Dynamics of Hermitian and Quasi-Hermitian Open Systems

Tutor
Prof. Luigi Solombrino

Ph.D. Student
Fabio Masillo

XXIII Course
To Fabiola
Foreword

Standard quantum mechanics is a well defined physical theory that is confirmed by innumerable experiments. Its standard application is related to closed quantum systems, i.e. systems that does not interact with the external environment. This last condition, useful for the understanding of quantum mechanical laws, appear too restrictive for physical applications. Open Quantum Dynamics is the theory that deals with systems interacting with an external (hidden to the observer) environment.

This thesis is devoted to the study of some, physically interesting, aspects of this theory that appear in the standard quantum mechanics as well as in some alternative descriptions. In particular, we will give special attention to pseudo and quasi-Hermitian systems, whose study has attracted many researchers in the last two decades.

Quantum systems will be described in the thesis by means of their density density operators; hence, their evolution will be associated with a map in the space of positive, (quasi-)Hermitian operators acting in the Hilbert space.

The main definitions and theorems, needed to deal with open systems, are included in chapter 1 and 2 to make the treatment, as possible, self contained. Chapters 3÷6 are mainly based on original results obtained in the course of the doctoral studies, and (almost all) already published on scientific journals.

Chapter 3, is devoted to the study of a consistent descriptions of multipartite quantum systems in pseudo Hermitian quantum mechanics. Here, we also present a solution for the inverse problem, i.e., we give a necessary and sufficient kinematical condition for compound quantum systems provided with alternative inner products to be described in terms of $k$ subsystems. In the last sections we propose some methods to characterize the dynamics ruled by quasi-Hermitian Hamiltonians associated with multipartite compound quantum systems.

The Quantum Brachistochrone Problem is an interesting application connecting the theory of open quantum systems and pseudo Hermitian operators. In particular in chapter 4 we try to clarify some different view points concerning the use of pseudo Hermitian Hamiltonian in order to realize ultra-
fast quantum computation, and show that the ultrafast pseudo-Hermitian
evolution can be simulated as an Hermitian open dynamics. An apparent
paradox arising in its application to quantum computation is also discussed
and solved.

In chapter 5 we study general properties of dynamical maps preserving
Hermiticity and quasi-Hermiticity. In particular we see that pseudo Hermi-
tian operators naturally arise even in the context of Hermitian maps. In fact
we prove that the generator of a dynamical semigroup is always a pseudo-
Hermitian operator. Moreover, improving a previous proposal by Jakob and
Stenholm, we introduce two new Lyapunov functionals for degenerate open
systems, and apply such results to a physical example.

The last chapter introduces an alternative way to think about dynamical
maps. These last, in fact, can be considered as composition of three different
maps: assignment map, unitary evolution and partial trace. In particular
the assignment map appears the key point to understand the evolution of
a physical system in relation with its correlations with an external environ-
ment. So we study the properties of general linear assignment maps, showing
that positivity axiom can be suitably relaxed, and propose a new class of dy-
namical maps (generalized dynamics). A puzzling result, arising in such a
context in quantum information theory, is also discussed.
# Contents

Foreword i

1 Introduction 1
   1.1 General Settings ................................. 1
       1.1.1 Standard Quantum Mechanics ................. 1
       1.1.2 Density operators ............................ 3
       1.1.3 Reduced Density Operators .................. 5
   1.2 Some relevant subsets of states .................. 6
   1.3 Open dynamics .................................. 7
       1.3.1 The Kraus-Stinespring form ................. 7
       1.3.2 An axiomatic approach ...................... 8
       1.3.3 One parameter semigroup .................... 10

2 Pseudo Hermitian Quantum Mechanics 13
   2.1 Some definitions ................................ 13
   2.2 Some useful theorems ............................ 14
       2.2.1 Spectrum of a pseudo Hermitian operator .... 15
       2.2.2 Antilinear symmetries ...................... 16
   2.3 Density operators in QHQM ...................... 17

3 Alternative descriptions and multipartite quantum systems 19
   3.1 Introduction .................................... 19
   3.2 Quasi-Hermitian descriptions of multipartite systems .... 20
       3.2.1 A kinematical characterization .............. 20
       3.2.2 A dynamical characterization ............... 22
   3.3 Quasi-Hermitian Hamiltonians on \(\mathbb{C}^8\) ........ 23
       3.3.1 A nondecomposable case ..................... 23
       3.3.2 A decomposable case ....................... 25
   3.4 A further dynamical characterization .............. 26
   3.5 Complete positivity in alternative descriptions .......... 27
# Contents

4 The Quantum Brachistochrone 29
  4.1 Introduction ................................................. 29
  4.2 Quantum Brachistochrone ................................. 30
  4.3 Ultrafast Pseudo-Hermitian Quantum Dynamics(?) ........ 31
  4.4 Time dependent Pseudo-Hermitian Hamiltonian .......... 32
  4.5 PH Quantum Dynamics as Hermitian open Dynamics ....... 33
  4.6 PH Quantum Dynamics as Hermitian open Dynamics II ... 39
  4.7 Some remarks on information theory ....................... 42
    4.7.1 Computation not interpretable ....................... 42
    4.7.2 Quantum NOT-Gate ................................. 43

5 Time evolution of quasi-Hermitian open systems 47
  5.1 Introduction ................................................. 47
  5.2 Hermitian maps and pseudo-Hermiticity .................... 49
  5.3 Quasi-Hermitian maps and pseudo-Hermiticity ............ 52
  5.4 Semigroup generators and pseudo-Hermiticity ............. 54
  5.5 Characterizing quasi-Hermitian maps ....................... 55
  5.6 Monotonically changing functionals ....................... 58
  5.7 Imaginary spectrum and unitary evolution ................. 61
  5.8 A physical example ........................................ 63
  5.9 Concluding remarks ........................................ 66

6 Assignment Maps 67
  6.1 Introduction ................................................. 67
  6.2 Relaxing Linearity .......................................... 69
  6.3 Relaxing Positivity ......................................... 71
    6.3.1 Some preliminary definitions ....................... 71
    6.3.2 Non CP maps ......................................... 73
    6.3.3 Physical Realization of Reduced Dynamics .......... 74
    6.3.4 Generalized Dynamics ............................... 77
  6.4 An example ................................................ 79
  6.5 Relaxing Consistency ....................................... 81
  6.6 Holevo Bound ............................................. 83

Conclusions 87

Bibliography 93
Chapter 1

Introduction

1.1 General Settings

Section 1.1 is essentially devoted to introduce the formalism of quantum mechanics. This is substantially a resume of the main axioms and tools necessary to deal with open dynamics. Definitions and theorems are usually quoted, if not explicitly stated otherwise, from the excellent book by Nielsen and Chuang [1].

1.1.1 Standard Quantum Mechanics

In Quantum mechanics the connection between physical objects and mathematical entities is completely specified by the following axioms:

A1 Every pure state of a physical system $S$ is represented by an element $|\psi\rangle$ (usually called ket) of an Hilbert space $\mathcal{H}$ associated to the system. Conversely, every ket in $\mathcal{H}$ represent a pure state. Moreover, two kets, $|\psi_1\rangle$ and $|\psi_2\rangle$ represent the same state if and only if $|\psi_1\rangle = \alpha |\psi_2\rangle$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.

A2 Given $n$ physical systems, $S_1, S_2, \ldots, S_n$, and the corresponding associated Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$, the Hilbert space that describes the compound quantum system $S = S_1 + S_2 + \ldots + S_n$ is given by:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n.$$  \hspace{1cm} (1.1)

Clearly, these axioms are not sufficient to give a complete description of a physical system. In fact, we need to connect the mathematical objects to the results that may occur during an experiment. This is done by the following axioms:
1. Introduction

A3 Given a physical system $S$ in the pure state $|\psi\rangle$, the probability of the result $m$ in a measure, described by the collection $\{M_m\}$ of measurement operators, is given by:

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

(1.2)

The measurement operators satisfy the completeness equation:

$$\sum_m M_m^\dagger M_m = I$$

(1.3)

where $I$ is the identity operator on $\mathcal{H}$.

A4 If a measure, performed on the physical system $S$ in the pure states $|\psi\rangle$, gives a result $m$, then the states immediately after the measure is the pure state:

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}.$$  

(1.4)

Remark 1.1. The measurement operators $M_m$ are maps

$$M_m : \mathcal{H} \rightarrow \mathcal{H'},$$

(1.5)

where $\mathcal{H}$ is generally different from $\mathcal{H'}$. Such operators are preferable with respect to the usual projective measurements because they give a mathematical description of cases in which the system of interest is destroyed or modified.

Now, it remains to establish how a physical system evolves when it is not subject to a measurement process:

A5 If an isolated physical system $S$ is in the state $|\psi(t_0)\rangle$ at the instant $t_0$, then the state $|\psi(t)\rangle$ that describes the system at the time $t$ is:

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$$

(1.6)

where $U(t, t_0)$ is an appropriate unitary operator.

It is possible to define axiom A5 in the following differential form:

A5' If an isolated physical system $S$ is in the state $|\psi(t_0)\rangle$ at the instant $t_0$, then the state $|\psi(t)\rangle$ that describes the system at the time $t$ is obtained solving the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle,$$

(1.7)

where $H$ is called Hamiltonian of the system and $H = H^\dagger$. 
1.1.2 Density operators

The Hilbert space formulation of Quantum Mechanics is an elegant theory that can be used in innumerable situations and whose correctness is confirmed by many experiments. We now introduce an alternative, but equivalent, description of Quantum Mechanics based on the concept of density operator, instead of ket. This tool is in particular useful when we deal with mixtures of states and open systems.

Let us define the density operator (or density matrix):

**Definition 1.1.** If a system is described by an ensemble of quantum states \{\mid \varphi_1 \rangle, \mid \varphi_2 \rangle, \ldots \mid \varphi_i \rangle \in \mathcal{H}^S\} with probabilities \{p_1, p_2, \ldots \mid 0 \leq p_i, \sum_i p_i = 1\} (proper mixture) then the density operator associated to the physical system is

\[ \varrho = \sum_i p_i \mid \varphi_i \rangle \langle \varphi_i \mid. \] (1.8)

Clearly, we have the following:

**Theorem 1.1 (Characterization of density operators).** An operator \( \varrho \) is a density operator associated to some ensemble \{p_i, \mid \psi_i \rangle\} if and only if it satisfies the conditions:

1. *(Trace condition)* \( \varrho \) has trace equal to one;
2. *(Positivity condition)* \( \varrho \) is a positive operator.

As it is evident the set of density operator is a convex set, i.e., if \( \varrho_1 \) and \( \varrho_2 \) are valid density operators then \( \varrho = p_1 \varrho_1 + p_2 \varrho_2 \) is a valid density operator for all \( p_1, p_2 \geq 0, p_1 + p_2 = 1 \). Extreme points of the convex set completely specify the convex set itself, in particular we call pure states the extreme points of the set of density operator. These states have the following, equivalent properties [2]:

1. a state is pure if and only if is a monodimensional projector;
2. \( \varrho \) is pure if and only if \( \text{Tr}(\varrho^2) = 1 \);
3. a state is pure if and only \( \varrho^2 = \varrho \).

According to Theorem 1.1, the density operator \( \varrho \) can be decomposed in the following form:

\[ \varrho = \sum_i \lambda_i |e_i \rangle \langle e_i| \] (1.9)
where $\sum_i \lambda_i = 1$, $\lambda_i \geq 0$ and $\langle e_i | e_j \rangle = \delta_{ij}$. This decomposition is clearly of the form (1.8), but it is not necessarily unique. The next theorem [3] deals with this problem:

**Theorem 1.2.** Let $\{ |e_1 \rangle, \ldots, |e_k \rangle \}$ with $k = \text{rank}(\varrho)$ be an eigen-ensemble of $\varrho$, i.e. $|e_i \rangle$ is eigenvector of $\varrho$ with eigenvalue $\lambda_i \neq 0$ and such that $\langle e_i | e_i \rangle = \lambda_i$.

Let $M_{ij}$ be an $r \times k$ matrix whose columns are $k$ orthonormal vectors in $\mathbb{C}^r$ ($r \geq k$), and let us put

$$|\psi_i \rangle = \sum_{j=1}^{k} M_{ij} |e_j \rangle, \ i =, \ldots, r. \quad (1.10)$$

Then the ensemble $\{ |\psi_1 \rangle, \ldots, |\psi_r \rangle \}$ has density matrix $\varrho$.

Conversely let $\{ |\phi_1 \rangle, \ldots, |\phi_s \rangle \}$ be any ensemble of pure states with density matrix $\varrho$. Then there exists an $s \times k$ matrix $N_{ij}$ whose columns are $k$ orthonormal vectors in $\mathbb{C}^s$ such that

$$|\phi_i \rangle = \sum_{j=1}^{k} N_{ij} |e_j \rangle, \ i =, \ldots, s. \quad (1.11)$$

Now it is simple to reformulate the axioms of the subsection 1.1.1 in terms of density operators:

**B1** Every state of a physical system $S$ is represented by a positive operator $\varrho$ with trace one, acting on the Hilbert space $\mathcal{H}$ associated to the system.

**B2** Given $n$ physical systems, $S_1$, $S_2$, $\ldots$, $S_n$, and the corresponding associated Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\ldots$, $\mathcal{H}_n$, the Hilbert space that describes the compound quantum system $S = S_1 + S_2 + \ldots + S_n$ is given by:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n. \quad (1.12)$$

**B3** Given a physical system $S$ in the state $\varrho$, the probability of the result $m$ in a measure, described by the collection $\{ M_m \}$ of measurement operators, gives $m$ is given by:

$$p(m) = \text{Tr}(M_m^\dagger M_m \varrho) \quad (1.13)$$

The measurement operators satisfy the completeness equation:

$$\sum_m M_m^\dagger M_m = \mathbb{I}. \quad (1.14)$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$. 
B4 If a measure, performed on the physical system $S$ in the state $\varrho$ gives a result $m$, then the states immediately after the measure is:

$$\varrho' = \frac{M_m \varrho M_m^\dagger}{\text{Tr}(M_m^\dagger M_m \varrho)}.$$  \hfill (1.15)

B5 If an isolated physical system $S$ is in the state $\varrho(t_0)$ at the instant $t_0$, then the state $\varrho(t)$ that describes the system at the time $t$ is:

$$\varrho(t) = U(t, t_0) \varrho(t_0) U^\dagger(t, t_0)$$  \hfill (1.16)

where $U(t, t_0)$ is an appropriate unitary operator.

It is possible to define axiom B5 in the following differential form:

B5' If an isolated physical system $S$ is in the state $\varrho(t_0)$ at the instant $t_0$, then the state $\varrho(t)$ that describes the system at the time $t$ is obtained solving the Liouville-Von Neumann equation:

$$i\hbar \frac{\partial}{\partial t} \varrho(t) = [H, \varrho(t)]$$  \hfill (1.17)

where $H$ is called Hamiltonian of the system and $H = H^\dagger$.

1.1.3 Reduced Density Operators

In section 1.1 we saw how to describe the evolution of an isolated system. But this is a very particular situation because in general every system is interacting with the outside world. To deal with this more general case we need to introduce a new tool: the reduced density operator.

Let us consider for example a composite quantum system $S + E$, with associated Hilbert space $\mathcal{H}^S \otimes \mathcal{H}^E$ (where $\mathcal{H}^S$ ($\mathcal{H}^E$) is the Hilbert space associated with the System (Environment)). By axiom B2, the state of $S+ E$ is described by the density operator $\varrho^{S+E}$. In this context, a description of the subsystem $S$ is given by the reduced density operator, defined as follows:

**Definition 1.2** (Reduced density operator). *Given a bipartite quantum system $S+E$ described by the density operator $\varrho^{S+E}$, the reduced density operator for system $S$ is defined by:*

$$\varrho^S \equiv \text{Tr}_E \varrho^{S+E}$$  \hfill (1.18)

where $\text{Tr}_E$ is the linear map between operators given by:

$$\text{Tr}_E(|e^S\rangle\langle f^S| \otimes |g^E\rangle\langle h^E|) = |e^S\rangle\langle f^S| \text{Tr}[|g^E\rangle\langle h^E|]$$  \hfill (1.19)

$(|e^S\rangle, |f^S\rangle \in \mathcal{H}^S$ and $|g^E\rangle, |h^E\rangle \in \mathcal{H}^E$).

It simple to see that the reduced density operator provides the correct statistical description for measures that involve only the subsystem $S$. 
1.2 Some relevant subsets of states

Theorem 1.1 gives a complete characterization of the set of density operators. Some relevant subsets are here introduced.

First of all, we observe that definition 1.1 connects the density operator $\varrho$ to a physical procedure of preparation of the state, i.e., if we indicate with $P_i$ the procedure that produces the pure state $|\varphi_i\rangle$, then the state $\varrho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ is obtained by activating randomly $P_i$ with probability $p_i$. The mixture so obtained is called proper mixture.

Let us consider now the quantum state

$$|\psi^{S+E}\rangle = \sum_i \sqrt{p_i} |i^S\rangle|i^E\rangle$$

where $\{|i^S\rangle\}$ is an orthonormal set in $\mathcal{H}^S$ and $p_i \geq 0$, $\sum_i p_i = 1$. Simple calculations show that

$$\varrho^S = \text{Tr}_E \varrho^{S+E} = \text{Tr}_E |\psi^{S+E}\rangle\langle\psi^{S+E}| = \sum_i p_i |i^S\rangle\langle i^S|$$

As it is evident, $\varrho^S$ does no longer represent a proper mixture but, on the contrary, an improper mixture, i.e., the coefficients $p_i$ do not admit an ignorance interpretation [4, 5]).

Two subsets of states play an important role in the study of open systems [6]:

- the set of separable states;
- the set of entangled states.

**Definition 1.3.** We say that a state $\varrho^{S+E}$ is separable if and only if

$$\varrho^{S+E} = \sum_{i,j} p_{ij} \varrho_i^S \otimes \varrho_j^E, \quad p_{ij} \geq 0, \quad \sum_{i,j} p_{ij} = 1,$$

where $\varrho_i^S \in \mathcal{S}(\mathcal{H}^S)$, $\varrho_j^E \in \mathcal{S}(\mathcal{H}^E)$.

The set of separable states for the compound system $S+E$ will be denoted by $\mathcal{S}^{S+E}$.

**Definition 1.4.** We say that a state $\varrho^{S+E}$ is entangled if and only if

$$\varrho^{S+E} \in \mathcal{E}^{S+E} = \mathcal{S}(\mathcal{H}^{S+E}) \setminus \mathcal{S}^{S+E}.$$

(1.23)
Note that an entangled state $\rho_{S+E}$ can always be written as

$$
\rho_{S+E} = \sum_{i,j} p_{ij} \rho_i^S \otimes \rho_j^E, \quad \Sigma_{i,j} p_{ij} = 1,
$$

(1.24)

but the coefficient $p_{ij}$ are not necessarily positive.

A relevant subset of $\mathcal{G}_{S+E}$ is the set of product states:

**Definition 1.5.** We say that a state $\rho_{S+E}$ is product state if and only if assumes the form

$$
\rho_{S+E} = \rho_S \otimes \rho_E.
$$

(1.25)

Note that definitions 1.3, 1.4, 1.5 mathematically characterize the physical states $\rho_{S+E}$. The connection between these properties and the physical concepts of proper and improper mixtures is given by the following trivial proposition [7]:

**Proposition 1.1.** $\rho_{S+E}$ is a proper mixtures of pure product states if and only if $\text{Tr}_E \rho_{S+E}$ is a proper mixture.

### 1.3 Open dynamics

In this section we introduce some standard results in open quantum dynamics. In particular we will consider the concepts of dynamical map, dynamical semigroup, positivity and complete positivity.

#### 1.3.1 The Kraus-Stinespring form

Now, we deal with the problem to describe the evolution of a system $S$ in interaction with an external environment $E$. Since the system $S + E$ is an isolated quantum system, we can apply axiom B5.

Let $\rho_{S+E}(0) = \rho_S(0) \otimes \rho_E(0)$ (where $\rho_E(0) = |e_1\rangle\langle e_1|$) be the state of the compound quantum system at time $t = 0$. So using axiom B5 the state $\rho_{S+E}(t)$ at time $t$ is given by

$$
\rho_{S+E}(t) = U(t,t_0) (\rho_S(0) \otimes \rho_E(0)) U^\dagger(t,t_0)
$$

(1.26)

for some unitary operator $U(t,t_0)$. Recalling the discussion of the previous section, we have that

$$
\rho_S(t) = \text{Tr}_E \rho_{S+E}(t).
$$

(1.27)

It is simple to show that the previous equation can be written in the following form [1]:

$$
\rho_S(t) = \sum_i E_i(t,t_0) \rho_{i}^S(0) E_i^\dagger(t,t_0)
$$

(1.28)
where $E_i(t, t_0) : \mathcal{H}^S \rightarrow \mathcal{H}^S$, $E_i(t, t_0) = \langle e_i | U(t, t_0) | e_1 \rangle$ ($\{|e_i\rangle\}_{i=1,...,n}$ is an orthonormal basis in $\mathcal{H}^E$). This last is called Kraus-Stinespring form (KS form).

It can appear that the study of an open system can always be reduced to the study of a wider isolated system. This is true only if we have a sufficient knowledge of the whole systems. So we must approach this problem in a different way. This is done in the next section.

### 1.3.2 An axiomatic approach

The problem to describe the evolution of an open quantum system is a non trivial one, and is essentially based on axioms derived by the analysis of equation (1.28). It is evident that (1.28) is a linear, Hermitian (i.e. Hermicity preserving), trace preserving map. This properties appear too weak to represent a physical process. In particular we want that a physical map transforms states into states. To convert this intuition in a mathematical framework we first need the following definitions:

**Definition 1.6.** Let $X : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be a linear map. We say that $X$ is a Positive map if and only if satisfies the following condition:

$$M \geq 0 \iff X[M] \geq 0.$$  

(1.29)

Moreover if we indicate with $\text{id}_A$ the identity operator on the space of matrices acting on $\mathcal{H}^A$, we have:

**Definition 1.7.** Let $X : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be a linear map. We say that $X$ is a Completely Positive map if and only if $X \otimes \text{id}_A$ is a positive map for all ancilla systems $A$.

These concept induce the identification of the physical evolution of a system $S$ with a map between states, this map will be called dynamical map:

**Definition 1.8.** An Hermitian map $m$ is said to be a dynamical map if and only if it satisfies the following axioms [8]:

**DA1** $m$ is a trace preserving map;

**DA2** $m$ is a convex linear map;

**DA3** $m \otimes \text{id}_A$ is a completely positive map.

Such requirements are justified on the basis of the following physical motivations:
1. The map must preserve the statistical character of $\varrho$.

2. The state of a physical system is univocally determined by its density matrix $\varrho$, then its evolution cannot depend on a particular convex decomposition of the density matrix itself.

3. For any system which has interacted in the past with another system $A$ not involved in the evolution described by $m$, the map $m \otimes \text{id}_A$ must be a positive map.

**Remark 1.2.** Sometimes axiom DA1 is replaced by the weaker axiom

DA1$'$: $\text{Tr} m[\varrho]$ is the probability that the process represented by $m$ occurs when $\varrho$ is the initial state. Thus, $0 \leq \text{Tr} m[\varrho] \leq 1$ for any state $\varrho$.

This axiom permits the treatment of systems subjected to dissipation

It is simple to prove the following [1]:

**Theorem 1.3.** If an Hermitian map $m$ satisfies axioms DA1$'$, DA2 and DA3, then $m$ assumes the KS form:

$$m[\varrho] = \sum_i E_i \varrho E_i^\dagger$$

(1.30)

where $\sum_i E_i^\dagger E_i \leq I$. The map $m$ satisfies axiom DA1 if and only if $\sum_i E_i^\dagger E_i = I$.

As it is evident, if one agrees to axiom DA3, only completely positive map can be used to describe open quantum dynamics. Yet positive, but not completely Positive, maps share some nice features that make them useful in many contexts.

Indeed, the following theorem holds [9, 10]:

**Theorem 1.4.** A state $\varrho \in S^{S+E}$ is entangled if and only if a positive map $X : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ exists such that $X \otimes \text{id}_E[\varrho]$ is not positive.

A typical example of positive, but not completely positive, map is the transposition $T$. In particular in low dimensional cases, theorem 1.4 can be specialized as:

**Theorem 1.5.** Let us consider a bipartite system $S_1 + S_2$, with $S_1$ a 2-level system and $S_2$ either a 2-level or a 3-level system; a state $\varrho \in S^{S_1+S_2}$ is entangled if and only if $T \otimes \text{id}_{S_2}[\varrho]$ is not positive.
So positive, but not completely positive, maps can be used to identify entanglement.

This capacity referred as entanglement detection plays an important role in quantum information theory: many of the key aspect of this theory are based on entangled states (quantum teleportation, quantum cryptography, etc.). So the detection and manipulation of entanglement is of great interest not only in physics but also in computation or more generally in mathematics.

1.3.3 One parameter semigroup

If the interaction between $S$ and $E$ is sufficiently weak, one expects that, on a typical time-scale, the dynamics of $S$ might be disentangled from that of the total system and efficiently described by a one-parameter dynamical semigroup of maps

**Definition 1.9.** A dynamical semigroup $\{\Lambda_t\}_{t \geq 0}$ is a continuous one parameter semigroup, i.e. $\Lambda_t \circ \Lambda_s = \Lambda_{t+s}, \forall t, s \geq 0$, of positive trace preserving linear maps.

In particular, we can regard $\{\Lambda_t\}_{t \geq 0}$ as the integrated form of the following Markovian master equation:

$$\frac{\partial}{\partial t} \rho = \mathcal{L} \rho$$

(1.31)

where $\mathcal{L}$ is a linear operator called generator of the dynamical semigroup.

**Remark 1.3.** We recall that the generator of a general continuous one parameter semigroup assumes the form:

$$\mathcal{L} : \rho \rightarrow \mathcal{L}[\rho] = -i[H, \rho] + \{G, \rho\} + \frac{1}{2} \sum_{i,j=1}^{n^2-1} C_{ij} \left\{ [F_i, \rho F_j^\dagger] + [F_i^\dagger, F_j] \right\}$$

(1.32)

where the matrix of coefficient $C_{ij}$ (Kossakowski matrix) is Hermitian, $H = H^\dagger$, $\text{Tr}H = 0$, $G = G^\dagger$ and the $F_j$ are such that $F_n^2 = 1/n$ and $\text{Tr}(F_j^\dagger F_k) = \delta_{jk}, 0 \leq j, k \leq n^2$.

Note that if we pose $G = 0$ and $C_{ij} = 0$, we reobtain equation 1.17, in particular we will call the map

$$\mathcal{H}[\cdot] = -i[H, \cdot]$$

(1.33)

Hamiltonian term of the generator $\mathcal{L}$, while we will call the map

$$\mathcal{D}[\cdot] = \{G, \cdot\} + \frac{1}{2} \sum_{i,j=1}^{n^2-1} C_{ij} \left\{ [F_i, \cdot F_j^\dagger] + [F_i^\dagger, F_j] \right\}$$

(1.34)
Dissipative terms of the generator $\mathcal{L}$.

Recalling the discussion of the previous section, we are interested in the case of dynamical semigroup composed by completely positive maps.

We have the following [11]:

**Theorem 1.6.** A linear operator $\mathcal{L}$ is the generator of a completely positive semigroup if it can be expressed in the form:

$$
\mathcal{L} : \rho \rightarrow \mathcal{L}[\rho] = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{n^2-1} C_{ij} \left\{ [F_i, \rho F_j^\dagger] + [F_i \rho, F_j^\dagger] \right\}
$$

where the matrix of coefficient $C_{ij}$ (Kossakowski matrix) is positive definite, $H = H^\dagger$, $\text{Tr}H = 0$ and the $F_j$ are such that $F_{n^2} = \mathbb{I}_n/\sqrt{n}$ and $\text{Tr}(F_j^\dagger F_k) = \delta_{jk}$, $0 \leq j, k \leq n^2$. 


Chapter 2

Pseudo Hermitian Quantum Mechanics

In the first chapter we saw that the dynamics of a closed system is described by the Schrödinger equation:

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle, \]

(2.1)

where \( H \) is an hermitian operator, i.e. \( H = H^\dagger \), called Hamiltonian of the system.

This axioms warrants that the integrated form (1.6) of (1.7) is a unitary map, so that in particular the probabilistic interpretation of the state vector (or of the density matrix) is valid.

Studies on pseudo-Hermitian and \( PT \)-symmetric Hamiltonians [12] proven that is possible to formulate a consistent quantum theory based on the subclass of quasi-Hermitian operators [13], so this chapter is devoted to give a short review of standard results concerning pseudo Hermitian and quasi Hermitian operators that will be used throughout this thesis.

2.1 Some definitions

Let us consider an Hilbert space \( \mathcal{H} \) endowed with the inner product \( \langle \cdot | \cdot \rangle \).

**Definition 2.1.** A linear operator \( A : \mathcal{H} \to \mathcal{H} \) is said weakly pseudo Hermitian if and only if an invertible linear operator \( \eta \) exists such that

\[ A^\dagger = \eta A \eta^{-1}. \]

(2.2)

Moreover
Definition 2.2. A linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is said pseudo Hermitian (or $\eta-$pseudo Hermitian) if and only if an invertible linear operator $\eta$ exists such that $\eta = \eta^\dagger$ and equation (4.8) holds.

In particular

Definition 2.3. A linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is said quasi Hermitian if and only if an invertible, linear, positive operator $\eta$ exists such that (4.8) holds.

Remark 2.1. Given an operator $A$ many linear operator $\eta$ can exist that satisfy the weakly pseudo Hermitian condition (4.8). The set of such operator will be denoted by $\Phi(A)$ i.e.:

$$\eta \in \Phi(A) \Leftrightarrow A^\dagger = \eta A \eta^{-1}. \quad (2.3)$$

In particular we have the following [14]:

Proposition 2.1. Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator, then if $\eta_1, \eta_2 \in \Phi(A)$ then $\eta_1^{-1} \eta_2$ is a symmetry of $H$, i.e. $[H, \eta_1^{-1} \eta_2] = 0$.

Let us indicate with $\mathcal{H}, \mathcal{Q}, \mathcal{P}$ and $\mathcal{W}$ respectively the set of Hermitian, quasi Hermitian, pseudo Hermitian and weakly pseudo Hermitian operators. Then we have immediately the following inclusions:

$$\mathcal{H} \subset \mathcal{Q} \subset \mathcal{P} \subset \mathcal{W}. \quad (2.4)$$

2.2 Some useful theorems

In this section we present some important results about (weakly)pseudo Hermitian operators that will be used in the sequel.

The first property connects $\eta-$pseudo Hermiticity with the existence of a (possibly indefinite) inner product invariant under time translation [15]:

Proposition 2.2. Let $\eta$ be an invertible Hermitian operator. The Hermitian indefinite inner product $\langle \cdot | \cdot \rangle_{\eta}$ defined by

$$\langle \psi_1 | \psi_2 \rangle_\eta = \langle \psi_1 | \eta | \psi_2 \rangle, \quad \forall \psi_1, \psi_2 \in \mathcal{H} \quad (2.5)$$

is invariant under the time translation generated by the Hamiltonian $H$ if and only if $H$ is $\eta-$pseudo Hermitian.
We note that if we indicate with $A^\dagger$ the adjoint of $A$ with respect to the alternative inner product (2.5) we have immediately that $A$ is $\eta-$pseudo Hermitian if and only if

$$A = A^\dagger.$$  \hspace{1cm} (2.6)

Let us consider a linear operator $H$ acting in a separable Hilbert space $\mathcal{H}$ and having a discrete spectrum. Moreover we shall assume that all the eigenvalues $E_n$ have finite algebraic multiplicity $g_n$ and that there is a basis of $\mathcal{H}$ in which $H$ is block diagonal with finite dimensional diagonal blocks. Then, a complete biorthonormal basis $\mathcal{B} = \{|\psi, a, i\rangle, |\phi, a, i\rangle\}$ exists such that the operator $H$ can be written in the following form [16, 17]:

$$H = \sum_n \sum_{a=1}^{d_n} \left( E_n \sum_{i=1}^{p_{n,a}} |\psi, a, i\rangle \langle \phi, a, i | + \sum_{i=1}^{p_{n,a}-1} |\psi, a, i\rangle \langle \phi, a, i + 1 | \right)$$  \hspace{1cm} (2.7)

where $d_n$ denotes the geometric multiplicity of $E_n$, $a$ is a degeneracy label and $p_{n,a}$ represents the dimension of the simple Jordan block $J_a(E_n)$ associated with the labels $n$ and $a$ (hence, $\sum_{a=1}^{d_n} p_{n,a} = g_n$).

Hence, $|\psi_n, a, 1\rangle$ (respectively, $|\phi_n, a, p_{n,a}\rangle$) is an eigenvector of $H$ (respectively, $H^\dagger$):

$$H|\psi_n, a, 1\rangle = E_n|\psi_n, a, 1\rangle, \quad H^\dagger|\phi_n, a, p_{n,a}\rangle = E_n^*|\phi_n, a, p_{n,a}\rangle$$  \hspace{1cm} (2.8)

and the following relations hold:

$$H|\psi_n, a, i\rangle = E_n|\psi_n, a, i\rangle + |\psi_n, a, i - 1\rangle, \quad i \neq 1,$$  \hspace{1cm} (2.9)

$$H^\dagger|\phi_n, a, i\rangle = E_n^*|\phi_n, a, i\rangle + n|\phi_n, a, i + 1\rangle, \quad i \neq p_{n,a}.$$  \hspace{1cm} (2.10)

The elements of the biorthonormal basis obey the relations

$$\langle \psi_m, a, i | \psi_n, b, j \rangle = \delta_{mn} \delta_{ab} \delta_{ij},$$  \hspace{1cm} (2.11)

$$\sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |\psi, a, i\rangle \langle \phi, a, i | = \sum_n \sum_{a=1}^{d_n} \sum_{i=1}^{p_{n,a}} |\phi, a, i\rangle \langle \psi, a, i | = \mathbb{I}.$$  \hspace{1cm} (2.12)

### 2.2.1 Spectrum of a pseudo Hermitian operator

In this section we analyze the spectral properties of pseudo Hermitian operators. Particular attention is given to the special subset of quasi Hermitian operators.
Theorem 2.1. Let $H$ be a linear operator acting in a Hilbert space $\mathcal{H}$. Suppose that the spectrum of $H$ is discrete, that its eigenvalues have finite algebraic multiplicity, and that (2.7) holds. Then the following conditions are equivalent:

1. the eigenvalues of $H$ are either real or come in complex-conjugate pairs and the geometric multiplicity and the Jordan dimension of the complex-conjugate pairs coincide;

2. $H$ is pseudo-Hermitian.

The previous theorem can be found in [18] (that generalizes a similar result for diagonalizable operator present in [15]). In particular the relation between Quasi Hermiticity and diagonalizability is determined by the following [18] that is again a generalization of a result present in [19]:

Theorem 2.2. Let $H$ be a linear operator with discrete spectrum. Then there exists a definite operator $\eta$ such that $H$ is $\eta$–quasi Hermitian if and only if $H$ is diagonalizable with real spectrum.

Indeed whenever $\eta$ is positive definite, we can define $\eta^{\frac{1}{2}}$ and the pseudo-Hermiticity condition implies

$$\eta^{-\frac{1}{2}}A\eta^{\frac{1}{2}} = \eta^{\frac{1}{2}}A\eta^{-\frac{1}{2}}.$$  

(2.13)

It is simple to note that

$$A' = \eta^{\frac{1}{2}}A\eta^{-\frac{1}{2}}$$  

(2.14)

is an Hermitian operator and that $A$ and $A'$ have the same spectrum.

2.2.2 Antilinear symmetries

A very interesting feature of pseudo Hermiticity is its connection with the existence of anti linear symmetries. As it is well known antilinear symmetries play an important role in physics (time reversal is associated with an anti linear operator). We present here a result obtained in [18], that generalizes [14]:

Theorem 2.3. Let $H$ be a linear operator. Suppose that the spectrum of $H$ is discrete, that its eigenvalues have finite algebraic multiplicity, and that (2.7) holds. Then the following conditions are equivalent:

- an antilinear invertible operator $\Omega$ exists such that $[H,\Omega] = 0$;

- $H$ is (weakly)pseudo-Hermitian;
2.3. Density operators in QHQM

- an antilinear involutory operator $\hat{\Omega}$ exists such that $[H, \hat{\Omega}] = 0$;
- a basis exists in which $H$ assumes a real form.

Note that the theorem only holds under the hypotheses of discrete spectrum, eigenvalues with finite algebraic multiplicity. If such hypotheses are removed the previous theorem can fail as it is shown in [20].

2.3 Density operators in QHQM

In this section an isomorphism between Hermitian and quasi Hermitian physical theories is exhibited at least in case of simple systems, which can provide a physically founded construction of density matrices for alternative descriptions in quasi-Hermitian quantum mechanics (QHQM).

Let us consider an Hilbert space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle$. Now let us consider a linear diagonalizable operator $H : \mathcal{H} \rightarrow \mathcal{H}$ with real spectrum. Then there exists an inner product $\langle \cdot | \cdot \rangle$ that renders $H$ self-adjoint (see theorem 2.2), defined as

$$\langle \cdot | \cdot \rangle = \langle \cdot | \eta \cdot \rangle$$

where $\eta : \mathcal{H} \rightarrow \mathcal{H}$ is a definite positive operator that satisfies the quasi Hermiticity condition (4.8), (recall that $\eta$ is not unique (see proposition 2.1)).

The physical Hilbert space $\mathcal{H}_{\text{phys}}$ whose dynamics is governed by the Hamiltonian operator $H$ can be obtained redefining $\mathcal{H}$, endowing it with the inner product $\langle \cdot | \cdot \rangle$ for some metric operator $\eta$ and completing the resultant inner product space. Clearly any linear operator $A$ acting on $\mathcal{H}_{\text{phys}}$ is self adjoint if and only if $A = A^\dagger$. These operators constitute the physical observables of the system.

The Hilbert space $\mathcal{H}_{\text{phys}}$ and the Hamiltonian $H$ define a pseudo-Hermitian system that can be equivalently described by the standard Hilbert space $\mathcal{H}$ and the Hermitian Hamiltonian $h = \eta^\frac{1}{2} H \eta^{-\frac{1}{2}}$. So we can realize an isomorphism between standard quantum mechanical observables $a$ (acting on $\mathcal{H}$) and the corresponding one in quasi Hermitian quantum mechanics $A$ (acting on $\mathcal{H}_{\text{phys}}$), by means of the following relations:

$$a \rightarrow A = \eta^{-\frac{1}{2}} a \eta^\frac{1}{2}$$

The action on the kets is fixed by requiring that the mean values are preserved:

$$|\psi\rangle \rightarrow |\psi\rangle = \eta^{-\frac{1}{2}} |\psi\rangle$$
while
\[ \langle \psi | \rightarrow \prec \psi | = \langle \psi | \eta^\frac{1}{2} \] (2.18)

So it is now simple to show that (2.16) still holds for density matrices of pure states: and in particular with the new bracket notation we have that a pure state assumes the nice form \(|\psi \rangle \prec \langle \psi|\), i.e. it is a self adjoint projector, with respect to the alternative inner product \(\langle \cdot | \cdot \rangle\). The set of all density matrices can be obtained, by analogy to the standard case, imposing the obvious conditions, namely, \(\varrho\) is a (Hermitian) positive trace-one operator with respect to the alternative inner product:

1. (Trace condition) \(\text{Tr} \varrho = 1\);
2. (Positivity condition) \(\varrho\) is a \(\eta_+\)-pseudo hermitian operator such that \(\varrho \eta\) is a positive operator.

The mean values of an observable \(a\) (which is represented by an Hermitian operator with respect to the inner product \(\langle \cdot | \cdot \rangle\)), is then obtained by the formula
\[ \langle a \rangle_\varrho = \text{Tr}(a \varrho) \] (2.19)
where the trace is taken in a orthonormal basis with respect to the inner product \(\langle \cdot | \cdot \rangle\). Using the isomorphism (2.16), (2.17) and (2.18), we see that an analogous equation holds in \(\mathcal{H}\):
\[ \langle A \rangle_\varrho = \text{Tr}(A \varrho) \] (2.20)
where the trace is taken in an orthonormal basis with respect to the standard inner product \(\langle \cdot | \cdot \rangle\).
Chapter 3

Alternative descriptions and multipartite quantum systems

3.1 Introduction

As we have already said in the previous chapter, studies on pseudo-Hermitian and $PT$-symmetric Hamiltonians [12] proven that it is possible to formulate a consistent quantum theory based on the subclass of quasi-Hermitian operators [13].

A remarkable picture of Hermitian and quasi-Hermitian dynamics, \[ i\frac{d}{dt}|\psi\rangle = H|\psi\rangle, \] (3.1)
is the existence of a (possibly infinite) set of dynamically invariant inner products $h_\eta$ associated with (positive) metric operators $\eta$ [15]. Generalizing our previous notation, we denote by $h_1(.,.)$ the standard or fiducial inner product in the Hilbert space $\mathcal{H}$, whose associated metric operator coincides with the identity $1$, and we define

\[ h_\eta(.,.) = h_1(.,\eta.). \] (3.2)

Hence, the possibility of alternative quantum descriptions naturally arises in this context.

Furthermore, we recall that a complete mathematical equivalence between quantum theories formulated in terms of quasi-Hermitian operators (QHQM) and the standard formulation of quantum mechanics (QM) was proven for simple quantum systems [21], [22].

Recently, QHQM has been extended to include compound (bipartite, finite-dimensional) quantum systems [23], and it has been shown that the
condition of quasi-Hermiticity of the Hamiltonian is no more sufficient to guarantee a proper (mathematical and physical) quantum mechanical description both of a compound system and of its component subsystems. In fact, it was proven that when one considers bipartite systems, the metric operator $\eta$ which realizes the quasi-Hermiticity condition must be the tensor product of two positive operators on the component spaces in order to guarantee such description; but this condition cannot be in general assumed for a quasi-Hermitian Hamiltonian [23].

Hence, reference [23] provides an early answer to an old question in literature; indeed, already in the standard quantum framework, the problem was raised on to analyze to what extent alternative quantum descriptions survive when one considers compound systems and interactions among them [24].

In this chapter, we will go more inside into this subject, by extending such results to multipartite, finite-dimensional compound quantum systems.

More explicitly, in section 3.2 we give a necessary and sufficient kinematical condition for compound quantum systems provided with the invariant inner product (3.2) to be described in terms of $k$ subsystems. Then, we outline a criterion to characterize the dynamics ruled by quasi-Hermitian Hamiltonians associated with multipartite compound quantum systems. In section 3.3, we illustrate these results by two examples. Finally, in the last section, we develop a new more systematic dynamical characterization method.

A different, physically interesting (from many point of view) insight into the problem of quasi-Hermitian, alternative descriptions will be given in the next chapter.

### 3.2 Quasi-Hermitian descriptions of multipartite systems

In this section, we expose the results presented in [23] and [25]. Note that all the statements for quasi-Hermitian operators also hold, with minimal and obvious changes, for the Hermitian ones.

#### 3.2.1 A kinematical characterization

Let us consider a compound quantum system described in a Hilbert space $\mathcal{H}^n$ of dimension $n = n_1 n_2 \ldots n_k$ provided with the alternative inner product, $h_\eta(.,.) = h_1(.,\eta.)$, where $h_1(.,.)$ denotes the standard inner product on $\mathcal{H}^n$.

Now, the following proposition [25] (which generalizes proposition 2 in reference [23]) gives a necessary and sufficient condition for the Hilbert spaces,
3.2. Quasi-Hermitian descriptions of multipartite systems

\( \mathcal{H}^{n_i} \) (\( i = 1, 2, \ldots, k \)), associated with \( k \) component systems, to be provided of suitable alternative inner products \( h_{\eta_i} \).

Denoting by \( U(n, \mathbb{C}, h_{\eta}) \) the unitary group which leaves invariant the inner product \( h_{\eta} \) (and analogously for the \( h_{\eta_i} \)’s), the following statement holds:

**Proposition 3.1.** The group \( U(n, \mathbb{C}, h_{\eta}) \) contains the transformations

\[
\bigotimes_i U_{\eta_i},
\]  

for any \( U_{\eta_i} \in U(n_i, \mathbb{C}, h_{\eta_i}) \), if and only if

\[
\eta = \bigotimes_i \eta_i.
\]

An equivalent statement [25], in terms of the quasi-Hermitian algebra \( \mathcal{H}_{\eta}^{n} \) associated with \( U(n, \mathbb{C}, h_{\eta}) \), (which generalizes proposition 3 in reference [23]) is the following:

**Proposition 3.2.** The set \( \mathcal{H}_{\eta}^{n} \) of quasi-Hermitian matrices of dimension \( n = n_1 n_2 \ldots n_k \) contains the subset

\[
K^{n} = \left\{ \bigotimes_i O_i \mid O_i \in \mathcal{H}_{\eta_i}^{n_i} \right\},
\]  

where \( \mathcal{H}_{\eta_i}^{n_i} \) represents the set of quasi-Hermitian matrices of dimension \( n_i \), if and only if

\[
\eta = \bigotimes_i \eta_i.
\]

**Proof.** Let us suppose \( \eta = \bigotimes_i \eta_i \). Then, trivially, the set \( K^{n} \) is constituted by quasi-Hermitian matrices, hence, \( K^{n} \cap \mathcal{H}_{\eta}^{n} \equiv K^{n} \). Conversely, let us suppose that for any positive operators \( \eta_i \), \( \eta \neq \bigotimes_i \eta_i \). Then, the set \( K^{n} \) is obviously an irreducible set of quasi-Hermitian matrices, then, by a known result in literature (see reference [13]), the metric is unique (up to a normalization factor), hence, it coincides with \( \bigotimes_i \eta_i \). Then, the set \( K^{n} \) contains some matrices that cannot be quasi-Hermitians with respect to the metric operator \( \eta_i \), hence, \( K^{n} \cap \mathcal{H}_{\eta_i}^{n_i} \subset K^{n} \).

Note that in the proof of proposition 3.2 we only use the irreducibility of the set \( K^{n} \), hence a similar statement can also be proven by substituting \( K^{n} \) with its minimal irreducible subset.
A direct consequence of proposition 3.2 is that the tensor product $\bigotimes_i O_i \in \mathcal{S}_n^\eta$ is certainly an observable if and only if $\eta = \bigotimes_i \eta_i$. Then, we can conclude that any quasi-Hermitian description of multipartite compound quantum systems admits a proper quantum mechanical interpretation in terms of component systems if and only if $\eta = \bigotimes_i \eta_i$.

**Remark 3.1.** Note that the most general quasi-Hermitian observable $O_\eta$ can be written as follows:

$$O_\eta = O \eta$$

(3.7)

where $O^\dagger = O$.

In fact, let $O_\eta$ be given. Then, from the invertibility of $\eta$ the solution of the previous equation reads

$$O = O_\eta \eta^{-1}.$$  

(3.8)

Now, from the $\eta$-pseudo-Hermiticity of $O_\eta$ we get

$$O^\dagger = \eta^{-1} O_\eta^\dagger = \eta^{-1} \eta O_\eta \eta^{-1} = O_\eta \eta^{-1} = O.$$  

(3.9)

(The converse is trivial).

Hence, the full class of observables $\{O_\eta\}$ (including the Hamiltonians) associated with multipartite compound quantum systems characterized by $\eta = \bigotimes_i \eta_i$ is constituted by the elements

$$O_\eta = O \bigotimes_i \eta_i$$

(3.10)

where $O$ runs on the full set of Hermitian operators.

### 3.2.2 A dynamical characterization

Let us consider now a compound multipartite quantum system whose dynamics is described on the Hilbert space $\mathcal{H}^m$ (of dimension $n = n_1 n_2 ... n_k$) by a time-independent Hamiltonian $H$ which satisfies

$$\eta H \eta^{-1} = H^\dagger, \quad \eta > 0.$$  

(3.11)

Then, the evolution operator $U$

$$U(t) = e^{-iHt}$$

(3.12)

satisfies

$$U^\dagger \eta U = \eta.$$  

(3.13)
3.3. Quasi-Hermitian Hamiltonians on $\mathbb{C}^8$

and the alternative inner product $h_\eta(\ldots) = h_1(\ldots, \eta_\cdot)$ is invariant under the dynamics generated by $H$.

Since there is an infinite class of dynamically invariant inner products associated with the metric operators $\eta$ satisfying the quasi-Hermiticity condition (see for instance [26], where the non-uniqueness of the inner product is also discussed), then, in order to properly describe the multipartite compound system under investigation, we may inquire if at least one of these inner products can be associated to a metric operator of the form $\eta = \bigotimes_i \eta_i$.

In order to do this, we can proceed directly by imposing the quasi-Hermiticity condition of $H$ in the unknown positive operator $\bigotimes_i \eta_i$,

$$\left( \bigotimes_i \eta_i \right) H - H^\dagger \left( \bigotimes_i \eta_i \right) = 0 \quad \text{(3.14)}$$

and by trying to solve, if possible, the corresponding system of equations.

This procedure has been applied to some examples of bipartite quantum systems in [23]. Let us show, in the next section, how this method works on a possibly multipartite quantum system.

3.3 Quasi-Hermitian Hamiltonians on $\mathbb{C}^8$

In this section, we will consider two quasi-Hermitian Hamiltonians on $\mathbb{C}^8$ admitting nonfactorizable and factorizable $\eta$ operators realizing the quasi-Hermiticity condition respectively. The Hamiltonian in the first example illustrate a physical situation where the whole vector space $\mathbb{C}^8$ cannot be decomposed neither as a tensor product of three Hilbert subspaces $\mathbb{C}^2$ nor as a product $\mathbb{C}^2 \otimes \mathbb{C}^4$ providing each one of the subspaces with suitable, alternative inner products. On the contrary, the Hamiltonian in the second example illustrates a physical situation where the whole vector space $\mathbb{C}^8$ can be decomposed as a product $\mathbb{C}^2 \otimes \mathbb{C}^4$ providing $\mathbb{C}^2$ and $\mathbb{C}^4$ with suitable inner products whereas a decomposition $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is forbidden.

3.3.1 A nondecomposable case

Let us consider a dynamics on the vector space $\mathbb{C}^8$, we suppose to be associated with a compound system described by the non-Hermitian Hamiltonian:

$$H = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & \frac{1}{3}D & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & \frac{1}{3}F \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}, \quad \text{(3.15)}$$
where

\[ D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad (3.16) \]

and

\[ F = \frac{1}{2} \begin{pmatrix} 1 + i & 2 \\ 2 & 1 - i \end{pmatrix} \quad (3.17) \]

A direct computation shows that \( H \) is quasi-Hermitian since it is diagonalizable with real spectrum.

Let us show that the Hamiltonian (3.15) cannot admit a positive \( \eta \) operator, satisfying the quasi-Hermiticity condition, of the form \( \eta_1 \otimes \eta_2 \otimes \eta_3 \), where \( \eta_1, \eta_2 \) and \( \eta_3 \) are positive operators on \( \mathbb{C}^2 \). In fact, writing

\[ \eta_1 = \begin{pmatrix} a & z \\ z^* & b \end{pmatrix}, \quad a, b \in \mathbb{R}, \quad z \in \mathbb{C}, \quad (3.18) \]

the conditions: \( \det \eta_1 > 0 \) (\( \Rightarrow ab > 0 \)) and \( \text{Tr} \eta_1 = a + b > 0 \), which ensure the positivity of \( \eta_1 \), imply that \( a, b \) must be non-zero, positive real numbers.

Then, let us consider the Kronecker product

\[ \eta_1 \otimes \eta_2 \otimes \eta_3 = \eta_1 \otimes \zeta = \begin{pmatrix} a \zeta & z \zeta^* \\ z^* \zeta^* & b \zeta^* \end{pmatrix} \quad (3.19) \]

where \( \eta_1 \) is given in equation (3.18), and impose the condition (3.14). We obtain the following matrix equations:

\[ \zeta L - L \zeta = 0 \quad (3.20) \]
\[ \zeta M - M^\dagger \zeta = 0 \quad (3.21) \]
\[ z(\zeta M - L \zeta) = 0 \quad (3.22) \]

According to equation (3.20), \( \zeta \) must have a diagonal form because it must commute with the diagonal nondegenerate matrix \( L \); but a direct computation shows that no diagonal \( \zeta \) can satisfy equation (3.21). In fact, it is easy to see that no diagonal (positive) operator can realize the quasi-Hermiticity condition for the matrix \( F \) given in equation (3.17). Hence, quasi-Hermiticity condition of \( H \) cannot be realized by an invertible, Hermitian positive operator \( \eta \) satisfying the condition \( \eta = \eta_1 \otimes \zeta \), which in turn implies that the same condition cannot be realized by an operator of the form \( \eta = \eta_1 \otimes \eta_2 \otimes \eta_3 \).

As a consequence, in this case, if we endow \( \mathbb{C}^8 \) with an alternative inner product which is invariant under the dynamics generated by the Hamiltonian (3.15), we obtain an Hilbert space which neither can be considered as a tensor
product of three Hilbert spaces $\mathbb{C}^2$ (provided each one with alternative inner products) nor as a product $\mathbb{C}^2 \otimes \mathbb{C}^4$ (providing $\mathbb{C}^2$ and $\mathbb{C}^4$ with suitable inner products). Then, the dynamics generated by $H$ in equation (3.15) neither can be associated with three interacting qubits nor to a qubit interacting with a four level system. Hence, non-Hermiticity of the Hamiltonian implies a one body structure of its associated quantum system; otherwise, equation (3.15) could describe multipartite compound quantum system dynamics without accessible informations about its component subsystems.

3.3.2 A decomposable case

Let us consider a dynamics on the vector space $\mathbb{C}^8$, associated with a compound system described by the non-Hermitian Hamiltonian:

$$H = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & F \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad (3.23)$$

where $D$ and $F$ are given in equations (3.16) and (3.17) respectively.

A direct computation shows that $H$ is quasi-Hermitian since it is diagonalizable with real spectrum.

The Hamiltonian (3.23) admits a positive $\eta$ operator, satisfying the quasi-Hermiticity condition, of the form $1 \otimes \eta_2$, where $1$ is the identity on $\mathbb{C}^2$ and $\eta_2$ is a positive operator on $\mathbb{C}^4$. In fact, a direct computation shows that

$$\eta = \begin{pmatrix} \eta_2 & 0 \\ 0 & \eta_2 \end{pmatrix}, \quad (3.24)$$

where $\eta_2$ is one of the operators satisfying $\eta_2 P \eta_2^{-1} = P^\dagger$, trivially satisfies the quasi-Hermiticity condition for $H$. Moreover, it is easy to show, by a straightforward calculation (analogous to the one performed in the previous subsection) that $\eta_2$ cannot be further factorized. Then, a decomposition $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ on the whole space $\mathbb{C}^8$ is forbidden in this case.

When one considers finite dimensional multipartite compound quantum systems in the standard formulation of quantum mechanics an ambiguity could arise, since several decompositions are in general allowed (for instance $\mathbb{C}^8$ can be decomposed as $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ or as $\mathbb{C}^2 \otimes \mathbb{C}^4$). The previous examples show that only some (possibly, a unique) decomposition of the whole space in terms of component subsystems permits a suitable description of a multipartite quasi-Hermitian quantum system (by means of a suitable alternative inner product), unitarily equivalent (see references [21], [22]) to
an Hermitian one. Hence, this peculiarity of quasi-Hermitian descriptions allows one to (partly) remove the above ambiguity.

Finally, we observe that an alternative description of a compound, Hermitian quantum system, associated with an inner product $h_\eta$ (see equation (3.2)), with $[H, \eta] = 0$, is untenable unless the metric operator $\eta$ is factorizable.

In the next section, we will give a more systematic characterization method to establish if a quasi-Hermitian Hamiltonian can describe the dynamics associated with multipartite compound quantum systems.

3.4 A further dynamical characterization

As we said in the previous sections, the dynamically invariant inner product, $h_\eta(\cdot, \cdot)$, associated with a quasi-Hermitian Hamiltonian $H$ is not unique [26].

More explicitly, if $H$ admits a nondegenerate spectrum, its spectral representation in terms of its biorthonormal eigenbasis, $\{|\psi_j\rangle, |\phi_j\rangle\}$, reads

$$H = \sum_j E_j |\psi_j\rangle\langle \phi_j|, \quad E_j \in \mathbb{R},$$

and the most general positive operator satisfying the pseudo-Hermiticity property is given by the well known expression

$$\eta = \sum_j p_j |\phi_j\rangle\langle \phi_j|, \quad p_j > 0.$$  \hspace{1cm} (3.25)

When $H$ admits a degenerate spectrum, the most general positive $\eta$ operator satisfying the pseudo-Hermiticity property can be computed, for instance, by following reference [23].

Note that for any $\eta$ (multiplying it, if necessary, by an uninfluential factor) we can always suppose $\text{Tr}\eta = 1$.

Now, an interesting result holds on density matrices [27], which we can adapt to our purposes, stating so the following proposition:

A positive matrix $\eta$ with $\text{Tr}\eta = 1$, which acts on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, corresponds to a tensor product of two positive matrices, and we say it to be factorizable, i.e.,

$$\eta = \eta_1 \otimes \eta_2$$  \hspace{1cm} (3.27)

where $\eta_1 = \text{Tr}_2 \eta$ and $\eta_2 = \text{Tr}_1 \eta$, if and only if

$$\text{Tr}(\eta \log \eta) = \text{Tr}_1(\eta_1 \log \eta_1) + \text{Tr}_2(\eta_2 \log \eta_2).$$  \hspace{1cm} (3.28)
As usual, $\text{Tr}_i$ denotes the partial trace with respect to the index $i$.

Then, given a quasi-Hermitian Hamiltonian $H$ associated with a bipartite compound quantum system, by computing the full class of positive $\eta$ operators (with $\text{Tr}\eta = 1$) satisfying the pseudo-Hermiticity property, we can use the previous result to establish if such class contains some factorizable operator, inducing so suitable alternative inner products on component spaces.

This criterion can be generalized to multipartite systems (composed by $k$-subsystems) as follows: we consider the more general positive $\eta$ (with $\text{Tr}\eta = 1$) associated with the whole system and, applying equation (3.28), we verify if it is the tensor product of a positive operator associated with the first subsystem and of another one associated with the $k-1$ remaining systems; then, we verify if the latter operator is the tensor product of a positive operator associated with the first of the $k-1$ subsystems and of another one associated with the remaining $k-2$ ones, and so on. In this way we have proven the following statement:

**Proposition 3.3.** A positive matrix $\eta$ with $\text{Tr}\eta = 1$, which acts on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_k$, corresponds to a tensor product of positive matrices $\eta_i$, i.e.

$$\eta = \bigotimes_i \eta_i$$

if and only if $\forall i = 1, 2, \ldots, k-1$

$$\text{Tr}_{I_i}(\eta_i \log \eta_i) = \text{Tr}_{I_{i+1}}(\eta_{I_{i+1}} \log \eta_{I_{i+1}}) + \text{Tr}_i(\eta_i \log \eta_i)$$

(3.30)

where $I = \{1, 2, \ldots, k\}$, $I_i = \{j \in I | j \geq i\}$, and moreover, if $S = \{s_1, s_2, \ldots, s_l\} \subseteq I$, $\text{Tr}_S = \text{Tr}_{s_1} \text{Tr}_{s_2} \ldots \text{Tr}_{s_l}$ and $\eta_S = \text{Tr}_{I-S}(\eta)$

Moreover, it is easy to show inductively that (3.30) is equivalent to the following condition:

$$\text{Tr}(\eta \log \eta) = \text{Tr}_{I-\{i\}}(\eta_{I-\{i\}} \log \eta_{I-\{i\}}) + \text{Tr}_i(\eta_i \log \eta_i).$$

(3.31)

### 3.5 Complete positivity in alternative descriptions

Now we are able to give a complete description of time evolution of density matrices in pseudo Hermitian quantum mechanics. Firstly let us consider closed quantum systems governed by the $\eta$—pseudo Hermitian time independent Hamiltonian $H$

$$H^\dagger = \eta H \eta^{-1}$$

(3.32)
Integrating equation (2.1) we obtain the following evolution for state vectors:

\[ |\psi_t\rangle = U(t, t_0)|\psi_{t_0}\rangle \]  

(3.33)

where \( U(t, t_0) = \exp[-iH(t - t_0)] \) (\( \hbar = 1 \)). It is now simple, using the isomorphism presented in section 2.3, to show that the time evolution of a density matrix in the alternative description provided by the \( \eta \)-pseudo Hermitian Hamiltonian \( H \), is described by the following equation:

\[ \frac{d}{dt} \rho = -i[H, \rho]. \]  

(3.34)

Note that this equation is the Liouville-von Neumann presented in section 1.1.2. As it was noted, in this case \( H = H^\dagger \). Analogously to the case of Hermitian dynamics, given an isolated bipartite quantum system it is possible to obtain the reduced dynamics of the subsystem of interest using the partial trace operation. A simpler way is to use the isomorphism presented in section 2.3. So we can obtain that a dynamical map assumes necessarily the form:

\[ m[\rho] = \sum_i E_i \rho E_i^\dagger. \]  

(3.35)

In an axiomatic approach to open dynamics of alternative quantum mechanical descriptions (see subsection 1.3.2), we can formulate the right axioms using the isomorphism presented in section 2.3. In particular, the following definition is useful:

**Definition 3.1.** An \( \eta \)-pseudo Hermitian map \( m \) is said to be a dynamical map if and only if the map

\[ m' : \rho \rightarrow m'[\rho] = \eta^{-\frac{1}{2}} E_i \rho E_i^\dagger \eta^{-\frac{1}{2}} \]  

(3.36)

is a Hermitian dynamical map.

Analogously we can obtain a generalized Lindblad-Kossakowski form for this alternative description:

\[ \mathcal{L} : \rho \rightarrow \mathcal{L}[\rho] = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{n^2-1} C_{ij} \left\{ [F_i, \rho F_j^\dagger] + [F_i \rho, F_j^\dagger] \right\} \]  

(3.37)

where the matrix of coefficient \( C_{ij} \) (Kossakowski matrix) is is positive definite, \( H = H^\dagger \), \( \text{Tr} H = 0 \) and the \( F_j \) are such that \( F_{n^2} = \mathbb{I}_n / \sqrt{n} \) and \( \text{Tr}(F_j^\dagger F_k) = \delta_{jk}, 0 \leq j, k \leq n^2 \).
Chapter 4

The Quantum Brachistochrone

4.1 Introduction

Quantum computation is the theory studying the possibility to use quantum dynamical processes in solving computational problems [1]. As in the standard computation case, one has to quantify the various costs (for example energy and time) that the desired calculation may require. The search for optimal time evolutions, with limited resources, is a natural problem that arises in such a context. The first part of this chapter is just devoted to recall some results in literature. In particular, in section 4.2 we give a precise mathematical formulation of this problem, the so called Quantum Brachistochrone, and expose the solution proposed in [28]. In section 4.3 we consider the same problem in pseudo Hermitian quantum mechanics [29] and the apparently paradoxical possibility of operating a computational process in an arbitrarily small time and with limited energy costs [30]. Yet, section 4.4 clarifies the impossibility to produce a transition between Hermitian and pseudo Hermitian quantum mechanics [31]. Then, in the second part of the paper, we will try to overcome this no-go theorem simulating a pseudo Hermitian dynamics by means of an Hermitian open dynamics (section 4.5). An alternative method, introduced in [32] for PT symmetric Hamiltonians, is analyzed and generalized to pseudo Hermitian systems in section 4.6.

Carefully discussing both methods, however, we show clearly that a faster computation can only be obtained at the cost of expend more energy and/or of increase the uncertainty in the results of the computation.

Finally, in section 4.7, some further troublesome aspects of the Brachistochrone Problem, related to information theory, are examined, showing that nor a NOT gate nor a controlled NOT gate or a controlled $U$ gate can be realized in such way, without a relevant loss of efficiency.
4. The Quantum Brachistochrone

4.2 Quantum Brachistochrone

The Quantum Brachistochrone Problem can be formulated as follows:

Given two quantum states $|\psi_i\rangle$ and $|\psi_f\rangle$, we want to find the (time-independent) Hamiltonian $H$ that performs the transformation

$$|\psi_i\rangle \to |\psi_f\rangle = e^{-i\frac{H\tau}{\hbar}}|\psi_i\rangle \quad (4.1)$$

in the minimal time $\tau$, for a fixed value of the difference of eigenvalues of $H$.

This problem was solved in [28] where it was shown that, if we put $\omega = |E_+ - E_-|$, and

$$|\psi_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_f\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad (|a|^2 + |b|^2 = 1), \quad (4.2)$$

the solution of the problem is given by the following Hermitian Hamiltonian:

$$H = \begin{pmatrix} s & \frac{\omega}{2} e^{-i\theta} \\ \frac{\omega}{2} e^{i\theta} & s \end{pmatrix}, \quad (4.3)$$

where

$$s = \frac{\omega \text{arg} a}{2 \text{arcsin} |b|} \quad (4.4)$$

and

$$\theta = \text{arg}(b) - \text{arg}(a) - \frac{\pi}{2}. \quad (4.5)$$

In particular if $a = |a|$ and $b = -i|b|$ we have that

$$H = \begin{pmatrix} 0 & \frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix}, \quad (4.6)$$

The minimal time to perform the required transformation is

$$\tau = \frac{2\hbar}{\omega} \arccos |\langle \psi_i | \psi_f \rangle| = \frac{2\hbar}{\omega} \arccos |a|. \quad (4.7)$$

In what follows we will put $\hbar = 1$. Note that $\tau$ is always a strictly positive quantity (except the trivial case $|\psi_f\rangle = |\psi_i\rangle$), moreover it is inversely proportional to $\omega$. These two observations make clear that to perform a faster transformation we need to use as initial and final states two non orthogonal states or to increase $\omega$, i.e. we need more energy.
4.3 Ultrafast Pseudo-Hermitian Quantum Dynamics

We recall definition 2.1:

**Definition 4.1.** A linear operator \( H : \mathcal{H} \rightarrow \mathcal{H} \) is said pseudo Hermitian (or \( \eta \)-pseudo Hermitian) if and only if an invertible linear hermitian operator \( \eta \) exists such that

\[
H^\dagger = \eta H \eta^{-1}.
\] (4.8)

Whenever \( \eta > 0 \), \( H \) is said quasi Hermitian.

We recall further (see definition 2.2) that the spectrum of a pseudo-Hermitian operator consists of complex conjugate pairs; in particular \( H \) is quasi Hermitian if and only if \( H \) is diagonalizable with real spectrum [18]. Note that, for a bidimensional pseudo Hermitian Hamiltonian \( H \) if \((E_+ - E_-) \in \mathbb{R} \setminus \{0\}\) then \( H \) is quasi Hermitian.

As it was shown in [29], the lower bound \( \tau \) can be made arbitrarily low if, with the same eigenvalues limitations, we allow to use pseudo Hermitian Hamiltonians. This would imply the realization of ultrafast computation with limited energy costs.

This paradox can be explained using the isomorphisms existing between the different, but physically equivalent, quasi Hermitian descriptions of quantum mechanics discussed in section 2.3.

In particular, in the pseudo-Hermitian viewpoint the Hamiltonian operator is the key tool for the definition of a metric in the state space, hence, the lower bound for the time \( \tau \) can be broken, for fixed states \( |\psi_i\rangle \) and \( |\psi_f\rangle \) because a metric always exists that reduces the distance between the given states. In other words, the use of a pseudo Hermitian Hamiltonian results in a reduction of the distance and consequently in a reduction of the time required for the transformation.

Let us consider for example, as distance measure the angle \( A \) [1]:

**Definition 4.2.** The angle \( A \) between two density matrices \( \varrho \) and \( \sigma \) is defined by:

\[
A(\varrho, \sigma) = \arccos F(\varrho, \sigma)
\] (4.9)

where \( F \) is the fidelity:

\[
F(\varrho, \sigma) = \text{Tr} \sqrt{\frac{1}{2} \varrho \sigma \frac{1}{2}}.
\] (4.10)
In particular the angle $A$ between the states in (4.2) is

$$A = \arccos \frac{\langle \psi_i | \eta | \psi_f \rangle \langle \psi_f | \eta | \psi_i \rangle}{\langle \psi_i | \eta | \psi_i \rangle \langle \psi_f | \eta | \psi_f \rangle} = \arccos |a|.$$  (4.11)

This last gives us the following

$$\tau = \frac{2}{\omega} A, \quad (4.12)$$

i.e., the velocity $v = \frac{\omega}{2}$ depends only on $\omega = |E_+ - E_-|$.

In particular the angle between the states $|\psi_i\rangle$ and $|\psi_f\rangle$, with respect to the metric induced by a $\eta$—pseudo Hermitian Hamiltonian, is

$$A = \arccos \frac{\langle \psi_i | \eta | \psi_f \rangle \langle \psi_f | \eta | \psi_i \rangle}{\langle \psi_i | \eta | \psi_i \rangle \langle \psi_f | \eta | \psi_f \rangle}. \quad (4.13)$$

By an appropriate choice of $\eta$ this distance can be made arbitrarily small, and in this way the time can be reduced to values near to zero. A possible choice is to put $\det \eta \to 0$, in fact when the determinant of a definite positive matrix $\eta$ goes to zero, the columns of $\eta$ become linearly dependent and so the initial and final states, with respect to the metric induced by $\eta$ become proportional, i.e. they coincide from a physical point of view.

This last observation means that if we could realize the passage from an Hermitian world to a pseudo Hermitian one we could reduce the transition time producing a “wormhole effect” in the state space. It remains to explore if we can produce this “wormhole effect” in Hermitian quantum mechanics.

### 4.4 Time dependent Pseudo-Hermitian Hamiltonian

The problem posed at the end of the previous section was first introduced in [31], (This paper was the center of an interesting debate between Mostafazadeh and Znojil [33, 34, 35, 36]). This problem can be synthetized in the following way.

Let us consider a system $S$. Let us suppose that the metric, that defines the geometry of the state space, is described by the time dependent operator $\eta(t)$. In particular

$$\eta(t) = \begin{cases} \mathbb{I} & \text{for } t \leq t_i, \\ \eta(t) & \text{for } t_i < t < t_f, \\ \bar{\eta} & \text{for } t \geq t_f, \end{cases} \quad (4.14)$$
4.5. PH Quantum Dynamics as Hermitian open Dynamics

We note that for \( t \leq t_i \) the dynamics is Hermitian while for \( t \geq t_f \) is \( \tilde{\eta} \)-pseudo Hermitian.

*How can we describe the evolution of the system between \( t_i \) and \( t_f \)?
We will call this problem “transition problem”.

The main assumption made in [31] is that a Schrödinger equation still holds:

\[ i \frac{d}{dt} U(t) = H(t) U(t) . \] (4.15)

To guarantee that \( H(t) \) is an observable \( \forall t \), it must be \( \eta(t) \)-pseudo Hermitian. Applying these two assumptions we obtain that the evolution is “unitary” (preserves the \( \eta(t) \)-norm) if and only if:

\[ H^\dagger(t) = \eta(t) H(t) \eta^{-1}(t) - i \eta(t) \frac{d}{dt} \eta^{-1}(t) , \] (4.16)

hence, \( H(t) \) is \( \eta(t) \)-pseudo Hermitian if and only if

\[ \eta(t) = \eta(0) . \] (4.17)

The previous equation implies that, in order to retain \( H(t) \) interpretable as the energy operator during the evolution, we must give up the unitary of the process or \( H(t) \) must be \( \eta(t) \) pseudo Hermitian for a time independent \( \eta \). This last possibility contradicts the assumption that \( \eta(t_f) \neq \eta(t_i) \).

4.5 PH Quantum Dynamics as Hermitian open Dynamics

In order to overcome the negative answer to the transition problem, in this section we will try to simulate a pseudo-Hermitian dynamics as an Hermitian open quantum dynamics. This point of view will induce the paradoxically consequence that to obtain a faster evolution, under the same eigenvalues limitations we have to use a dissipative process. This apparent contradiction shows that the eigenvalues of a pseudo Hermitian Hamiltonian cannot always be interpreted as energy values. We will clarify our viewpoint at the end of the section.

Let us consider a bidimensional Hilbert space \( \mathcal{H} \). We recall that a generic \( \eta \)-quasi Hermitian Hamiltonian \( H : \mathcal{H} \to \mathcal{H} \) can always be written in the form:

\[ H = \eta^{-\frac{1}{2}} h \eta^\frac{1}{2} , \] (4.18)

for some Hermitian operator \( h : \mathcal{H} \to \mathcal{H} \).
4. The Quantum Brachistochrone

Let us consider an orthonormal basis such that $\eta$ is in diagonal form, and without loss of generality we can suppose that

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \tag{4.19}$$

(in fact if $H$ is $\eta$-quasi Hermitian then it is $\alpha \eta$-quasi Hermitian for all $\alpha \in \mathbb{R}$). Moreover we can suppose that

$$h = \begin{pmatrix} a & d \\ d & c \end{pmatrix}, \tag{4.20}$$

with $a, c, d \in \mathbb{R}$ (if $d = |d| e^{i\theta}$, we can operate a change of basis by the unitary matrix $\text{diag}(1, e^{-i\theta})$). So we can consider only quasi Hermitian Hamiltonian of the form:

$$H = \begin{pmatrix} a & \lambda d \\ \lambda^{-1}d & c \end{pmatrix}, \tag{4.21}$$

with $\lambda > 0$.

Let us now consider the standard decomposition of a matrix in Hermitian and anti Hermitian part:

$$H = H_1 + iH_2, \tag{4.22}$$

where $H_1, H_2$ are Hermitian matrix. We have:

$$H_1 = \begin{pmatrix} a & \frac{\lambda + \lambda^{-1}}{2}d \\ \frac{\lambda + \lambda^{-1}}{2}d & c \end{pmatrix} \tag{4.23}$$

and

$$H_2 = \begin{pmatrix} 0 & \frac{\lambda - \lambda^{-1}}{2i}d \\ -\frac{\lambda - \lambda^{-1}}{2i}d & 0 \end{pmatrix} \tag{4.24}$$

Note that $H_2$ is a definite matrix, the null matrix, if and only if $H$ is Hermitian, i.e. if $d = 0$ or $\lambda = 1$.

Moreover $H_2$ is a traceless matrix, so the eigenvalues of $H_2$ $\mu_1$ and $\mu_2 (\leq \mu_1)$ satisfy the following relation:

$$\mu_1 = -\mu_2 = \frac{|\lambda - \lambda^{-1}|}{2} |d|. \tag{4.25}$$

Let us now consider the semigroup generated by $H$, where $H$ is a quasi Hermitian operator. We have the following equation:

$$\varrho(t) = e^{-iHt} \varrho e^{iH^\dagger t}. \tag{4.26}$$
Differentiating both sides of the previous equation we obtain a Lindblad-Kossakowsky type equation:

$$\frac{d}{dt} \rho = -i[H_1, \rho] + \{H_2, \rho\}. \quad (4.27)$$

Note that if $H_2 = 0$ the previous equation reduce to the Liouville-von Neumann equation, so it is natural to call $-i[H_1, \cdot]$ hamiltonian term.

It is simple to show that:

$$\frac{d}{dt} \text{Tr}(\rho) = \text{Tr}\{H_2, \rho\} = 2\text{Tr}(H_2 \rho). \quad (4.28)$$

As simple consequence we obtain that:

**Proposition 4.1.** The semigroup generated by a quasi Hermitian Hamiltonian is not trace preserving. In particular it is not strictly trace decreasing or strictly trace increasing.

Then a non trivial problem arises. In fact if we accept the probabilistic interpretation of the density matrix we cannot give a well defined interpretation of probabilities greater than one.

A simple way to overcome these difficulties is to normalize by hand the resulting density operator. This solution appears problematic because it introduces a non linear operation that cannot be extended to arbitrary density operator. In fact the greater obstacle to non linear evolution in quantum theory is that it produces different evolutions for different decompositions of the same density operator (this last problem is not so problematic if we consider only pure states, because they admit a unique decomposition).

Another way to solve this problem is to introduce a global dissipative term in the Hamiltonian:

$$H' = H - iD, \quad (4.29)$$

with $D = \mu_1 I$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator on $\mathcal{H}$, and $\mu_1$ is the maximal eigenvalue of $H_2$. In this way the density matrices obtained using $H'$ as generator of the semigroup:

1. are proportional to the original ones, i.e.

$$e^{-iHt} \rho e^{+iHt} = ke^{-iH't} \rho e^{+iH't}, \quad (4.30)$$

with $k = e^{-2\mu_1 t}$;

2. $\text{Tr}(\rho(t)) \leq 1$ for all $t \geq 0$. 

Note that $D = \mu_1 \mathbb{I}$ is the minimal dissipative term that satisfies the previous conditions.

In particular because the dissipative term $D$ is a multiple of the identity $\mathbb{I}$ we can separate the process in a deterministic evolution governed by the $\eta$–quasi Hermitian Hamiltonian $H$ and a purely dissipative process caused by $D$. The minimal time $\tau$ to operate the transformation

$$\ket{\psi_i} \rightarrow \ket{\psi_f},$$

(4.31)
can be obtained by the analysis of the equation:

$$\ket{\psi_i} \rightarrow \ket{\psi_f} = e^{-iHt}\ket{\psi_i},$$

(4.32)
where $H = \eta^{-\frac{1}{2}} h \eta^\frac{1}{2}$. This transformation can be written in the form:

$$\ket{\psi_i'} \rightarrow \ket{\psi_f'} = e^{-iht}\ket{\psi_i'},$$

(4.33)
where

$$\ket{\psi_i'} = \frac{\eta^{\frac{1}{2}} \ket{\psi_i}}{\sqrt{\langle \psi_i \eta \psi_i \rangle}} \text{ and } \ket{\psi_f'} = \frac{\eta^{\frac{1}{2}} \ket{\psi_f}}{\sqrt{\langle \psi_f \eta \psi_f \rangle}}.$$ (4.34)

Note that the transformation (4.33) is a unitary one generated by the Hermitian operator $h$, so to obtain the minimal time to perform the transformation (4.33) we can use the results obtained in [29]: the minimal time $\tau$ to perform the transformation

$$\ket{\psi_i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \ket{\psi_f'} = \begin{pmatrix} a \\ b \end{pmatrix},$$

(4.35)
with $h$ Hermitian operator with fixed $\omega$, is

$$\tau = \frac{2}{\omega} \arccos |a|.$$ (4.36)

To quantify the effect of the dissipative term, we operate a basis transformation, so we can consider

$$\eta^\frac{1}{2} = \begin{pmatrix} 1 \\ g^* \\ f \end{pmatrix},$$

(4.37)
where $f \in \mathbb{R}$ and $f - gg^* > 0$ (we recall that $(\eta^\frac{1}{2})_{11}$ can be always posed equal to 1 multiplying $\eta^\frac{1}{2}$ by an irrelevant factor $\alpha \in \mathbb{R}$).

Then

$$\eta^{-\frac{1}{2}} = \frac{1}{f - gg^*} \begin{pmatrix} f \\ -g^* \\ 1 \end{pmatrix},$$

(4.38)
and
\[ \eta = \begin{pmatrix} 1 + gg^* & g(1 + f) \\ g^*(1 + f) & f^2 - gg^* \end{pmatrix}. \] (4.39)

As particular case, let us consider two orthogonal states
\[ |\psi_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\psi_f\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (4.40)

Recalling equation (4.33), we need to find
\[ |\psi'_i\rangle = \eta^{\frac{1}{2}} |\psi_i\rangle \sqrt{\langle \psi_i \eta | \psi_i \rangle} \text{ and } |\psi'_f\rangle = \eta^{\frac{1}{2}} |\psi_f\rangle \sqrt{\langle \psi_f \eta | \psi_f \rangle}. \] (4.41)

So we have:
\[ |\psi'_i\rangle = \frac{1}{\sqrt{1 + gg^*}} \begin{pmatrix} 1 \\ g^* \end{pmatrix} \text{ and } |\psi'_f\rangle = \frac{1}{\sqrt{f^2 + gg^*}} \begin{pmatrix} g \\ f \end{pmatrix}. \] (4.42)

In particular we have that
\[ |a'| = \sqrt{|\langle \psi'_i | \psi'_f \rangle|^2} = \sqrt{1 - \frac{(f - gg^*)^2}{(1 + ee^*)(f^2 + gg^*)}}. \] (4.43)

Operating a basis transformation, represented by the unitary matrices $U^\dagger$, where
\[ U = \frac{1}{\sqrt{1 + gg^*}} \begin{pmatrix} 1 & g \\ g^* & -1 \end{pmatrix} \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix} \] (4.44)
we can obtain, by an appropriate choice of $\gamma_1$ and $\gamma_2 \in \mathbb{R}$:
\[ |\psi'_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\psi'_f\rangle = \begin{pmatrix} a' \\ ib' \end{pmatrix} \] (4.45)
with $a', b' \in \mathbb{R}^+$. 

The right $\eta$—pseudo Hermitian Hamiltonian is then given, in the original basis, by
\[ h = Uh'U^\dagger, \] (4.46)
where, using equation (4.6),
\[ h' = \begin{pmatrix} 0 & \frac{\eta}{2} \\ \frac{\eta}{2} & 0 \end{pmatrix}. \] (4.47)
The original Hamiltonian $H$ has then the form:

$$H = \eta^{-\frac{1}{2}} U h U^\dagger \eta^{\frac{1}{2}} =$$

$$= \frac{\omega}{2} \begin{pmatrix} e^{-i\gamma_1} & 0 \\ 0 & e^{-i\gamma_2} \end{pmatrix} \begin{pmatrix} \frac{e^{(1+f)}}{f - g'g'^*} & 1 + \frac{g'^*}{f - g'g'^*} + \frac{(1+g')(1+f)}{1+g'g'^*} \\ 1 + \frac{1-g'}{f-g'g'^*} & \frac{e^{(1+f)}}{f - g'g'^*} \end{pmatrix} \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix}$$

(4.48)

where $g' = ge^{i(\gamma_2 - \gamma_1)}$. Then, let us consider $H_1 = \frac{H + H^\dagger}{2}$ and $H_2 = \frac{H - H^\dagger}{2}$. Note that $H_1$ is a traceless matrix and so has two symmetric eigenvalues $E_+ = E$ and $E_- = -E$. In particular we obtain that in the limit $\tau \to 0$, i.e. $f \to gg^*$, we have that

$$(2E)^2 = (E_+ - E_-)^2 = (\text{Tr} H_1)^2 - 4\text{Det} H_1 \to \infty. \quad (4.49)$$

Moreover the analysis of the dissipative factor $\mathfrak{d}(g,f)$ gives, in the limit $gg^* \to f$, the following results:

$$\mathfrak{d}(f) = \lim_{gg^* \to f} \mathfrak{d}(g,f) = \lim_{gg^* \to f} \frac{\langle \psi_f | \psi_f \rangle}{\langle \psi_i | \psi_i \rangle} = \frac{1}{\sqrt{f}} e^{-(\frac{1}{2}+f)}. \quad (4.50)$$

A graphical representation of the dissipative factor $\mathfrak{d}$ vs. $f$ is given in figure 4.1.

We can observe that the dynamics is strongly dissipative (the probability of revelation is reduced to values less than 20%). Equation (4.49) justifies a posteriori the usual identification of $H_1$ with the Hamiltonian of the system and its eigenvalues can be considered as energy values. In fact if we want realize the transformation generated by the Hamiltonian $H$ as an open dynamics we implicitly assume that an auxiliary system $A$ exists such that $S + A$ is a closed systems governed by the Hermitian Hamiltonian $H^{S + A}$.

Now, in general, an open dynamics is the reduced dynamics of a broader open system so the time to operate the transformation (4.32), performed by the dynamical semigroup generated by $H$ is the same as the time to perform the closed evolution

$$|\psi_I^{(S)} \rangle \otimes |\phi_I^{(A)} \rangle \to |\psi_F^{(S)} \rangle \otimes |\phi_F^{(A)} \rangle, \quad (4.52)$$

1 $H$ is a traceless matrix, $H_2$ is a traceless matrix (see equation (4.24)) and so we have the thesis.
by a new Hermitian operator $H^{S+A}$. In particular under the energy constraints discussed, this time can not be made arbitrarily small because the distance between the initial and final states in the compound system $S+A$ is always greater than the distance between the reduced initial and final states (the distance cannot increase by operating a partial trace operation [1]).

To conclude, in this section we shown that the pseudo Hermitian evolution dynamics can be obtained by means of an open dynamics in the Hermitian world, but such process is strongly dissipative, so that all this has an energy cost. The same argument holds if we use instead of a pseudo Hermitian Hamiltonian $H$ a more general non Hermitian Hamiltonian subjected to the only restriction that the difference of the eigenvalues is real and fixed (see [37],\(^2\)).

### 4.6 PH Quantum Dynamics as Hermitian open Dynamics II

In this section we analyze a different method presented in [32], and here generalized to quasi Hermitian Hamiltonians $H$, in order to simulate a pseudo-Hermitian evolution by means of a Hermitian open dynamics. As we said above, any $\eta$–quasi Hermitian Hamiltonian $H : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ can be written as

$$H = \eta^{-1/2}h\eta^{1/2},$$

(4.53)

where $h = h^\dagger$. Without loss of generality, we can put $\text{Tr}h = 0$ (this last condition relies on the fact that the only important quantity is the difference of the eigenvalues $\omega$). The aim of reference [32] was to provide a way to obtain

---

\(^2\)In this paper we show that the faster then Hermitian evolution of non Hermitian dynamics is obtained increasing the energy cost, while in [37] this aspect was neglected.
the pseudo Hermitian evolution in $\mathcal{H}^2$ starting from the Hermitian dynamics of a larger system $\mathcal{H}^2 \oplus \tilde{\mathcal{H}}^2$, i.e. realizing the following transformation:

$$\phi_i \rightarrow \phi(t) = U(t)\phi_i,$$

(4.54)

where $U(t)$ is a unitary transformation (generated by a Hermitian Hamiltonian $H$), and

$$\phi(t) = \begin{pmatrix} \psi(t) \\ \chi(t) \end{pmatrix},$$

(4.55)

with the requirement that

$$\psi(t) = e^{-iH}\psi_i,$$

(4.56)

for all $t \geq 0$. In other words, the state vector $|\psi\rangle$ represents only partially the state of the system. The unobserved part, represented by $\chi$ lies in the space $\tilde{\mathcal{H}}^2$. We will call $\tilde{\mathcal{H}}^2$ unobserved space and its elements unobserved states.

To realize equation (4.54), let us consider the eigenvectors of $H$ and $H^\dagger$; in particular, if we denote with $|e_+\rangle$ and $|e_-\rangle$ the (normalized) eigenvector of $h$ relative to the eigenvalues $\omega_2$ and $-\omega_2$ respectively, we have immediately

$$H\eta^{-\frac{1}{2}}|e_\pm\rangle = \pm \frac{\omega}{2}\eta^{-\frac{1}{2}}|e_\pm\rangle$$

(4.57)

and

$$H^\dagger\eta^{\frac{1}{2}}|e_\pm\rangle = \pm \frac{\omega}{2}\eta^{\frac{1}{2}}|e_\pm\rangle$$

(4.58)

The non orthonormal set

$$\{\eta^{-\frac{1}{2}}|e_+\rangle, \eta^{-\frac{1}{2}}|e_-\rangle, \eta^{\frac{1}{2}}|e_+\rangle, \eta^{\frac{1}{2}}|e_-\rangle\}$$

can be transformed into an orthonormal set if we dilate the Hilbert space $\mathcal{H}^2$ into $\mathcal{H}^2 \oplus \tilde{\mathcal{H}}^2$ by the so called Naimark dilation.

**Remark 4.1.** In the subsequent discussion we use as bases for the matrix representation in $\mathcal{H}^2$ the orthonormal set $\{|e_+\rangle, |e_-\rangle\}$. In particular the column of $\eta^{-\frac{1}{2}}(\eta^{\frac{1}{2}})$ are the eigenvectors of $H$ ($H^\dagger$ respectively) and then we have:

$$H\eta^{-\frac{1}{2}} = \eta^{-\frac{1}{2}}\tilde{E}$$

(4.59)

and

$$H^\dagger\eta^{\frac{1}{2}} = \eta^{\frac{1}{2}}\tilde{E},$$

(4.60)

where $\tilde{E} = \text{diag}(\omega, -\omega)$. 
4.6. PH Quantum Dynamics as Hermitian open Dynamics II

Denoting by \( V = [|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle] \) the matrix of the extended vectors we have

\[
V = f \begin{pmatrix} \eta^{-\frac{1}{2}} & \eta^{\frac{1}{2}} \\ X & Y \end{pmatrix}, \tag{4.61}
\]

where \( f \) is a normalization factor and \( X, Y \) are \( 2 \times 2 \) matrices. Imposing the orthonormalization of the column of \( V \) it is simple to show that \( \det \eta = 1 \) (this condition can always be supposed following similar arguments following equation (4.20)) \( X = \eta^{\frac{1}{2}} \) and \( Y = -\eta^{-\frac{1}{2}} \), \( f = \frac{1}{\sqrt{\text{Tr}(\eta)}} \). To obtain the matrix \( H \) let us impose that it is possible to recover the original model when we restrict to the first two row of \( V \) in the new model. In particular from equation (4.59), (4.60) and an ansatz \( f[H\eta^{-\frac{1}{2}}, H^\dagger\eta^{\frac{1}{2}}] = f[\eta^{-\frac{1}{2}}, \eta^{\frac{1}{2}}]E \) where \( E = (\omega_2, -\omega_2, \omega_2, -\omega_2) \), it is possible to show that

\[
H = f^2 \begin{pmatrix} H\eta^{-1} + \eta H & H - H^\dagger \\ H^\dagger - H & H\eta^{-1} + \eta H \end{pmatrix}, \tag{4.62}
\]

where \( f = \sqrt{\cos \frac{\alpha}{2}} \). In order to reproduce equation (4.56) the initial state must be chosen as

\[
\phi_i = \begin{pmatrix} 
\psi_i \\
\chi_i 
\end{pmatrix} = \begin{pmatrix} 
\psi_i \\
\eta \psi_i 
\end{pmatrix}, \tag{4.63}
\]

In particular for this initial choice the time evolution of the system is described by

\[
\phi(t) = U(t)\phi_i = \begin{pmatrix} 
U(t) & 0 \\
0 & \eta U(t)\eta^{-1} 
\end{pmatrix} \begin{pmatrix} 
\psi_i \\
\chi_i 
\end{pmatrix}. \tag{4.64}
\]

The subsequent discussion in [32] makes evident that the procedure used by the authors is a method to reduce the distance between the starting and final states, in particular in the limit \( \tau = 0 \) the two states appear indistinguishable.

In fact by the results contained in the previous section, \( \tau \) vanishes when the columns of \( \eta \) becomes linearly independent (see equation (4.43)). In particular this means that to preserve condition \( \det \eta = 1 \) we have to substitute \( \eta \) with \( \eta' = \frac{1}{\det \eta} \); then, \( \det \eta \to 0 \) implies \( \frac{\langle \psi_i | \psi_i \rangle}{\langle \chi_i | \chi_i \rangle} \to 0 \), so that the dominant term in \( \phi_i \) is the hidden state \( |\chi_i\rangle \). Analogous conclusion holds for \( \phi_f \). Moreover, in this limit, \( |\chi_i\rangle \approx |\chi_f\rangle \) and so the two states becomes indistinguishable.

We note that this last remark agrees with our previous discussions at the end of section 4.2.

In conclusion, the methods proposed in this section and in section 4.5 share the nice feature to solve the transition problem between Hermitian and pseudo Hermitian evolution presented in section 4.4 (in fact the evolution
is governed by the Hermitian Hamiltonian $H$. Yet, as it is evident, the advantage in time produced by such methods is deleted by the introduction of a dissipative effect or by a reduction in the probability of revelation of the final state.

4.7 Some remarks on information theory

As already said in the introduction, in [29], the authors suggest that pseudo Hermitian Hamiltonians can be used to realize ultrafast computation with limited energy cost (in particular as a way of carrying out a NOT gate, i.e., a transformation between orthogonal states).

According our previous analysis this ultrafast computation can be realized only in two ways:

- increasing the difference of the energy values (see section 4.5);
- reducing the distance between the initial and final states (see section 4.6).

This last possibility can be called “Computation over a non orthogonal computational basis”.

In this section we will point out some troublesome implications of such computation in information theory.

4.7.1 Computation not interpretable

In this subsection we analyze some kinematical aspects on the use of non orthogonal computational bases. Let us consider the bidimensional case, and choose as computational basis $\mathcal{B}$ the states

\[ |\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_0\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \]

with $|a|^2 + |b|^2 = 1$. A possible Positive Operator Valued Measure [1] that can be used to establish the results of the computation in this basis is composed by the operator elements:

\[ E_0 = \begin{pmatrix} b^* & -a^* \\ -a & b \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix} = \begin{pmatrix} |b|^2 & -ab^* \\ -a^*b & |a|^2 \end{pmatrix}, \]

\[ E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]
and
\[ E_2 = I - \frac{1}{1 + |a|}(E_0 + E_1), \quad (4.68) \]
associated respectively to the results 0, 1, 2: if the result of the measure is
0(1), the final states can only be \(|\psi_0\rangle\) (respectively \(|\psi_1\rangle\)), while if we obtain
as result the value 2 we cannot discriminate between the two possible final
states. In particular the probability \(P_{|\psi_i\rangle}(2)\) to obtain as result 2 when the
input state is \(|\psi_i\rangle\) is:
\[ P_{|\psi_0\rangle}(2) = P_{|\psi_1\rangle}(2) = |a|. \quad (4.69) \]
This last represents the probability that the result of the computation cannot
be interpreted. Comparing equation (4.69) with the time \(\tau\) in equation (4.36):
\[ \tau = \frac{2}{\omega} \arccos |a|, \quad (4.70) \]
we see that if we decrease the time \(\tau\) needed to operate the transformation
\(|\psi_0\rangle \rightarrow |\psi_1\rangle\), we increase the probability to obtain a non interpretable result.

### 4.7.2 Quantum NOT-Gate

As it was said above, some authors suggest to apply the Brachistochrone
Problem in the construction of quantum NOT-gates. To increase the speed
of the computation we agree to use a non orthogonal computational basis:
\[ |\psi_0\rangle, |\psi_1\rangle, \langle \psi_0 | \psi_1 \rangle \neq 0. \quad (4.71) \]
Without loss of generality, we can put
\[ |\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_0\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} \end{pmatrix} \quad (4.72) \]
So the state \(|\psi_1\rangle\) is identified in the Bloch sphere by the angles \(\theta\) and \(\phi = \frac{\pi}{2}\). Let us suppose that the transformation is generated by an Hermitian
Hamiltonian \(H\). If we require that, after a time \(\tau\) the transformation \(e^{-iH\tau}\)
produces the desired transformations:
\[ |\psi_0\rangle \rightarrow |\psi_1\rangle, \quad (4.73) \]
and
\[ |\psi_1\rangle \rightarrow |\psi_0\rangle, \quad (4.74) \]
we obtain that the faster hamiltonian \(H\) that realize the transformation (4.73)
doesn’t realize (4.74), unless \(\langle \psi_0 | \psi_1 \rangle = 0\). In fact it is simple to show that
for the states (4.72), we must have:
\[ e^{-iH\tau} = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (4.75) \]
and then
\[
e^{-iH\tau}|\psi_1\rangle = \begin{pmatrix} \cos \frac{\theta}{2} & -i\sin \frac{\theta}{2} \\ -i\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ -i\sin \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos \theta \\
-i\sin \theta \end{pmatrix} |\psi_0\rangle
\]

More generally, this dynamical problem, connected to the use of non-orthogonal computational bases, puts a natural question:

Is there an Hermitian Hamiltonian $H$ that realizes the NOT gate in this non-orthogonal basis in a minimal time $\tau$?

Using the Bloch sphere representation, [38], it is possible to show that an arbitrary unitary evolution corresponds to a rotation in the Bloch sphere. Moreover, if we suppose that some unitary transformation $U(\tau)$ realizes the transformations (4.73) and (4.74) in the time $\tau$, then the unitary transformation $U(2\tau)$ is the identity transformation and we can conclude that

\[
\frac{\omega}{2}2\tau = \pi.
\]

Comparing this last with the time needed by the faster Hamiltonian that produces the transformation $|0\rangle \rightarrow |1\rangle$ (we need only to put $\theta = \pi$) we obtain that the use of a nonorthogonal base doesn’t produce any advantages.

One can try to overcome this last problem using a controlled NOT gate (see figure 4.2). As it is evident the action of this gate is to perform the transformation $|\psi_1\rangle \rightarrow |\psi_0\rangle$ in the time $\tau$, while $|\psi_0\rangle \rightarrow |\psi_1\rangle$ is automatically performed (in a time $\tau = 0$). As it is simple to note this last corresponds to the total transformations:

\[
|\psi_1\rangle|\psi_1\rangle \rightarrow |\psi_1\rangle|\psi_0\rangle
|\psi_0\rangle|\psi_1\rangle \rightarrow |\psi_0\rangle|\psi_1\rangle
\]

Such transformations, however, are not realizable by a unitary transformation. Indeed the validity of (4.78) appears strictly connected to the no cloning
4.7. Some remarks on information theory

Figure 4.3: Bloch sphere

Theorem [1], i.e. the impossibility to copy non-orthogonal states using completely positive maps. In fact by an appropriate unitary transformation acting only on the first (or second) subsystem it would be possible to realize the following transformations:

\[
\begin{align*}
|\psi_1\rangle|\psi_1\rangle & \rightarrow |\psi_1\rangle|\psi_1\rangle, \\
|\psi_0\rangle|\psi_1\rangle & \rightarrow |\psi_0\rangle|\psi_0\rangle,
\end{align*}
\]

that is to copy the first qubit. Hence, also the possibility of realizing a controlled NOT gate is excluded. Finally we can try to use a standard control–U operation of the form

\[
V = |e_1\rangle\langle e_1| \otimes U + |e_0\rangle\langle e_0| \otimes I,
\]

with \(\langle e_0|e_1\rangle = 0\) and \(U = e^{-i\tau H}\) with \(H\) given in equation (4.3). It is simple to prove that the resulting quantum operation is

\[
\varepsilon[\rho_{\text{input}}] = \text{Tr}_{\text{input}}(V \rho_{\text{input}} \otimes |e_1\rangle\langle e_1| V^\dagger).
\]

In particular we obtain

\[
\varepsilon[|\psi_1\rangle\langle \psi_1|] = p|\psi_0\rangle\langle \psi_0| + (1 - p)|\psi_1\rangle\langle \psi_1|,
\]

and

\[
\varepsilon[|\psi_0\rangle\langle \psi_0|] = (1 - q)|\psi_0\rangle\langle \psi_0| + q|\psi_1\rangle\langle \psi_1|,
\]

where \(p = |\langle \psi_1|e_1\rangle|^2\) and \(q = |\langle \psi_0|e_0\rangle|^2\). Now recalling that the angle \(A\) (see equation (4.9)) is a distance measure, we obtain that

\[
\frac{\pi}{2} = A(|e_0\rangle\langle e_0|, |e_1\rangle\langle e_1|) \leq A(|e_0\rangle\langle e_0|, |\psi_0\rangle\langle \psi_0|) + A(|\psi_0\rangle\langle \psi_0|, |\psi_1\rangle\langle \psi_1|) + A(|\psi_1\rangle\langle \psi_1|, |e_1\rangle\langle e_1|),
\]
that gives
\[ \frac{\pi}{2} \leq \arccos \sqrt{p} + \arccos |\langle \psi_0 | \psi_1 \rangle| + \arccos \sqrt{q}. \]

Note that if \( \langle \psi_0 | \psi_1 \rangle \neq 0 \) then \( p \) and/or \( q \) are necessarily less then 1. Alternatively, the same result can be proved making resort to the Bloch sphere representation. In fact \( 2A(|a\rangle \langle a|, |b\rangle \langle b|) \) is the arc between the two pure states \(|a\rangle, |b\rangle \) in the Bloch sphere. In particular to minimize the probability of error we need to have the four states \(|e_0\rangle, |e_1\rangle, |\psi_0\rangle \) and \(|\psi_1\rangle \) on the same diameter. The zero error case is obtained only when \(|\psi_0\rangle \) and \(|\psi_1\rangle \) are orthogonal. This last observation proves that an inevitable error exists connected to the use of non orthogonal basis for the computation. We stress finally that these results suggest to convert the equation (4.7) into an informational inequality that connects physical quantity:

\[ \Delta t \geq \frac{2\hbar}{\Delta E \varepsilon}, \]

where \( \Delta t \) represents the time needed to perform the transformation, \( \Delta E \) is the difference between the maximal and minimal energy values and \( \varepsilon \) is a parameter that characterize the efficiency of the computational process (that depends on the geometry of the problem).

**Conclusions**

In this paper we have analyzed the Quantum Brachistochrone problem in Hermitian and quasi Hermitian quantum mechanics, focusing our attention on the physical realizability of these processes. In particular this study shows that the violation for the bound for the minimal time obtained in quasi Hermitian quantum mechanics is only apparent, and can be justified by geometrical arguments. A simulation of PH dynamics by Hermitian open dynamics was analyzed in sections 4.5 and 4.6. This possibility is physically realizable but it has an energy and efficiency cost. In section 4.7 these results applied to simple quantum gates, suggest that equation (4.7) has an informational meaning that is quasi Hermitian invariant. In fact (4.82) gives an explicit connection between efficiency, energy and time needed to do a computational process.
Chapter 5

Time evolution of quasi-Hermitian open systems

In this chapter are presented the results appeared in [39]. Some equations are modified to make the treatment compatible with the isomorphism exposed in section 2.3.

5.1 Introduction

When one describes a quantum system by a density operator, as we made in the preceding chapters, its time evolution is usually associated with a map on the set of Hermitian, positive operators having unit trace. Such a map indeed preserves the probabilistic character of the density operator (which provides a probability distribution on the observables of the system), allowing a coherent physical interpretation of the theory. Analogously, when one deals with a compound system, time evolution of a subsystem can be viewed as a map of its reduced density matrix (obtained by contracting on the indices of the other subsystems).

In particular, an open system can be considered as a system $S$ influenced by an unobserved, huge environment $E$; hence, the time evolution of the reduced density matrix $g_S$ pertaining to the system is described by a dynamical map (respecting the above restrictions):

$$g_S \rightarrow \varepsilon[g_S]. \quad (5.1)$$

In other words,

$$g_S(t) = \text{Tr}_E[U g_{SE}(0) U^\dagger] = \varepsilon[g_S(0)], \quad (5.2)$$

where $g_{SE}$ represents the state of $S + E$, $U$ is its unitary evolution operator and $\text{Tr}_E$ denotes the partial trace on the degrees of freedom of $E$. When,
The map $\varepsilon$ is completely positive (CP), and it preserves the positivity of the reduced density matrices [1]. However, if the initial state $\varrho_{SE}(0)$ is not a factorizable state (or a classically correlated state [40, 41]) the map $\varepsilon$ is characterized to be an Hermitian map; this map transforms reduced density matrices into Hermitian matrices without preserving necessarily their positivity [42, 43].

Usually, under condition of short environmental correlation times one may neglect any memory effect and formulate the reduced system dynamics by means of a quantum dynamical semigroup [44], whose generator $\mathcal{L}$ (Liouville operator or Liouvillian) satisfies a master equation of the form [45, 46, 11]

$$\frac{d}{dt} \varrho_{S} = \mathcal{L}[\varrho_{S}]$$

The operator $\mathcal{L}$ acts on the space $H$ of the density matrices; when in particular $\mathcal{L}$ is of Lindblad form [46], then the corresponding dynamical map is CP (see subsection 1.3.3).

In this chapter, we extend from many points of view some previous results by Jakob and Stenholm. Firstly, we shall prove rigorously that any operator on $H$ representing a dynamical map, hence in particular $\mathcal{L}$, is always a pseudo-Hermitian operator (we will consider throughout the chapter finite dimensional Hilbert spaces). Secondly, we will consider quantum systems described by pseudo-Hermitian Hamiltonians, extending all the above concepts and properties to such systems, and in particular to a relevant subclass of them, the quasi-Hermitian ones.

After characterizing dynamical maps associated with system-environment closed dynamics, ruled by Hamiltonians whether Hermitian (in section 5.2) or quasi-Hermitian (in section 5.3), we apply such concepts to the particular case of dynamical semigroups, proving rigorously in section 5.4 that the Liouvillian $\mathcal{L}$ is a pseudo-Hermitian operator. We further characterize in section 5.5 the subclass of pseudo-Hermitian operators representing quasi-Hermitian maps, and a necessary and sufficient condition for complete positivity of such maps is also given.

In conclusion, sections 5.2–5.5 are all devoted to discuss and characterize Hermitian and quasi-Hermitian maps. All these results, however, constitute a necessary premise to sections 5.6–5.9, which deals with the definition and the properties of a monotonic functional. Indeed, a crucial point in the theory of open quantum systems is the construction of a functional (Lyapunov functional) which depends monotonically on the time, showing the approach of the system towards an ultimate steady state. In classical physics, where the system evolves towards a state of thermal equilibrium, such role is played by the entropy; yet, the von Neumann entropy, which is usually adopted
in quantum physics, does not display a monotonic time behavior, in other words, it cannot be considered as the quantum counterpart of the Boltzmann function.

A new variational function \( \Omega \) has been recently introduced in literature by Jakob and Stenholm [47, 48, 49, 49], which can be used to define a generalized entropy functional. Thus, in section 5.6, we propose a modified version of the functional introduced in reference [49], that also works in case of degenerate spectrum of \( \mathcal{L} \), overcoming then some difficulties in the approach of Jakob and Stenholm, and further refine it by means of a new, strictly decreasing, state-dependent functional. Starting from some considerations about the effectiveness of these functionals, we enlighten in section 5.7, by means of some useful theorems, the connection between imaginary eigenvalues of \( \mathcal{L} \) and unitary evolution. Next, the results in section 5.6 are applied in section 5.8 to a physical example of a Liouvillian with degenerate spectrum, already studied in reference [48]. Finally, section 5.9 contains some concluding remarks and a comparison between our method and the former one by Jakob and Stenholm.

### 5.2 Hermitian maps and pseudo-Hermiticity

We denote with \( \mathcal{H} \) the Hilbert space of states of our physical system and we assume \( \dim \mathcal{H} = n < \infty \). Let \( \mathbf{L}(\mathcal{H}) \) be the set of endomorphisms on \( \mathcal{H} \). Denoting with \( \mathcal{S}(\mathcal{H}) = \{ \rho \in \mathbf{L}(\mathcal{H}) | \rho = \rho^\dagger, \rho \geq 0, \text{Tr} \rho = 1 \} \) the set of Hermitian density matrices, the Hermitian maps

\[
\varepsilon : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{H}),
\]

(5.5)

describing the dynamics of an open system, usually satisfies the following properties:

**i** \( \varepsilon \) is convex linear, i.e., \( \varepsilon[\sum_k p_k \rho_k] = \sum_k p_k \varepsilon[\rho_k], \forall \rho_k \in \mathcal{S}(\mathcal{H}) \) and \( \forall p_k \in \mathbb{R}^+ \) with \( \sum_k p_k = 1 \);

**ii** \( \varepsilon \) is CP;

**iii** \( \text{Tr}\varepsilon[\rho] = \text{Tr}\rho, \forall \rho \in \mathcal{S}(\mathcal{H}) \).

Then \( \varepsilon \) assumes the form [1]:

\[
\varepsilon[\rho] = \sum_{J=1}^{n^2} G_J \rho G_J^\dagger,
\]

(5.6)
50 \quad 5. \ Time \ evolution \ of \ quasi-Hermitian \ open \ systems

where \ \{G_J\} \ is \ an \ orthonormal \ basis \ in \ \textbf{L}(\mathcal{H}).

According \ to \ some \ authors \ (see \ for \ instance \ Refs. \ [50, \ 51, \ 52]), \ it \ can \ be \ preferable \ to \ replace \ CP \ condition \ with \ the \ weaker

\textbf{ii}’ \ \varepsilon \ preserves \ Hermiticity, \ i.e., \ \varepsilon[A^\dagger] = (\varepsilon[A])^\dagger, \ \forall A \in \textbf{L}(\mathcal{H}).

If \ \varepsilon \ satisfies \ i, \ ii’ \ and \ iii \ then \ it \ assumes \ a \ Kraus-type \ form \ [42]:

\begin{equation}
\varepsilon[\varrho] = \sum_{J=1}^{n^2-k} G_J \varrho G^\dagger_J - \sum_{J=n^2-k+1}^{n^2} G_J \varrho G^\dagger_J, \quad (5.7)
\end{equation}

with \ \textbf{L}(\mathcal{H}), \ \text{Tr}(G^\dagger_J G_J) = 0 \ if \ I \neq J \ and

\begin{equation}
\sum_{J=1}^{n^2-k} G^\dagger_J G_J - \sum_{J=n^2-k+1}^{n^2} G^\dagger_J G_J = \mathbb{I}. \quad (5.8)
\end{equation}

As \ usual \ in \ literature \ (see \ for \ instance \ [53]) \ let \ us \ associate \ with \ any \ density \ matrix \ \varrho \ a \ vector \ \varrho \ in \ a \ Hilbert \ space \ \mathbf{H} \ of \ dimension \ n^2, \ by \ the \ mapping

\begin{equation}
\varrho \rightarrow \varrho \ (5.9)
\end{equation}

where \ the \ \textbf{n}^2 \ components \ \{(\varrho)\}_i \ are \ obtained \ with \ the \ following \ rule: \ \{(\varrho)\}_1 = \varrho_{nn}, \ \{(\varrho)\}_2 = \varrho_{n-1n}, \ldots, \ \{(\varrho)\}_n = \varrho_{n1}, \ \{(\varrho)\}_{n+1} = \varrho_{n-1n}, \ldots, \ \{(\varrho)\}_{n^2} = \varrho_{11},

and \ let \ us \ put \ \textbf{S}(\mathcal{H}) = \{(\varrho)\}.

The \ inner \ product \ between \ two \ elements \ of \ \textbf{S}(\mathcal{H}) \ coincides \ with \ the \ product \ of \ the \ corresponding \ elements \ in \ \textbf{S}(\mathcal{H}):

\begin{equation}
\langle\langle \varrho_1 | \varrho_2 \rangle\rangle = \langle \varrho_1, \varrho_2 \rangle = \text{Tr}(\varrho_1^\dagger \varrho_2). \quad (5.10)
\end{equation}

Left-right linear transformations of the matrix \ \varrho \ of \ the \ form

\begin{equation}
A \varrho B, \quad A, B \in \textbf{L}(\mathcal{H}) \quad (5.11)
\end{equation}

induce \ the \ corresponding \ linear \ transformations \ of \ the \ vector \ \varrho \ in \ \mathbf{H} \ of \ the \ form

\begin{equation}
A \otimes B^T \varrho \ (5.12)
\end{equation}

where \ T denotes \ matrix \ transposition. \ From \ the \ linearity \ of \ \varepsilon \ on \ \textbf{L}(\mathcal{H}) \ (see \ equation \ (5.7)), \ we \ immediately \ get

\begin{equation}
\varepsilon = \sum_{I,J=1}^{n^2} |K_I\rangle \varepsilon_{IJ} \langle\langle K_J | \quad (5.13)
\end{equation}

where \ \varepsilon_{IJ} \in \mathbb{C}, \ and \ \{ |K_I\rangle \}_{I=1, \ldots, n^2} \ is \ an \ orthonormal \ basis \ in \ \mathbf{H}.
5.2. Hermitian maps and pseudo-Hermiticity

In order to prove that $\varepsilon$ in (5.13) is a pseudo-Hermitian operator, let us study its antilinear symmetries.

Let

$$\mathcal{G} : \mathcal{H} \rightarrow \mathcal{H}$$

be the antilinear involutory map defined as

$$\mathcal{G} : |A\rangle \rightarrow |A^\dagger\rangle.$$  \hspace{1cm} (5.14)

An explicit representation of $\mathcal{G}$ can be given in terms of the orthonormal vectors

$$|E_I\rangle \equiv |E_{ii'}\rangle$$ \hspace{1cm} (5.16)

in $\mathcal{H}$, corresponding to the orthonormal matrices

$$|e_i\rangle\langle e_{i'}|,$$  \hspace{1cm} (5.17)

where $\{|e_i\rangle\}_{i=1,\ldots,n}$ denotes the standard basis in $\mathcal{H}$. In fact, putting

$$\mathcal{G} = \sum_I |E_{\alpha(I)}\rangle \ast \langle E_I|$$ \hspace{1cm} (5.18)

where $\alpha(I) = \alpha(ii') = (i'i)$ and $\ast$ denotes the complex conjugation operator, it is immediate to verify that the operator (5.18) satisfies condition (5.15).

Then, by using property (ii') we get

$$\mathcal{G}\varepsilon|A\rangle = \varepsilon \mathcal{G}|A\rangle,$$  \hspace{1cm} (5.19)

As a consequence of the above commutation between $\varepsilon$ and its antilinear symmetry $\mathcal{G}$, recalling the equivalence between pseudo-Hermiticity of a linear operator and the existence of an antilinear symmetry (see subsection 2.2.1), the following proposition immediately follows:

**Proposition 5.1.** Every Hermitian map is represented by a pseudo-Hermitian operator acting on $\mathcal{H}$.

(We stress that this result survives also in the infinite dimensional case, provided that only finite degeneracies occur in the spectrum of $\varepsilon$ [20]). Moreover, as a direct consequence of the pseudo-Hermiticity property and the peculiarity of the spectrum of pseudo-Hermitian operators [15], the following equivalent statement holds [18]:

_The eigenvalues of an Hermitian map are real or appear in complex conjugate pairs and the geometric multiplicity and the Jordan dimension of the_
5. Time evolution of quasi-Hermitian open systems

Complex conjugate pairs are the same.

Then, let us denote by $|X^\alpha, 1\rangle\rangle$ the eigenvectors of $\varepsilon$ pertaining to the (possibly) degenerate eigenvalues $\lambda_I^\alpha$:

$$
\varepsilon |X^\alpha, 1\rangle\rangle = \lambda_I^\alpha |X^\alpha, 1\rangle\rangle.
$$

(5.20)

A complete biorthonormal basis $\{|X^N, i\rangle\rangle, |Y^N, i\rangle\rangle\}$ exists in $\mathcal{H}$ in which $\varepsilon$ is block diagonal and assumes the form [18]:

$$
\varepsilon = \sum_N \left[ \lambda_N^0 \sum_{i=1}^{p_N} |X^0_N, i\rangle\rangle \langle\langle Y^0_N, i| + \sum_{i=1}^{p_N-1} |X^0_N, i\rangle\rangle \langle\langle Y^0_N, i+1| \right]
+ \sum_M \left[ \lambda_M^+ \sum_{i=1}^{p_M} |X^+_M, i\rangle\rangle \langle\langle Y^+_M, i| + \sum_{i=1}^{p_M-1} |X^+_M, i\rangle\rangle \langle\langle Y^+_M, i+1| \right]
+ \lambda_M^- \sum_{i=1}^{p_M} |X^-_M, i\rangle\rangle \langle\langle Y^-_M, i| + \sum_{i=1}^{p_M-1} |X^-_M, i\rangle\rangle \langle\langle Y^-_M, i+1| \right]
$$

(5.21)

where $\lambda_N^0 \in \mathbb{R}$, $\lambda_M^+ = (\lambda_M^-)^*$ and the superscripts $\pm$ denote the sign of $\text{Im}(\lambda_M^\alpha)$, while by $p_N$ we denote the dimensions of the simple Jordan blocks. Moreover

$$
\varepsilon |X^\alpha, i\rangle\rangle = \lambda_I^\alpha |X^\alpha, i\rangle\rangle + |X^\alpha, i-1\rangle\rangle \quad i \neq 1,
$$

(5.22)

and the elements of the biorthonormal basis obey the usual relations

$$
\sum_{\alpha=0,+,-} \sum_I \sum_i |X^\alpha, i\rangle\rangle \langle\langle Y^\alpha, i| = \mathcal{I}
$$

(5.23)

and

$$
\langle\langle Y^\alpha, i|X^\beta, j\rangle\rangle = \delta_{\alpha\beta} \delta_{IJ} \delta_{ij}
$$

(5.24)

where $\mathcal{I}$ denotes identity operator on $\mathcal{H}$.

5.3 Quasi-Hermitian maps and pseudo Hermiticity

In this section, we still consider dynamical maps associated with open quantum systems but we assume now that the closed system-environment dynamics is ruled by a quasi-Hermitian Hamiltonian $H$. The quasi-Hermiticity condition reads

$$
\zeta H \zeta^{-1} = H^\dagger,
$$

(5.25)

where however the condition $\zeta > 0$ is no more sufficient. Indeed, whenever one considers bipartite systems, if $\eta$ and $\xi$ denote positive operators which
characterize alternative inner products in the Hilbert spaces of the subsystems, one must necessarily assume that the whole Hamiltonian is quasi-Hermitian with respect to a factorizable \( \zeta \):

\[
\zeta = \eta \otimes \xi \tag{5.26}
\]

with \( \eta > 0 \) and \( \xi > 0 \), in order to allow a consistent description of the subsystems [23, 25].

In this context, density states are represented by (normalized) quasi-Hermitian operators \( \rho' \) [23]. By recalling that \( \eta^2 \) induces a unitary mapping between \( \eta \)-quasi-Hermitian systems and the Hermitian ones (see section 2.3) we immediately obtain \( \rho' = \eta^{-\frac{1}{2}}g\eta^\frac{1}{2} \) where \( g \in \mathcal{S}(\mathcal{H}) \).

Then, the set \( \mathcal{S}'(\mathcal{H}) \) of density matrices associated with the system are represented by quasi-Hermitian operators with unit trace:

\[
\mathcal{S}'(\mathcal{H}) = \{ \rho' \in \mathcal{L}(\mathcal{H}) | \rho' = \eta^{-\frac{1}{2}}g\eta^\frac{1}{2}, g \geq 0, \eta = \eta^\dagger > 0, \text{Tr}\rho' = 1 \}. \tag{5.27}
\]

On the analogy from the case of Hermitian maps, we assume that quasi-Hermitian maps \( \varepsilon' \) acting on quasi-Hermitian density matrices, satisfy the following conditions:

\begin{itemize}
  \item [i] \( \varepsilon' \) is convex linear;
  \item [ii] \( \varepsilon' \) preserves quasi-Hermiticity;
  \item [iii] \( \varepsilon' \) is trace preserving.
\end{itemize}

As a direct consequence of above-mentioned unitary mapping and equation (5.7) in section 5.2, maps satisfying i, ii, iii assume the following form:

\[
\varepsilon'[\rho'] = \eta^{-\frac{1}{2}}\varepsilon[\eta^{\frac{1}{2}}g\eta^{-\frac{1}{2}}]\eta^{\frac{1}{2}} = \sum_{J=1}^{n^2-k} G_J'gG_J'^\dagger - \sum_{I=n^2-k+1}^{n^2} G_I'gG_I'^\dagger \tag{5.28}
\]

where

\[
G_I' = \eta^{-\frac{1}{2}}G_I\eta^\frac{1}{2}, \tag{5.29}
\]

\[
G_I'^\dagger := \eta^{-1}G_I'^\dagger \eta \tag{5.30}
\]

and \( \text{Tr}(G_I'^\dagger G_J') = \text{Tr}(G_I^\dagger G_J) = 0 \) if \( I \neq J \).

Resorting to the one-to-one correspondence between matrices and vectors (see equation (5.9) in section 5.2) we put:

\[
\hat{\mathcal{G}}|\hat{A}\rangle := |\eta^{-1}\hat{A}^\dagger\eta\rangle. \tag{5.31}
\]

It is easy to verify that:
5. Time evolution of quasi-Hermitian open systems

a) \( \tilde{G} \) is antilinear;

b) \( \tilde{G} \) is involutory;

c) \( \tilde{G} \) commutes with \( \varepsilon' \).

Then the following proposition can be immediately proven by paraphrasing the proof of proposition 5.1:

**Proposition 5.2.** Every quasi-Hermitian map is represented by a pseudo-Hermitian operator acting on \( H \).

As a direct consequence of proposition 5.2 the following equivalent statement holds [18]:

The eigenvalues of quasi-Hermitian maps are real or appears in complex conjugate pairs and the geometric multiplicity and the Jordan dimension of the complex conjugate pairs are the same.

Hence, we can conclude that the form of the spectral decomposition of \( \varepsilon' \) is the same as the one in equation (5.21) of section 5.2.

5.4 Semigroup generators and pseudo-Hermiticity

It is well known [11, 45, 46] that under condition of short environmental correlation times one may neglect any memory effect and formulate the reduced system dynamics by means of a quantum dynamical semigroup [44], whose generator \( L \) satisfies a master equation of the form (5.3) in section 5.1. Then, dynamical semigroups constitute a subset of Hermitian or quasi-Hermitian dynamical maps (for quasi-Hermitian systems, see [54]). Hence, as a direct consequence of propositions 5.1 and 5.2 in section 5.2 and 5.3 respectively, the Liouvillian \( L \) is represented in \( H \) by a pseudo-Hermitian operator.

Moreover, dynamical semigroups are composed by contraction operators (that is the norm of the evolved state is a decreasing function in time)[45] whose eigenvalues have negative real part. Hence, again, the spectral decomposition of \( L \) assumes the form of equation (5.21) in section 5.2 with \( \text{Re}(\lambda_{n}) \leq 0 \).

Now, by using the correspondence between matrices and vectors (see section 5.2) and the property that dynamical semigroups are trace preserving, we get

\[
\text{Tr}(L[\varrho]) = \langle \langle I | L[\varrho] \rangle \rangle = 0, \quad \forall \varrho \in L(H).
\] (5.32)
5.5. Characterizing quasi-Hermitian maps

On the basis of equation (5.32), we are now able to prove a property, that we will be useful in the following. Let us consider an eigenstate \(|X_\alpha^\omega, i⟩⟩\) of \(L\), and let us suppose that the corresponding eigenvalue is non zero: \(\lambda_\alpha^\omega \neq 0\). Then, applying equation (5.32) to such state, it arises to be traceless. Applying now the same equation to the other elements of the Jordan chain (see equation (5.22) of section 5.2), we obtain

\[
0 = \langle\langle \mathbb{1}|L|X_\alpha^\omega, i⟩⟩ = \lambda_\alpha^\omega \langle\langle \mathbb{1}|X_\alpha^\omega, i⟩⟩ = \lambda_\alpha^\omega \operatorname{Tr}(|X_\alpha^\omega, i⟩⟩),
\]

i.e., all \(|X_\alpha^\omega, i⟩⟩\) are traceless. Yet, \(\{|X_\alpha^\omega, i⟩⟩\}\) is a basis on \(L(H)\), hence it cannot be formed by traceless matrices only. We can then conclude that at least one eigenvector exists pertaining to the eigenvalue \(\lambda = 0\).

Summarizing, we have that,

- if \(\lambda\) is an eigenvalue of \(L\) then \(\operatorname{Re}(\lambda) \leq 0\);

and in finite dimensions (or else in the infinite dimensional case, provided that only finite degeneracies occur in its spectrum) we proven the following properties of \(L\):

- the eigenvalue \(\lambda = 0\) always belongs to the spectrum of \(L\);
- if \(\lambda\) is an eigenvalue of \(L\), then \(\lambda^*\) too is an eigenvalue of \(L\).

Note that these last properties are conjectured in [48] on the ground of physical motivations, but not proven.

5.5 Characterizing quasi-Hermitian maps

We have seen, in the previous sections, that quasi-Hermitian maps are necessarily represented by pseudo-Hermitian operators acting on \(H\). In this section, we will go deep into this subject, considering the inverse problem, that is we will characterize the subclass of pseudo-Hermitian operators representing quasi-Hermitian maps.

First of all, we observe that the most general quasi-Hermitian map \(\varepsilon'\) assumes necessarily the form:

\[
\varepsilon'[\varrho'] = \sum_{I,J} C_{IJ} F_I \varrho' \eta^{-1} F^\dagger_J \eta
\]
where the matrix $C_{IJ}$ is Hermitian and $\{F_J\}$ is an orthonormal basis in $L(H)$.

In fact, any $\eta$–quasi-Hermitian matrix $\varrho'$ can be decomposed as $\varrho' = \varrho \eta$ where the Hermitian matrix $\varrho$ is unique. Now the mapping

$$\mathcal{M} : \varrho' \rightarrow \varrho = \varrho' \eta^{-1}$$

(5.35)
is linear and invertible. Recalling that an arbitrary Hermitian map $\varepsilon$ has the following form

$$\varepsilon : \rho \rightarrow \varepsilon[\rho] = \sum_{I,J} C_{IJ} F_I \rho F_J^\dagger$$

(5.36)
with $C_{IJ} = C_{JI}^*$, and putting

$$\varepsilon' = \mathcal{M}^{-1} \circ \varepsilon \circ \mathcal{M}$$

(5.37)
equation (5.34) immediately follows.

Let us consider the operator $\Upsilon : S(H) \rightarrow S(H)$ defined by

$$\Upsilon A := \eta^{-1} A \eta = (A^\dagger)^\dagger.$$ 

(5.38)
Then, $\Upsilon$ is an Hermitian operator. In fact, trivially,

$$(A, \Upsilon B) = \text{Tr} (A^\dagger \eta^{-1} B \eta) = \text{Tr} (\eta A^\dagger \eta^{-1} B) = (\Upsilon A, B).$$

(5.39)
Hence, from (5.34) we obtain

$$\varepsilon'[\varrho'] = \sum_{I,J} C_{IJ} F_I \varrho' \Upsilon F_J^\dagger = \sum_{I,J} C_{IJ} F_I \varrho' \Upsilon F_J^\dagger.$$ 

(5.40)
Denoting by $\Upsilon_{IJ}$ the matrix associated with $\Upsilon$ in the orthonormal basis $\{F_J^\dagger\}$:

$$\Upsilon_{IJ} = \left(F_I^\dagger, \Upsilon F_J^\dagger\right)$$

(5.41)
we can also get

$$\varepsilon'[\varrho'] = \sum_{I,J} C_{IJ} F_I \varrho' \Upsilon_{KJ} F_K^\dagger = \sum_{I,J} \tilde{C}_{IJ} F_I \varrho' F_J^\dagger,$$

(5.42)
where $\tilde{C}_{IJ} = \sum_K C_{IJ} \Upsilon_{KJ}$ or, in matrix form, $\tilde{C} = C \Upsilon^T$ (hence, $\tilde{C}$ is pseudo-Hermitian).

In particular, putting $F_I = E_I$ (see equation (5.7) of section 5.2), we obtain the following explicit form of the matrix $\Upsilon$:

$$\Upsilon_{jm,st} = \left(E_j^\dagger, \Upsilon E_s^\dagger\right) = \eta^{-1}_{mj} \eta_{sj}.$$ 

(5.43)
We have then proven the following proposition:
Proposition 5.3. The map

\[ \varepsilon'[\varrho'] = \sum_{I,L} \tilde{C}_{IL} E_I \varrho' E_L^\dagger \]  

(5.44)

preserves quasi-Hermiticity if and only if \( \tilde{C} \) is pseudo-Hermitian and admits a decomposition of the form

\[ \tilde{C} = C \Upsilon^T \]  

(5.45)

with \( C = C^\dagger \) and \( \Upsilon \) of the form (5.43).

We stress that, as a consequence of the mapping (5.35), \( \varepsilon' \) in equation (5.44) is CP if and only if the map \( \varepsilon \) in equation (5.36) is CP. In such case indeed \( \varepsilon' \otimes \text{Id} \) transforms density states into density states (belonging to the tensor product of Hilbert spaces endowed with alternative inner products characterized by positive operators \( \eta \) and \( \xi \), respectively).

Let us consider now the inverse problem. Let be given the pseudo-Hermitian operator on \( \mathbf{H} \)

\[ \hat{\varepsilon}[\varrho] = \sum_{I,J} C_{IJ} E_I \varrho \chi^{-1} E_J^\dagger \chi, \]  

(5.46)

where \( \chi \) is a linear invertible operator with \( \chi \neq \chi^\dagger \) and \( C_{IJ} = C_{JI}^* \). Let us show that \( \hat{\varepsilon} \) in (5.46) cannot preserve property ii in section 5.3.

It is immediate to verify that \( \hat{\varepsilon} \) is a pseudo-Hermitian operator. In fact, \( \hat{\varepsilon} \) commutes with the involutory, antilinear operator

\[ \mathcal{F}[\varrho] = \chi^{-1\dagger} \varrho^\dagger \chi. \]  

(5.47)

However, by expliciting equation (5.46) we get

\[ \hat{\varepsilon}[\hat{\varrho}] = \sum_{il,jm,st} C_{il,jm} \Phi_{jm,st} |e_i \rangle \langle e_l| \varrho (|e_s \rangle \langle e_t|)^\dagger \]  

(5.48)

where

\[ \Phi_{jm,st} = \chi_{js} \chi^{-1}_{tm}, \]  

(5.49)

and the operator

\[ \sum_{jm} C_{il,jm} \Phi_{jm,st} \]  

(5.50)

cannot be, in general, pseudo-Hermitian, as the following counterexample shows.
Let us put indeed

\[
\chi = \begin{pmatrix} 2 & i \\ 0 & 1 \end{pmatrix}, \quad \chi^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ 0 & 1 \end{pmatrix}.
\]

(5.51)

Then,

\[
\Phi = \begin{pmatrix} 1 & 0 & i \frac{1}{2} & 0 \\ -i & 2 & -\frac{1}{2} & i \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -i \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\]

(5.52)

and

\[
C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.
\]

(5.53)

By a direct computation it is immediate to verify that \(C\Phi\) cannot admit eigenvalues appearing in complex conjugate pairs so that it cannot be pseudo-Hermitian (see theorem 2.1).

5.6 Monotonically changing functionals

As we said in the Introduction, a crucial point in the theory of open quantum systems is the construction of a functional (Lyapunov functional) which describes the approach of the system towards an ultimate steady state monotonically in time. In classical physics, the entropy production rate provides a Lyapunov functional of nonequilibrium states. Yet, in quantum theory, neither von Neumann entropy nor Fano entropy display a monotonic time behavior.

In this section, we will reconsider the Fano-type variational function \(\Omega\), recently introduced by Jakob and Stenholm, which can be used to define a generalized entropy functional \([47, 48, 49]\), and extend it to the case of nondiagonalizable \(\mathcal{L}\).

When \(\mathcal{L}\) is nondiagonalizable its spectral decomposition reads (see section
5.6. Monotonically changing functionals

5.4) \[ L = \sum_N \left[ \lambda_N^0 \sum_{i=1}^{P_N} \left| X_N^0, i \right\rangle \left\langle Y_N^0, i \right| + \sum_{i=1}^{P_N-1} \left| X_N^0, i + 1 \right\rangle \left\langle Y_N^0, i \right| \right] 
+ \sum_M \left[ \lambda_M^+ \sum_{i=1}^{P_M} \left| X_M^+, i \right\rangle \left\langle Y_M^+, i \right| + \sum_{i=1}^{P_M-1} \left| X_M^+, i + 1 \right\rangle \left\langle Y_M^+, i \right| \right] 
+ \lambda_M^- \sum_{i=1}^{P_M} \left| X_M^-, i \right\rangle \left\langle Y_M^-, i \right| + \sum_{i=1}^{P_M-1} \left| X_M^-, i + 1 \right\rangle \left\langle Y_M^-, i \right| \right]. \quad (5.54) \]

Then, \[ L^\dagger = \sum_N \left[ \lambda_N^0 \sum_{i=1}^{P_N} \left| Y_N^0, i \right\rangle \left\langle X_N^0, i \right| + \sum_{i=1}^{P_N-1} \left| Y_N^0, i + 1 \right\rangle \left\langle X_N^0, i \right| \right] 
+ \sum_M \left[ (\lambda_M^+)^* \sum_{i=1}^{P_M} \left| Y_M^+, i \right\rangle \left\langle X_M^+, i \right| + \sum_{i=1}^{P_M-1} \left| Y_M^+, i + 1 \right\rangle \left\langle X_M^+, i \right| \right] 
+ (\lambda_M^-)^* \sum_{i=1}^{P_M} \left| Y_M^-, i \right\rangle \left\langle X_M^-, i \right| + \sum_{i=1}^{P_M-1} \left| Y_M^-, i + 1 \right\rangle \left\langle X_M^-, i \right| \right] \quad (5.55) \]

where \[ L^\dagger \left| Y_\alpha^0, p_I \right\rangle = (\lambda_I^\alpha)^* \left| Y_\alpha^0, p_I \right\rangle \] \quad (5.56)

and \[ L^\dagger \left| Y_\alpha^0, i \right\rangle = (\lambda_I^\alpha)^* \left| Y_\alpha^0, i \right\rangle + \left| Y_\alpha^0, i + 1 \right\rangle \quad (i < p_I). \] \quad (5.57)

Then, the function \[ M_\Omega[\varrho] := \left\langle \varrho | \Omega \varrho \right\rangle \]

with \[ \Omega := \sum_N \left| Y_N^0, p_N \right\rangle \left\langle Y_N^0, p_N \right| + \sum_M \left[ \left| Y_M^+, p_M \right\rangle \left\langle Y_M^+, p_M \right| \right] 
+ \left| Y_M^-, p_M \right\rangle \left\langle Y_M^-, p_M \right| \right] = \Omega^\dagger, \quad (5.58) \]
is suitable for our purposes and extends the definition in [48] to the non-diagonalizable case too.

In fact, putting \[ \left| \varrho \right\rangle = \sum_\alpha \sum_I \sum_i c_\alpha^0 I_i \left| X_\alpha^0, i \right\rangle, \] \quad (5.59)

and observing that \[ \left\langle \left\langle \varrho | L^\dagger \Omega | \varrho \right\rangle \right\rangle = \left\langle \left\langle \varrho | \Omega L | \varrho \right\rangle \right\rangle^* = \sum_N \lambda_N^0 |c_N^0|_p^2 + \sum_M \lambda_M^+ |c_M^{+, p_M}|^2 + \sum_M \lambda_M^- |c_M^{-, p_M}|^2, \] \quad (5.60)
5. Time evolution of quasi-Hermitian open systems

we obtain

$$\frac{d}{dt} M_\Omega[\rho] = \frac{d}{dt} \langle \langle \rho | (L^\dagger \Omega + \Omega L) \rho \rangle \rangle = 2 \sum_N \lambda_N^0 |c_{N,pN}^0|^2 + 2 \sum_M \text{Re}(\lambda_M^+) |c_{M,pM}^+|^2$$

$$+ 2 \sum_M \text{Re}(\lambda_M^-) |c_{M,pM}^-|^2 \leq 0. \quad (5.61)$$

As it is evident the operator $\Omega$ is closely dependent by the dynamics [48].

Let us discuss now some properties of $\Omega$. By definition, $\Omega$ is an Hermitian, positive operator with respect to the inner product

$$\langle \langle \rho_1 | \rho_2 \rangle \rangle = \text{Tr}(\rho_1^\dagger \rho_2). \quad (5.62)$$

Putting

$$\| \rho \|_\Omega = \sqrt{M_\Omega[\rho]} \quad (5.63)$$

it is immediate to verify that $\| \cdot \|_\Omega$ is a seminorm, i.e.,

i. $\| \rho \|_\Omega \geq 0, \forall \rho$;

ii. $\| \lambda \rho \|_\Omega = |\lambda| \| \rho \|_\Omega, \forall \rho, \lambda \in \mathbb{R}$;

iii. $\| \rho_1 + \rho_2 \|_\Omega \leq \| \rho_1 \|_\Omega + \| \rho_2 \|_\Omega$;

iv. $p_1 \sqrt{M_\Omega[\rho_1]} + p_2 \sqrt{M_\Omega[\rho_2]} \geq \sqrt{M_\Omega[p_1 \rho_1 + p_2 \rho_2]}, \forall p_1, p_2 \geq 0, p_1 + p_2 = 1$.

Finally, $\Omega$ is an invertible operator if and only if $L$ is diagonalizable (see equation (5.58)).

The functional $\Omega$ introduced above, although it allows to overcome the relevant problem of degeneracy in the spectrum of $L$, presents however two troublesome features. Indeed, it is immediate to verify that the function $\langle \langle \rho | \Omega \rho \rangle \rangle$ vanishes on the subspace generated by the vectors $|X_\alpha^i, p_I\rangle \rangle$ with $p_I < p_I; \text{ moreover, it is constant on the subspace generated by the eigenvectors of } L \text{ pertaining to purely imaginary eigenvalues.}$

As regards the former, we propose here a refinement of the method, defining a new monotonic functional which no more exhibits such feature. To this end, we introduce a partition of the set of states such that on each state, a state-dependent decreasing functional can be defined. In order to make that, let us note that any element of $H$ is characterized by the set \{c_{i,i}^\alpha \} of its coefficients, when decomposed along the basis \{|X_\alpha^0, i\rangle \rangle, |Y_\alpha^0, i\rangle \rangle\}. Thus, we associate with each state $| \rho \rangle \rangle$ the numbers $m(\alpha, I)[\rho] = \max \{1, i |c_{i,i}^\alpha \neq 0\}$, and we say that $\rho_1$ and $\rho_2$ belong to the same equivalence class if $m(\alpha, I)[\rho_1] = m(\alpha, I)[\rho_2], \forall \alpha, I.$
Then, for the set of states belonging to the same equivalence class, a corresponding state dependent monotonic functional can be defined as:

\[
\Omega[\varrho] = \sum_N \langle Y_0^N(m(0,N)[\varrho]) \rangle \langle \langle Y_0^N, m(0,N)[\varrho] \rangle \rangle + \sum_M \langle Y_0^M(m(+,M)[\varrho]) \rangle \langle \langle Y_0, m(+,M)[\varrho] \rangle \rangle + \langle Y_0^-M(m(+,M)[\varrho]) \rangle \langle \langle Y_0^M, m(+,M)[\varrho] \rangle \rangle.
\] (5.64)

The effectiveness of the above definition will be discussed in section 5.8, by means of an example. Note, however, that \(\Omega[\varrho]\) too is constant in time on the subspace corresponding to the purely imaginary eigenvalues of \(L\). Such problem will be approached in the next section.

5.7 Imaginary spectrum and unitary evolution

Let us begin by considering a very particular case with

\[
L^\dagger = -L.
\] (5.65)

In this case the dynamical semigroup is constituted by unitary transformations. In fact, by using the anti-Hermiticity condition we immediately get

\[
\frac{d}{dt} \langle \langle \varrho_1|\varrho_2 \rangle \rangle = \langle \langle L\varrho_1|\varrho_2 \rangle \rangle + \langle \langle \varrho_1|L\varrho_2 \rangle \rangle = \langle \langle \varrho_1|L^\dagger\varrho_2 \rangle \rangle + \langle \langle \varrho_1|L\varrho_2 \rangle \rangle = 0.
\] (5.66)

More generally, we can show now that if \(L\) admits imaginary spectrum, then it can generate only unitary transformations. As a preliminary step we prove the following:

**Lemma 5.1.** Let \(L\) be a dynamical semigroup generator. Then, for every imaginary or null eigenvalue of \(L\), the algebraic and the geometric multiplicity coincide.

**Proof.** In the Heisenberg picture, evolution of an observable \(A\) is described by:

\[
\frac{d}{dt} A(t) = L^\dagger [A(t)].
\] (5.67)

Let us consider an imaginary (or null) eigenvalue \(\lambda = ir\) \((r \in \mathbb{R})\) of \(L^\dagger\) and let us suppose, by absurd, that the dimension of its Jordan block is \(\geq 2\)
In particular, putting $t \to \infty$ while the evolution of $|A(0)\rangle = |Y_I, p_I - 1\rangle$ will be given by:

$$|A(0)\rangle = |Y_I, p_I - 1\rangle \Rightarrow |A(t)\rangle = e^{irt}|Y_I, p_I - 1\rangle + te^{irt}|Y_I, p_I\rangle$$  \hspace{1cm} (5.68)

while the evolution of $|A^\dagger(0)\rangle$ (if different) will be:

$$|A^\dagger(0)\rangle = |(Y_I, p_I - 1)^\dagger\rangle \Rightarrow |A^\dagger(t)\rangle = e^{irt}|(Y_I, p_I - 1)^\dagger\rangle + te^{irt}|(Y_I, p_I)^\dagger\rangle.$$  \hspace{1cm} (5.69)

In particular, putting $t_k = \frac{2\pi k}{\tau} \ (k \in \mathbb{N})$, one obtains:

$$|A(t_k)\rangle \pm |A^\dagger(t_k)\rangle = |Y_I, p_I - 1\rangle \pm |(Y_I, p_I - 1)^\dagger\rangle + t(|Y_I, p_I\rangle \pm |(Y_I, p_I)^\dagger\rangle)$$  \hspace{1cm} (5.70)

Of course, above matrices cannot vanish simultaneously. Suppose then

$$|A(t_k)\rangle + |A^\dagger(t_k)\rangle \neq 0$$  \hspace{1cm} (5.71)

and let $|e\rangle$ be an eigenstate of $(|Y_I, p_I\rangle + |(Y_I, p_I)^\dagger\rangle)$ pertaining to the eigenvalue $a \in \mathbb{R}$. Hence,

$$\text{Tr}[(A + A^\dagger)e^a] = \text{Tr}[(A(t) + A^\dagger(t))g(0)] = |A(t_k)\rangle + |A^\dagger(t_k)\rangle = \langle\langle g(0)|Y_I, p_I - 1\rangle\rangle + \langle\langle g(0)|(Y_I, p_I - 1)^\dagger\rangle\rangle + t\langle\langle g(0)|((Y_I, p_I)\rangle + |(Y_I, p_I)^\dagger\rangle)$$  \hspace{1cm} (5.72)

where $g(0) = |e\rangle \langle e|$, and putting $t = t_k$,

$$\text{Tr}[(A + A^\dagger)e^a] = \alpha + t_k a$$  \hspace{1cm} (5.73)

where $\alpha = \langle\langle g(0)|Y_I, p_I - 1\rangle\rangle + \langle\langle g(0)|(Y_I, p_I - 1)^\dagger\rangle\rangle < \infty$. Clearly, in the limit $k \to \infty$ we obtain $|\text{Tr}[(A + A^\dagger)e^a]| \to \infty$ and a contradiction arise. \hspace{1cm} $\square$

We are now able to prove the following

**Proposition 5.4.** If the generator $\mathcal{L}$ of a dynamical semigroup has a purely imaginary spectrum, then the semigroup is formed by unitary transformations.

**Proof.** Let us consider an auxiliary operator $\tilde{\mathcal{L}}$ with the same biorthogonal decomposition as $\mathcal{L}$ and eigenvalues $\lambda_K = iq_K$ with $q_K \in \mathbb{Q}$, i.e $q_K = \frac{m_K}{n_K}$ ($m_K, n_K \in \mathbb{Z}$). It is easy to show that the semigroup generated by $\tilde{\mathcal{L}}$ is periodic, with period $\tilde{T} = 2\pi \prod_K n_K$. Then

$$\exp[(\tilde{T} - t)\tilde{\mathcal{L}}] = \exp[-t\tilde{\mathcal{L}}] = \tilde{\Lambda}^{-t}, \ \forall t.$$  \hspace{1cm} (5.74)
5.8. A physical example

We now build up a (Cauchy) sequence \( q_K \to r_K \) (with \( q_K \in \mathbb{Q} \)), and note that in such limit we have \( T \to \infty, \hat{L} \to \hat{L} \). Hence, the L.H.S. of (5.74) tends to a positive operator, and so the R.H.S. In conclusion, every element of the semigroup admits a positive inverse and by theorem 6 of [55] we conclude that the semigroup is constituted by unitary transformations only.

As a consequence of Proposition 5.4, in the subspaces associated with imaginary eigenvalues of \( \hat{L} \), the semigroup is constituted by unitary transformations. This peculiarity explains the constant (in time) behavior of \( \Omega_{[\rho]} \) on the elements of these subspaces.

5.8 A physical example

We now apply the general formalism developed till now to an open quantum system already discussed in [48] for a diagonalizable Liouvillian \( \hat{L} \). Let us briefly resume it.

The system consists of a two-level atom driven by a classical field of frequency \( \omega \). The atom is exposed to radiative damping, due to an interaction with a reservoir. In the interaction picture and the rotating wave approximation, the dynamics is determined by the following master equation:

\[
\frac{d}{dt} \begin{pmatrix} \rho_{22} \\ \rho_{21} \\ \rho_{12} \\ \rho_{11} \end{pmatrix} = -i \begin{pmatrix} -i\Gamma & -V & V & 0 \\ -V & -i\gamma & 0 & V \\ V & 0 & -i\gamma & -V \\ i\Gamma & V & -V & 0 \end{pmatrix} \begin{pmatrix} \rho_{22} \\ \rho_{21} \\ \rho_{12} \\ \rho_{11} \end{pmatrix}, \tag{5.75}
\]

where \( \Gamma \) and \( \gamma \geq \frac{\Gamma}{2} \) are the damping rates of populations and coherences, respectively. The coupling is determined by the parameter \( V = |d \cdot E_0| \) where \( d \) denotes the dipole moment of the atomic transition. We assume that the difference between the atomic transition frequency and the driving field frequency is zero, so that the classical driving field resonates with the atomic transition.

The eigenvalues of \( \hat{L} \) are given by:

\[
\lambda_1 = 0 \tag{5.76}
\]
\[
\lambda_2 = -\frac{1}{2}[\Gamma + \gamma + i\sqrt{16V^2 - (\Gamma - \gamma)^2}] \tag{5.77}
\]
\[
\lambda_3 = -\frac{1}{2}[\Gamma + \gamma - i\sqrt{16V^2 - (\Gamma - \gamma)^2}] \tag{5.78}
\]
\[
\lambda_4 = -\gamma. \tag{5.79}
\]
5. Time evolution of quasi-Hermitian open systems

Since we are interested to examine a non diagonalizable Liouvillian, we put now \(16V^2 - (\Gamma - \gamma)^2 = 0\), so that \(\lambda_2 = \lambda_3\), hence \(\mathcal{L}\) is pseudo-Hermitian according with the result in section 5.4. For instance if \(4V = \Gamma - \gamma\) the Liouvillian reads

\[
\mathcal{L} = \begin{pmatrix}
-\Gamma & \frac{i}{4}(\Gamma - \gamma) & -\frac{i}{4}(\Gamma - \gamma) & 0 \\
\frac{i}{4}(\Gamma - \gamma) & -\gamma & 0 & \frac{i}{4}(\Gamma - \gamma) \\
-\frac{i}{4}(\Gamma - \gamma) & 0 & -\gamma & \frac{i}{4}(\Gamma - \gamma) \\
\Gamma & -\frac{i}{4}(\Gamma - \gamma) & \frac{i}{4}(\Gamma - \gamma) & 0
\end{pmatrix}
\] (5.80)

and its Jordan decomposition \(J\) is given by

\[
J = A^{-1} \mathcal{L} A
\] (5.81)

where

\[
J = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}(\Gamma + \gamma) & 1 & 0 \\
0 & 0 & -\frac{1}{2}(\Gamma + \gamma) & 0 \\
0 & 0 & 0 & -\gamma
\end{pmatrix},
\] (5.82)

\[
A = \begin{pmatrix}
\frac{(\Gamma - \gamma)^2}{\Gamma + \gamma} & -1 & 0 & 0 \\
-\frac{2i}{\Gamma + \gamma} & i & \frac{2i}{\Gamma - \gamma} & 1 \\
\frac{2i}{\Gamma + \gamma} & -i & -\frac{2i}{\Gamma - \gamma} & 1 \\
\frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} & 0 & 0 & 0
\end{pmatrix}
\] (5.83)

and

\[
(A^{-1})^\dagger = \begin{pmatrix}
\frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} & \frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} & \frac{(\Gamma - \gamma)(3\Gamma + \gamma)}{\Gamma + \gamma} & 0 \\
0 & 0 & \frac{i}{2}(\Gamma - \gamma) & \frac{1}{2} \\
\frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} & 0 & 0 & 0
\end{pmatrix}
\] (5.84)

From equations (5.83) and (5.84) we respectively obtain

\[
|X_1\rangle = \begin{pmatrix} \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma} \\ -\frac{2i}{\Gamma + \gamma} \\ \frac{2i}{\Gamma + \gamma} \\ 1 \end{pmatrix}, \quad |X_2\rangle = \begin{pmatrix} -1 \\ i \\ -i \\ 1 \end{pmatrix}
\] (5.85)

\[
|X_3\rangle = \begin{pmatrix} 0 \\ 2i \\ -\frac{2i}{\Gamma - \gamma} \\ 0 \end{pmatrix}, \quad |X_4\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}
\] (5.86)
5.8. A physical example

Figure 5.1: Comparison between \( \log \Omega \) (dashed line) and \( \log \Omega_\varrho \) (continuous line) for the following set of parameters \( C_1 = \frac{1}{18}, C_2 = \frac{0.1}{18}, C_3 = C_4 = 0, \) and \( \gamma = \frac{\Gamma}{2} \).

\[ |Y_1\rangle = \left( \begin{array}{c} \frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} \\ 0 \\ 0 \\ \frac{\Gamma^2 + 6\Gamma \gamma + \gamma^2}{2(\Gamma + \gamma)^2} \end{array} \right) \quad |Y_2\rangle = \left( \begin{array}{c} 0 \\ 0 \\ \frac{(\gamma - \gamma)^2}{2(\Gamma + \gamma)^2} \end{array} \right), \quad (5.87) \]

\[ |Y_3\rangle = \left( \begin{array}{c} \frac{1}{2}(\Gamma - \gamma) \\ \frac{i}{4}(\Gamma - \gamma) \\ -\frac{i}{4}(\Gamma - \gamma) \\ \frac{(\gamma - \gamma)^2}{4(\Gamma + \gamma)^2} \end{array} \right) \quad |Y_4\rangle = \left( \begin{array}{c} 0 \\ \frac{1}{\gamma} \\ \frac{1}{\gamma} \\ 0 \end{array} \right). \quad (5.88) \]

Applying now the results in section 5.6, the following functional \( \Omega \) is suitable to study the evolution of the system \( |\varrho\rangle = \sum_i C_i |X_i\rangle \) with \( C_3 \neq 0 \):

\[ \Omega = |Y_1\rangle \langle Y_1| + |Y_3\rangle \langle Y_3| + |Y_4\rangle \langle Y_4| \quad (5.89) \]

while, whenever \( C_3 = 0 \), the state-dependent functional to be used is

\[ \Omega_{|\varrho\rangle} = |Y_1\rangle \langle Y_1| + |Y_2\rangle \langle Y_2| + |Y_4\rangle \langle Y_4|. \quad (5.90) \]

Figure 5.1 shows clearly the different behaviors of the two functionals in such case.
5.9 Concluding remarks

We note that the functional in equation (5.89) of previous section coincides with the one obtained by Jakob and Stenholm in [48] and [49]. There are however some relevant differences between our method and the one adopted by these authors.

Jakob and Stenholm, indeed, approach the degenerate case introducing some auxiliary parameters which subsequently go to zero. These techniques can be justified (as they do) in case of driven systems, conjecturing for instance a weak, variable external influence, but appear somewhat artful in case of spontaneously decaying systems. Moreover, in this last case, these authors shown by a particular example that the restriction of the Liouvillian to the subspace corresponding to a degenerate eigenvalue is diagonalizable. Yet, the value of such eigenvalue in their example is 0, and we proven in the Lemma of section 5.7 that this property actually holds for such eigenvalue, but cannot be assumed in general. Hence, we believe that our proposal supplements and improves the original one by Jakob and Stenholm.
Chapter 6

Assignment Maps

6.1 Introduction

An old, still open, problem in literature concerns the search for necessary and sufficient conditions in order that a map can be considered a Dynamical Map [8, 56] (i.e. a map describing the evolution of a (reduced) density matrix \( \varrho^S \), obtained from \( \varrho^{S+E} \) by partial trace on the indexes pertaining to the environment \( E \)). We already introduced such concept in subsection 1.3.2; we recall that usually an Hermitian map \( m \) is said a Dynamical Map if and only if it satisfies the following axioms [8]:

**A1** \( m \) is a convex linear map;

**A2** \( m \) is a trace preserving map;

**A3** \( m \otimes I \) is a positive map.

Such requirements are justified on the basis of the following physical motivations:

1. The state of a physical system is univocally determined by its density matrix \( \varrho \), then its evolution cannot depend on a particular convex decomposition of the density matrix itself.

2. The map must preserve the statistical character of \( \varrho \).

3. For any system which has interacted in the past with another system \( A \) not involved in the evolution described by \( m \), the map \( m \otimes \text{Id}_{A} \) must be a positive map (note that **A1** and **A3** imply that \( m \) is Completely Positive (CP)).
Adopting a different point of view, a dynamical map can be considered as the composition of three maps [51]:

\[ m[\rho] = T_E \circ U \circ A[\rho] \] (6.1)

where (denoting by \( \mathcal{H}^E \) the Hilbert space associated with the environment \( E \) and by \( \mathcal{H}^{S+E} \) the one associated with the compound system \( S+E, S(\mathcal{H}^S) \) the set of states on \( \mathcal{H}^S \) and with \( B(\mathcal{H}^S) \) the set of bounded operators)

- \( A : S(\mathcal{H}^S) \to B(\mathcal{H}^{S+E}) \) is said Assignment Map;
- \( U : \varrho^{S+E} \in B(\mathcal{H}^{S+E}) \mapsto U\varrho^{S+E}U^\dagger \in B(\mathcal{H}^{S+E}) \), with \( U \) unitary;
- \( T_E : \varrho^{S+E} \in B(\mathcal{H}^{S+E}) \mapsto \text{Tr}_E(\varrho^{S+E}) \in B(\mathcal{H}^S) \) is a partial trace on the indexes corresponding to \( E \).

It is easily seen that \( T_E \) and \( U \) are CP, linear and trace preserving maps.

Hence, the problem posed at the beginning of this section can be translated to an equivalent question for the Assignment Map \( A \).

It is usual in literature (see for instance [51] for a physical discussion) to require that

\begin{align*}
B1 & \quad A \text{ is a Linear Map, } A[\Sigma_i p_i \varrho_i] = \Sigma_i p_i A[\varrho_i]. \\
B2 & \quad A \text{ is a Consistent Map, } \text{Tr}_E A[\varrho] = \varrho; \\
B3 & \quad A \text{ is a Positive Map, } A : S(\mathcal{H}^S) \to S(\mathcal{H}^{S+E});
\end{align*}

Yet, the following theorem [51], due originally to Pechukas, states that

**Theorem 6.1.** If the map \( A \) satisfies B1, B2 e B3 then

\[ A[\varrho] = \varrho \otimes \tau \] (6.2)

with \( \tau \geq 0 \) and \( \text{Tr}(\tau) = 1 \).

As it is evident, axioms B1 ÷ B3 do not permit the description of systems initially correlated with an environment[40, 41, 42, 43]. Hence, many attempts have been made in literature in order to overcome such difficulties, relaxing some of above axioms. In this chapter we intend just inquire into the properties of an assignment map such that the map \( m \) in (6.1) is a dynamical map. To this end, we discuss in section 6.2 the linearity axiom, relaxing it to a convex linearity, and we show that Pechukas’ Theorem still holds under such weakened hypothesis. Subsequently, in section 6.3, we carefully examine the controversial axiom about positivity. In particular, we consider the
general form of a (linear, consistent, not CP) assignment map and suggest a new, wider definition of dynamical map, that we call Generalized Dynamics. By the way, some properties of it and an interesting result about the minimal dimension of the Hilbert space $H^E$ are also proven. In section 6.4 the methods and concepts exposed in section 6.3 are exemplified and widely discussed together some their consequences, while section 6.5 contains some remarks about consistency axiom. Next section 6.6 is devoted to settle a seemingly paradoxical question, raised by the use of generalized dynamics rather than the usual dynamical maps, and concerning the Holevo bound on the mutual information exchanged between two systems.

6.2 Relaxing Linearity

We consider now an Assignment Map $A$ that satisfies axioms B2 and B3 but not B1.

We recall that if a system is described by an ensemble of quantum states $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots |\varphi_i\rangle \in H^S\}$ with probabilities $\{p_1, p_2, \ldots |0 \leq p_i, \sum p_i = 1\}$ (proper mixture) then the appropriate density operator is

$$\varrho = \sum_i p_i |\varphi_i\rangle \langle \varphi_i| \quad (6.3)$$

About the evolution of such a mixture, it is physically reasonable to assert that the following axiom holds:

**B1’** $A$ is a convex linear map, i.e., $A[\sum_i p_i \rho_i] = \sum_i p_i A[\rho_i]$ for all $\rho_i \in S(H^S)$ and $p_i \geq 0$ such that $\sum p_i = 1$.

We note that B’1 implies A1. Yet, it is a simple matter to prove the next Proposition:

**Proposition 6.1.** An Assignment map $A$ that satisfies B2, B3 and B1’ assumes the form:

$$A[\varrho] = \varrho \otimes \tau \quad (6.4)$$

with $\tau$ positive trace-one operator.

**Proof.** Firstly, we note that B2 and B3 imply that for all pure states $P$

$$A[P] = P \otimes \tau(P). \quad (6.5)$$

In fact, if $A[P]$ is pure, then its Schmidt decomposition must consist in only one term, in order to avoid any contradiction with B2, so (6.5) holds. If $A[P]$
is not a pure state, we can always consider an auxiliary system $E'$ and a pure state $\varrho^{S+E+E'}$ such that $\text{Tr}_{E'} \varrho^{S+E+E'} = \mathcal{A}[P]$. By the previous reasoning we have $\varrho^{S+E+E'} = P \otimes \tau^{E+E'}(P)$, and then $\mathcal{A}[P] = \text{Tr}_{E'}(P \otimes \tau^{E+E'}(P)) = P \otimes \text{Tr}_{E'}(\tau^{E+E'}(P))$.

Let us now consider the totally mixed bidimensional state and two distinct decomposition of it

$$\varrho = \frac{1}{2} \mathbb{I}_2 = \frac{1}{2} P_1 + \frac{1}{2} P_2 = \frac{1}{2} Q_1 + \frac{1}{2} Q_2$$

with $\{P_1, P_2\} \cap \{Q_1, Q_2\} = \emptyset$. We must have:

$$\sum_i \mathcal{A}[P_i] = \sum_j \mathcal{A}[Q_j] \quad (6.6)$$

and then

$$\sum_i P_i \otimes \tau(P_i) = \sum_j Q_j \otimes \tau(Q_j) \quad (6.7)$$

Let us suppose $\tau(Q_1) \neq \tau(Q_2)$, and consider a matrix $\sigma$ such that

$$(\sigma, \tau(Q_2)) \equiv \text{Tr}\sigma\tau(Q_2) = 0 \neq \text{Tr}\sigma\tau(Q_1). \quad (6.8)$$

By direct calculation of

$$\text{Tr}_{E}(\mathcal{A}[\varrho] \mathbb{I} \otimes \sigma) \quad (6.9)$$

we obtain

$$c_1 P_1 + c_2 P_2 = c_3 Q_1 \quad (6.10)$$

that can never be satisfied, so that finally $\tau(Q_1) = \tau(Q_2)$ and

$$\mathcal{A}[\varrho] = \varrho \otimes \tau(Q_1). \quad (6.11)$$

for all pure states belonging to the line $Q_1 \lor Q_2$. We note now that the Assignment on the line $Q_1 \lor Q_2$ depends only on $\tau(Q_1)$; since for every pure state a line exists that contains the state itself and $Q_1$ we have the proof. \(\square\)

Hence, while in general convex linearity does not imply linearity, previous proposition shows clearly that axioms B1 and B1' lead to the same evolution on the set of product states (with $\tau$ fixed); now we observe that axioms B1' implies that every elements $\varrho^{S+E}$ of this set can be seen as a convex combination

$$\varrho^{S+E} = \sum_i p_i |e_i \rangle \langle e_i| \otimes \tau \quad (6.12)$$
6.3. Relaxing Positivity

for some \( \{|e_i\}\subset \mathcal{H}^E \), and then we can conclude that \( \text{Tr}_E \rho^{S+E} \) is a proper mixture; in other words, we can state \textit{a posteriori} the equivalence of B1 and B1’ if we limit ourselves to consider such a mixtures. In particular we can say that:

\textit{A particular choice of axioms for the assignment map } \mathcal{A} \textit{corresponds to a selection of subsets of } \mathcal{S}(\mathcal{H}^{S+E}) \textit{for which the dynamics has a physical interpretation.}

To understand this last point, let as consider a different kind of mixture (\textit{improper mixture} see section 1.2), which arises in the description of subsystems of a compound system.

In such case, indeed,

\[ \rho^S = \text{Tr}_E \rho^{S+E} \quad (6.13) \]

does not longer represent a proper mixture since the coefficients \( p_i \) do not admit an \textit{ignorance interpretation}, and the physical motivation of axiom B1’ are now feeble.

But, we agree that \textit{“giving up linearity is not desiderable; it would disrupt quantum theory in a way that is not experimentally supported”} [57]. Hence, the introduction of an assignment map \( \mathcal{A} \) (which corresponds to fix the environmental state and the initial correlations) is a ticklish problem we deal with in the next sections.

### 6.3 Relaxing Positivity

#### 6.3.1 Some preliminary definitions

We recall the following definitions [50]:

**Definition 6.1** \((P_m)\). \textit{We call Positivity Domain of } \(m\) \textit{ (and we write } \mathcal{P}_m \textit{) the set of all the density matrices } \rho \textit{ such that } m[\rho] \textit{ is a density matrix.}

**Definition 6.2** \((C_A)\). \textit{We call Compatibility Domain of the assignment map } \mathcal{A}, \textit{ and we write } \mathcal{C}_A, \textit{ the set of all the density matrices } \rho \textit{ such that } \mathcal{A}[\rho] \textit{ is a density matrix.}

The Compatibility Domain of \( \mathcal{A} \) is clearly the Positivity domain of \( \mathcal{A} \). The different terminology is connected with the different role that \( \mathcal{A} \) and \( m \) have in the description of open systems.

Moreover, if \( m[\rho] = \text{Tr}_E(U\mathcal{A}[\rho]U^\dagger) \) then the following inclusions hold (see figure6.1) [50]:

\[ \mathcal{S}(\mathcal{H}) \supseteq \mathcal{P}_m \supseteq \mathcal{C}_A \quad (6.14) \]

Finally, coherently with the analysis in reference [58], where non CP maps were studied, we give the following
Definition 6.3. We say that an Hermitian, trace preserving map

\[ m : \mathcal{S}(\mathcal{H}^S) \rightarrow \mathcal{B}(\mathcal{H}^S) \]  \hspace{1cm} (6.15)

is a Reduced Dynamics if and only if a system \(E\), a state \(\varrho^{S+E} = \varrho^S \otimes \varrho^E + \sigma^C\), (where \(\varrho^{S(E)} = \text{Tr}_{E(S)} \varrho^{S+E}\)) and a unitary transformation \(U\) exist such that

\[ m[\varrho] = \text{Tr}_E U [\varrho \otimes \varrho^E + \sigma^C] U^\dagger \quad \forall \varrho \in \mathcal{S}(\mathcal{H}^S). \]  \hspace{1cm} (6.16)

Comparing (6.16) and (6.1) we can recognize in the previous definition the Assignment Map

\[ \mathcal{A}[\varrho_i] = \varrho_i \otimes \varrho^E + \sigma^C \]  \hspace{1cm} (6.17)

that is clearly linear and consistent, and has a non empty compatibility domain, since by hypothesis a state exists of the form \(\varrho^{S+E} = \varrho^S \otimes \varrho^E + \sigma^C\). It is easily seen that

A necessary condition in order that a map \(m\) is a Reduced Dynamics is that

\[ m[\varrho] = \Lambda[\varrho] + A \text{Tr}(\varrho) \]  \hspace{1cm} (6.18)

where \(\Lambda\) is a completely positive, trace preserving map and \(A\) is an Hermitian, traceless matrix.

In fact using eq. (6.16) we have

\[ m[\varrho] = \text{Tr}_E U [\varrho \otimes \varrho^E + \sigma^C] U^\dagger = \text{Tr}_E U [\varrho \otimes \varrho^E] U^\dagger + \text{Tr}_E U [\sigma^C] U^\dagger \]  \hspace{1cm} (6.19)

Clearly

\[ \text{Tr}_E U [\varrho \otimes \varrho^E] U^\dagger = \Lambda[\varrho] \]  \hspace{1cm} (6.20)

with \(\Lambda\) completely positive, and

\[ \text{Tr}_E U [\sigma^C] U^\dagger = A \]  \hspace{1cm} (6.21)

is a traceless operator.
6.3. Relaxing Positivity

6.3.2 Non CP maps

Recalling the definitions given in the previous subsection, we can now state the following:

**Proposition 6.2.** Given a set \( \{ \varrho_i \} \) of linear independent density matrices, an Assignment map \( A \) that satisfies \( B1 \) and \( B2 \) assumes necessarily the form:

\[
A[\varrho_i] = \varrho_i \otimes \tau_i + \sigma^c_i
\]

where \( \text{Tr}_E \tau_i = 1 \) and \( \text{Tr}_{E(S)} \sigma^c_i = 0 \).

**Proof.** In fact a linear map is univocally determined by its action on a basis. So let \( \{ \varrho_i \} \) be a set of linearly independent density matrices, then

\[
A[\varrho_i] = \varrho_i \otimes \varrho^S + \varrho^E \quad \forall \varrho \in \mathcal{S}(\mathcal{H}^S)
\]

As it is well known, it is always possible [1] to consider the system \( S + E \) as a subsystem of a system \( S + E + E' \) such that

\[
\text{Tr}_E \varrho^S + \varrho^E = \varrho^S + \varrho^E
\]

where \( \varrho^S + \varrho^E \) is pure. Now it is simple to verify that \( U' = U \otimes \text{Id}_{E'} \) is a unitary transformation that satisfies (6.16). \( \square \)

We note explicitly that the assignment map in (6.17) is a particular case of equation (6.22). Moreover we have the following:

**Proposition 6.3.** If \( A \) is an assignment map of the type (6.17) then a pure state \( \varrho^S + \varrho^E \) and a unitary transformation \( U^S + E \) exist such that:

\[
m[\varrho] = \text{Tr}_E U^{S + E} [\varrho \otimes \varrho^E + \varrho^C] U^{S + E \dagger} \quad \forall \varrho \in \mathcal{S}(\mathcal{H}^S)
\]

where \( \varrho^C = \text{Tr}_S \varrho^S + \varrho^E \).

**Proof.** By hypothesis, a state \( \varrho^S + \varrho^E = \varrho^S \otimes \varrho^E + \varrho^C \) (not necessarily pure) and a unitary transformation \( U \) exist such that:

\[
m[\varrho] = \text{Tr}_E U[\varrho \otimes \varrho^E + \varrho^C] U^\dagger \quad \forall \varrho \in \mathcal{S}(\mathcal{H}^S).
\]

As it is well known, it is always possible [1] to consider the system \( S + E \) as a subsystem of a system \( S + E + E' \) such that

\[
\text{Tr}_{E'} \varrho^{S + E + E'} = \varrho^{S + E}
\]

where \( \varrho^{S + E + E'} \) is pure. Now it is simple to verify that \( U' = U \otimes \text{Id}_{E'} \) is a unitary transformation that satisfies (6.16). \( \square \)
Proposition 6.3 assures that any assignment map of type (6.17) can be associated with a reduced dynamics, and that in particular $\rho^E$ can always be considered as a partial trace of a pure state $\rho^{E+S}$. We can conclude that such a map is not necessarily a CP map. However, a sufficient condition can be easily stated: indeed, (see also [58])

If $\rho^C$ commutes with $U$ (for instance, if $\rho^C = 0$) then the reduced dynamics is a CP map:

$$m[\rho] = \text{Tr}_E U[\rho \otimes \rho^E] U^\dagger.$$  \hspace{1cm} (6.25)

As we will show in the next sections, this condition is not necessary in order to obtain CP maps. In particular we will see in section 5.5 that CP maps exist which are obtained from Assignment maps that do not satisfy Pechukas’ Theorem.

### 6.3.3 Physical Realization of Reduced Dynamics

We will show now how we can realize a Reduced Dynamics on a finite dimensional Hilbert space. We recall firstly a well known result for CP Maps [1]:

A trace preserving CP map can always be modelled as a reduced dynamics with assignment map

$$\mathcal{A}[\rho] = \rho \otimes \tau$$  \hspace{1cm} (6.26)

where $\tau \in S(\mathcal{H}^E)$, $\rho \in S(\mathcal{H}^S)$ and $\dim(\mathcal{H}^E) \leq \dim(\mathcal{H}^S)^2$.

An analogous results holds for Reduced Dynamics obtained from Assignment Maps of type (6.17):

**Proposition 6.4.** A Reduced Dynamics on $S$ can be modeled on a space $\mathcal{H}^{S+E}$ with $\dim \mathcal{H}^E \leq d^3$ where $d = \dim \mathcal{H}^S$.

**Proof.** By Proposition 6.3 we can limit ourselves to consider Reduced Dynamics generated from pure states:

$$|\alpha\rangle = \sum_{i=1}^{d} \sqrt{p_i} |i^S\rangle |i^E\rangle$$  \hspace{1cm} (6.27)

where $p_i \geq 0$ and $\sum_{i=1}^{d} p_i = 1$.

Clearly we must have:

$$m[\rho] = \text{Tr}_E U[\rho \otimes g^E + \rho^C] U^\dagger \forall \rho \in S(\mathcal{H}^S)$$  \hspace{1cm} (6.28)

where $g^E = \sum_{i=1}^{d} p_i |i^E\rangle \langle i^E|$. We observe that the Reduced Dynamics is univocally determined by the vectors belonging to the set $\{U |j^S\rangle |i^E\rangle\}$ whose
6.3. Relaxing Positivity

Cardinality is \( \leq d^2 \). Moreover, resorting to the Schmidt decomposition of the pure state \( U|j^S\rangle|i^E\rangle \) we can put in general (with obvious notation)

\[
U|j^S\rangle|i^E\rangle = \sum_{k=1}^{d} q_k |e_k^S(i,j)\rangle |f_k^E(i,j)\rangle.
\] (6.29)

Since the cardinality of \( \{|f_k^E\rangle\} \) is less than \( d^3 \), we have \( \dim \text{span}\{|f_k^E\rangle\} \leq d^3 \).

It is simple to verify that

\[
\mathfrak{m}[\varrho] = \text{Tr}_E U[\varrho \otimes \varrho^E + \varrho^C] U^\dagger = \text{Tr}_E (\text{Id}_S \otimes U_E) U[\varrho \otimes \varrho^E + \varrho^C] U^\dagger (\text{Id}_S \otimes U_E^\dagger)
\] (6.30)

for any \( \varrho \in \mathcal{S}(\mathcal{H}^S) \) and for all unitary matrices \( U_E \). Since we can always construct a unitary transformation \( U_E \) such that the \( |i^E\rangle \)'s are all contained in the span\( \{U_E|f_k^E(i,j)\rangle\} \) the statement easily follows.

Finally, we recall now some results regarding reduced dynamics of type (6.17).

Let us observe that the action of the transposition \( T \) on an Hermitian operator consists in a complex conjugation of the elements of the matrices. \( T \) can be associated to an \( \mathbb{R} \)–linear, positive map in \( \mathcal{S}(\mathcal{H}^S) \), which however never can assume the form (6.16). Let us start with the bidimensional case:

**Proposition 6.5.** The transposition in 2-dimension is not a Reduced Dynamic.

**Proof.** Let us introduce the operators

\[
F_1 = |e_1\rangle\langle e_1|, \quad F_2 = |e_2\rangle\langle e_2|, \quad F_3 = |e_1\rangle\langle e_2|, \quad F_4 = |e_2\rangle\langle e_1|.
\] (6.31)

Then

\[
T[\varrho] = \sum_{ij} c_{ij} F_i \varrho F_j^\dagger
\] (6.32)

where

\[
c_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\] (6.33)

We need to have a form of the type:

\[
T[\varrho] = \Lambda[\varrho] + A \text{Tr}(\varrho).
\] (6.34)
So let us suppose that exists such a \( A \) then
\[
A = |f_1\rangle\langle f_1| - |f_2\rangle\langle f_2|
\] (6.35)

with
\[
|f_1\rangle = \lambda \frac{1}{2} (|e_1\rangle - d^*|e_2\rangle),
\] (6.36)
\[
|f_2\rangle = \lambda \frac{1}{2} (d|e_1\rangle + |e_2\rangle),
\] (6.37)

\( \lambda \in \mathbb{R}^+, d \in \mathbb{C} \). Clearly
\[
\langle f_1|f_2\rangle = 0, \quad \langle f_1|f_1\rangle = \langle f_2|f_2\rangle = \lambda (1 + |d|^2).
\] (6.38)

So we can write
\[
T[\varrho] = T[\varrho] - A(\text{Tr} \varrho) + A\text{Tr}\varrho.
\] (6.39)

The term:
\[
\Lambda[\varrho] = T[\varrho] - A(\text{Tr} \varrho) = \sum_{ij} c'_{ij} F_i \varrho F_j^\dagger
\] (6.40)

with
\[
c'_{ij} = \begin{pmatrix}
1 + a & 0 & 0 & b \\
0 & 1 - a & b^* & 0 \\
0 & b & a & 1 \\
b^* & 0 & 1 & -a
\end{pmatrix}
\] (6.41)

where \( a = \lambda (1 - |d|^2) \) and \( b = -2\lambda d \) is an Hermitian map that can never be Completely Positive for any choice of \( A \) (the matrix \( c'_{ij} \) is never positive).

This proof can be easily extended to higher dimension:

**Proposition 6.6.** The transposition in \( d \)-dimension \((d > 2)\) is not a Reduced Dynamic.

**Proof.** Let us suppose that exist \( \Lambda \) CP and \( A \) traceless Hermitian matrix such that:
\[
T[\varrho] = \Lambda[\varrho] + A\text{Tr}(\varrho)
\] (6.42)

Consider the set of number:
\[
M = \{ \langle e_i|A|e_i\rangle \}_{i=1,2,...,d}
\] (6.43)

Let us consider the two greatest element of the set \( M \) and the corresponding vectors, say \( |e_1\rangle \) and \( |e_2\rangle \). Consider the CP map \( \pi \):
\[
\pi[\varrho] = (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) \varrho (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)
\] (6.44)
Let us consider the composite map:

\[ \pi \circ T \circ \pi = \pi \circ \Lambda \circ \pi[\varrho] + \pi[A]\text{Tr}\pi[\varrho] \]  

(6.45)

We restrict ourselves to the density operators on the subspace generated by \( |e_1\rangle \) and \( |e_2\rangle \). In this case the map \( \pi \circ T \circ \pi \) is the transposition \( T_2 \) in this bidimensional subspace and then the previous equation become:

\[ T_2 = \Lambda'[\varrho] + A'\text{Tr}(\varrho) \]  

(6.46)

where \( \Lambda' = \pi \circ \Lambda \circ \pi \) is a CP map and \( A' = \pi[A] \) is an Hermitian operator with \( \text{Tr}A' = \alpha \geq 0 \). So the previous equation can be rewritten in the form:

\[ T_2 = \Lambda''[\varrho] \]  

(6.47)

with \( A'' = A' - \frac{1}{2}\alpha I \) and \( \text{Tr}(A'') = 0 \). It is easy to show that

\[ T_2 = \Lambda''[\varrho] + A''\text{Tr}(\varrho) \]  

(6.48)

where \( \Lambda''[\varrho] = \Lambda'[\varrho] + \frac{1}{2}\alpha I\text{Tr}(\varrho) \) is a CP map. But then, by the previous proposition, we have a contradiction.

\[ \square \]

Remark 6.1. The proof can be extended in a trivial way to the case of antiunitary maps, i.e. map \( \Upsilon \) of the form:

\[ \Upsilon[\varrho] = UT[\varrho]U^\dagger \]  

(6.49)

We can conclude that

The set of positive map is not a subset of the set of Reduced Dynamics of type (6.17)(hence, there are positive maps that do not belong to the set of Reduced Dynamics).

Finally, we recall another well-known result [1] about the connection between the assignment and the form of the dynamics:

If a dynamical map is CP then exists an assignment of the form (6.4).

6.3.4 Generalized Dynamics

We are now ready to propose a generalization of the map of type (6.17):

Definition 6.4. We call Generalized Dynamics the maps \( \mathcal{M} \) of the form:

\[ \mathcal{M} = \mathcal{T}_E \circ \mathcal{U} \circ \mathcal{A}, \]  

(6.50)

where \( \mathcal{T}_E \) is the trace with respect to the environment, \( \mathcal{U} \) is a unitary map on the combined space \( S + E \), and \( \mathcal{A} \) is a linear, consistent but not necessarily positive assignment map.
We easily obtain:

**Proposition 6.7.** A map $\mathcal{M}$ is a Generalized Dynamics if and only if $\text{Tr}(\mathcal{M}[\varrho]) = \text{Tr}(\varrho) \forall \varrho$.

**Proof.** If $\mathcal{M}$ is a Generalized Dynamics then it is simple to verify that $\text{Tr}(\mathcal{M}[\varrho]) = \text{Tr}(\varrho) \forall \varrho$.

Conversely, let us suppose that $\mathcal{M}$ is a linear, trace preserving map. Clearly a linear map is univocally determined by its action on a basis, i.e. by the matrices $\mathcal{M}[P_i] = M_i$ where $\{P_i\}$ is a set of linearly independent projectors.

Since any Hermitian operator can always be written as the difference of two positive operator, let us put in particular

\[
M_i = M_i^+ - M_i^-
\]

where $M_i^+, M_i^- \geq 0$. Moreover, given a density operator $\rho^S$ of $S$, it is always possible to consider an auxiliary system $E$ and a pure state $\rho^{S+E}$, such that $\rho^S = \text{Tr}_E \rho^{S+E}$. Of course $\frac{M_i^\alpha}{\text{Tr}(M_i^\alpha)}$, $(\alpha = \pm)$, are density operators, then it is possible to associate to them a pure state of the whole system as previously described. In particular, given the set $\{M_i^\alpha\}$, it is possible to construct the pure states associated to every element, $\rho[M_i^\alpha]$, in such a way that they are mutually orthogonal.

Let us consider now a set of mutually orthogonal projectors $\{Q_i^\alpha\}$ in the auxiliary system, and choose an assignment map of the following form:

\[
\mathcal{A}[P_i] = \text{Tr}(M_i^+) P_i \otimes Q_i^+ - \text{Tr}(M_i^-) P_i \otimes Q_i^- = P_i \otimes (\text{Tr}(M_i^+) Q_i^+ - \text{Tr}(M_i^-) Q_i^-)
\]

Clearly $\mathcal{A}$ is a linear and consistent map, then it satisfies axioms B1 and B2. Since a unitary map certainly exists connecting these sets of mutually orthogonal states

\[
U P_i \otimes Q_i^\alpha U^\dagger = \rho[M_i^\alpha],
\]

the statement easily follows. $\Box$

Note that, trivially, if we denote by $U_E$ the transformation

\[
U_E[\rho] = (I^S \otimes U')^\dagger \rho (I^S \otimes U')
\]

it is possible to generate the same sub dynamics (i.e. with the same compatibility domain) associated with same fixed $\mathcal{A}$ and $U$, by making the following substitutions:

\[
U \rightarrow U \circ U_E \quad (6.54)
\]

\[
\mathcal{A} \rightarrow U_E^\dagger \circ \mathcal{A} \quad (6.55)
\]
In the next section we will give an explicit example of a generalized dynamics together with some remarks and comments about its features.

We note that, using the same arguments in the proof of Proposition 6.4 the environment can be modelled by a space whose dimension is \( \leq d^3 \) \((d = \dim S)\).

To conclude this section, we recall that, as it is proven in the Theorem in the Introduction, axioms B1, B2 and B3 are not useful to describe systems initially correlated with an external environment. On the other hand, would we reject at all any positivity requirement, we could construct Hermitian map of the form (6.50) that might not be interpreted as a physical evolution (the assignment map might have an empty compatibility domain). So, on the basis of previous discussions, we can relax axiom B3 with:

**B3’** \( A \) has a non empty compatibility domain.

Then, one easily sees that for every Hermitian map \( m \) there exists an assignment \( A \) that satisfies B1, B2 and B3’ if and only if \( P_m \) is non empty.

Indeed, given such an \( A \) (which has in particular a non empty compatibility domain), \( m \) is a generalized dynamics:

\[
m[\rho] = \text{Tr}_E(UA[\rho]U^\dagger).
\]  

(6.56)

that has clearly a non void positivity domain. On the contrary, if \( m \) has a positivity domain then, following the construction used in the previous Proposition, we have the proof.

Moreover, following the construction of \( A \) in the proposition 6.7 it is simple to prove that for every Hermitian map \( m \) an assignment \( A \) exists such that the convex sets \( P_m \) and \( C_A \) have the same dimension.

In particular, to define a linear map on \( S(H^S) \), we need to know its action at least on \( d^2 \) linearly independent density matrices.

So we can conclude that if we require that the map \( m \) is the result of a physical process,

*the set of positive maps is a proper subset of the set of Generalized Dynamical Maps.*

### 6.4 An example

As we anticipated in subsection 6.3.4, we will give in this section an explicit example of generalized dynamics. To this end, we reconsider a physical system, already studied in literature (see reference [50]). In their paper, Jordan *et al.* consider a bipartite system composed by two qubits, described
by two sets of Pauli matrices $\Sigma_i$ and $\Xi_i$ respectively, and subject to the evolution generated by the Hamiltonian

$$H = \omega \frac{1}{2} \Sigma_3 \otimes \Xi_1.$$  \hspace{1cm} (6.57)

In particular after a time $\omega t = \frac{1}{2} \pi$ the evolution of the system (in the Heisenberg Picture), is completely determined by the mean values:

$$\langle \Sigma_1 \rangle' = a_1 \quad \langle \Sigma_2 \rangle' = a_2 \quad \langle \Sigma_3 \rangle' = \langle \Sigma_3 \rangle,$$  \hspace{1cm} (6.58)

where $a_1 = -\langle \Sigma_2 \Xi_1 \rangle$ and $a_2 = \langle \Sigma_1 \Xi_1 \rangle$ and the prime indicates mean values evaluated at the time $t = \frac{\pi}{2 \omega}$. We consider, as in reference [50], $a_1$ and $a_2$ fixed parameters of our system. Then, the resulting map, extended by linearity, induces the following transformations:

$$m[I] = I + a_1 \Sigma_1 + a_2 \Sigma_2$$  \hspace{1cm} (6.59)

$$m[\Sigma_1] = 0$$  \hspace{1cm} (6.60)

$$m[\Sigma_2] = 0$$  \hspace{1cm} (6.61)

$$m[\Sigma_3] = \Sigma_3$$  \hspace{1cm} (6.62)

Note that the map so constructed can be considered a Generalized Dynamics that satisfies the consistency condition. It is indeed enough to put $U = e^{iHt}$, and choose as assignment map:

$$\mathcal{A}[\varrho_i] = \varrho_i \otimes \varrho_E^\prime + \sigma_i^C$$  \hspace{1cm} (6.63)

where $\sigma_i^C = \sigma_i^P - a_1 \Sigma_2 \otimes \Xi_1 + a_2 \Sigma_1 \otimes \Xi_1$ and $\varrho_i \otimes \varrho_E^\prime + \sigma_i^P$ is a linear combination of $\{ \Sigma_i \otimes \Xi_j | \text{Tr}(\Sigma_i \otimes \Xi_j \Sigma_1 \otimes \Xi_1) = \text{Tr}(\Sigma_i \otimes \Xi_j \Sigma_2 \otimes \Xi_1) = 0 \}$

These assignments are all compatible with the dynamic generated by (6.57).

Alternatively we can give more assignments and unitary evolutions that reproduce the same dynamics, by using exactly the same procedure described in the previous section. Putting $P_1 = \frac{1}{2}(1 + \Sigma_1)$, $P_2 = \frac{1}{2}(1 - \Sigma_1)$, $P_3 = \frac{1}{2}(1 + \Sigma_2)$, $P_4 = \frac{1}{2}(1 + \Sigma_3)$, we obtain indeed:

$$m[P_1] = \frac{1}{2}(I + a_1 \Sigma_1 + a_2 \Sigma_2)$$

$$m[P_2] = m[P_1]$$

$$m[P_3] = m[P_1]$$

$$m[P_4] = \frac{1}{2}(I + a_1 \Sigma_1 + a_2 \Sigma_2 + \Sigma_3)$$
6.5 Relaxing Consistency

Let us suppose that $0 \neq a_1^2 + a_2^2 \leq 1$. Noticing that an Hermitian matrix
\[ \sigma = \frac{1}{2}(I + c_1 \Sigma_1 + c_2 \Sigma_2 + c_3 \Sigma_3) \]
is positive if and only if $c_1^2 + c_2^2 + c_3^2 \leq 1$, one easily sees that $m[P_1] \geq 0$, so that
\[ m[P_1] = \alpha_1 |e_1\rangle\langle e_1| + \alpha_2 |e_2\rangle\langle e_2| \]
where $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, whereas $m[P_4] \not\geq 0$, so that
\[ m[P_4] = \beta_1 |f_1\rangle\langle f_1| - \beta_2 |f_2\rangle\langle f_2| \]
where $\beta_1, \beta_2 \geq 0$, $\beta_1 - \beta_2 = 1$ ($\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ are orthonormal bases in the space $\mathcal{H}^1$ associated to the first qubit). Following the procedure exposed in the proof of Proposition 6.7, we easily obtain a possible Assignment $A$ and a corresponding unitary transformation $\mathcal{U}$ that satisfy $m = \mathcal{T}_E \circ \mathcal{U} \circ A$.

We stress here that the choice of the set $\{P_i\}$ influences the subsequent dynamics for the compound system, as it is evident from the formulas above, and different choices lead to different dynamics of whole system. It is simple to show that at least two different assignment $A_1$ and $A_2$ and unitary transformation $\mathcal{U}_1$ and $\mathcal{U}_2$ exist for the same dynamics (that are not trivially connected as in (6.54), (6.55)). This non uniqueness in the possible representations for the same dynamics is clearly connected with the loss of knowledge about the environment and possible correlations with it, owing to the operation of partial trace, which can be interpreted as the removal of the environment in a possible measurement on the system of interest. We will return on this point in section 6.6.

6.5 Relaxing Consistency

In this section we limit ourselves to some observations on non consistent, linear, positive maps. Some examples of such maps have been recently studied [57] in connection with the evolution of classically correlated states. In particular, the following Assignment Map is introduced in reference [57]:
\[ \mathcal{A}[Q] = \sum_i \text{Tr}[Q \Pi_i] \Pi_i \otimes \tau_i \quad (6.64) \]
where $\Pi_i$ is a set of orthonormal projectors on $\mathcal{H}^S$. As it is evident $\text{Tr}_E \mathcal{A}$ provides the components of $Q$ on the $n-$dimensional subspace generated by the orthonormal set $\{\Pi_i\}$. Clearly, the set
\[ \mathcal{E}_A = \{ \varrho \in \mathcal{S}(\mathcal{H}) | \text{Tr}_E \mathcal{A}[\varrho] = \varrho \} , \quad (6.65) \]
that we call *consistency domain* of $\mathcal{A}$, is a proper subset of $\mathcal{S}(\mathcal{H})$ and in particular it has the form

$$\mathcal{C}_A = \{ \varrho = \sum_i p_i |\Pi_i| \varrho \geq 0, \sum_i p_i = 1 \}, \quad (6.66)$$

*i.e.*, $\mathcal{C}_A$ is a convex set generated by mutually orthogonal pure states. The Assignment map (6.64), as it is evident, is not only positive but also CP, so that the composition:

$$m[\varrho] = \text{Tr}_E(U\mathcal{A}[\varrho]U^\dagger). \quad (6.67)$$

is a CP map itself. In the following, we limit ourself to Completely Positive Assignment Maps, studying their consistency domain.

We recall that by *Shauder’s fixed point theorem* [1], every completely positive assignment map has a non empty consistency domain. Then we can prove the following Proposition, which generalizes to some extent the above observation on the structure of $\mathcal{C}_A$:

**Proposition 6.8.** If the consistency domain of a CP Assignment Map $\mathcal{A}$ contains two non orthogonal pure states $P, Q$, then it contains every pure state belonging to the line $P \lor Q$.

**Proof.** Let $P = |e_0\rangle\langle e_0|$ and $Q = |e_1\rangle\langle e_1|$ be two pure states. We note that every CP Assignment map can be written as [1]:

$$\mathcal{A}[\varrho] = \text{Tr}_{E'}U'[|\varrho \otimes \varrho^{E'}\rangle \otimes \varrho^{E'}]U'^\dagger \quad (6.68)$$

where $E$ and $E'$ are two auxiliary systems and $\varrho^E \in \mathcal{S}(\mathcal{H}^E)$ and $\varrho^{E'} \in \mathcal{S}(\mathcal{H}^{E'})$ are fixed pure states.

Obviously the consistency domain of $\mathcal{A}$ is the same of the consistency domain of

$$\mathcal{A}'[\varrho] = U'[|\varrho \otimes \varrho^E \otimes \varrho^{E'}\rangle U'^\dagger \quad (6.69)$$

We must have, imposing consistency,

$$\mathcal{A}'[|e_0\rangle\langle e_0|] = |e_0\rangle\langle e_0| \otimes |p\rangle\langle p|, \quad (6.70)$$

and

$$\mathcal{A}'[|e_1\rangle\langle e_1|] = |e_1\rangle\langle e_1| \otimes |q\rangle\langle q|, \quad (6.71)$$

for some (normalized) $|p\rangle, |q\rangle \in \mathcal{H}^{E+E'}$. Moreover, we have (using (6.68))

$$\text{Tr}(\mathcal{A}'[|e_0\rangle\langle e_0|]\mathcal{A}'[|e_1\rangle\langle e_1|]) = |\langle e_0|e_1\rangle|^2 \quad (6.72)$$

while using (6.70) and (6.71)

$$\text{Tr}(\mathcal{A}'[|e_0\rangle\langle e_0|]\mathcal{A}'[|e_1\rangle\langle e_1|]) = |\langle p|q\rangle|^2 |\langle e_0|e_1\rangle|^2 \quad (6.73)$$
whence we deduce

$$|\langle p|q\rangle||\langle e_0|e_1\rangle| = |\langle e_0|e_1\rangle|. \quad (6.74)$$

Since, by hypothesis, $|\langle e_0|e_1\rangle| \neq 0$ then we must have $\langle p|q\rangle = e^{i\alpha}$ and the thesis follows at once.

(Note that, on the contrary, if $|\langle e_0|e_1\rangle| = 0$ then we can have $\langle p|q\rangle \neq e^{i\alpha}$, and some elements on the line $P \lor Q$ can exist that do not belong to the consistency domain).

Previous proposition shows that, as in the example at the beginning of the section, the consistency domain of the map $\mathcal{A}$ can be generated by convex combination from a set of pure states $\mathcal{E}_{\mathcal{A}}^P$ that can be partitioned in mutually orthogonal sets $(\mathcal{E}_{\mathcal{A}}^P)_i$ such that if $P_1 \in (\mathcal{E}_{\mathcal{A}}^P)_i$ and $P_2 \in (\mathcal{E}_{\mathcal{A}}^P)_j$ then $P_1 \lor P_2 \subseteq (\mathcal{E}_{\mathcal{A}}^P)_i$ if and only if $i = j$.)

\section{6.6 Holevo Bound}

In section 6.3 we saw that an Hermitian map with non empty positivity domain can be always seen as a dynamical map. We examine now the consequence of this fact in the transmission of information through quantum channels. The main quantity of interest for such a problem is the Holevo Bound. We recall the following:

\textbf{Theorem 6.2} (The Holevo Bound). [1] Suppose that Alice prepares a state $\rho_X$, where $X = 0, \ldots, n$, with probabilities $p_0, \ldots, p_n$. Bob performs a measurement described by POVM elements $\{E_y\} = \{E_0, \ldots, E_n\}$ on the state, with measurement outcome $Y$. The Holevo bound states that for any such measurement Bob may do:

$$H(X : Y) \leq S(\rho) - \sum_x p_x S(\rho_X) \quad (6.75)$$

where $\rho = \sum_X p_X \rho_X$.

The right side of inequality (6.75) is called Holevo Bound or Holevo quantity and constitutes an upper bound on the accessible information $H(X : Y)$.

Firstly, we observe that a Positive Operator Valued Measure (POVM) is a trace preserving CP map. So if the system evolves according to a CP map, the Holevo bound cannot increase: this fact can be interpreted as the natural corruption of information during the time. Yet as is simple to verify, there are many dynamical maps, originated by linear, consistent, but not positive assignment maps that appear to violate this upper bound. How can we solve this seeming contradiction?
Let us suppose that the system evolves accordingly to a (non CP) map $m$. If the map $m$ is really a “physical” evolution then an assignment $A$ and a unitary transformation $U$ exist such that:

$$m[\rho] = \text{Tr}_E(UA[\rho]U^†). \quad (6.76)$$

So we can say that Alice prepares a state $A[\rho_X]$ where $X = 0, \ldots, n$ with probabilities $p_0, \ldots, p_n$, and the state evolves according to $U$. Recalling that the unitary evolution, the partial trace and the POVM described by the elements $\{E_i\}$ are all CP maps we can apply the Holevo Bound theorem and obtain an upper bound on the accessible information:

$$H(X : Y) \leq S(A[\rho]) - \sum_x p_X S(A[\rho_X]). \quad (6.77)$$

On the other hand, since discarding systems by partial trace cannot increase mutual information

$$S(\varrho) - \sum_x p_X S(\varrho_X) \leq S(A[\rho]) - \sum_x p_X S(A[\rho_X]). \quad (6.78)$$

Hence, we see that equation (6.75) cannot be deduced from (6.77) and (6.78) and then a violation of Holevo bound theorem is possible, when we consider generalized dynamics.

However, for such dynamics we can give a new upper bound on the accessible information $H(X : Y)$. In fact as it is well known

$$S(\varrho) - \sum_X p_X S(\varrho_X) \leq H(X) \quad (6.79)$$

where $H(X)$ is the Shannon Entropy of the random variable $X$. It is simple to show that, given an Hermitian map $m$, there exist an assignment $A$ which maps the $\varrho_X$’s into orthogonal states, and a unitary map $U$ such that:

$$m[\rho] = \text{Tr}_E(UA[\rho]U^†),$$

so that [1]

$$S(A[\rho]) - \sum_x p_X S(A[\rho_X]) = H(X).$$

We can conclude that if a quantum channel violates the Holevo bound, then the system under study exhibits with certainty the existence of some correlation with an external environment, owing to its initial preparation. On the contrary the inequality $H(X : Y) \leq H(X)$ can never be violated by a
physical process so the quantity $H(X)$ can be considered as the upper bound for the accessible information for correlated systems.

As it is evident, to obtain the previous results it is necessary to assume the existence of correlations between the system under study and an auxiliary system, which implies that $\rho_X$ does not represent in general a proper mixture but, on the contrary, an improper mixture. As a collateral results, we obtain then that the potential information content of an improper mixture can be sensibly greater than the one of a proper mixture described by the same operator.

On the grounds of such discussion, we can conclude that generalized dynamics (characterized by non CP assignment maps) are then able to describe open systems, which interacted with an external environment.

Indeed, despite their non uniqueness and even if partial tracing on the environment degrees of freedom seemingly removes any information about preexistent correlation, such class of dynamics exhibits in a clear way the existence of initial correlations (if any) of the studied system since a greater amount of accessible information is associated with them. Besides, this fact points out the need to retain the epistemological distinction between proper and improper mixtures.
Conclusions

The aim of this thesis was to present some problems that appear in the treatment of open systems in standard and alternative quantum mechanics.

Pseudo Hermitian operator have their main applications in the study of alternative descriptions of quantum mechanics. As we saw in chapter 2, these last are proven (at least in case of simple systems) to be equivalent to the standard one by means of the unitary isomorphism introduced in sec. 2.3.

The first problem, presented in chapter 3, deals with the description of multipartite systems. Indeed, severe limitations arise for compound quantum systems to be described in terms of $k$ subsystems. We proposed some criterions to single out when such decomposition is allowed, and the aforesaid equivalence still hold.

In chapter 4 we analyzed a possible application of pseudo-Hermitian systems to ultrafast quantum computation. Some errors that can appear in a careless use of pseudo Hermitian operator have been pointed out, together with the (energy) cost of this procedure. Surprisingly, Pseudo Hermitian operator can have applications in the treatment of Hermitian quantum systems as exposed in chapter 5. In particular, we shown that the generator of a dynamical semigroup is represented by a pseudo Hermitian operator and this last property ensures the possibility to introduce monotonically changing functional that characterize the irreversible behavior of the dynamics. These observations have been generalized to quasi Hermitian open systems and two new Lyapunov functional have been introduced, which can be useful in the (yet doubtful) case of degenerate systems.

Finally, in chapter 6, we extended the set of possible dynamical map relaxing, by physical arguments, the axioms used as standard starting point for the treatment of open dynamics in order to overcome some criticisms of this axiomatic approach, due to Pechukas’s theorem, and extended its applicability to correlated systems. Finally we give a set of weaker axioms that we use to define a larger set of physical interpretable evolutions (generalized dynamics). The application of this map introduce a paradox in quantum
information theory (violation of the Holevo Bound) that we clarify by means of physical arguments.
Bibliography


Bibliography


