstrained solid, which is the flat metric in spherical coordinates 4.4. Using the metric of the natural manifold would be a mistake, because solutions (2.57) and (2.58) were find inside the classical theory of elasticity, where the manifold does not participate in the deformation of the body. Here stands the great difference between the classical and cosmological relativistic approaches to the problem I wish to tackle. In classical elasticity, one chooses a set of coordinates to describe the solid body and observes the deformation from an external point of view, since the space surrounding the body is not affected by the deformation. So the metric always remains the flat one anyhow: this is how I proceeded in section 2.6 where the metric tensor was in fact not even mentioned. In the cosmological interpretation the manifold, whose curvature is an intrinsic property, is all we have and all we see, and the embedding space is only an abstract mathematical entity. Imagine we live on the manifold, and when it curves we perceive it from the inside: the subsequent description of our world after deformation is unavoidably affected by the metric attained by our space. Hence, the metric to use is always the one of the natural manifold.
In this spirit, using 4.4, 2.57 and 2.58 become

\[ \varepsilon_{rr} = \frac{pb^3}{2\mu r^3} \]

\[ \varepsilon_{\vartheta\vartheta} = \frac{\varepsilon_{\varphi\varphi}}{\sin^2 \vartheta} = -\frac{pb^3}{4\mu r} \]

Comparing these values with 5.4 and 5.5

\[ \varepsilon_{rr} = \frac{f - w'^2}{2} \]

\[ \varepsilon_{\vartheta\vartheta} = \frac{r^2 - w^2}{2} \]

we get

\[ w = \sqrt{\frac{2r^3 \mu + pb^3}{2\mu r}} \tag{5.26} \]

\[ f = \frac{8pb^3 r^3 \mu + 9p^2 b^6 + 16r^6 \mu^2}{8\mu r^3 (2r^3 \mu + pb^3)} \tag{5.27} \]

In practice, I have rewritten the exact solutions in terms of \( f \) and \( w \). With these values the Lagrangian density 5.16 becomes

\[ \mathcal{L}(r) = \frac{3}{8} b^6 \frac{p^2}{r^4 \mu^2} \sqrt{\frac{8pb^3 r^3 \mu + 16r^6 \mu^2}{(9b^3 p^2 + 8b^3 pr^3 \mu + 16r^6 \mu^2)^2}} \]

\[ (27b^12 p^4 \mu + 27 \lambda b^{12} p^4 + 48b^9 p^3 r^3 \mu^2 + 160b^6 p^2 r^6 \mu^3 + 432b^6 p^3 r^4 \mu^2 + 256b^3 pr^9 \mu^4 - 768b^3 pr^7 \mu^3 + 256r^{12} \mu^5 - 3840r^{10} \mu^4) \]

I will now effect the variation on \( \mathcal{L}(r) \), so as to find that \( \delta \mathcal{L}(r) = 0 \).

\[ \diamond \textbf{Remark}: \text{ equation } 5.25, \text{ i.e. } \delta \mathcal{L} = 0, \text{ holds up to higher order terms. This fact is} \]

\[ \diamond \text{ apparent if we consider that the SST is not linear, but the values for } f \text{ and } w \text{ I cast} \]

\[ \diamond \text{ inside are obtained from a linear theory. Thus, } \delta \mathcal{L} \text{ must be } 0 \text{ only until first order} \]

\[ \diamond \text{ terms. In conclusion, let me reformulate } 5.25 \text{ as} \]

\[ \delta \mathcal{L} = \sum_{n>1} c_n \alpha^n(r, \lambda, \mu, b) \tag{5.28} \]

where \( \alpha^n(r, \lambda, \mu, b) > 0 \) is a set of infinitesimal functions, and \( c_n \) only a constant. In this way, it is also proved that any perturbation introduced in the Lagrangian density
functional leads to bigger functionals, thus reinforcing the fact that the Lagrangian is a minimum (and so an extremum).

The variation will now be effected on the strain tensor in covariant form $\varepsilon_{\mu\nu}$. In the most general case, the variation is represented by a tensor, say $\alpha_{\mu\nu}(r)$, which sums to $\varepsilon_{\mu\nu}(r)$. However, the elements off the diagonal are always zero because of symmetry reasons, and $\varepsilon_{\vartheta\vartheta}$ and $\varepsilon_{\varphi\varphi}$ differ only by the factor $\sin^2 \vartheta$ which does not vary because it is not function of $r$. Then,

$$
\alpha_{\mu\nu}(r) = \begin{pmatrix}
\alpha_{rr}(r) & 0 & 0 \\
0 & \alpha_{\vartheta\vartheta}(r) & 0 \\
0 & 0 & \alpha_{\varphi\varphi}(r) \sin^2 \vartheta
\end{pmatrix}
$$

(5.29)

Hence, I will consider the variation of the Lagrangian only in the two cases $\alpha_{rr}(r)$ and $\alpha_{\vartheta\vartheta}(r)$.

1. Set $\alpha_{rr}(r) = \alpha(r) \neq 0$ and $\alpha_{\vartheta\vartheta}(r) = 0$. Proceeding as before we have

$$
\frac{r^2 - w^2}{2} = \frac{pb^3}{4\mu} \frac{1}{r} \quad \text{and} \quad \frac{f - w^2}{2} = \frac{pb^3}{2\mu} \frac{1}{r^3} + \alpha(r)
$$

leading to

$$
f = \left( \frac{8pb^3 r^3 \mu + 9p^2 b^6 + 16r^6 \mu^2 + 32\alpha(r) \mu^2 r^6 + 16\alpha(r) \mu r^3 pb^3}{8\mu r^3 (2r^3 \mu + pb^3)} \right)
$$

$$
w = \sqrt{\left( \frac{2r^3 \mu + pb^3}{2\mu r} \right)^2}
$$

Plugging this values into $\bar{L}(r)$, and expanding with respect to powers of $\alpha(r)$, I found that 5.28 perfectly holds.

2. Set $\alpha_{rr}(r) = 0$ and $\alpha_{\vartheta\vartheta}(r) = \alpha(r) \neq 0$. Repeating the same procedure we get

$$
f = \left( \frac{8pb^3 r^3 \mu + 9p^2 b^6 + 16r^6 \mu^2 + 16\alpha(r) \mu^3 pb^3 + 64\alpha(r) \mu^2 r^6 + 32r^7 \mu^2 \alpha'(r) - 8pb^3 r^4 \alpha'(r)}{8\mu r^3 (2r^3 \mu + pb^3 + 4\alpha(r) \mu r^3)} \right)
$$

$$
w = \sqrt{\left( \frac{2r^3 \mu + pb^3 + 4\alpha(r) \mu r^3}{2\mu r} \right)^2}
$$

Again, inserting these perturbed solutions into $\bar{L}(r)$, 5.28 holds. Then, it is true that the two perturbations lead to a bigger Lagrangian (see also figure 5.5)

Finally, I managed to prove that the SSC accounts also for solutions of problems regarding three-dimensional continua typical of the classical theory of elasticity.
5.5 The Issue of the Lagrange Equations

The SST gives birth to equations that describe several phenomena, among which stands the problem I am discussing here. Recall that when applied to space-time, this theory provides the famous Hubble parameter 4.35. On the other hand, the result I obtained in the previous section proves the prediction I made, but the path followed to get here was not smooth. In particular one big issue remains and it is still a little astonishing: neither the Lagrange equations do have the expected behaviour, nor their linearization yields promising solutions. Why does it happen? Here I will try to give further explanations.

Firstly, as I have already proposed, the (non-)linearity plays an important role. Equations 5.18 and 5.19 are surely more general than the Navier-Cauchy equations 2.46, from which formulas 2.56 to 2.60 were found. Albeit this is undoubtedly true, it should weigh more on the attempt to solve the equations rather than on satisfying the general equations with known (even if linear) solutions: in fact the linear solutions should figure among the numerous other solutions. But they do not. Then, another reason can be suggested.

The approach I followed after building the Lagrangian density is defined as ‘point Lagrangian’. In other words, the Lagrangian density 5.16 is the most general possibility for a Lagrangian: actually, the same cannot be stated for the Lagrange equations 5.18 and 5.19 subsequently derived. This is because, in order to derive the equations I imposed the constraint of using a particular line element, obeying to the spherical symmetry. This greatly simplifies the computations (even though they remain quite long and intricate!), but leads to the inevitable loss of solutions. And the solutions I was looking for might have been gone! In the most general case the metric tensor contains six independent parameters. They are only reduced by one since the scalar curvature $R$ establishes a relation among the elements of the metric. Hence, five variables appear in the most general equations, whereas I only had one, $f(r)$.

By the way, the most general case involves very long computations, which are not needed to prove my thesis: in fact, the extremization of the Lagrangian is indeed enough. To reinforce the validity of the use of the Lagrangian density only, I am showing that it passes some tests and agrees with observational data. In other words
we expect that the curvature and the deformation energy decrease as we go farther from the point of maximum stress, where the loading is applied (i.e. at the radius $b$ of the hollow sphere, taking the problem of the spherical cavity as an example). Choosing steel as material we have for example that

$$\mu = 76\,923 \text{ GPa} \quad \lambda = 115\,385 \text{ GPa}$$

Moreover, set the radius of the cavity $b = 1 \text{ m}$ and the pressure $p = 10 \text{ bar} = 10^6 \text{ Pa}$. Then, plotting the curvature and the Lagrangian density I obtained the graphs of figures 5.4 and 5.5 (red line) respectively, accordingly to what I expected. In 5.5 the green line is a perturbation of the Lagrangian density, which proves to be always bigger than the unperturbed one.

Figure 5.4. Variation of curvature with the radius
5.6 The Cylindrical Case

This theory holds also for the problem involving the cylindrical cavity, explored in section 2.6.2. Here I am quickly reporting some important results, since the procedure is exactly the same. The metric of a cylindrical manifold in its unstrained state is

\[ ds_0^2 = dr^2 + r^2d\theta^2 + dz^2 \]

(5.30)
Then, I shall write down the metrics of the natural and the reference manifold respectively:

\[ ds^2 = f(r) \, dr^2 + r^2 \, d\theta^2 + dz^2 \]  
\[ g_{\mu\nu} = \begin{pmatrix} f(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ ds_f^2 = w^2(r) \, dr^2 + w^2(r) \, d\theta^2 + dz^2 \]  
\[ \eta_{\mu\nu} = \begin{pmatrix} w^2(r) & 0 & 0 \\ 0 & w^2(r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

The curvature of the strained manifold, again computed with the aid of Maxima, is

\[ R = \frac{f'}{rf^2} \]  

The strain tensor and the second-order scalars needed to compute the Lagrangian density are

\[ \varepsilon_{rr} = \frac{f - w'^2}{2} \]  
\[ \varepsilon_{\theta\theta} = \frac{r^2 - w^2}{2} \]  
\[ \varepsilon_{zz} = 0 \]  
\[ \varepsilon^r = \frac{f - w'^2}{2f} \]  
\[ \varepsilon^\theta = \frac{r^2 - w^2}{2r^2} \]  
\[ \varepsilon = \frac{f - w'^2}{2f} + \frac{r^2 - w^2}{2r^2} \]  
\[ \varepsilon^{rr} = \frac{f - w'^2}{2f^2} \]  
\[ \varepsilon^{\theta\theta} = \frac{r^2 - w^2}{2r^4} \]
\[ \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} = \left( \frac{f - w'}{f^2} \right)^2 + \left( \frac{r^2 - w^2}{4r^4} \right)^2 \] (5.42)

And the Lagrangian density is:

\[ L = \left( \frac{f'}{ff} + \frac{1}{2} \lambda \left( \frac{f - w'^2}{2f} + \frac{r^2 - w^2}{2r^2} \right)^2 + \mu \left( \frac{(f - w'^2)^2}{4f^2} + \frac{(r^2 - w^2)^2}{4r^4} \right) \right)^2 \sqrt{fr} \] (5.43)

In the same spirit as for the spherical case, we shall cast into 5.43 the exact solutions from section 2.6.2, i.e.

\[ f = \frac{pb^2r^2\mu + p^2b^4 + r^4\mu^2}{\mu r^2(r^2\mu + pb^2)} \] (5.44)

\[ w = \sqrt{\frac{r^2\mu + pb^2}{\mu}} \] (5.45)

\[ L = \frac{1}{8} b^4 p^2 \frac{1}{r^3 \mu^2} \sqrt{\frac{1}{b^2 p^2 + b^2 p r^2 \mu + r^4 \mu^2}} \left( \frac{b^2 p^2 + b^2 p r^2 \mu + r^4 \mu^2}{(b^2 p^2 + b^2 p r^2 \mu + r^4 \mu^2)^2} \right)^2 \left( 2b^8 p^4 \mu + \lambda b^8 p^4 + 4b^6 p^3 r^2 \mu^2 + 8b^4 p^2 r^4 \mu^3 + 8b^2 p r^6 \mu^4 - 16b^2 p r^4 \mu^3 + 4r^8 \mu^5 - 32r^6 \mu^4 \right) \]

Applying the perturbation method the predictions are verified.

And for the perturbation on \( \varepsilon_{r} \), we have

\[ f = \left( \frac{pb^2 r^2 \mu + p^2 b^4 + r^4 \mu^2 + 2\alpha(r) r^4 \mu^2 + 2\alpha(r) r^2 \mu pb^2}{\mu r^2(r^2 \mu + pb^2)} \right) \]

\[ w = \sqrt{\left( \frac{r^2 \mu + pb^2}{\mu} \right)} \]

For the perturbation on \( \varepsilon_{\theta} \), we have

\[ f = \left( \frac{pb^2 r^2 \mu + p^2 b^4 + r^4 \mu^2 + 4\alpha(r) r^4 \mu^2 + 2r^5 \mu^2 \frac{\partial \alpha(r)}{\partial r} + 2\alpha(r) r^2 \mu pb^2}{\mu r^2(r^2 \mu + pb^2 + 2\alpha(r) r^2 \mu)} \right) \]

\[ w = \sqrt{\left( \frac{r^2 \mu + pb^2 + 2\alpha(r) r^2 \mu}{\mu} \right)} \]
Conclusion

Many different topics, belonging to different branches of physics, were used to fulfil my purpose. They range from classical physics to the modern Einstein’s relativity, and forth to newly proposed cosmological theories. I myself had to learn, and in a few cases only review, much of this knowledge. However, I tried to put together as much material as possible in order to make my treatise accessible to students, like me before undertaking this ‘research’, who are neither acquainted with, but interested in relativistic physics and cosmology, nor are they familiar with the burden their mathematical language represents. Nonetheless, I believe that the mathematics lying underneath all these pages is the glue which keeps together the great variety of subjects treated, establishing a continuity from the beginning to the end.

I intended to ‘kick off’ with an introduction on tensor analysis, to give a good and satisfactory idea of the whole field, and to affirm that it is a matter of great importance which we cannot do without. I did not pretend to teach, but just to explain a series of notions that can be exercised and broadened in a proper book on tensors: the bibliography I consulted is vast, and some of the texts are referenced in the last pages (I even followed a course by professor Tartaglia, named ‘Tensor fields in physics’). Then, the following chapters increasingly show that tensors are tools of fundamental importance: for instance, general relativity could not be formulated otherwise, thus my thesis absolutely needs them to be fully understood and proved.

On the other hand I appreciate the theory I worked on. The ideas of the Strained State Cosmology, which I learnt from the articles by Tartaglia and his collaborators, has proved to be really interesting to me, as well as stimulating on scientific grounds. It is admirable for the fact that it strictly follows the Occam’s razor, being a theory which does not require to devise new entities to explain the unexplainable. The description of space-time as a material continuum is arguable, but indeed original.
Finally, my contribution to this scientific field is of course meagre, as it should appear from a reading of my work. Nonetheless, it covers the function to prove the generality carried by the Strained State Theory. Even though minimal, my activity on the problem was neither easy nor quick. Partly because I was, and I am, not mastering the subject, and partly since critical issues showed up during the computations. All these complications are not reported in the treatise, but I must say that they took much of my time during the production of the thesis. The part I eventually reckoned to write is just the final step of my tough job, which is indeed the richer one, other than being correct.
Appendix A

Differential Operators

A.1 Cartesian Coordinates \((x,y,z)\)

Given \(f : \mathbb{R}^3 \to \mathbb{R}\), \(f : (x,y,z) \mapsto f(x,y,z)\), and \(F : \mathbb{R}^3 \to \mathbb{R}^3\), \(F : (x,y,z) \mapsto F(x,y,z) = F_x(x,y,z)e_x + F_y(x,y,z)e_y + F_z(x,y,z)e_z\):

\[
\text{grad } f = \nabla f = \frac{\partial f}{\partial x}e_x + \frac{\partial f}{\partial y}e_y + \frac{\partial f}{\partial z}e_z
\]

\[
\text{div } F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

\[
\text{curl } F = \nabla \times F = \begin{vmatrix}
    e_x & e_y & e_z \\
    \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
    F_x & F_y & F_z
\end{vmatrix}
\]

\[
\Delta f = \nabla^2 f = \nabla \cdot \nabla = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]

A.2 Spherical Polar Coordinates \((r,\vartheta,\varphi)\)

Given \(f : \mathbb{R}^3 \to \mathbb{R}\), \(f : (r,\vartheta,\varphi) \mapsto f(r,\vartheta,\varphi)\), and \(F : \mathbb{R}^3 \to \mathbb{R}^3\), \(F : (r,\vartheta,\varphi) \mapsto F(r,\vartheta,\varphi) = F_r(r,\vartheta,\varphi)e_r + F_\vartheta(r,\vartheta,\varphi)e_\vartheta + F_\varphi(r,\vartheta,\varphi)e_\varphi\):
A – Differential Operators

Figure A.1. Spherical coordinate system \((r, \vartheta, \varphi)\)

\[
x = r \sin \vartheta \cos \varphi \\
y = r \sin \vartheta \sin \varphi \\
z = r \cos \vartheta
\]

\[
\text{grad } f = \nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \vartheta} e_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial f}{\partial \varphi} e_\varphi
\]

\[
\text{div } F = \nabla \cdot F = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (F_\vartheta \sin \vartheta) + \frac{1}{r \sin \vartheta} \frac{\partial F_\varphi}{\partial \varphi}
\]

\[
\text{curl } F = \nabla \times F = \frac{1}{r \sin \vartheta} \left( \frac{\partial}{\partial \vartheta} (F_\varphi \sin \vartheta) - \frac{\partial F_\vartheta}{\partial \varphi} \right) e_r + \frac{1}{r} \left( \frac{1}{\sin \vartheta} \frac{\partial F_r}{\partial \vartheta} - \frac{\partial F_\varphi}{\partial r} \right) e_\vartheta + \frac{1}{r \sin \vartheta} \left( \frac{\partial}{\partial r} (r F_\varphi) - \frac{\partial F_\vartheta}{\partial \varphi} \right) e_\varphi
\]

\[
\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial f}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 f}{\partial \varphi^2}
\]

A.3 Cylindrical Coordinates \((r, \vartheta, z)\)

Given \(f : \mathbb{R}^3 \to \mathbb{R}, f : (r, \vartheta, z) \mapsto f(r, \vartheta, z)\), and \(F : \mathbb{R}^3 \to \mathbb{R}^3, F : (r, \vartheta, z) \mapsto F(r, \vartheta, z) = F_r(r, \vartheta, z)e_r + F_\vartheta(r, \vartheta, z)e_\vartheta + F_z(r, \vartheta, z)e_z\):

\[
x = r \cos \vartheta \\
y = r \sin \vartheta \\
z = z
\]

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Figure A.2. Cylindrical coordinate system \((r, \theta, z)\)

\[
\text{grad } f = \nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z
\]

\[
\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial F_r}{\partial r} \right) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}
\]

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \mathbf{e}_\theta + \frac{\partial}{r} \left( \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \mathbf{e}_z
\]

\[
\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\]
Appendix B

Conversion Formulas for Elastic Parameters

Given:

- $\lambda$: Lamé’s first parameter;
- $\mu$: Lamé’s second parameter or shear modulus;
- $E$: Young’s modulus;
- $\nu$: Poisson’s ratio;
- $K$: bulk modulus.

In terms of $\lambda$ and $\mu$:

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad \nu = \frac{\lambda}{2(\lambda+\mu)} \quad K = \lambda + \frac{2\mu}{3}$$

In terms of $E$ and $\nu$:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)} \quad K = \frac{E}{3(1-2\nu)}$$

In terms of $E$ and $\mu$:

$$\lambda = \frac{\mu(E-2\mu)}{4\mu-E} \quad \nu = \frac{E}{2\mu} - 1 \quad K = \frac{E\mu}{3(3\mu-E)}$$

In terms of $K$ and $\lambda$:

$$E = \frac{9K(K-\lambda)}{3K-\lambda} \quad \mu = \frac{3(K-\lambda)}{2} \quad \nu = \frac{\lambda}{3K-\lambda}$$
In terms of $K$ and $\mu$:

$$E = \frac{9K\mu}{3K+\mu} \quad \lambda = K - \frac{2\mu}{3} \quad \nu = \frac{3K-2\mu}{2(3K+\mu)}$$

In terms of $\lambda$ and $\nu$:

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} \quad \mu = \frac{\lambda(1-2\nu)}{2\nu} \quad K = \frac{\lambda(1+\nu)}{3\nu}$$

In terms of $\mu$ and $\nu$:

$$E = 2\mu(1 + \nu) \quad \lambda = \frac{2\mu\nu}{1-2\nu} \quad K = \frac{2\mu(1+\nu)}{3(1-2\nu)}$$

In terms of $K$ and $\nu$:

$$E = 3K(1 - 2\nu) \quad \lambda = \frac{3K\nu}{1+\nu} \quad \mu = \frac{3K(1-2\nu)}{2(1+\nu)}$$

In terms of $K$ and $E$:

$$\lambda = \frac{3K(3K-E)}{9K-E} \quad \mu = \frac{3KE}{9K-E} \quad \nu = \frac{3K-E}{6K}$$
Appendix C

Theorems

**Definition 13** (Homogeneous function). A function \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) is positive homogeneous of degree \( k \) iff
\[
f(\alpha x) = \alpha^k f(x)
\]
for all \( \alpha > 0 \), and \( k \in \mathbb{C} \)

**Theorem 8** (Euler’s). Let \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a continuously differentiable function. Then \( f \) is positive homogeneous of degree \( k \) iff
\[
x \cdot \nabla f(x) = x_i \frac{\partial f(x)}{\partial x_i} = kf(x)
\]
where Einstein’s summation convention has been used.

**Proof.** It suffices to differentiate both sides of the equality in definition 13 with respect to \( \alpha \) to get:
\[
k\alpha^{k-1} f(x) = \frac{\partial f(x)}{\partial x'} \cdot \frac{\partial x'}{\partial \alpha}
\]
where \( x' = \alpha x \). Since
\[
\frac{\partial x'}{\partial \alpha} = x
\]
setting \( \alpha = 1 \) and \( \frac{\partial f}{\partial x} = \nabla f(x) \) we get
\[
x \cdot \nabla f(x) = kf(x)
\]
\( \square \)


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5.1 Euclidean and non-Euclidean subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

5.2 Various possible strategies of deformation of a flat solid continuum

5.3 Various possible strategies of deformation of a flat solid continuum

5.4 Variation of curvature with the radius

5.5 Variation of the Lagrangian density with the radius (red) and perturbation (green)

A.1 Spherical coordinate system $(r, \vartheta, \varphi)$

A.2 Cylindrical coordinate system $(r, \vartheta, z)$