Imbalanced Holographic Superconductors

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Introduction

Our ability to extract results from a quantum field theory mostly relies on perturbation theory. In this framework the physical observables are usually evaluated as an expansion in powers of the coupling constant, i.e. a dimensionless parameter, $g$, which measures the departure from a free field theory. When $g$ is small, i.e. $g \ll 1$, perturbation theory provides a reliable tool for computing physical quantities. When $g$ becomes of order 1, this approach is bound to fail, since we cannot view the field theory as a small deformation of the free one. In this case we shall say that the field theory is strongly interacting.

In physics there are many examples of strongly interacting quantum field theories. In the realm of high energy physics a prototypical example is quantum chromodynamics (QCD). The asymptotically-free nature of QCD [1, 2] makes perturbation theory reliable at high energy. On the other hand, at low energies, QCD becomes strongly coupled so that relevant phenomena such as confinement and chiral symmetry breaking are non-perturbative in nature. These features make the analytic study of low energy QCD very difficult.

Interesting regimes which cannot be captured by perturbation theory also occur at finite temperature and finite baryon density. For instance, hadrons, formally bound states of quarks and gluons, deconfine at high temperature leading to a new phase of matter: the quark-gluon plasma (QGP). Experimental evidence of the QGP has been observed at the Relativistic Heavy Ion Collider (RHIC) in Brookhaven (USA) see e.g. [3] and it is under investigation at the Large Hadron Collider (LHC) at CERN. The main property of the QGP is that it seems to behave as a strongly coupled fluid rather than a weakly coupled gas. Hence the investigation of its equilibrium and non-equilibrium properties from a theoretical point of view needs non-perturbative tools.

The only powerful non-perturbative first-principle approach to QCD, is based on a reformulation of the theory on a discrete Euclidean spacetime Lattice [4] and on Monte Carlo numerical analysis. However, this method is not well suited to describe finite quark density regimes and real time issues.

As an alternative, people has developed phenomenological effective field theories which
are believed to reproduce some aspects of the infra-red (IR) physics of QCD. Relevant examples are the chiral lagrangian for chiral symmetry breaking and Nambu-Jona-Lasinio (NJL) models (see e.g. references in [5]) for finite density issues.

Other paradigmatic examples of strongly coupled systems arise in the realm of condensed matter physics. In some cases, as suggested by Sachdev [6], the strong interaction nature is due to the appearance of quantum critical points at zero temperature. These points are actually described by scale invariant quantum field theories because of the infinite correlation length which arises.

Traditional condensed matter tools, based on weakly interacting quasiparticles, such as Landau-Fermi liquid theory and BCS theory (see e.g. [7, 8]), provide extremely successful descriptions of standard materials displaying superconductivity or superfluidity. However, these standard methods do not give reliable theoretical descriptions of unconventional systems for which, thus, a quasiparticle interpretation is lacking.

Examples of strongly coupled regimes appear in the description of the physics of gases of cold trapped atoms (see e.g. [9]). In the experimental setups in which they are realized, there is the possibility of tuning some external parameter so that the system goes from a weakly coupled BCS regime to a strongly coupled Bose-Einstein condensate (BEC) one. The physics at the crossover between the two regimes is governed by a strongly coupled scale invariant theory.

Another example is found within unconventional superconductors, such as high-$T_c$ ones, displaying superconductivity below a relatively high critical temperature $T_c$. Their phase diagram is often conjectured to include a quantum critical point. Both the superconducting and normal phase developing around quantum critical points (i.e. in the so-called quantum-critical region) require in principle non-standard theoretical tools, namely non-BCS and non-Fermi liquid theories, to be employed.

In the last years a relevant non-standard tool to address non-perturbative questions in field theory has been developed in the realm of string theory. The tool goes under the name either of AdS/CFT or gauge/gravity or holographic correspondence [10, 11, 12]. In brief, it is based on a conjectured duality between certain strongly coupled regimes of ordinary quantum field theories in $d$ spacetime dimensions and classical (i.e. weakly coupled) theories of gravity in at least $d+1$ dimensions. As a result, the correspondence

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1 While conventional phase transitions occur at finite temperature, when the growth of random thermal fluctuation leads to a change in the physical state of a system, quantum phase transitions, which take place at absolute zero, are driven by quantum fluctuations.

2 The term duality indicates a correspondence between two theories in different regimes of their couplings.

3 The necessary extra dimension on the gravity side is mapped in to the Renormalization Group energy
maps difficult quantum problems on the field theory side into easier, classical ones on the gravity side. In its simplest form, the correspondence relates a strongly coupled conformal field theory (CFT) to classical gravity on Anti-de Sitter (AdS) backgrounds.

Differently from other non-perturbative approaches, the holographic correspondence is well suited to study not only equilibrium physics but also real-time processes, phases with non-zero fermionic densities, transport coefficients and response to perturbations. The main limitation of this approach is that, at present, realistic field theories like QCD cannot be directly explored. However, despite its limitation to toy models, the correspondence has provided valuable insights at both the quantitative and the qualitative level on properties of strongly coupled systems realized in nature (paradigmatic examples are provided by the transport properties of the QGP, see [13] as a review).

Applications of this duality in the realm of condensed matter physics can be found in the context of unconventional superconductors. Assuming that a conformally invariant quantum critical point develops in their phase diagram, one can in principle map this one into an AdS gravity background, implementing the holographic correspondence in its simplest form. Perturbations within the quantum critical region can be simply accounted for by the dual gravity setups, too. For example one can easily go to a finite temperature regime, which on the gravity side amounts to place a black hole at the center of the AdS spacetime. Analogously one can vary other external parameters, like chemical potential, magnetic field etc. in a precisely controlled way from the dual gravity perspective. Many attempts to use the AdS/CFT approach to model strongly coupled superconductors have been recently made by Hartnoll et al. in [14, 15, 16]. The $U(1)$ symmetry breaking phase, which characterizes superconductivity, is mapped into dual charged black hole solutions exhibiting a non-trivial profile for a charged scalar field dual to Cooper-like condensates. Transport properties, such as conductivity, can then be extracted without referring to microscopical details of the dual field theory model.

The present thesis fits into this research line and its main goal is to investigate, within the holographic approach, whether certain features predicted by the weakly coupled analysis extend to the strong coupling regime. With the aim of focusing on a particular issue, we have decided to consider the behavior of superconductivity in the presence of a chemical potential imbalance $\delta\mu$ between the fermionic species condensing into Cooper-like pairs.

The occurrence of superconductive phases where two fermionic species are involved with different populations, or different chemical potentials, is an interesting possibility relevant both in condensed matter and in finite density QCD contexts. A chemical potential mismatch is naturally implemented in QCD setups due to differences between the quark

scale of quantum field theories.
species (see e.g. [17]). In metallic superconductors the imbalance can be realized by means of the Zeeman coupling of an external magnetic field with the spins of the electrons. At weak coupling, imbalanced Fermi mixtures are expected to develop novel inhomogeneous superconducting phases, where the Cooper pairs have non zero total momentum. This is the case of the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase [18]. The latter can develop provided the chemical potential mismatch is not too large (otherwise the system reverts to the normal non-superconducting phase) and not below a limiting value $\delta \mu = \delta \mu_1$ found by Chandrasekhar-Clogston [19]. At this point, at zero temperature, the system experiences a first order phase transition between the standard superconducting and the LOFF phase.

The experimental occurrence of such inhomogeneous phases is still unclear, and establishing their appearance in strongly-coupled unconventional systems from a theoretical point is a challenging question.

With the aim of providing some toy-model-based insights on this issue, we have studied the simplest holographic realization of strongly coupled imbalanced superconductors. Motivated by the experimental evidence that high $T_c$ superconductors are effectively layered, and so describable in terms (2+1)-dimensional quantum field theories (around their critical point), we have considered gravitational dual models in 3+1 dimensions. The breaking of a $U(1)_A$ symmetry characterizing superconductivity is driven, on the gravity side, by the appearance of a non trivial profile for a scalar field charged under a $U(1)_A$ Maxwell field in an asymptotically $AdS_4$ black hole background as in [15, 16]. The chemical potential mismatch is accounted for in the gravity setup by turning on the temporal component of another Maxwell field $U(1)_B$ under which the scalar field is uncharged.

The model depends on two parameters, namely the charge of the scalar field and its mass. For a particular choice of the latter, aimed on implementing a condensate of canonical dimension 2, we will show that the critical temperature below which a superconducting homogeneous phase develops decreases with the chemical potential mismatch, as is expected in weakly coupled setups. However, the phase diagram arising from the holographic model shows many differences with respect to its weakly coupled counterparts. In particular there is no sign of a Chandrasekhar-Clogston bound at zero temperature and the phase transition is always second order. Moreover, it seems that there is no evidence of a LOFF phase. A different situation arises for different choices of the parameters which seem to allow for Chandrasekhar-Clogston bounds at zero temperatures.

This work is organized as follows. In chapter 1 we will firstly provide a basic introduction of the two sides of the holographic correspondence, namely conformal field and $AdS$ backgrounds. Then we will get through the statement and the main implications of the AdS/CFT correspondence. In chapter 2 we will develop the main generalizations of the
duality useful in the condensed matter realm, namely we will extend it to finite temperatures and finite chemical potential regimes. In chapter 3 we will provide some condensed matter background, focusing on (imbalanced) superconductors. We will first review their properties within the BCS theory and then we will briefly report on some aspects of unconventional superconductivity, providing some motivations for applying holographic tools to these systems. In chapter 4 we will introduce our holographic model for imbalanced superconductivity at strong coupling and discuss its main features, based on both analytic and numerical methods. We will end up in with few concluding remarks and a list of future developments.

### Notation

| Planck units | $\hbar = c = 1$ |
| flat metric | $\eta_{\mu\nu} = \text{diag}(-1,1,\ldots,1)$ |
| vector in $d$-dimensional space | $x_\mu = (t,\vec{x})$, or simply $x$ |
| labels of the boundary fields | greek indices $\mu,\nu\ldots$ |
| labels of the bulk fields | latin indices $a,b,\ldots$ |
Chapter 1

AdS/CFT correspondence

The AdS/CFT correspondence is a conjectured equivalence between conformal quantum field theories (CFT) and higher dimensional theories of quantum gravity (strings) in asymptotically Anti-de Sitter (AdS) backgrounds. The original statement [10, 11, 12] specifically involves the $SU(N_c)$ Super Yang-Mills theory with four supersymmetries ($\mathcal{N} = 4$ SYM) in four dimensions and the type IIB superstring theory in a curved $AdS_5 \times S^5$ background. The remarkable aspect of the correspondence is a duality map between different regimes of the two theories. The $\mathcal{N} = 4$ SYM is a scale invariant theory characterized by the Yang-Mills coupling $g_{YM}$ and the number of colors $N_c$. On the other side there is a closed string theory (see e.g. [20]) characterized by the string’s length $l_s$ and the string coupling $g_s$. Explicit calculations, see [21, 22] for a review, show that the dimensionless parameters on both sides are related in the following way

$$g_s = \frac{g_{YM}^2}{4\pi}, \quad \frac{L^4}{l_s^4} = g_{YM}^2 N_c,$$  \hspace{1cm} (1.1)

where $L$ is the radius of curvature of the $AdS_5$ and $S^5$ spaces. Let us now consider the $g_s \to 0$ limit, so that all the quantum corrections due to string loops are suppressed. Furthermore let us take the low energy limit $E \ll l_s^{-1}$, so that strings can be considered as effectively point-like objects. The resulting theory is just a classical theory of (supersymmetric type IIB) gravity in ten dimensions, see [20]. One can also use the ten-dimensional Newton constant $G_{10} = \frac{(2\pi)^7}{16\pi} g_s^2 l_s^8 \sim l_p^8$, where $l_p$ is the Planck length, in place of $g_s$ in (1.1) and obtain equivalently

$$G_{10} \frac{L^8}{l_p^8} = \frac{\pi^4}{2N_c^2}, \quad \frac{L^4}{l_s^4} = g_{YM}^2 N_c.$$  \hspace{1cm} (1.2)

The above mentioned independent limits on the string model can now be rewritten as

$$\frac{L^8}{l_p^8} \sim N_c^2 \gg 1, \quad \frac{L^4}{l_s^4} = \lambda = g_{YM}^2 N_c \gg 1,$$  \hspace{1cm} (1.3)
where $\lambda = g_{\text{YM}}^2 N_c$ is the 't Hooft coupling. The first limit corresponds in the conformal quantum field theory side to the large $N_c$ limit at fixed $\lambda$

$$N_c \to \infty, \quad g_{\text{YM}}^2 \to 0 \quad \text{with} \quad \lambda = g_{\text{YM}}^2 N_c \quad \text{fixed}, \quad (1.4)$$

and the second limit to the strong 't Hooft coupling. We conclude that the $\lambda \to \infty$ and large $N_c$ limit in the quantum field theory side are mapped into the region of parameters $g_s \to 0$ and $E \ll l_s^{-1}$, where the full string theory reduces to a classical theory of gravity, namely a type IIB supergravity in ten dimensions, as sketched in figure 1.1.

Even if the correspondence has not been proved yet at the mathematical level, it passed a considerable number of checks and it is believed to be true in the whole range of the parameters. However, beyond the limits in (1.3), getting some valuable insights on both sides of the duality becomes harder and harder as the gravity theory becomes highly quantum or deeply involved in the stringy realm.

As we have just mentioned the original Maldacena’s statement for the correspondence [10] involves string theory, and to understand it one has to get first through several additional technologies such as supersymmetry, supergravity, etc. However, for the aims of this thesis, we will adopt another point of view trying to justify the AdS/CFT correspondence without referring to any particular stringy realization.

The outline of this chapter is the following. In section 1.1 and 1.2 we will present the two players of the duality, namely conformal field theories and AdS backgrounds. These
sections have to be intended as basic introductions of some notions which will be useful in the following. In section 1.3 we will try to give some arguments supporting the plausibility of the \( \text{AdS}/\text{CFT} \) correspondence, giving along the way a picture of the validity regime of the duality. In section 1.4 we will discuss how to relate quantities on both sides of the duality and how to compute correlation functions of quantum field theories from the equations of motion of classical gravity theories in \( \text{AdS} \) backgrounds.

Standard reviews on the \( \text{AdS}/\text{CFT} \) correspondence involving explicit realizations in the string theory realm are given by [21, 22, 23]. However, [13, 24, 25, 26] are good references from which we based the outline of this section.

### 1.1 Conformal field theories

Conformal field theories have quite peculiar properties. In addition to Poincaré invariance they have a scaling symmetry linking physics at different scales. This feature is in contrast with the existence of asymptotic states, since given a state with a definite mass one can construct a continuous spectrum of states with mass ranging from zero to infinity. This does not allow for the standard definition of an S-matrix formalism. However, one can use conformal invariance to strictly constrain observables of the theory such as the correlation functions.

Many interesting theories, like Yang-Mills theory in four dimensions, are classically scale-invariant; but generally this scale invariance does not extend to the quantum theory whose definition requires a cutoff which breaks scale invariance. There are, however, some special cases in which scale invariance is preserved at the quantum level. This is the case of finite theories, such as the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory \( (\mathcal{N} = 4 \ \text{SYM}) \) in four dimensions, and theories at fixed points of the renormalization group flow. Moreover scale invariance is a common feature of quantum critical points in condensed matter models. These are points at zero temperature in the phase diagram at which a certain quantum phase transition happens by tuning some external parameter. Thus, studying scale-invariant theories is relevant for various physical applications both in the realm of high energy and in condensed matter physics.

In this section we will just review some basics of the conformal group and its implications for field theories, focusing on features which will be useful in the contest of the \( \text{AdS}/\text{CFT} \) correspondence. Good reviews can be found, e.g. in [27, 28].
1.1.1 Classical scale invariance

A classical field theory without scales or dimensionful parameters is invariant under dilatations, i.e. under simultaneous rescaling of coordinates and fields. The simplest example of a conformal field theory in 3+1 dimensions is the one containing a single scalar field with a quartic interaction. The action

\[ S = \int d^4x \left( (\partial \phi)^2 + \frac{\lambda}{4!} \phi^4 \right) \]  

is invariant under the transformation of the field induced by the transformation on the spacetime coordinates \( x \rightarrow ax \)

\[ \phi(x) \rightarrow a\phi(ax). \]  

(1.6)

The coupling constant \( \lambda \) is dimensionless, which ensures the scaling invariance of the theory at the classical level. A mass term in the action would break this invariance explicitly.

More generally the transformation of a generic field \( \Psi \) under dilatations is

\[ \Psi(x) \rightarrow a^\Delta \Psi(ax), \]  

(1.7)

without involving any Lorentz index. The field gets simply rescaled by a power of \( a \) given by the scaling dimension \( \Delta \) of the field. The latter corresponds, at the classical level, to the canonical dimension in mass derived from the free action by dimensional analysis.

A further example of classical scale invariance is given by the Yang-Mills (YM) action in 3+1 dimensions

\[ S_{YM} = -\int d^4x \frac{1}{4g_{YM}^2} Tr(F_{\mu\nu}F^{\mu\nu}). \]  

(1.8)

Here \( g_{YM}^2 \) is the dimensionless gauge coupling and

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \]  

are matrices transforming in the adjoint representation of an \( SU(N) \) group. Classical scale invariance is also retained by coupling this theory with massless scalars and fermions.

Scaling symmetry is naturally broken by quantum corrections, this happens for the two examples above. However, there are cases still supporting a notion of scale invariance at the quantum level as we will see in paragraph 1.1.4.

1.1.2 The conformal group

Before going through the quantum version of a scale invariant theory let’s see how this simple dilatation symmetry can be enhanced, under general assumptions, to a larger symmetry group, i.e. the conformal group.
1.1. CONFORMAL FIELD THEORIES

Conformal transformations are those which leave the metric invariant up to an arbitrary function of the spacetime coordinates, i.e. a conformal weight $\Omega(x)$

$$ds^2 = dx_\mu dx^\mu \rightarrow \Omega^2(x)dx_\mu dx^\mu.$$  \hfill (1.10)

When the spacetime dimension is $d = 2$ the conformal group is infinite-dimensional and corresponds to all possible holomorphic transformations on a complex plane. These kind of transformations leave the angles between vectors on a plane invariant. This is the kind of symmetry present on string’s worldsheet, and it is useful to compute string scattering amplitudes, see [29] for a review.

Conformal field theories interesting to us leave in a higher dimensional spacetime, then let us focus on the cases with $d > 2$, in which the conformal group is finite.

From (1.10) we see that the conformal group is a generalization of the usual Poincaré group. In fact, when $\Omega^2(x) = 1$ the metric is left completely invariant and we are dealing with Lorentz transformations and translations. When $\Omega^2(x) =$ const. the metric gets rescaled by an overall constant factor. The novelty are the special conformal transformations which change the metric by an overall factor $\Omega^2(x)$ strictly dependent on the spacetime coordinates $x$.

The content of the conformal group can be investigated in detail by looking at the conformal algebra. Parameterizing the infinitesimal transformations of the coordinates and of the metric by

$$x'_\mu = x_\mu + V_\mu(x) + \ldots$$

$$\Omega(x) = 1 + \frac{\omega(x)}{2} + \ldots,$$

and using (1.10) we find the condition

$$\partial_\nu V_\mu + \partial_\mu V_\nu = \omega(x)\eta_{\mu\nu}.$$ \hfill (1.12)

Taking the trace of (1.12) we find $\omega(x) = 2\frac{\partial_\nu V_\nu}{d}$ and finally the relation

$$\partial_\nu V_\mu + \partial_\mu V_\nu = 2\frac{(\partial_\rho V_\rho)}{d}\eta_{\mu\nu}.$$ \hfill (1.13)

In order to satisfy this condition the general infinitesimal displacement $V_\mu$ should be at most quadratic in the coordinates [28]. Plugging such ansatz in (1.13) one finds the independent parameters for the infinitesimal conformal transformations

$$\delta x^\mu = \begin{cases} \begin{array}{c} a^\mu \\
\omega^{\mu\nu} x_\nu \\
a x^\mu \\
b^\mu x^2 - 2x^\mu (b \cdot x) \end{array} \end{cases} \begin{bmatrix} [a^\mu] \\
[\omega^{\mu\nu}] = -[\omega^{\nu\mu}] \\
[a] \\
[b^\mu] \end{bmatrix} \begin{bmatrix} P_\mu \\
J_{\mu\nu} \\
D \\
K_\mu \end{bmatrix}.$$ \hfill (1.14)
These are \(d\) independent translations labeled by \(a_\mu\), \(\frac{d}{2}(d-1)\) independent Lorentz transformations labeled by the antisymmetric parameter \(\omega_{\mu\nu}\), \(d\) special conformal transformations labeled by the vector \(b_\mu\) and 1 independent parameter \(a\) for the dilatations. All together there are \(\frac{1}{2}(d+2)(d+1)\) independent infinitesimal parameters; to each of them there corresponds a generator of the conformal algebra on the right hand side of (1.14). By an explicit isomorphism one can relate the conformal generators to the generators of the group \(SO(2,d)\). This is the group of the transformations preserving the linear element in \(R^{(2,d)}\) with two time directions and \(d\) spatial ones

\[
 ds^2 = -dx_0^2 - dx_{d+1}^2 + dx_1^2 + \ldots + dx_d^2. \tag{1.15}
\]

Another conformal group transformation is given by the inversion, i.e. a discrete transformation

\[
 x_\mu \rightarrow \frac{x_\mu}{x^2} \tag{1.16}
\]

which as well leaves the metric invariant up to a conformal factor

\[
 ds^2 = \frac{ds^2}{x^4}. \tag{1.17}
\]

Then we shall denote a general conformal group \(Conf(d)\) in \(d > 2\) Lorentz spacetime by its isomorphic version \(O(2,d)\). When the starting spacetime is euclidean the conformal group is \(O(1,d+1)\), the group of transformations which leaves invariant a linear element analogue to (1.15) but with only one time direction and \(d+1\) spatial ones.

### 1.1.3 Scale invariance and conformal invariance

At this point let’s see in more detail how scale and Poincaré invariance may imply the full conformal invariance in a field theory under certain technical assumptions.

Take first a scalar field theory invariant under translations; Noether’s theorem implies the existence of a conserved current

\[
 J_\mu = T_\mu^\nu a^\nu. \tag{1.18}
\]

with a conserved stress-energy tensor

\[
 T_\nu^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \delta_\nu^{\mu} \mathcal{L} \quad \text{with} \quad \partial_\mu T^{\mu\nu} = 0. \tag{1.19}
\]

Analogously, to each spacetime symmetry is associated a conserved current which can be set to the form

\[
 J_\mu = T_\mu^\nu \delta x^\nu. \tag{1.20}
\]
Here the stress energy-tensor $T_{\mu\nu}$ is not generally the same of (1.19). However, since the conserved current is defined up to an antisymmetric tensor $C^{\mu\nu}$

$$J^\mu = J^\mu + \partial_\nu C^{\mu\nu}$$

(1.21)

$$\partial_\mu J^\mu = 0 \rightarrow \partial_\mu J'^\mu = 0,$$

the new stress-energy tensor in (1.20) can be suitably obtained from (1.19) through the Belifante procedure [30]. The new stress-energy tensor is the so-called Belifante tensor and it is symmetric. In fact, when the infinitesimal displacement of the coordinates is due to a Lorentz transformation $\delta x^\nu = \omega^{\nu\rho} x_\rho$ the associated conserved current reads

$$\partial_\mu J'^\mu = \partial_\mu (T_{\mu\nu} \omega^{\nu\rho} x_\rho) = T_{\mu\nu} \omega^{\nu\mu} = 0,$$

(1.22)

with a symmetric stress-energy tensor. When the theory is invariant under dilatations the conserved current is $J_\mu = T_{\mu\nu} \lambda x^\nu$, and the associated stress-energy tensor is traceless under certain technical assumptions

$$\partial_\mu J_\mu = \lambda T_{\mu}^{\mu} = 0.$$  

(1.23)

The current for conformal transformations is $J_\mu = T_{\mu\nu} V^\nu$. If the theory is invariant under the Poincaré group, then, as we have seen above, it admits a conserved symmetric stress-energy tensor. The derivative of the conformal current simplifies to

$$\partial_\mu J'^\mu = (\partial_\mu T_{\mu\nu}) V^\nu + T_{\mu\nu} \partial_\nu V^\nu = T_{\mu\nu} (\partial_\mu V^\nu + \partial_\nu V^\mu) = \frac{\partial_\nu V_\rho}{d} T_{\rho}^{\mu}$$

(1.24)

where in the last equality we have used the relation (1.13). If the theory is also scale invariant and the additional technical hypothesis are satisfied the stress-energy tensor is also traceless and (1.24) is identically zero. This brings us to the desired result: a Poincaré and scale invariant theory with particular assumptions is also invariant under the whole conformal group. The particular conditions we have referred to about can be easily realized in most reasonable classical and quantum field theories, although exotic counterexamples exist.

### 1.1.4 Quantum field theory and conformality

What happens as a classically conformal theory is quantized? After quantization one also needs to renormalize the theory by introducing a new energy scale $\mu$. This procedure

$$\partial_\mu J'^\mu = x^\nu T_{\mu\nu} + \frac{\partial_\nu}{\sigma(\partial_\nu, \phi)} (\Delta \phi),$$

where $T_{\mu\nu}$ is the Belifante conserved stress-energy tensor and $(\Delta \phi)$ is the global variation of the scalar field. To obtain a traceless stress-energy tensor one can add an antisymmetric superpotential $T^{\nu\sigma} = T^{\nu\sigma} + \frac{1}{2} \delta_\rho \delta_\sigma X^{\nu\rho\sigma}$. The new stress-energy tensor is traceless only if $X^{\nu\rho\sigma}$ is written in terms of a suitable combination of $\sigma^{\mu\nu}$ [31], where $V_\nu = \frac{\partial_\nu}{\sigma(\nu, \phi)} (iS_{\nu} + \eta_{\nu} \Delta) = \partial_\mu \sigma^{\mu\nu}$, $S_{\mu\nu}$ is the spin operator and $\Delta$ the scaling dimension of the field. See also [28] for a review.
breaks scale invariance and the scale symmetry is said to be anomalous. The couplings are generally running $g(\mu)$, and their variation under $\mu$ is governed by the equation

$$\mu \frac{d}{d\mu} g(\mu) = \beta(g),$$  \hspace{1cm} (1.25)

where $\beta(g)$ is the beta function.

Since quantum field theories must be independent on the renormalization scale $\mu$, one can derive an equation describing the evolution of the n-point correlation functions with the energy scale. This is the Callan-Symanzick equation and can also be seen as the Ward identity for dilatations [32].

We saw that at the classical level a scale invariant theory has a traceless stress-energy tensor. After quantization the theory exhibits the so called trace anomaly, since roughly speaking

$$T_{\mu}^\mu \sim \beta(g).$$  \hspace{1cm} (1.26)

Furthermore the canonical dimension $\Delta$ of the fields gets corrected by an anomalous dimension $\gamma$

$$\Delta \rightarrow \Delta + \gamma(g), \quad \gamma = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z,$$  \hspace{1cm} (1.27)

where $Z$ is the renormalization constant of the fields. However it is immediately seen from (1.26) that there are cases in quantum field theory in which scale (conformal) invariance is still a symmetry of the theory. This can happen in two ways:

- at fixed points $g^*$ of the renormalization group (RG) flow, where the couplings are not running $\beta(g^*) = 0$, and the trace anomaly is zero $T_{\mu}^\mu = 0$,

- in finite theories for which $\beta(g) = 0$ for each $g$. In this case there are no divergences and no RG flow at all.

These are the cases we have in mind when we refer to conformal quantum field theories (CFT). Fixed points of the RG flow can be generally UV or IR if they are situated in the high or low energy domain, and can be at strong or weak coupling. For our applications we will be mostly concerned with quantum critical points, which are fixed points of the RG flow at zero temperature, with a divergent coherence length $\xi$ and with a strongly coupled dynamics.

A well known example of a finite theory in 3+1 dimensions is the Yang-Mills theory with four supersymmetries ($N = 4$ SYM), see [22] for a review. This theory is an ordinary Yang-Mills theory coupled to 4 Weyl fermions and 6 real scalars all in the adjoint representation of the gauge group. This precise amount of fields leads to an exact compensation inside the beta function between the contributions of the gluons, fermions and scalars. The
resulting beta function is vanishing up to third loop, but there are arguments saying it should be vanishing at all loops [33].

### 1.1.5 Representations of the conformal algebra

To define the quantities of interest in a conformal field theory it is necessary to study the representations of the conformal algebra. It contains the Poincaré algebra and some more relations between the special conformal and dilatation generators

\[
\begin{align*}
[J_{\mu\nu}, J_{\alpha\beta}] &= i \eta_{\mu\alpha} J_{\nu\beta} - i \eta_{\mu\beta} J_{\nu\alpha} + i \eta_{\nu\beta} J_{\mu\alpha} - i \eta_{\nu\alpha} J_{\mu\beta}, \quad (1.28) \\
[J_{\mu\nu}, P_\rho] &= i \eta_{\mu\rho} P_\nu - i \eta_{\nu\rho} P_\mu, \quad (1.29) \\
[P_\mu, P_\nu] &= 0, \quad (1.30) \\
[J_{\mu\nu}, K_\rho] &= i \eta_{\mu\rho} K_\nu - i \eta_{\nu\rho} K_\mu, \quad (1.31) \\
[J_{\mu\nu}, D] &= 0, \quad (1.32) \\
[D, P_\mu] &= i P_\mu, \quad (1.33) \\
[D, K_\mu] &= -i K_\mu, \quad (1.34) \\
[P_\mu, K_\nu] &= 2i J_{\mu\nu} + 2i \eta_{\mu\nu} D. \quad (1.35)
\end{align*}
\]

First of all one should note that now \( m^2 = P_\mu P^\mu \) is not a Casimir operator. The representations of the conformal algebra cannot be identified with multiplets of the same mass and spin. Even the whole S-matrix formalism is no more suitable since we cannot define asymptotic states.

The conformal group \( O(2,d) \) has a non-compact subgroup \( SO(1,d-1) \times SO(1,1) \) with generators \( J_{\mu\nu} \) and \( D \), isomorphic to the Lorentz times the dilatation group. If we specialize to the case of four dimensions, irreducible representations of such group are labeled by the couple \((j_1, j_2)\) and by \( \Delta \). Hence, interesting representations of the conformal group can be the ones involving fields as eigenfunctions of the non-compact subgroup \( SO(1,d-1) \times SO(1,1) \). This means that fields \( \mathcal{O}_\Delta^{(j_1,j_2)} \) have defined properties under the scaling symmetry and Lorentz transformations. In fact the commutation relations at the origin of the coordinates read

\[
\begin{align*}
[D, \mathcal{O}_\Delta^{(j_1,j_2)}(0)] &= -i \Delta \mathcal{O}_\Delta^{(j_1,j_2)}(0), \quad (1.36) \\
[J_{\mu\nu}, \mathcal{O}_\Delta^{(j_1,j_2)}(0)] &= \Sigma_{\mu\nu} \mathcal{O}_\Delta^{(j_1,j_2)}(0), \quad (1.37)
\end{align*}
\]

where \( \Sigma_{\mu\nu} \) are the matrices of a finite dimensional representation of the Lorentz group.

The commutation relations (1.33) and (1.34) imply that the operator \( P_\mu \) raises the scaling (conformal) dimension of the field by one unit \((\Delta + 1)\), and \( K_\mu \) lowers it \((\Delta - 1)\). This implies that each representation of the conformal theory contains several fields with...
different scaling dimensions. Is there a lower bound? Yes. A general field theory shouldn’t admit states with negative norm, i.e. the theory should be unitary. A conformal field theory satisfies this request only under particular conditions depending on the fields and the spacetime dimensions $d$. For example [34] for scalar fields the conformal dimension should be above the unitarity bound

$$\Delta \geq \frac{(d-2)}{2},$$

(1.38)
i.e. it must be greater than the scaling dimension of a free scalar field. Therefore each representation of the conformal group must have some field with the lowest scaling dimension $\Delta$, annihilated by $K_\mu$. Such fields are called primary fields

$$[K_\mu, O^{(j_1,j_2)}(0)] = 0.$$  

(1.39)

Since there is no upper bound, each representation is infinite. The content is a primary field and an infinite set of composite fields obtained through the rising operator $P_\mu$. Without loss of generality we may say that representations are labeled by the primary operators which form the spectrum of the theory.

Some representations are special. Let us consider for example the representation with a primary scalar field. The first elements of the representation are

$$\{\phi, P_\mu \phi, P_\mu P^\mu \phi = \Box \phi, \ldots\}.$$  

(1.40)

One can show [34] that the third operator above has negative norm when the unitarity bound (1.38) is saturated

$$\Delta = \frac{(d-2)}{2}.$$  

(1.41)

In this case we shall set that operator to zero $\Box \phi = 0$ and this corresponds to a free scalar field with conformal dimension (1.41). The surprising fact is that we can say that a free scalar field has always a defined conformal dimension (1.41) equal to the canonical one which doesn’t acquire an anomalous dimension as in (1.27). Such operators are called protected against renormalization and their representation is a short representation. This feature is satisfied by all the free fields and also by conserved currents [34].

### 1.1.6 Correlation functions

Since the $O(2,d)$ conformal group is much larger than the Poincaré group, it severely restricts the correlation functions of primary operators, which must be invariant under conformal transformations. Using conformal algebra one may show [28] that one point functions are vanishing on the CFT vacuum

$$<O_\Delta> = 0.$$  

(1.42)
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Two point functions of fields with different conformal dimension vanish

\[ \langle O_{\Delta_1}(x) O_{\Delta_2}(y) \rangle = A \frac{\delta_{\Delta_1, \Delta_2}}{|x - y|^{2\Delta_1}}. \]  (1.43)

Three-point functions are also determined up to a constant

\[ \langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) O_{\Delta_3}(x_3) \rangle = c_{\Delta_1, \Delta_2, \Delta_3} \frac{c_{\Delta_1, \Delta_2, \Delta_3}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_1 - x_3|^{\Delta_1 + \Delta_3 - \Delta_2} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1}}. \]  (1.44)

Similar expressions arise for non-scalar fields.

1.2 Anti-de Sitter spaces

The AdS/CFT correspondence maps conformal field theories into higher dimensional theories of gravity (or strings) on Anti-de Sitter backgrounds. In order to understand its content it is thus necessary to describe what an AdS space is. The AdS_{d+1} space is the maximally symmetric solution of the Einstein’s equations in d+1 spacetime dimensions with a negative cosmological constant \( \Lambda \). The Einstein’s action reads

\[ S = \frac{1}{2k_{d+1}^2} \int d^{d+1} x \sqrt{-g} (R - \Lambda), \]  (1.45)

where \( k_{d+1}^2 \) is the gravitational constant in \( d+1 \) dimensions, related to the Newton constant \( G_{d+1} \) by \( 16\pi G_{d+1} = 2k_{d+1}^2 \). \( R \) is the Ricci scalar and \( \Lambda \) is the cosmological constant. The Ricci scalar \( R \) as \( \Lambda \) contains second order derivatives in the metric, hence it has dimension \( l^{-2} \). In order for the action to be dimensionless the gravitational constant must have dimension \( k_{d+1}^2 \sim l^{d-1} \). In Planck units the only scale is the Planck length, thus

\[ k_{d+1}^2 \sim l_p^{d-1}. \]  (1.46)

The Einstein’s equations of motion read

\[ G_{\mu\nu} = -\frac{\Lambda}{2} g_{\mu\nu}, \]  (1.47)

where the Einstein tensor is given by

\[ G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \frac{1}{2} R. \]  (1.48)

Taking the trace of (1.47) we get a relationship between the cosmological constant and the Ricci scalar

\[ R = \frac{(d+1)}{(d-1)} \Lambda. \]  (1.49)
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If we further require the Ricci tensor to be proportional to the metric
\[ R_{\mu\nu\rho\lambda} = \frac{\Lambda}{d(d-1)} (g_{\nu\lambda}g_{\mu\rho} - g_{\nu\rho}g_{\mu\lambda}) \]  
we have a maximally symmetric solution, see [35], i.e. the number of linearly independent killing vectors is maximal, equal to \(\frac{1}{2}(d+1)(d+2)\).

In Euclidean signature, the maximally symmetric solution with positive cosmological constant is the sphere \(S^{d+1}\) with isometry \(SO(d+2)\) and the one with negative curvature is the hyperboloid \(H^{d+1}\) with isometry \(SO(1,d+1)\). In Minkowskian signature the maximally symmetric solution with \(\Lambda > 0\) is called the de-Sitter space \((dS^{d+1})\) and the one with \(\Lambda < 0\) is called Anti-de-Sitter \((AdS_{d+1})\). All these spaces can also be realized as the set of solutions of quadratic equations embedded in a \((d+2)\)-dimensional flat space with a suitable signature.

For example \(AdS_{d+1}\) can be represented as an hyperboloid
\[ x_0^2 + x_{d+1}^2 - x_1^2 - \ldots - x_{d-2}^2 = L^2, \]  
in the flat \(R^{2,d}\) which has a line element
\[ ds^2 = -dx_0^2 - dx_{d+1}^2 + dx_1^2 + \ldots + dx_d^2. \]
The parameter \(L\) is called the \(AdS\) radius, and it is connected with the cosmological constant of the space
\[ \Lambda = -\frac{d(d-1)}{L^2}. \]
By construction, the space has an isometry group \(O(2,d)\) identical to the conformal group in \(d\) dimensions. Equation (1.51) can be solved by setting the global parametrization
\[ x_0 = L\cosh\rho \cos \tau \]
\[ x_{d+1} = L\cosh\rho \sin \tau \]
\[ x_i = L\sinh\rho \theta_i, \quad \sum_{i=1}^{d} \theta_i^2 = 1. \]
Substituting this into (1.52), we obtain the metric on \(AdS_{d+1}\) as
\[ ds^2 = L^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{(d-1)}^2), \]  
where \(d\Omega_{(d-1)}^2\) is the line element of a \((d-1)\)-dimensional sphere. The parameters \(\rho \in [0, \infty)\) and \(\tau \in [0, 2\pi]\) cover the Minkowskian hyperboloid exactly once, for this reason \((\rho, \tau, \theta_i)\) are called global coordinates of \(AdS\). Notice that time is periodic and therefore we have closed time-like curves. To avoid this situation and obtain a causal spacetime, we can
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simply take the universal covering of this space where \( \tau \in (-\infty, \infty) \) is decompactified. From now on, when we refer to \( \text{AdS} \), we only consider this universal covering space.

In addition to the global parametrization (1.54) of \( \text{AdS} \), there is another set of local coordinates \( (t, \vec{x}, u) \) with \( u > 0 \) which will be useful for our purposes. It is defined by

\[
\begin{align*}
x_0 &= \frac{1}{2u} (1 + u^2 (L^2 + \vec{x}^2 - t^2)) \\
x_i &= L u x_i \quad i = 1, \ldots, d \\
x_d &= \frac{1}{2u} (1 - u^2 (L^2 - \vec{x}^2 + t^2)) \\
x_{d+1} &= L u t.
\end{align*}
\]

These coordinates cover only one half of the hyperboloid (1.51). Substituting this into (1.52), we obtain another useful form of the \( \text{AdS}_{d+1} \) metric

\[
ds^2 = L^2 \left( u^2 dx_\mu dx^\mu + \frac{du^2}{u^2} \right).
\] (1.57)

In this form of the metric the subgroups \( \text{ISO}(1, d-1) \) and \( \text{SO}(1, 1) \) of the isometry group \( \text{O}(2, d) \) are manifest, where \( \text{ISO}(1, d-1) \) is the group of Poincaré transformations on \( (t, \vec{x}) \) and \( \text{SO}(1, 1) \) is the scale transformation which leaves the metric (1.57) invariant

\[
(t, \vec{x}, u) \rightarrow (at, a \vec{x}, a^{-1} u), \quad a > 0.
\] (1.58)

This means that the \( \text{AdS} \) space is foliated by \( d \)-dimensional Minkowskian spaces over \( u \) which run from zero to infinity. For this reason \( (t, \vec{x}, u) \) are called Poincaré coordinates. Every Minkowskian slice is multiplied by a warp factor \( u^2 \), whose meaning is that an observer living on the flat slice sees all lengths rescaled by a factor \( u \) according to its position in the \( d+1 \) dimension. Note that the metric at \( u = \infty \) blows up. Through a conformal transformation we can obtain a conformally equivalent metric \( ds^2 = \frac{dr^2}{u^2} \) which is equivalent to \( R^{1,d-1} \) at \( u = \infty \). For this reason the plane at \( u = \infty \) is called the conformal boundary of the \( \text{AdS} \) space. The plane at \( u = 0 \) is instead an horizon because the killing vector \( \frac{\partial}{\partial t} \) has zero norm \( (g_{00} = 0) \) at \( u = 0 \). However since the parametrization is not global the metric can be extended beyond the horizon, thus \( u = 0 \) doesn’t correspond to a true singularity of the metric.

There are further forms of the \( \text{AdS} \) metric commonly used. They only differ by a redefinition of the coordinate \( u \). For example redefining \( r = L^2 u \) one obtains

\[
ds^2 = \frac{r^2}{L^2} dx_\mu dx^\mu + \frac{L^2}{r^2} dr^2,
\] (1.59)

where now the \( r \) coordinate has the dimension of a length, the horizon is at \( r = 0 \) and the conformal boundary at \( r = \infty \). Another possibility is to set \( z = \frac{1}{u} = \frac{L^2}{r} \). The metric (1.57) takes the form

\[
ds^2 = L^2 \frac{2}{z^2} (dx_\mu dx^\mu + dz^2),
\] (1.60)
Figure 1.2: The $AdS$ space is foliated by several copies of Minkowski space. The lengths increase with the warp factor $u^2$. $u = \infty$ is the conformal boundary of the space, while $u = 0$ is the horizon.

where the conformal boundary is now set at $z = 0$ and the horizon at $z = \infty$. This metric is invariant under the transformations analogous to (1.58)

$$(t, \bar{x}, z) \rightarrow a(t, \bar{x}, z), \quad a > 0.$$  

(1.61)

To conclude this brief excursion on the geometry of $AdS_{d+1}$ let’s consider the Euclidean continuation of its metric (1.59). We can go to an Euclidean signature by performing a Wick rotation on the time coordinate $t \rightarrow -it_E$. The resulting metric is then

$$ds^2 = \frac{r^2}{L^2} (dt_E^2 + d\bar{x}^2) + \frac{L^2}{r^2} dr^2. \quad (1.62)$$

Every slice of the $AdS$ space is now a flat $R^d$ plane. In particular at $r = \infty$ the Minkowskian conformal boundary is replaced by an euclidean plane $R^d$. On the other hand the $r = 0$ plane, which was an horizon, i.e. a plane of null vectors, is now a point. In fact in Euclidean space the only vectors with zero norm are zero vectors. Thus we now shall speak about the center of the space in $r = 0$ instead of an horizon.

### 1.2.1 Gravity in an $AdS$ vacuum

The $AdS$ metric (1.59) solves the equations of motion following from the action (1.45), but it could also be the vacuum\(^2\) of a more general gravity theory containing interacting fields, such as scalars or vectors, which we will refer to as bulk fields in $d+1$ dimensions. A general action writes

\(^2\)The vacuum of a theory of gravity is obtained by setting all the additional fields to zero.
\[ S(g_{ab}, A_a, \phi, \ldots) \sim \frac{1}{2k_{d+1}^2} \int d^{d+1}x \sqrt{-g} \left( \mathcal{R} - \Lambda + Tr(F^2) + (\partial \phi)^2 + V(\phi) + \ldots \right). \] (1.63)

The dots other than further bulk fields, may in general contain higher powers of curvature, and terms coming from the dimensional reduction of a higher dimensional string theory. The gravity theory is classical when such terms are suppressed. This happens when the theory is considered at large volumes and when the strings are effectively point-like. These are exactly the limits in (1.3), which we report here for completeness

\[ \frac{L}{\ell_p} \gg 1, \quad \frac{L}{\ell_s} \gg 1. \] (1.64)

The classical gravity action leads to second order differential equations of motion for the bulk fields. To determine the solution one then needs to specify two boundary conditions, one in the interior of the \( AdS \) space \( r = 0 \) \( (z = \infty) \) and one at the conformal boundary \( r = \infty \) \( (z = 0) \). The latter boundary conditions will play a crucial role in the contest of the \( AdS/CFT \) correspondence.

1.3 Motivating the duality

In this section we will try to give some motivations to the \( AdS/CFT \) correspondence without going into the string theory realm. A first clue follows from the previous sections: \( d \)-dimensional conformal field theories and \( AdS_{d+1} \) spaces have common symmetries. In particular the conformal group \( O(2, d) \) coincides with the group of isometries of the \( AdS_{d+1} \) metric (1.59). Moreover the \( AdS/CFT \) correspondence provides [36] an explicit realization of the holographic principle (see [37] for a review), which states that the number of degrees of freedom of a gravity theory matches the number of degrees of freedom of a lower dimensional quantum field theory. Finally the additional spatial dimension of the gravity theory \( r \) may be seen as a geometrical realization of the RG energy scale of the dual field theory. Let us briefly discuss these points.

1.3.1 The holographic principle

This principle states that a theory of gravity, say in \( d+1 \) dimensions, in a region of space has a number of degrees of freedom which scales like that of a quantum field theory on the boundary of that region. This is a direct consequence of black hole thermodynamics. The basic fact is that to a black hole it must be assigned an entropy to preserve the second law of thermodynamics, otherwise the entropy of some in-falling stuff would
disappear. Hawking confirmed the Bekenstein conjecture \[38\] that this black hole entropy is proportional to the area of the event horizon

\[
S_{BH} = \frac{A}{4G_{d+1}},
\]

(1.65)

where \(G_{d+1}\) is Newton’s constant in Planck units. The point is that the black hole entropy is the maximal entropy of anything else in the same volume. Therefore every region of space has a maximum entropy scaling with the area of the boundary and not with the enclosed volume as one may think. This is much smaller than the entropy of a local quantum field theory in the same space, which would have a number of states \(N \sim e^V\), and the maximum entropy \(S \sim \log N\) would have been proportional to the volume \(V\). The maximum entropy in a region of space can instead be related to the number of degrees of freedom \(N_d\) of a local quantum field theory living in fewer dimensions

\[
S = \frac{A}{4G_{d+1}} = N_d.
\]

(1.66)

This is the full statement of the holographic principle \[37\].

The \(AdS/CFT\) correspondence is a particular realization of this principle where the gravity theory lives in an \(AdS_{d+1}\) vacuum, and its degrees of freedom are encoded on the conformal boundary of the space. We will use the general statement that a \(CFT_d\) lives on the boundary of the \(AdS_{d+1}\) space, bearing in mind that this is not completely correct. What is true, as we will see in the following, is that the \(AdS_{d+1}\) degrees of freedom are sources for the \(CFT_d\) degrees of freedom.

The holographic principle (1.66) applied to the particular case of \(AdS/CFT\) correspondence \[36\] tells us something about the regime of validity of the correspondence. The area of the boundary of an \(AdS_{d+1}\) space is

\[
A = \int_{r \to \infty, \text{ fixed } t} d^{d-1}x \sqrt{g_{(d-1)}} = \int_{r \to \infty} d^{d-1}x \frac{r^{d-1}}{L_{d-1}},
\]

(1.67)

where \(g_{(d-1)}\) is the determinant of the \(AdS_{d+1}\) metric (1.59) embedded on the boundary \(r = \infty\), and calculated on slices of constant time

\[
ds^2_{(d-1)} = \frac{r^2}{L^2} dx^2, \quad \text{as } r \to \infty.
\]

(1.68)

The integral (1.67) must be regularized because it suffers from divergences coming both from the integral over \(d^{d-1}x\) and from the fact that we are taking \(r \to \infty\). Thus we shall integrate not up to \(r = \infty\) but rather up to a cutoff \(r = R\). Moreover we will trade the integral over the space coordinate by a volume \(V_{d-1}\). Given this, (1.67) becomes

\[
A = \left(\frac{R}{L}\right)^{d-1} V_{d-1}.
\]

(1.69)
The maximum entropy in the bulk is then

$$ \frac{A}{4G_{d+1}} \sim \frac{V_{d-1}}{4G_{d+1}} \left( \frac{R}{L} \right)^{d-1}. $$

(1.70)

The dual quantum field theory in \(d\) dimensions is also UV and IR divergent. Regularize it the same way by introducing a box of volume \(V_{d-1}\), and a short distance cutoff \(a\) (i.e. a high energy cutoff \(a^{-1}\)). It is sensible to say that this UV cutoff in the field theory corresponds to an IR cutoff in the dual gravity side, i.e. we can safely take \(a^{-1} \sim \frac{R^2}{L^2}\).\(^3\)

The total number of degrees of freedom \(N_d\) of a quantum field theory in \(d\) dimensions is given by the number of spatial cells \(\frac{V_{d-1}}{a^{d-1}} \sim V_{d-1} \frac{R^{d-1}}{L^2(a-1)}\) times the number of degrees of freedom per lattice site. For example a quantum field theory with matrix fields \(\Phi^{ab}\) in the adjoint representation of the symmetry group \(U(N)\) has a number of degrees of freedom per point equal to \(N^2\), see [25]. Thus

$$ N_d \sim V_{d-1}R^{d-1} \frac{N^2}{L^2(d-1)}. $$

(1.71)

Using then (1.66) and the result (1.70) we obtain, up to numerical factors

$$ \frac{L^{d-1}}{G_{d+1}} \sim \left( \frac{L}{l_p} \right)^{d-1} \sim N^2, $$

(1.72)

where in second equality we have written the gravitational constant in Planck units \(G_{d+1} \sim \frac{l^{d-1}}{l_p}\). This relation connects the parameters on the gravity theory side to the parameters in the dual conformal field theory only by means of the holographic principle. From the first limit in (1.64) and (1.72) it follows that the gravity theory in an \(AdS\) vacuum with radius \(L\) is classical when the number of degrees of freedom \(N^2\) per site of the conformal field theory is large

$$ \left( \frac{L}{l_p} \right)^{d-1} \sim N^2 \gg 1. $$

(1.73)

In explicit realizations of the correspondence, when one refers to particular stringy backgrounds such as that of the original Maldacena’s paper [10], one can exactly verify [36] the holographic principle by taking the exact matching of the parameters (1.2).

### 1.3.2 Geometrizing the renormalization group flow

Consider a \(d\)-dimensional quantum field theory. A possible way to describe such a theory is to organize the physics in terms of lengths or energy scales [39]. If one is interested in the properties of the theory at a large length scale \(z \gg a\), where \(a\) is the spacing of the lattice degrees of freedom or a possible cutoff of the theory, instead of using the bare theory

\(^3\)Recall that the coordinate in \(AdS\) space with dimension of an energy is \(u = \frac{r}{L}\).
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Figure 1.3: The extra dimension $z = \frac{L^2}{r}$ of the bulk theory is the resolution scale of the field theory. The left figure indicates a series of block spin transformations. The right figure is a cartoon of AdS space, which organizes the field theory information from UV physics near the conformal boundary to the IR physics near the event horizon. Figure taken from [25].

defined at a microscopic scale $a$, it is more convenient to integrate-out short distance degrees of freedom and obtain an effective field theory at a scale $z$. One can proceed further and define an effective field theory at a scale $z' \gg z$. This procedure defines a renormalization group (RG) flow and gives rise to a continuous family of effective theories in $d$-dimensional Minkowski spacetime labeled by the RG scale $z$. A remarkable fact is that the RG equations are local in $u = \frac{1}{z}$ interpreted as an energy scale. This means that we don’t need to know the behavior of the physics deeply in the UV or in the IR to understand how things are changing in $u$. At this point we can visualize this continuous family of effective theories as a single theory in $d + 1$ dimensions with the RG scale $z$ becoming a spatial coordinate.

From this discussion it follows the already mentioned organizing principle: the UV/IR connection. From the view point of the gravity theory, physics near the conformal boundary $z = 0$ is the large volume physics, i.e. IR physics. Near the horizon $z = \infty$ is instead the short distance UV physics. In contrast, from the view point of the quantum field theory, physics at small $z$ corresponds to short distance UV physics and vice versa.

1.4 Statement of the duality

The previous section was mainly involved to suggest that two apparently different theories could be actually connected one to another. The motivations we gave are really far from being demonstrations. The deepest clues of Maldacena’s argument [10] are provided by a
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lot of quantitative checks (though a rigorous mathematical proof is still lacking) which we
will not review here due to lack of space. Remember that we are interested in the classical
gravity limit where computations become mathematically tractable. This corresponds
from (1.3) in the dual field theory side to large-$N$ and strong-coupling regime. A possible
statement for the correspondence in this limit can be the following (see e.g. [40]):

$$(d+1)\text{-dimensional classical gravity theories on } \text{AdS}_{d+1}\text{ vacuum}
\text{ are equivalent to}
\text{the large } N \text{ limit of strongly coupled } d\text{-dimensional CFTs in flat space.}$$

Now that we have established the equivalence we must provide a prescription [11, 12] to
relate the degrees of freedom of both sides of the duality. The idea is that to every field
in $\text{AdS}$ should correspond a local gauge invariant operator in CFT. To anticipate some
results of this section

fields in $\text{AdS} \longleftrightarrow$ local operators in CFT

spin $\longleftrightarrow$ spin

mass $\longleftrightarrow$ scaling dimension $\Delta$.

Hence to a scalar field in the bulk corresponds a scalar operator, to a gauge field in the
bulk a conserved current in the boundary and to the bulk metric a conserved stress-energy
tensor in CFT:

$$\psi \longleftrightarrow O$$
$$A_a \longleftrightarrow J_\mu$$
$$g_{ab} \longleftrightarrow T_{\mu\nu}$$

Moreover the field theory’s partition function is connected with the exponential of the
euclidean continuation\footnote{We will not be interested into real-time correlators in the following.} of the renormalized gravity action evaluated on shell

$$Z_{\text{CFT}} \longleftrightarrow e^{-S_{\text{E-\,on-shell}}}.$$ 

Therefore, correlation functions may be easily derived by deriving right hand side of the
previous equation with respect to the sources.

1.4.1 The field-operator correspondence

First of all we need a prescription to relate bulk fields to operators in the conformal field
theory, which we will call from now on boundary fields. Only in this way will it be possible
to compare physical quantities of both sides of the correspondence.
Consider a conformal field theory lagrangian $\mathcal{L}_{\text{CFT}}$. It can be perturbed by adding arbitrary functions, namely sources $h^A(x)$ of local operators $\mathcal{O}_A(x)$

$$\mathcal{L}_{\text{CFT}} \to \mathcal{L}_{\text{CFT}} + \sum_A \mathcal{O}_A(x)h^A(x), \quad (1.74)$$

where $A$ stands for the set of all the quantum numbers of the boundary field. This is a UV perturbation because it is a perturbation of the bare lagrangian by local operators. In $AdS$ space, it corresponds to a perturbation near the boundary $z = 0$. Thus the perturbation by a source $h(x)$ of the CFT will be encoded in the boundary condition on the bulk fields.

Take now the source and extend it to the bulk side $h(x) \to h(x_\mu, z)$ with the extra coordinate $z$. Fields in the boundary will be denoted with coordinates $x$, and bulk fields will be dependent on the coordinates $(x_\mu, z)$. Suppose $h(x_\mu, z)$ to be the solution of the equations of motion in the bulk with boundary condition

$$h(x_\mu, z)|_{z=0} = h(x), \quad (1.75)$$

and another suitable boundary condition at the horizon to fix $h(x_\mu, z)$ uniquely. As a result we have a one to one map between bulk fields and boundary fields [11, 12]. In fact, to each local operator $\mathcal{O}(x)$ corresponds a source $h(x)$, which is the boundary value in $AdS$ of a bulk field $h(x_\mu, z)$.

In order to deduce which field should be related to a given operator symmetries come in help, because there is no completely general recipe. For instance conserved currents in a quantum field theory theory, corresponding to global symmetries, should be dual to gauge fields in order to construct gauge invariant perturbations to the conformal field theory. Take for example a conserved vector current $J^\mu(x)$. Its source is an external background gauge field $A_\mu(x)$

$$\int d^d x J^\mu(x)A_\mu(x) \quad (1.76)$$

Note that this perturbation is gauge invariant when the current $J_\mu$ is conserved. In fact under the gauge transformation of the field $A_\mu \to A_\mu + \partial_\mu f$ the extra term

$$\int d^d x J_\mu \partial^\mu f = \int d^d x (\partial^\mu (J_\mu f)) = 0 \quad (1.77)$$

contains a total derivative and a term that is identically zero. Thus conserved currents couple to gauge invariant sources, which in the interpretation of the $AdS$/CFT correspondence can be extended to the bulk into dynamical gauge fields $A_\mu(x_\mu, z)$ [13].

Another important example is that of the conserved stress-energy tensor $T_{\mu\nu}$. The source should be a tensor $g_{\mu\nu}$. To have a gauge invariant coupling

$$\int d^d x T_{\mu\nu}(x)g^{\mu\nu}(x) \quad (1.78)$$
$g^{\mu\nu}(x)$ should be the boundary value of a gauge field corresponding to the local translational invariance. The field we are talking about is of course the metric tensor $g_{ab}(x_\mu, z)$ with boundary value

$$g^{ab}(x_\mu, z)|_{z=0} = g^{\mu\nu}_{z=0}(x).$$

(1.79)

The right-hand side of the previous equation is to be intended as the embedding of the bulk metric on the boundary of the $AdS$ space at $z = 0$, so that the $zz$ component vanishes.

It is important to note that on the gravity side the global symmetries arise as large gauge transformations. In this sense there is a correspondence between global symmetries in the gauge theory and gauge symmetries in the dual gravity theory.

### 1.4.2 Mass-dimension relation

Having in mind the field-operator correspondence let’s see how the conformal dimension of an operator is related to properties of the dual bulk field. For illustration take a massive scalar bulk field $\psi$, dual to some scalar gauge invariant operator $O$ in the boundary theory. The Euclidean bulk classical action may be written as

$$S_E = -\frac{1}{2k^{d+1}} \int d^{d+1}x \sqrt{g}(g^{ab}\partial_a\psi\partial_b\psi + m^2\psi^2)$$

(1.80)

where $g$ is the determinant of the euclidean version of the $AdS$ metric (1.60). The scalar field has been rescaled using the gravitational constant $k_{d+1}$ in order to make it dimensionless. Then (1.80) writes

$$S_E = -\frac{1}{2k_{d+1}} \int d^d x_\mu dz \frac{L^{d+1}}{z^{d+1}} \left( \frac{z}{L^2}(\partial_z\psi)^2 + \frac{z^2}{L^2}(\partial_\mu\psi)^2 + m^2\psi^2 \right).$$

(1.81)

The resulting equation of motion is

$$z^{d+1}\partial_z \left( \frac{1}{z^{d-1}}\partial_z \psi \right) + z^{d+1}\partial_\mu \left( \frac{1}{z^{d-1}}\partial_\mu \psi \right) = m^2L^2\psi.$$  

(1.82)

Since the bulk spacetime is translationally invariant along the $x_\mu$ directions, it is convenient to introduce a Fourier decomposition in these directions by writing

$$\psi(x_\mu, z) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x}\psi(k_\mu, z).$$  

(1.83)

In terms of these Fourier modes the equation of motion for $\psi$ writes

$$z^{d+1}\partial_z (z^{-(d-1)}\partial_z \tilde{\psi}) - k^2z^2\tilde{\psi}^2 - m^2L^2\tilde{\psi} = 0.$$  

(1.84)
Near the boundary $z \sim 0$ the second term in (1.84) can be neglected and the equation can be readily resolved [24] by finding the particular solution $\tilde{\psi} \sim z^{\Delta}$ with $\Delta$ satisfying the relation

$$\Delta(\Delta - d) = m^2 L^2,$$  
(1.85)

the two roots of which are

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}$$  
(1.86)

Note that $\Delta_{+} + \Delta_{-} = d$, thus we can set $\Delta = \Delta_{+}$ and $\Delta_{-} = d - \Delta$. The general form of the solution to the equation of motion (1.84) becomes

$$\tilde{\psi}(k,z) \sim \tilde{C}_1(k)z^{d-\Delta} + \tilde{C}_2(k)z^{\Delta} \quad \text{as} \quad z \to 0.$$  
(1.87)

Fourier transforming this solution back into the coordinate space leads to

$$\psi(x_\mu, z) \sim C_1(x)(z^{d-\Delta} + \ldots) + C_2(x)(z^\Delta + \ldots) \quad \text{as} \quad z \to 0.$$  
(1.88)

There are then two independent linear solutions which start from $z = 0$ with some power of $z$ and corrections given by going away from the boundary.

Note that the exponents in (1.88) are real provided that

$$m^2 L^2 \geq -\frac{d^2}{4}$$  
(1.89)

This is the so called Breitenlohner-Freedman (BF) bound [41], below it has been shown that the theory becomes unstable. This tells us that also negative values of the mass are allowed provided that they are not "too negative".

However in the stable region above the BF-bound one must still distinguish between two regions [42] (see also [13] for a review)

- in the finite interval $-\frac{d^2}{4} \leq m^2 L^2 \leq -\frac{d^2}{4} + 1$ both of the terms in (1.88) are normalizable with respect to the scalar product

$$\langle \psi_1, \psi_2 \rangle = -i \int_{\Sigma} d\vec{x} dz \sqrt{g} g^{\mu\nu}(\psi_1^* \partial_\mu \psi_2 - \psi_2^* \partial_\mu \psi_1).$$  
(1.90)

Assuming $\psi \sim z^\Delta$ the scalar product has a boundary behavior like $z^{2\Delta + 2-d}$ as $z \sim 0$, and the integral results finite only when

$$\Delta \geq \frac{d - 2}{2}$$  
(1.91)

which resembles exactly the unitarity bound (1.38).
1.4. STATEMENT OF THE DUALITY

Figure 1.4: Plot of the mass dimension relation for scalar fields in \(d = 3\). Unitarity bound in the conformal field theory also defines the domain of stability of bulk fields. When \(-\frac{9}{4} < m^2 L^2 < -\frac{5}{4}\) there are two normalizable modes, when \(m^2 L^2 > -\frac{5}{4}\) there is only one normalizable mode.

- when \(m^2 L^2 \geq -\frac{d^2}{4} + 1\) the first term in (1.88) is always non-normalizable and encodes the leading behavior of the solution as \(z \to 0\). The non-normalizable mode corresponds to a source of a given operator in the field theory, while the normalizable mode to the expectation value of that operator (see [25] for a review)

\[
\langle O\rangle = (2\Delta - d)C_2(x).
\]  

(1.92)

For scalar fields one can then plot (1.85) including the BF-bound (1.89) and the unitarity bound (1.38) to see the domain of stability of the field.

Let us now come back to equation (1.82) and look at the boundary conditions we must impose.

1. Conformal boundary \(z = 0\).

The boundary condition here can be set using the \(AdS/CFT\) correspondence. We saw that the boundary value of a bulk field should be identified with the source of the corresponding operator as in (1.75). The solution (1.88) for the scalar field \(\psi\) tells us that when a non-normalizable mode is present, the leading behavior near the conformal boundary is controlled by it. We should then require \(\psi(x_\mu, z)|_{z=0} = z^{d - \Delta}C_1(x)|_{z=0} = h(x)\); however this would lead to a zero source \(h(x)\). In order to have a finite source we should define the boundary condition as [11, 12]

\[
\lim_{z \to 0} z^{d - \Delta}\psi(x_\mu, z) = h(x),
\]  

(1.93)
which identifies the source \( h(x) \) with the first coefficient \( C_1(x) \) of the solution (1.88)

\[
C_1(x) = h(x). \tag{1.94}
\]

With this observation we shall modify (1.75) to a more suitable form in which we extract the singular behavior \( f(z) \)

\[
\lim_{z \to 0} f(z) h(x_{\mu}, z) = h(x). \tag{1.95}
\]

In the range \(-\frac{d^2}{4} \leq m^2 L^2 \leq -\frac{d^2}{4} + 1\) where both terms in (1.88) are normalizable either one can be used to be the source. From this two different boundary theories can be constructed in which the dimensions of the operator are \( \Delta \) or \( d - \Delta \). We shall use the convention in which the slower falloff is identified with the source because it corresponds to the leading behavior as \( z \sim 0 \).

2. \textit{Interior of the AdS space} \( z \to \infty \).

The behavior of this point is different whether the spacetime is Euclidean or Minkowskian.

- \textit{Euclidean AdS}: \( z = \infty \) is the center point of the space.
  
  One should require regularity of the solution. Once this has been done \( C_2(x) \) is completely determined as a functional of \( C_1(x) \). Since this coefficient is fixed by the other boundary condition (1.94) we are lead to a uniquely defined regular solution \( \psi(x_{\mu}, z) \) which extends inside the whole AdS space.

- \textit{Minkowskian AdS}: \( z = \infty \) is an horizon rather than a singular point.
  
  It turns out that in this case we are dealing with incoming and outgoing waves. Driven by the fact that nothing should escape from an horizon, a suitable boundary condition is to keep only incoming waves, see [43] for a review. We will not consider the Minkowskian version of the correspondence here.

To summarize let us write again the solution (1.88) using (1.94) and (1.92)

\[
\psi(x_{\mu}, z) \simeq h(x) z^{d-\Delta} + \frac{\langle \mathcal{O} \rangle}{(2\Delta - d)} z^{\Delta} \quad \text{as} \quad z \to 0. \tag{1.96}
\]

This expression states that the leading term of the solution is related to the source of the dual field and the subleading term to its expectation value. At this point, \( \Delta \) in (1.85) can be identified with the conformal dimension in mass of the boundary field \( \mathcal{O} \) dual to the bulk field \( \psi \). In fact, from (1.96) dimensional analysis tells us that \( h(x) \) should have dimension \( l^{\Delta - d} \) with \( \mathcal{O} \) having dimension \( l^{-\Delta} \).
1.4. STATEMENT OF THE DUALITY

Similar formulas to (1.85) relating the mass of a bulk field and the dimension of the associated operator can be obtained for general bulk fields. For $p$-forms equation, see [21], (1.85) generalizes to

$$(\Delta - p)(\Delta + p - d) = m^2 L^2,$$ (1.97)

which implies a further generalization of equation (1.96). For example for a massive gauge field $A_a$ ($p = 1$) in $AdS$

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2} \sqrt{(d-2)^2 + 4m^2 L^2}.$$ (1.98)

In the massless case $\Delta(j_\mu) = d - 1$, i.e. the dimension of a conserved current in a CFT. Finally, for massless spin 2 fields, like $g_{ab}$, $\Delta = d$ consistently with the protected dimension of the dual stress-energy tensor $T_{\mu \nu}$.

The normalizable modes arise only when (1.91) is satisfied. Thus the local operators in the boundary theory satisfy the unitarity bound (1.38). The general message in all this construction of the $AdS$/CFT correspondence is that we start with a local lagrangian in the bulk and declare that all the fields correspond to operators of a boundary theory. This boundary theory is compatible with all the general rules of a conformal field theory such as locality, unitarity, etc. The inverse route is not always possible, not all the conformal field theories admit a gravitational dual, see e.g. [44].

1.4.3 Euclidean correlation functions of local operators

Here we see how to compute correlation functions of local gauge-invariant operators of the conformal field theory in terms of the gravity theory. In view of the field-operator correspondence it is natural to postulate [11, 12] that the Euclidean partition functions of the two theories must agree upon the identification (1.95). The proposal for the correspondence is simply

$$Z^E_{\text{CFT}} \{h(x)\} = Z^E_{\text{gravity in AdS}} \{h(x_\mu, z)\},$$ (1.99)

where $\{h(x)\}$ is the collection of all the sources associated to each local operator in the field theory side, and $\{h(x_\mu, z)\}$ is the collection of the bulk fields. However we don’t have a very useful idea of what is the right hand side of this equation, except in the limits (1.64) where this gravity theory becomes classical. In these limits we can do the path integral by a saddle point approximation since the gravity action

$$S^E_{\text{gravity}} \sim \frac{L^{d-1}}{G_{d+1}} I_{\text{dimensionless}} \sim N^2 I_{\text{dimensionless}},$$ (1.100)

where in the second equality we have used (1.72), and $I_{\text{dimensionless}}$ is the dimensionless action of the on-shell classical gravity. The superscript $^E$ reminds us that we are considering
the analytic continuation in Euclidean space of such action. Then the gravity generating
functional drastically simplifies to

\[ Z^{E_{\text{gravity in AdS}}}[[h(x, z)]] \sim e^{-S^{E_{\text{gravity in AdS}}}}(\hat{h}(x, z))], \quad (1.101) \]

inserting the last expression into (1.99) we are lead to the simplified form of the AdS/CFT prescription

\[ Z^{E_{\text{CFT}}}[\{h(x)\}] = e^{-W^{E_{\text{AdS}}}[[h(x)]]} \simeq e^{-S^{E_{\text{gravity in AdS}}}}(\hat{h}(x, z)). \quad (1.102) \]

The saddle point \( \{\hat{h}(x, z)\} \) is the solution of the equations of motion. Boundary conditions (1.95) imply that it is a function of the sources \( \{h(x)\} \) of the CFT. Thus both sides of (1.102) depend upon the same variables.

The on-shell action needs to be renormalized because for instance it typically suffers from infinite-volume (i.e. IR) divergences due to the integration region near the boundary of AdS. These divergences are dual to UV ones in the gauge theory, consistently with the UV/IR connection. The procedure to remove such divergences is well understood and goes under the name of holographic renormalization, see e.g. [45].

At this point, using the AdS/CFT prescription (1.102), we can compute [21, 22] connected correlation functions of a conformal field theory

\[ \langle O(x_1) \ldots O(x_n) \rangle_c = \left. \frac{\delta^n W^{E_{\text{AdS}}}[\{h(x)\}]}{\delta h(x_1) \ldots \delta h(x_n)} \right|_{h=0}, \quad (1.103) \]

which simply become functional derivatives of the on-shell, classical gravity action

\[ \langle O(x_1) \ldots O(x_n) \rangle_c = \left. \frac{\delta^n S^{E_{\text{gravity in AdS}}}}{\delta h(x_1) \ldots \delta h(x_n)} \right|_{h=0}. \quad (1.104) \]

### 1.4.4 An example: the massless scalar field

It is useful to understand the above mentioned concepts by going through an explicit example. The simplest one is a theory of gravity with only a massless scalar. Equation (1.85) implies that the dual conformal operator should have scaling dimension \( \Delta = d \).

Let’s compute one point and two point functions in the dual theory [24]. First of all we must find the equation of motion from the action (1.80) but with \( m^2 = 0 \). From the result (1.84) in momentum space it reads

\[ z^{d-1} \partial_z (\frac{1}{z^{d-1}} \partial_z \tilde{\psi}) - k^2 \tilde{\psi} = 0. \quad (1.105) \]

Now replace \( \tilde{\psi} \rightarrow kz\phi(kz) \). Equation (1.105) reduces to the Bessel equation

\[ (kz)^2 \phi''(kz) + (kz)\phi'(kz) - (d + (kz)^2)\phi(kz) = 0, \quad (1.106) \]
whose general solution is a combination of two generalized Bessel functions

\[ \phi(kz) = A(k)I_{d-2}(kz) + B(k)K_{d-2}(kz). \]  (1.107)

We know the asymptotic behaviors of these functions. In the center of the AdS space

\[ I_{d-2} \to \infty, \quad K_{d-2} \to 0 \quad \text{as} \quad z \to \infty. \]  (1.108)

Requiring regularity of the solution one must impose \( A(k) = 0 \). We are left then with the non-normalizable solution which near the conformal boundary goes like

\[ K_{d-2} \simeq \frac{1}{(kz)^{d-2}} (1 + \ldots + C_{d-2}(kz)^{d-2} \log(kz)) \quad \text{as} \quad z \to 0, \]  (1.110)

where the dots indicate linear terms in \((kz)\). Now let us impose the second boundary condition (1.93) reminding that a massless scalar field has \( \Delta = d \). It is useful to introduce a cutoff \( z = \epsilon \)

\[ \tilde{\psi}(k, \epsilon) = \tilde{h}(k) \]  (1.111)

where \( \tilde{h}(k) \) is the source in the momentum space of the dual operator to the scalar field \( \psi \)

\[ \tilde{h}(k) = \int d^d x e^{-ik \cdot x} h(x). \]  (1.112)

Finally we obtain a solution which is function of the source

\[ \tilde{\psi}(k, z) = \left( \frac{(kz)^2}{(k \epsilon)^2} \right) K_{d-2}(kz) \tilde{h}(k). \]  (1.113)

Now let us evaluate the on-shell action to find the generating functional of the connected Green’s functions using (1.102). The computation can be simplified using a trick. Integrating by parts the action we are lead to a boundary term and a term containing the equation of motion which is identically zero

\[ W[h] = S_{\text{gravity in AdS}}(\tilde{h}) = \int d^d x dz \partial_z \left( \sqrt{g} \tilde{\psi}^*_g \partial_z \tilde{\psi} \right) = \int d^d x \left( \frac{1}{z^{d-1}} \psi \partial_z \psi \right)_{z=\epsilon}^{z=\infty}. \]  (1.114)

Using (1.83) and the solution (1.113) to find the generator in the momentum space, we find

\[ W[\tilde{h}] = \int d^d x \left( \frac{d^d k}{(2\pi)^d} \right) \left( \frac{d^d q}{(2\pi)^d} \right) \left( \frac{1}{z^{d-1}} \right) \left( \frac{2(kz)kK_{d-2}(kz)+(kz)^2kK'_{d-2}(kz)}{(k \epsilon)^2K_{d-2}(k \epsilon)} \right) \tilde{h}(k)_{z=\epsilon}^{z=\infty} \]  (1.115)

\[ \left( \frac{(qz)^2K_{d-2}(qz)}{(q \epsilon)^2K_{d-2}(q \epsilon)} \right)_{z=\epsilon}^{z=\infty} e^{-ik \cdot x} e^{+iq \cdot x} \tilde{h}(q) = \]

\[ = \int d^d k \left( \frac{d^d q}{(2\pi)^d} \right) \delta^d(k + q) \left( 4k^d \log(k \epsilon) + \sum_k \frac{1}{k \epsilon} \right) (\text{polynomial in } p) + \mathcal{O}(\epsilon) \tilde{h}(k) \tilde{h}(q), \]
where we used the expansion (1.110). At the end of this computation we shall impose \( \epsilon \to 0 \). The divergent terms \( \sim \frac{1}{\epsilon} \) can be avoided as in usual quantum field theory adding counterterms in the Lagrangian through the procedure of renormalization, this is the content of holographic renormalization [45]. Thus keeping only the logarithmic term we are ready to compute the two point function in the Fourier space

\[
< \mathcal{O}(p_1) \mathcal{O}(p_2) > = \frac{\delta^d W[\tilde{h}]}{\delta \tilde{h}(p_1) \tilde{h}(p_2)} |_{\tilde{h}=0} = (2\pi)^d \delta^d(k+q) 4k^d \log(k \epsilon).
\]  

(1.116)

Going back to the real space and using that for the massless scalar \( \Delta = d \) we obtain the two point function

\[
< \mathcal{O}(x_1) \mathcal{O}(x_2) > = \frac{1}{|x_1 - x_2|^{2\Delta}}.
\]  

(1.117)

This result is consistent with (1.43) coming from the conformal field theory. Therefore this computation confirms our interpretation of \( \Delta \) as the conformal dimension of the conformal operator \( \mathcal{O} \) dual to the massless scalar field \( \psi \). This result can be generalized to massive scalar fields through a similar calculation [21].

The Euclidean one point-function can be obtained in a similar manner, however the regularization procedure is more complicated. Through the unique procedure of holographic renormalization one obtains the general formula (1.92) and for the scalar massless field

\[
< \mathcal{O} > = dC_2(x),
\]  

(1.118)

where \( C_2(x) \) is the second coefficient of the solution for generic scalar field (1.88).

### 1.5 Summary

The AdS/CFT correspondence is a conjectured equivalence between two different theories, which however exhibit the same global symmetry group. We may think of the two theories as two formalisms describing the same underlying theory. The importance of the correspondence is that the two theories overlap in two different domains of validity leading to the so-called strong/weak duality. One can then gain some information on strongly coupled conformal field theories by studying weakly coupled dual gravity theories.

The non trivial aspect is the connection between the spectrum of the conformal field theory and the fields in the gravity side. To each field in the bulk there corresponds an operator whose source is the boundary value of the bulk field as in (1.95). The quantum numbers on both sides are related, as in (1.85) and (1.98). Then the prescription to compute physical quantities like correlation functions is given by (1.102). We often deal with divergent gravity actions, but these IR divergences are related to UV divergences in
the field theory and we can get rid of them by the useful tool of holographic renormalization. The AdS/CFT prescription has been summarized in table 1.1.

Remember that we have investigated a particular limit (1.3) of the correspondence where the gravity theory is classical and the conformal field theory is strongly coupled and at large $N$.

<table>
<thead>
<tr>
<th>$CFT$ side</th>
<th>$Gravity$ side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (RG-scale)</td>
<td>$r$ (radial coordinate)</td>
</tr>
<tr>
<td>$T = 0$ vacuum</td>
<td>$AdS$ background</td>
</tr>
<tr>
<td>$\mathcal{L} = \mathcal{L}_{CFT} + \mathcal{O}(x)h(x)$</td>
<td>$\lim_{x \to 0} f(z)h(x, z) = h(x)$</td>
</tr>
<tr>
<td>$\mathcal{O}, J_{\mu}, T_{\mu\nu}, \ldots$</td>
<td>$\psi, A_a, g_{ab}, \ldots$</td>
</tr>
<tr>
<td>$e^{-W_E[h(x)]}$</td>
<td>$e^{S_{AdS}[h(x, z)]}$</td>
</tr>
<tr>
<td>global symmetry</td>
<td>local symmetry</td>
</tr>
</tbody>
</table>

Table 1.1: AdS/CFT prescription
Chapter 2

Thermal $AdS$/CFT

The $AdS$/CFT correspondence, whose main features have been described in the previous chapter, provides a very useful tool for studying strongly interacting field theories. Several applications arose in the last years, both in the high energy and in condensed matter realm. Within these applications useful techniques came out to enlarge the original statement of the duality and to provide a larger connection between more generic, non-conformal field theories at strong coupling and their geometrical description. Already in 1998, E. Witten in [46] extended the just born correspondence to conformal quantum field theories at finite temperature and to a confining Yang-Mills gauge theory. Such generalizations of the original $AdS$/CFT statement go sometimes under the name of gauge/gravity duality.

The first applications were found in the high energy realm. In particular, inspired by the phenomenology of ultra-relativistic heavy ion collisions, the $AdS$/CFT approach has been applied to provide toy models for the quark-gluon plasma (QGP), a fluid of strongly coupled quarks and gluons. The simplest toy model is $\mathcal{N} = 4$ SYM theory at finite temperature, see [13] for a review. By means of computations in the dual gravity side, one can find analytically both thermodynamical and transport properties of this phase of matter, like the shear viscosity over the entropy density ratio $\frac{\eta}{s} = \frac{h}{4\pi k_B}$ [47], see [43] for a review. This value is remarkably close to that of the QCD plasma as deduced from the experiments.

Recently, the correspondence has been applied to model condensed matter systems, e.g. at finite charge density, lacking of a weakly coupled quasiparticle interpretation [14, 15, 16, 48, 49]. Further generalizations are required within the applications to condensed matter systems. Some of these systems are intrinsically non relativistic and the basic gauge/gravity duality statement can be enlarged to consider field theories which do not exhibit the full conformal invariance, but only invariance under translations, spatial
rotations and dilatations.

In view of these developments, and specifically having in mind applications to condensed matter physics, in this chapter we will focus on the extensions of the $AdS/CFT$ correspondence to finite temperature (in section 2.1) and finite charge density (in section 2.2) field theories. As we will see in a moment they have a simple dual description in terms of (charged) black holes in a higher dimensional spacetime. The non relativistic case will be only mentioned in section 2.3 for completeness. Good reviews on these topics are [25, 40, 50, 51].

### 2.1 Finite temperature

The partition function of a quantum field theory in thermodynamic equilibrium at finite temperature $T$ can be rewritten as an Euclidean path integral on closed Euclidean time paths of length $\beta = \frac{1}{T}$ with periodic (antiperiodic) boundary conditions on bosonic (fermionic) fields. For example for a scalar field $\phi$ we have

$$Z_E[\beta] = \text{Tr}(e^{-\beta \hat{H}}) = \int_{\phi(t_E+\beta,\vec{x})=\phi(t_E,\vec{x})} D\phi e^{-SE[\phi]},$$

(2.1)

With a compactified time, Minkowskian space has been rewritten as $R^{d-1} \times S^1$. Let us now see how this picture can be implemented in the dual gravity side.

Since we are interested on CFT vacua, let us consider the simplest Einstein gravity action (1.45) with the cosmological constant given in (1.53)

$$S = \frac{1}{2k_{d+1}^2} \int d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{L^2} \right).$$

(2.2)

There are in principle two solutions of the resulting Einstein’s equations which realize the $R^{d-1} \times S^1$ geometry at the boundary and are asymptotically $AdS$. One is given by the thermal $AdS$ solution, i.e. the usual $AdS$ space (1.62) with the Euclidean time compactified on a circle of period $\beta_0$

$$ds^2_{\text{thermal}} = \frac{\gamma^2}{L^2} dt_E^2 + \frac{\gamma^2}{L^2} d\vec{x}^2 + \frac{L^2}{\gamma^2} dr^2, \quad \text{with} \quad t_E = t_E + \beta_0.$$  

(2.3)

The other is given by a black hole solution in $AdS$, whose metric we postpone to the next paragraph. As we will see in a moment, the two descriptions are non equivalent and the preferred choice is given by the black hole solution because the thermal $AdS$ solution is in a sense trivial.
2.1. FINITE TEMPERATURE

Figure 2.1: Schematic representation of the Schwarzschild-AdS black hole in \((d+1)\) dimensions. In the interior of the space there is a planar horizon in \(r_H\), at the boundary resides the conformal field theory.

2.1.1 Schwarzschild-AdS black hole

The black hole solution rewritten in Minkowskian signature (see [40] for a review) is

\[
ds^2 = \frac{L^2}{z^2} \left( - f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right), \quad f(z) = 1 - \left( \frac{z}{z_H} \right)^d, \tag{2.4}\]

or equivalently, in terms of the \(r\)-coordinate

\[
ds^2 = \frac{r^2}{L^2} \left( - f(r) dt^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2 f(r)} dr^2, \quad f(r) = 1 - \left( \frac{r_H}{r} \right)^d. \tag{2.5}\]

The metrics above describe an object which goes under several names, such as planar black hole, Poincaré black hole, black brane and many others. Let’s call it Schwarzschild-AdS black hole to remind that it has the same features of an ordinary Schwarzschild black hole but it is asymptotically AdS instead of being asymptotically flat. In fact as \(z \to 0\), and respectively \(r \to \infty\), \(f \to 1\) and the AdS metric is recovered near the boundary. Deeply in the interior the space has been cut by placing an horizon at \(z = z_H\) (or \(r = r_H\)), where the blackening factor \(f(z)\) (or \(f(r)\)) vanishes linearly. The horizon is not a sphere as in the common Schwarzschild black hole, but an entire plane \(R^{d-1}\). The external regions are then \((z_H, 0)\) or \((r_H, \infty)\). Note that this time the horizon is an actual horizon in the sense that an asymptotic observer doesn’t receive information from the region inside the horizon. Instead the AdS metric’s (1.60) horizon at \(z = \infty\) is an artifact of the local coordinates, in fact the metric can be continued inside the horizon for example using global coordinates. The deformation of the metric is in the interior of the space, hence finite temperature is an IR (macroscopic) deformation in the dual field theory.
2.1.2 Temperature of the black hole

A first consequence of the holographic map in the present context is the identification of the black hole temperature with that of the dual field theory. The temperature of the \( \text{AdS} \) black hole (see e.g. [36]) can be easily computed by continuing the metric in Euclidean spacetime. Performing a Wick rotation to the metric (2.4) we get

\[
\begin{align*}
\frac{ds^2_E}{z^2} &= \left( \frac{L^2}{z^2} + f(z)dt^2_E + dx^2 + \frac{dz^2}{f(z)} \right). 
\end{align*}
\]  

(2.6)

A study of the near horizon geometry will give us the relationship between the temperature of the black hole and the parameter \( z_H \). The near horizon \( z \sim z_H \) metric can be found by taking \( z \simeq z_H + \tilde{z} \) with \( \tilde{z} \to 0 \), and taking the Taylor series expansion of \( f(z) \)

\[
\begin{align*}
f(z) &\simeq f(z_H) + f'(z_H)\tilde{z} + \ldots \simeq f'(z_H)\tilde{z}, 
\end{align*}
\]  

(2.7)

since \( f(z_H) \) is identically zero. This leads to

\[
\begin{align*}
\frac{ds^2_{E,\text{near horizon}}}{z_H^2} &\approx \frac{L^2}{z_H^2} \left( f'(z_H)\tilde{z}dt^2_E + dx^2 + \frac{d\tilde{z}^2}{f'(z_H)\tilde{z}} \right). 
\end{align*}
\]  

(2.8)

Defining new coordinates

\[
\begin{align*}
\rho &= 2\sqrt{\frac{\tilde{z}}{|f'(z_H)|}}, & \tau &= \frac{1}{2}|f'(z_H)|t_E, 
\end{align*}
\]  

(2.9)

we finally find the near horizon geometry

\[
\begin{align*}
\frac{ds^2_{E,\text{near horizon}}}{z_H^2} &\sim \frac{L^2}{z_H^2} \left( \rho^2 d\tau^2 + d\rho^2 + dx^2 \right). 
\end{align*}
\]  

(2.10)

This metric looks like a plane \( \mathbb{R}^{d-1} \) times an euclidean plane in polar coordinates \( (\rho, \tau) \). There is a deficit angle which leads to a conical singularity in \( \rho = 0 \) unless \( \tau \) is periodic according to

\[
\tau = \tau + 2\pi. 
\]  

(2.11)

The corresponding period for the euclidean time coordinate is

\[
\beta = \frac{4\pi}{|f'(z_H)|}. 
\]  

(2.12)

What does this period stand for in the dual field theory? Remember (see section 1.4.1) that the value of the bulk metric \( g_{ab} \) on the conformal boundary is related to the boundary metric \( g_{\mu\nu} \) by

\[
\lim_{z \to 0} \frac{z^2}{L^2} g_{ab}(x_\mu, z) = g_{\mu\nu}. 
\]  

(2.13)
2.1. **FINITE TEMPERATURE**

where the boundary metric has to be interpreted as the pull back of the bulk metric so that there is no $g_{zz}$ component. From (2.6) and (2.13) we see that the boundary metric is simply the Euclidean flat metric

$$ds_E^2 = dt_E^2 + d\vec{x}^2$$  \hspace{1cm} (2.14)

where the Euclidean time coordinate $t_E$ is periodically identified with period (2.12). The inverse of the periodicity, as previously discussed, should be identified with the temperature $T$ of the field theory. Thus

$$T = \frac{1}{\beta} = \frac{|f'(z_H)|}{4\pi} = \frac{d}{4\pi z_H},$$  \hspace{1cm} (2.15)

and using the $r$-coordinate

$$T = \frac{|f'(r_H)|r_H^2}{4\pi L^2} = \frac{dr_H}{4\pi L^2}.$$  \hspace{1cm} (2.16)

Notice that setting a conformal invariant field theory at finite temperature doesn’t uniquely define the equilibrium temperature. In fact there is no other scale to which we can compare it. All nonzero temperatures are equivalent. This is manifest by the invariance of the bulk metric (2.4) under the residual scale transformations

$$(t, \vec{x}, z) \rightarrow a (t, \vec{x}, z) \quad z_H \rightarrow az_H.$$  \hspace{1cm} (2.17)

By means of this transformation $z_H$ can be eliminated from equation (2.15) by setting $a = \frac{1}{z_H}$. In the new coordinates $z' = \frac{z}{z_H}$ and the horizon is simply $z'_H = 1$. Thus a scale invariant theory only admits two independent temperatures: zero and nonzero.

### 2.1.3 Thermodynamical quantities

**Free energy**

Given the notion of temperature we can try to find other thermodynamical quantities. It suffices to evaluate the free energy $F$ which, from standard thermodynamics, is given by

$$Z_{\text{CFT}} = e^{-\beta F} \rightarrow F = -T \log Z_{\text{CFT}},$$  \hspace{1cm} (2.18)

where $Z_{\text{CFT}}$ is the field theory partition function and we set the Boltzmann constant $k_B$ to one. From the AdS/CFT prescription (1.102) we can relate the CFT partition function, at large $N$ and strong coupling, to the saddle-point approximation of the gravity generating functional. The free energy (2.18) is then proportional to the on-shell gravity action for the black hole solution $\hat{g}_{ab}$

$$F = TS_{\text{gravity}}(\hat{g}_{ab}).$$  \hspace{1cm} (2.19)
The latter is given more precisely not only by the euclidean continuation of (2.2) which is the standard Hilbert-Einstein action

\[ S_{HE} = -\frac{1}{2k_{d+1}^2} \int d^{d+1}x \sqrt{g} \left( \mathcal{R} + \frac{d(d - 1)}{L^2} \right), \]  

(2.20)

but also by an additional boundary term. It is the Gibbons-Hawking [52] term

\[ S_{GH} = \frac{1}{2k_{d+1}^2} \int_{r=\infty} d^dx \sqrt{\gamma} (-2K), \]  

(2.21)

which is needed to cancel boundary terms coming from the variation of the Einstein-Hilbert action with respect to the metric. \( K \) is the trace of the extrinsic curvature of the boundary

\[ K = \gamma^{\mu\nu} \nabla_\mu n_\nu = \frac{1}{\sqrt{g}} \partial_r \left( \frac{\sqrt{g}}{\sqrt{g_{rr}}} \right), \]  

(2.22)

where \( n^\nu = \frac{\delta^r}{\sqrt{g_{rr}}} \) is an outward pointing unit normal vector to the boundary at \( r = \text{const.} \), and \( \gamma \) is the induced metric on the boundary. Let us rewrite the Schwarzschild-AdS black hole metric in Euclidean space with a new blackening factor

\[ ds^2 = g(r)dt_E^2 + \frac{r^2}{L^2} dx^2 + \frac{dr^2}{g(r)} \quad \text{with} \quad g(r) = \frac{r^2}{L^2} \left( 1 - \frac{r_H^d}{r^d} \right). \]  

(2.23)

Introducing a large \( R \) cutoff the boundary metric \( \gamma \) reads

\[ ds^2 \approx g_{\mu\nu} dx^\mu dx^\nu = g(R)dt_E^2 + \frac{R^2}{L^2} dx_i dx_i \quad \text{as} \quad R \to \infty. \]  

(2.24)

There is no intrinsic curvature of the boundary metric since it is flat.

The saddle point is the analytic continuation of the Schwarzschild-AdS metric to the Euclidean spacetime (2.23). Evaluating the terms (2.20) and (2.21) on the solution (2.23), and imposing a cutoff at large \( r \to R \), one finds (performing, where present, the integration over \( r \), ranging from \( r_H \) to \( R \))

\[ S_{HE} = \frac{1}{2k_{d+1}^2} \int d^dx \left( \frac{2R^d}{L^{d+1}} - \frac{2r_H^d}{L^{d+1}} \right), \]  

(2.25)

\[ S_{GH} = \frac{1}{2k_{d+1}^2} \int d^dx \left( - (d-1) \frac{R^{d-2}}{L^{d-1}} - (d-1) \frac{R^{d-1}}{L^{d+1}} g'(R) \right). \]  

(2.26)

Using the expression on the right hand side of (2.23) for the blackening factor and expanding around \( r = R \to \infty \) we obtain

\[ S_E = S_{HE} + S_{GH} = \frac{1}{2k_{d+1}^2} \int d^dx \left( (d - 2) \frac{r_H^d}{L^{d+1}} - 2(d - 1) \frac{R^d}{L^{d+1}} \right). \]  

(2.27)

(2.28)
Notice that the second term in such expression is divergent for \( R \to \infty \). One can cancel it by adding a suitable local counterterm in a covariant fashion. It is easily seen that an additional term of the form

\[
S_{\text{ct.}} = \frac{1}{2k_{d+1}^2} \int_{r=\infty} d^d x \sqrt{\gamma} \frac{2(d-1)}{L} = \frac{1}{2k_{d+1}^2} \int d^d x \frac{2(d-1)}{L^d} R^{d-1} \sqrt{g(R)},
\]

(2.29)
does the right job. Notice that there will still be a divergent factor coming from the integration over the \( d-1 \) space coordinates. However, we will be ultimately concerned with densities and thus these divergences will drop out. The Euclidean action \( S_E \) evaluated on the Schwarzschild-AdS black hole metric (2.23) will be given by

\[
S_E = S_{\text{HE}} + S_{\text{GH}} + S_{\text{ct.}} = \frac{1}{2k_{d+1}^2} \int d^d x \frac{r_H^d}{L^{d+1}} = \frac{1}{2k_{d+1}^2} V_{d-1} \frac{r_H^d}{L^{d+1}} = -\frac{(4\pi)^d L^{-d-1}}{2k_{d+1}^2} V_{d-1} T^{d-1},
\]

(2.30)

where in the last equivalence we used the relation (2.16). \( V_{d-1} \) is the infinite spatial volume and \( T = \frac{1}{\beta} \) is the temperature of the black hole, i.e. the inverse of the length of the time circle. The above method of adding local counterterms in a covariant fashion to the Euclidean action has been largely exploited and goes under the name of holographic renormalization, see [45] for a review. Notice that in the thermal AdS case, where there is no horizon \( r_H = 0 \), this method would give a trivial result \( S_E = 0 \).

Another possibility to regularize the euclidean action is given by the Hawking-Page prescription [53], implemented by Witten in [46]. The method consists in subtracting to the Einstein-Hilbert (2.20) and Gibbons-Hawking (2.21) terms evaluated on the black hole solution (2.23) the analogous contributions evaluated on the thermal AdS background (2.3). The Hilbert-Einstein (2.20) and Gibbons-Hawking (2.21) terms evaluated on the thermal AdS solution (2.3) are given by

\[
S_{\text{thermal}}^{\text{HE}} = \frac{1}{2k_{d+1}^2} \int d^d x \frac{2 R^d}{L^{d+1}}, \quad (2.31)
S_{\text{thermal}}^{\text{GH}} = \frac{1}{2k_{d+1}^2} \int d^d x \left( - \frac{2d R^d}{L^{d+1}} \right). \quad (2.32)
\]

Note that in (2.32) the integration over the \( r \) coordinate has been performed over the range \((0, R)\) since thermal AdS solution (2.3) does not have an horizon. The whole euclidean action results

\[
S_{\text{thermal}}^{\text{HE}} + S_{\text{thermal}}^{\text{GH}} = \frac{1}{2k_{d+1}^2} V_{d-1} \beta_0 \frac{2 R^d}{L^{d+1}} 2(1-d). \quad (2.33)
\]

Now, to compare these contributions to the analogous ones evaluated on the Schwarzschild-AdS black hole solution (2.25-2.26), one must relate the two periods \( \beta \) and \( \beta_0 \) requiring
that the metrics asymptotically match at \( r = R \to \infty \). Now
\[
g_u(R) = g(R) \quad \text{for the black hole metric,} \tag{2.34}
g_{\text{thermal}}(R) = \frac{R^2}{L^2} \quad \text{for the thermal AdS solution.} \tag{2.35}
\]
Thus the two periods are related by
\[
\beta_0 = \beta \frac{\sqrt{g_u(R)}}{\sqrt{g_{\text{thermal}}(R)}} = \beta \frac{\sqrt{g(R)}}{L} \frac{L}{R}. \tag{2.36}
\]
Inserting this relation into (2.33) one finds
\[
S^\text{thermal}_{HE} + S^\text{thermal}_{GH} = - \frac{1}{2k^2} V_{d-1} \beta 2(d - 1) \frac{R^{d-1}}{L^d} \sqrt{g(R)} = - S_{\text{ct.}}, \tag{2.37}
\]
which has exactly the opposite value of the local counterterm (2.29). Thus, following Hawking-Page prescription, one obtains the same final result as in the previous case where we made use of local counterterms because
\[
S_{HE} + S_{GH} - S^\text{thermal}_{HE} - S^\text{thermal}_{GH} = S_{HE} + S_{GH} + S_{\text{ct.}}. \tag{2.38}
\]
Let us finally compute the free energy density \( F = \frac{F}{V_{d-1}} \) by using (2.19)
\[
F = - \frac{1}{2k_{d+1}^2} r_H^d = - (4\pi)^d L^{d-1} \frac{k^2}{2k_{d+1}^2 d^d} T^{d-1}. \tag{2.39}
\]
As we have previously discussed the thermal AdS solution (2.3) could be in principle another possibility for a dual to a finite temperature CFT. What makes us choose the black hole solution (2.5) with respect to the thermal AdS one here, is that the former gives always a lower value of the free energy. In fact the free energy of the thermal AdS solution is actually zero and the black hole free energy (2.39) is always negative.

Entropy

The entropy density \( s \) is given by
\[
s = \frac{S}{V_{d-1}} = - \frac{\partial F}{\partial T} = \frac{(4\pi)^d L^{d-1}}{2k_{d+1}^2 d^d} T^{d-1}. \tag{2.40}
\]

The power of \( T \) could have been anticipated. It follows from dimensional analysis, since the temperature is the only scale of a thermal CFT. Thanks to (1.72) we find that the entropy density is proportional to the number \( "N^2" \) of degrees of freedom of the dual field theory
\[
S \sim N^2 T^{d-1}. \tag{2.41}
\]
2.1. FINITE TEMPERATURE

We may check our computation by using directly the Bekenstein-Hawking formula (1.65). The area of the black hole horizon is

$$A = \int_{r=r_H, \text{ fixed}} \sqrt{\gamma_{(d-1)}} d^{d-1}x = \left(\frac{r_H}{L}\right)^{d-1} V_{d-1}$$

(2.42)

where $\gamma_{(d-1)}$ is the induced metric at the horizon (at fixed $t$ and $r = r_H$)

$$ds^2_{(d-1)} = \frac{r_H^2}{L^2} dx^2.$$ 

(2.43)

Therefore the entropy is

$$S = \frac{A}{4G_N} = \frac{r_H^{d-1}}{L^{d-1}} \frac{V_{d-1}}{4G_N} = \frac{2\pi}{k_{d+1}} \frac{r_H^{d-1}}{L^{d-1}} V_{d-1}.$$ 

(2.44)

By using the relation between the horizon radius and the temperature of the black hole (2.15) one finds the same result (2.40) obtained via holographic methods.

A remarkable result of such computations comes from explicit realizations of the $AdS$/CFT correspondence. In the particular case of thermal $\mathcal{N} = 4$ SYM in $d = 4$ where $N = N_c$ is the number of colors of the $SU(N_c)$ gauge group, the relation (1.72) becomes\(^1\)

$$\frac{L^3}{4G_5} = \frac{N_c^2}{2\pi},$$

(2.45)

which leads to an entropy

$$s_{\lambda=\infty} = \frac{\pi^2}{2} N_c^2 T^3.$$ 

(2.46)

This result is reliable in the classical gravity limit with no stringy corrections, hence, as seen from (1.3) in the large $N_c$, strong coupling $\lambda = \infty$ limit of the conformal quantum field theory. One may also compute, by standard perturbative methods of quantum field theory, the entropy density at weak coupling $\lambda = 0$, see [13] for a review. The ratio between the entropies at strong and weak coupling is then

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4}.$$ 

(2.47)

This is a very interesting result because even if the coupling changes within a very wide range, the thermodynamic properties change very mildly. This observation, though not the precise $\frac{3}{4}$ ratio, seems to be a generic phenomenon for field theories with gravitational dual. The transport properties, which directly depend on the couplings, change instead dramatically. At weak coupling we deal with an ideal gas-like plasma of quasiparticles, and at strong coupling we have a nearly ideal liquid with no quasiparticles at all.

\(^1\)From the compactification of the 10-dimensional gravity theory in $AdS_5 \times S^5$ on the 5-dimensional sphere $S^5$ one can relate the two gravity constants $G_{10}$ to $G_5$, i.e. $\frac{L^3 Vol(S^5)}{16\pi G_{10}} = \frac{1}{16\pi G_5}$. Using (1.2) and considering that $Vol(S^5) = \pi^3$ we are lead to the desired result (2.45).
Pressure and energy density

At zero chemical potential the pressure is given by minus the free energy density\(^2\), thus taking (2.39)

\[
p = -\mathcal{F} = \frac{(4\pi)^d L^{d-1}}{2k_{d+1}^2 d^d} T^d. \tag{2.48}
\]

The energy density can be then obtained from the thermodynamic relation

\[
\mathcal{F} = -p = \epsilon - Ts \tag{2.49}
\]

hence taking (2.40) the energy density is given by

\[
\epsilon = (d - 1) \frac{(4\pi)^d L^{d-1}}{2k_{d+1}^2 d^d} T^d. \tag{2.50}
\]

One can verify the holographic result by identifying \(\epsilon\) with the ADM energy density of the black hole solution. The ADM energy is given by (see e.g. [54])

\[
E_{\text{ADM}} = -\frac{1}{k^2_{d+1}} \sqrt{g_{tt}} \int_{r \to \infty} t=\text{fixed} \ d^{d-1}x \sqrt{(\gamma_{(d-1)}(K - K^{\text{thermal}}))}, \tag{2.51}
\]

where \(\gamma_{(d-1)}\) is the determinant of the induced metric on the hypersurface at fixed time and fixed \(r\) coordinate (2.43). Notice that it coincides for both black hole (2.23) and thermal \(AdS\) (2.3) solution, since there is no time component. \(K\) are the extrinsic curvatures of the black hole and thermal \(AdS\) solutions. At constant time hypersurfaces they are given by

\[
K = \frac{1}{\sqrt{g_{(d)}}} \partial_r \left( \frac{\sqrt{g_{(d)}}}{\sqrt{g_{rr}}} \right) \tag{2.52}
\]

where this time \(g_{(d)}\) is the determinant of the induced metric of the black hole and thermal \(AdS\) solution in \(d\)-dimensional space at fixed time

\[
ds_{(d)}^2 = \frac{r^2}{L^2} dx^2 + \frac{dr^2}{g(r)} \quad \text{for the black hole solution}, \tag{2.53}
\]

\[
ds_{(d)}^2 = \frac{r^2}{L^2} dx^2 + \frac{L^2}{r^2} dr^2 \quad \text{for the thermal AdS solution}. \tag{2.54}
\]

Using such metrics in (2.52) and taking a cutoff \(R\) one obtains

\[
E_{\text{ADM}} = \frac{1}{k_{d+1}^2} \sqrt{g(R)} V_{d-1} \frac{R^{d-1}}{L^{d-1}} \left( \frac{(d - 1) \sqrt{g(R)}}{R} - \frac{(d - 1)}{L} \right). \tag{2.55}
\]

\(^2\)The internal energy of a system is given by \(\epsilon = Ts - p + \mu \rho\), where \(T\) is the temperature, \(s\) the entropy density, \(p\) the pressure, \(\mu\) the chemical potential and \(\rho\) the charge density. The free energy density is given by \(\mathcal{F} = \epsilon - Ts\). Combining the two expressions at zero chemical potential one obtains the desired result.
taking the limit $R \to \infty$ and the expression for the blackening factor given in (2.23) one is lead to

$$E_{\text{ADM}} = \frac{1}{2k_{d+1}^2} (d-1)V_{d-1} \frac{r_H^d}{L^{d+1}} = (d-1)V_{d-1} \frac{(4\pi)^d L^{d-1}}{2k_{d+1}^2 d^d} T^d,$$  \hspace{1cm} (2.56)

which exactly matches with our previous holographic result (2.50) when rescaled by $V_{d-1}$.

**The heat capacity density and speed of sound**

The heat capacity density at constant volume reads, from the previous result (2.50)

$$c_V = \left. \frac{\partial \epsilon}{\partial T} \right|_V = (d-1) \frac{(4\pi)^d L^{d-1}}{2k_{d+1}^2 d^d} T^{d-1}. \hspace{1cm} (2.57)$$

The speed of sound at zero chemical potential is obtained using (2.40) and (2.57)

$$v_s^2 = \frac{s}{c_V} = \frac{1}{(d-1)}, \hspace{1cm} (2.58)$$

which is the expected result for a $d$-dimensional conformal field theory at finite temperature. For example in $d = 4$ we get the standard result $v_s^2 = \frac{1}{3}$. Notice that $v_s^2 = \frac{dp}{d\epsilon}$ consistently, since we can write the pressure $p$ (2.48) in terms of the energy density (2.50)

$$p = \frac{1}{(d-1)} \epsilon. \hspace{1cm} (2.59)$$

This also means that the trace of the CFT’s stress-energy tensor at equilibrium $T_{\mu\nu} = \text{diag}(-\epsilon, p, \ldots, p)$ is zero

$$\text{tr}T_{\mu\nu} = -\epsilon + (d-1)p = 0, \hspace{1cm} (2.60)$$

consistently with the conformal invariance of the dual field theory.

To summarize the story so far: conformal and strongly coupled quantum field theories in equilibrium at finite temperature can be mapped into Schwarzschild-AdS black hole backgrounds. All the thermodynamical quantities can be computed analytically using the prescription (1.102) of the $AdS$/CFT correspondence.

### 2.2 Finite chemical potential

Condensed matter systems commonly exhibit a $U(1)$ symmetry. This could be an electromagnetic symmetry which is of course gauged. However, in many condensed matter processes dynamical photons can be neglected for at least two reasons. One is that electromagnetic interaction is usually observed to be weak. Secondly, the electromagnetic interaction is screened in a charged medium. Thus in some condensed matter problems
we can neglect dynamical photons by considering an effective field theory description involving effective degrees of freedom that are charged fields but with no gauge bosons. From this point of view the electromagnetic symmetry can be threaten as a global symmetry. An electromagnetic current $J_\mu$ can be induced adding to the Lagrangian of the field theory a source term of the form $\mathcal{L} = J_\mu(x)A^\mu(x)$.

### 2.2.1 Reissner-Nordstrom-AdS black hole

So what is the gravity dual of a field theory with a global $U(1)$ symmetry? We have already seen that to each global symmetry there corresponds a gauge symmetry in the bulk. The connection between bulk fields and boundary operators that we have established in (1.76) tells us that the background field $A_\mu(x)$ should be the boundary value of a bulk gauge invariant field $A_a(x_\mu, z)$ as in (1.95). Thus in order to describe a global symmetry in the field theory side we need a Maxwell field in the bulk.

The minimal framework capable of describing the physics of a massless gauge field $A_a$ in the bulk is the Einstein-Maxwell theory, where we keep again a non vanishing negative cosmological constant in order to have an AdS vacuum

$$S = \frac{1}{2k_{d+1}^2} \int d^{d+1}x \sqrt{g} \left( R + \frac{d(d-1)}{L^2} - \frac{1}{4} F_{ab} F^{ab} \right).$$

(2.61)

Here $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field strength. The Maxwell coupling $g^2$ has been absorbed in the redefinition of the Maxwell field and the latter results dimensionless.

Consider first the case with no magnetic field. Let us look at the solutions of the Einstein-Maxwell theory (2.61) together with an homogeneous ansatz for the time component of the Maxwell field

$$A = A_t(r)dt.$$  

(2.62)

The Einstein’s equations of motion are

$$G_{\mu\nu} - \frac{d(d-1)}{2L^2} g_{ab} = -\frac{1}{2} T_{ab}.$$  

(2.63)

with the stress-energy tensor

$$T_{ab} = \frac{1}{4} g_{ac} F_{cd} F^{cd} - F_{ac} F^c_b.$$  

(2.64)

Maxwell’s equations are

$$\nabla_a F^{ab} = 0.$$  

(2.65)

The charged black hole solution is the Reissner-Nördstrom (RN)-AdS black hole. The form of the metric is the same as (2.23) but with a different blackening factor $g(r)$; in
2.2. **FINITE CHEMICAL POTENTIAL**

Minkowskian space it writes

\[
\begin{align*}
    ds^2 &= -g(r)dt^2 + \frac{r^2}{L^2}d\vec{x}^2 + \frac{dr^2}{g(r)}, \\
g(r) &= \frac{r^2}{L^2} \left(1 - \frac{r^d_{\text{H}}}{r^d}\right) + \frac{1}{2} \frac{\mu^2 (r_H/r)^{2(d-2)} (1 - \left(\frac{r}{r_H}\right)^{d-2})}{2(d-1)}, \\
    A_t &= \mu \left(1 - \left(\frac{r_H}{r}\right)^{d-2}\right),
\end{align*}
\]

(2.66, 2.67, 2.68)

where \( r = r_H \) is now an outer planar horizon, which hints a further inner horizon. The two horizons coincide in the limit of zero temperature. In this case the black hole is said to be extremal. The solution contains a nonzero gauge field. Its profile is found by requiring that it should be vanishing at the horizon \( A_t(r_{\text{H}}) = 0 \). This condition can be retained (see e.g. [55]) by requiring the norm of the vector field \( g^{tt} \sqrt{g} A_t A_t = \frac{1}{g(r)} \frac{r^2}{L^2} A_t A_t < \infty \).

The charge density of the dual field theory \( <J_t> = \rho \) is related to the subleading behavior of the bulk Maxwell field analogously to (1.96), see [25]. Using the suitable generalization of (1.92) and \( \Delta(j_{\mu}) = d - 1 \), we have

\[
    \rho = \frac{1}{2k_{d+1}} \frac{(d-2)\mu^{d-2}}{r_H^{d-1}},
\]

(2.69)

which can also be given by

\[
    \rho = -\frac{\delta \mathcal{L}}{\delta(\partial_r A_t)},
\]

(2.70)

where \( \mathcal{L} \) is the bulk lagrangian density. This implies that we can rewrite

\[
    A_t = \mu - \frac{\rho L^{d-1} 2k_1^2}{(d-2)r^{d-2}}.
\]

(2.71)

Sometimes we will deal with the grancanonical ensemble where the number of particles can vary and the chemical potential \( \mu \) is fixed. The leading term in (2.71) will be the source of the conserved charge in the field theory side and the coefficient in \( \frac{1}{r^{d-2}} \) will be the response. The canonical ensemble will instead threat \( \rho \) as the source and \( \mu \) as the response. The use of one or the other description will depend upon the particular problem under investigation.

The RN-AdS black hole solution is characterized by two scales: the chemical potential

\[
    \mu = \lim_{r \to \infty} A_t(r)
\]

(2.72)

and the horizon radius \( r_{\text{H}} \). From the dual field theory perspective it is more physical to think in terms of the temperature instead than the horizon radius. The temperature can be obtained [40] by using (2.16) and (2.67)

\[
    T = \frac{r_H}{4\pi L^2} \left( d - \frac{\gamma^2 \mu^2 L^2}{r_H^2} \right), \quad \text{with} \quad \gamma^2 = \frac{1}{2} \frac{(d-2)^2}{(d-1)}.
\]

(2.73)
CHAPTER 2. THERMAL ADS/CFT

Figure 2.2: Schematic representation of the Reissner-Nördstrom-AdS black hole. In the interior of the space there is the outer horizon. On the boundary there is the charge density of the dual field theory.

Differently from the Schwarzschild-AdS black hole even if we can again scale out $r_H$ from (2.73), the temperature can go continuously to zero. The physical behavior of the theory now depends on the dimensionless ratio $\frac{T}{\mu}$.

2.2.2 Thermodynamical quantities

In the grand canonical ensemble the partition function is given by

$$Z_E = \text{Tr}(e^{-\beta(H-\mu N)}) = e^{-\beta \Omega},$$  \hspace{1cm} (2.74)

where $N$ is the number of particles and $\Omega$ is the Gibbs free energy. Using the holographic approach one may compute $\Omega$ by evaluating on the RN-AdS black hole solution (2.66-2.67) the euclidean continuation of the action (2.61)

$$S_E = -\frac{1}{2k_{d+1}^2} \int d^{d+1}x \sqrt{-g}(R + \frac{d(d-1)}{L^2} + \frac{1}{4} F_{ab} F^{ab}).$$  \hspace{1cm} (2.75)

To regularize such action no additional counterterms other than (2.29) are needed (see e.g. [40]) because the Maxwell field falls off sufficiently quickly near the boundary in the dimensions of interest $d \geq 3$. The Gibbs free energy density is given by

$$\omega = \frac{T S_E}{V_{d-1}},$$  \hspace{1cm} (2.76)

hence explicitly evaluating the on-shell regularized action $S_E$ as we did above, one finds

$$\omega = -\frac{1}{2k_{d+1}^2} \frac{r_H^d}{L^{d+1}} \left(1 + \frac{\gamma^2 \mu^2 L^2}{d-2 \frac{r_H^2}{L^2}} \right).$$  \hspace{1cm} (2.77)
2.2. *FINITE CHEMICAL POTENTIAL*

By solving equation (2.73) for the outer horizon \( r_H \)

\[
r_H = \frac{2\pi L^2}{d} T + \frac{\gamma \mu L}{\sqrt{d}} \sqrt{1 + \frac{4\pi^2 L^2 T^2}{d\gamma^2 \mu^2}},
\]

(2.78)

and expanding in \( T \) one obtains the Gibbs free energy density as a function of the temperature

\[
\omega \simeq -a \mu^d - b \mu^{d-1} T - c \mu^{d-2} T^2 - \ldots.
\]

(2.79)

where \( a, b \) and \( c \) are constants depending on \( \mu, L \) and \( d \).

We can recompute the charge density of the field theory by

\[
\rho = -\frac{\partial \omega}{\partial \mu},
\]

(2.80)

using the formula for the Gibbs free energy (2.77) and reminding that the outer horizon also depends on the chemical potential \( \mu \) we find exactly the same expression in (2.69).

The pressure is given by

\[
p = -\omega = \frac{1}{2k_{d+1}^2} \frac{r_H^d}{L^{d+1}} \left( 1 + \frac{\gamma^2 \mu^2 L^2}{d-2} \frac{r_H^2}{r_H^2} \right) \simeq \frac{a \mu^d + b \mu^{d-1} T + c \mu^{d-2} T^2 + \ldots}{2k_{d+1}^2} \]

(2.81)

The entropy density is given by

\[
s = \frac{\partial p}{\partial T}|_\mu,
\]

(2.82)

which exactly matches with the entropy density given by the area law (2.44) when one takes the horizon radius function of the temperature (2.78).

The remaining thermodynamical observables easily follow. In the canonical ensemble, the Helmholtz free energy density \( F \) is given by \( F = \epsilon - Ts = \omega + \mu \rho \). Using the same procedure of the previous section it is easy to verify holographically that the stress-energy tensor is still traceless.

### 2.2.3 Near horizon geometry

The Reissner-Nordstrom geometry is interesting as \( T \to 0 \). In this limit the horizon has a fixed value at

\[
r_H^2 = \frac{1}{2d} \frac{(d-2)^2 L^2 \mu^2}{(d-1)}.
\]

(2.83)

To find the near horizon metric take the series Taylor expansion of the blackening factor

\[
g(r) \simeq g(r_H) + g'(r_H) \tilde{r} + \frac{1}{2} g''(r_H) \tilde{r}^2
\]

(2.84)
where again \( r = r_H + \tilde{r} \) with \( \tilde{r} \to 0 \). We find that

\[
g(r_H) = 0, \quad g'(r_H) \sim T = 0, \quad g''(r_H) = \frac{2d(d-1)}{L^2}.
\] (2.85)

The near horizon metric reads then

\[
ds^2_{\text{near horizon}} \simeq -\frac{\tilde{r}^2}{L^2} dt^2 + \frac{r_H^2}{L^2} d\vec{x}^2 + \frac{L^2}{d(d-1)r^2} d\tilde{r}^2,
\] (2.86)

from which we recognize the \( \text{AdS}_2 \times \mathbb{R}^{d-1} \) metric. The presence of the \( \text{AdS} \) factor means that this IR region of the geometry is scale invariant.

### 2.3 Non relativistic gauge/gravity duality

Many condensed matter systems at their quantum critical points exhibit scale invariance. Sometimes this is only a non relativistic version of the scale invariance involved in the conformal transformations which treat space and time coordinates at the same level. Assuming spatial isotropy, in general we can have the following scaling transformation

\[
t \to a^z t, \quad \vec{x} \to a \vec{x}.
\] (2.87)

where \( z \) is the dynamical critical exponent. The symmetry algebra of such group of transformations contains the generators of the translations \( P_i \), rotations \( M_{ij} \), and dilatations \( D \). The algebra is sometimes called the Lifshitz algebra [40] and contains the standard commutation relations for the operators \((M_{ij}, P_i, H)\) together with the action of dilatations

\[
[D, M_{ij}] = 0, \quad [D, P_i] = iP_i, \quad [D, H] = iz H.
\] (2.88)

This symmetry can be realized geometrically in a higher dimensional spacetime by taking the following metric

\[
ds^2 = L^2 \left( -\frac{dt^2}{z^{2z}} + \frac{d\vec{x}^2}{z^2} + \frac{dz^2}{z^2} \right).
\] (2.89)

The case of \( z = 1 \) is the standard Anti de Sitter space, where the scale invariant theory is enhanced to a conformal invariant theory since; beside rotations, translations and dilatations, the theory enjoys Lorentz boosts and special conformal symmetries. For \( z > 1 \) these spaces are candidate duals to non-relativistic field theories.

Such generalization of the \( \text{AdS/CFT} \) is relevant to condensed matter systems at non relativistic scale invariant quantum critical points. An example is given by ultracold trapped atoms, see e.g. [9] for a review. Such systems exhibit a typical crossover from two regimes: one where fermionic atoms pair up into bosonic molecules (the BCS phase),
and the other where the binding mechanism is very strong and the bosonic molecules are tightly bound enough to Bose-Einstein condense (the BEC phase). The system behaves as a nearly ideal fluid. The gauge/gravity tools have been applied [48, 49] in order to investigate the main equilibrium and transport properties of such systems.

2.4 Summary

In this chapter we have discussed the holographic realization of field theories at finite temperature and finite chemical potential. At thermal equilibrium these systems are described in the contest of the gauge/gravity duality by means of dual black hole solutions. Conformal field theories at \( T \neq 0 \) have a dual description in terms of AdS-Schwarzschild black holes (2.4). Turning on a chemical potential in the dual field theory corresponds to taking an asymptotically AdS Reissner-Nördstrom black hole (2.66)-(2.67).

The temperature of the black hole can be easily found by requiring the Euclidean metric to be regular in the interior of the space. The free energy of the dual field theory is given by the euclidean bulk action (2.19). Computations of such kind must be taken carefully. The bulk action suffers from infinities and must be regularized. One possibility is to consider additional boundary counterterms such as (2.29). On the other hand one may compute the difference between the black hole solution and the thermal solution. Other thermodynamical quantities are easily found.

Applications to condensed matter systems have been recently developed in the direction of non relativistic theories. Here the scaling symmetry differs by the dynamical critical exponent \( z \neq 1 \) with respect to the relativistic case. The dual geometries are thus described by a metric of the form (2.89).
Chapter 3

Imbalanced superconductors

Many condensed matter models rely on the existence of quasiparticles, dressed particles where the neglected interactions have been absorbed in the dressing. The best known examples are the Landau-Fermi liquid theory, and the BCS theory of superconductivity (see e.g. [56]). However there are relevant systems in condensed matter physics where standard tools are not reliable. Among them there are unconventional superconductors. These materials can not be described by the standard BCS theory because either the interactions are not mediated by phonons as in the usual BCS theory, or the system is inherently strongly coupled. The latter possibility may happen when the onset of superconductivity occurs in the vicinity of a quantum critical point (see e.g. [51, 62]), which exhibits scale invariance. In the last years many experimental efforts have been devoted to the study of unconventional superconductors, giving some (still not definitive) experimental evidences in support to the idea of the existence of relativistic quantum critical points within their phase diagram, see [57] for a review.

A general question is, thus, whether phenomena predicted at weak coupling (within, say, the BCS theory), are to be expected in unconventional cases too. In this thesis we have decided to focus on a very specific example: the case in which a chemical potential imbalance is implemented among the different fermionic species in the model.

For a superconductor, such an imbalance among the two spin species (up, down) can be induced by the Zeeman coupling with an external magnetic field. At weak coupling when the applied field is large enough, a novel phase of inhomogeneous superconductivity can develop, namely the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase [18]. This is a phase where Cooper pairs have non zero total momentum. By further increasing the field it may become more energetically favorable for the system to turn to the normal phase. The occurrence of a LOFF phase is hard to be experimentally detected because of the dominance of the orbital coupling of the external magnetic field with respect to the
coupling to the spins. Thus, the superconductive phase may be destroyed too early, before an appreciable inhomogeneous phase could have been seen. The orbital coupling results negligible when the geometry of such superconductors is layered. This is the case, as we will see in the following, of unconventional superconductors, where, however, it is not clear whether the weak coupling predictions can apply. It is thus interesting to ask whether a LOFF phase may occur in the phase diagram of unconventional superconductors. In the following chapter we will try to answer this question for an holographic toy model and using AdS/CFT tools.

In fact gauge/gravity duality comes in the game as a novel tool to explore properties of strongly interacting systems. In condensed matter these are systems at quantum critical points, effectively described by field theories in various dimensions with scale invariance. Differently from, e.g., the case of $\mathcal{N} = 4$ SYM where a precise gravity (or string) description is known, in the condensed matter contexts we have in mind, we actually miss the precise microscopic details of the quantum field theory and hence the detailed map with a dual gravity background. For this reason the holographic approach in these cases relies on an effective bottom-up procedure: one tries to implement the main ingredients in the game, such as scales, symmetries and their breaking, in a dual gravity description with the aim to extract some universal property which should thus not depend on the particular microscopic QFT model.

In this chapter we want to introduce some basic condensed matter background in view of the application of the AdS/CFT discussed in the next chapter. Hence, in the first section we will briefly review the standard theory of superconductivity starting from the main qualitative aspects, going through the London equations, the Ginzburg-Landau theory and ending with the standard BCS theory. This section is mainly based on [7, 8, 58, 59]. In section 3.2 we will present the basic features of imbalanced superconductors, and focus on the occurrence of the LOFF phase (good reviews on this topic are [17, 60]). In section 3.3, mainly based on [6, 40, 61, 62], we will review the basic aspects of quantum critical points and their possible occurrence in unconventional superconductors.

### 3.1 An overview of superconductivity

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes [63] by using the just discovered liquid helium as a refrigerant to reach temperatures of a few degrees of Kelvin. He observed that the electrical resistance of various metals disappeared completely below some critical temperature $T_c$, characteristic of the material. Thus perfect conductivity ($\sigma = \infty$) is the first traditional hallmark of superconductivity. Above $T_c$ the resistivity has the standard form of a normal metal, $\rho(T) = \rho_0 + BT^5$. Below $T_c$ the resistivity drops
3.1. AN OVERVIEW OF SUPERCONDUCTIVITY

Abruptly to zero and currents can flow in a superconductor with no discernible dissipation of energy. The next peculiar characteristic of perfect diamagnetism was discovered in 1933 by Meissner and Ochsenfeld \[64\]. They found that not only a magnetic field is excluded from entering a superconductor, as might appear explained by perfect conductivity, but also that a field in an originally normal sample above \(T_c\) is expelled as it is cooled down to the superconducting phase. Perfect conductivity would instead trap the magnetic flux in.

The existence of such Meissner effect implies that superconductivity will be destroyed by a critical magnetic field \(H_c(T)\). In fact as the magnetic field is turned on, a certain amount of energy is required to reset it in the interior of the superconductor through the screening currents. If the applied field is large enough it would be energetically favorable for the sample to turn back to the normal phase allowing the field to penetrate. The thermodynamic critical field \(H_c(T)\) is determined by equating the energy \(\frac{H^2(T)}{8\pi}\) per unit volume, associated to holding the magnetic field out the sample, with the difference of the free energy densities between the normal and the superconducting phase

\[
f_s(T) + \frac{H^2(T)}{8\pi} = f_n(T).
\]

(3.1)

Empirically it was found that \(H_c(T)\) is quite well approximated by a parabolic law

\[
H_c(T) \approx H_c(0) \left(1 - \frac{T}{T_c}\right)^2.
\]

(3.2)

At zero magnetic field the normal to superconducting phase transition is second order, while in the presence of an external field the order can differ.

It was discovered in 1957 by Abrikosov \[65\] that the manner in which penetration of the magnetic field occurs when increasing field strength depends on the type of the superconductor.

- **Type I**: Below the critical field value \(H_c(T)\) there is no penetration flux. When the applied field strength exceeds the critical value \(H_c(T)\) the entire sample reverts suddenly to the normal phase where the field penetrates perfectly, and the transition is first order.

- **Type II**: Below a lower critical value \(H_{c1}(T)\) there is no penetration flux. However, there is a region \(H_{c1}(T) < H < H_{c2}(T)\) in which the system exhibits a partial penetration flux, corresponding to a rather complicated state which goes under the name of mixed state. Above the upper critical field \(H_{c2}(T)\) the normal phase is favorable and the transition is second order.

A normal conductor at low temperatures has a specific heat of the form \(c \approx AT + BT^3\). Below the critical temperature this behavior is substantially altered. As the temperature
is lowered below $T_c$, the specific heat suddenly jumps to a higher value, and then decreases slower than the normal conductor’s specific heat pattern. The dominant low-temperature behavior is of the form $e^{-\frac{\Delta}{kT}}$, which is the characteristic behavior of a system whose excited levels are separated from the ground state by an energy gap of $2\Delta$.

### 3.1.1 The London theory

In 1935 the brothers F. and H. London [66] examined for the first time in a quantitative way the main features of superconductors. They proposed two phenomenological equations based on a two-fluid model of Gorter and Casimir [67]. The crucial assumption is to model the superconducting material as composed by two fluids: a normal and a superconducting one. Thus the total electron density is given by the sum of the superconducting and the normal electron densities

$$n = n_s(T) + n_n(T).$$

The density $n_s(T)$ of the superconducting electrons approaches the full electronic density $n$ by lowering enough the temperature, but it drops to zero as the temperature reaches $T_c$. The remaining fraction of electron density $n_n$ constitutes the normal fluid, which cannot carry an electric current without normal dissipation.

To each fraction of electrons one must associate a different current density. The normal fluid is governed by the Ohm’s law, where the current is proportional to the electric field $J_n = \sigma E$ and $\sigma$ is the conductivity. The superconducting electrons should instead behave in a different manner. The postulate is that their current is increasing with respect to the electric field\(^1\)

$$\frac{dJ_s}{dt} = \frac{n_s e^2}{m} E = \frac{c^2}{4\pi \lambda^2} E.$$ \[(3.4)\]

where $\lambda$ is a phenomenological parameter connected to the mass $m$, the charge $e$ of the electron and the density of the superconducting electrons. Moreover London brothers observed that the perfect diamagnetism property could be captured by a further equation for the supercurrent

$$\nabla \times J_s = -\frac{c}{4\pi \lambda^2} B,$$ \[(3.5)\]

which, together with (3.4), constitutes the London equations. The Maxwell equation $\nabla \times B = \frac{4\pi}{c} J_s$ and the London equation (3.5) imply that

$$\nabla^2 B = \frac{1}{\lambda^2} B, \quad \nabla^2 J_s = \frac{1}{\lambda^2} J_s,$$ \[(3.6)\]

\(^1\)We restore here the speed velocity constant $c$ to obtain the London equations in the original fashion.
which tell us that currents and magnetic fields in superconductors are exponentially screened from the interior of the sample with penetration depth $\lambda$. In fact a particular solution of (3.6) for a magnetic field parallel to the surface of the material is an exponentially decreasing field in the interior of the superconductor

$$B(x) = B(0)e^{-\frac{x}{\lambda}},$$

(3.7)

where $x$ is measured from the surface. This is precisely the Meissner effect.

### 3.1.2 The Ginzburg-Landau theory

In 1950 Ginzburg and Landau [68] developed a theory describing the transition from a normal to a superconducting phase as a spontaneous symmetry breaking. Generalizing the original Landau theory of phase transitions [69], they asserted that the superconducting phase could be characterized by a complex order parameter $\psi(x)$, whose magnitude measures the density of superconducting electrons $n_s$ at the position $x$

$$|\psi(x)|^2 = n_s.$$

(3.8)

This quantity should be non zero in the superconducting phase and vanishing in the normal phase.

The starting point of the theory is the expression of the free energy of the superconductor as a function of the order parameter $\psi(x)$. According to the Landau theory, this is found by expanding the free energy near the transition point in powers of the small order parameter $\psi$ and its first derivatives with respect to the coordinates. The complex order parameter $\psi = |\psi|e^{i\varphi}$ is defined up to a phase $\varphi$. Physical quantities must not be affected by this arbitrariness. This feature excludes odd powers of $\psi$ in the expansion of the free energy. In the absence of an electromagnetic field the free energy reads

$$f = f_n - \frac{1}{2m^*}|\nabla\psi|^2 + \alpha(T)|\psi|^2 + \frac{1}{2}\beta|\psi|^4.$$  

(3.9)

Here $f_n$ is the free energy density of the normal phase when $\psi = 0$. $\beta$ is a positive coefficient, otherwise the minimum free energy would occur for arbitrarily large values of $|\psi|^2$, and it does not depend on the temperature. In order to have a non vanishing order parameter at $T < T_c$ one must ask to $\alpha$ to depend on the temperature as follows

$$\alpha(T) = -\alpha'(1 - \frac{T}{T_c}), \quad \text{with} \quad \alpha' > 0.$$  

(3.10)

Let us now take an homogeneous superconductor, with no external field. The order parameter is constant in space and the expression (3.9) reduces to

$$f_s - f_n = \alpha(T)|\psi|^2 + \frac{1}{2}\beta(T)|\psi|^4.$$  

(3.11)
The equilibrium value $\psi_{eq}$ of the order parameter, i.e. the value realizing the minimum of (3.11), is vanishing when $T > T_c$

$$|\psi|_{eq}^2 = 0, \quad \text{for } T > T_c. \quad (3.12)$$

Instead, below the critical temperature it is given by

$$|\psi|_{eq}^2 = -\frac{\alpha(T)}{\beta} = -\alpha'(1 - \frac{T}{T_c}), \quad \text{for } T < T_c. \quad (3.13)$$

This means that the superconducting electron density decreases when the critical temperature is approached, and drops to zero at $T = T_c$.

When a magnetic field is present, the expression for the free energy (3.9) has to be modified. First we must add the energy density due to the presence of the magnetic field $B_8/8\pi$. Secondly the gradient has to be modified in order to satisfy the requirement of gauge invariance. Thus the free energy (3.9) writes now

$$f = f_n + \frac{B^2}{8\pi} - \frac{1}{2m}|(\nabla - ieA)|\psi|^2 + \alpha(T)|\psi|^2 + \frac{1}{2}\beta|\psi|^4. \quad (3.14)$$

By minimizing this functional with respect to the independent functions $\psi$, $\psi^*$, $\vec{A}$, one finds the differential equations which determine the distribution of the wave function $\psi$ and the magnetic field, i.e. the Ginzburg-Landau equations

$$\alpha \psi + \beta|\psi|^2\psi - \frac{1}{2m}(\nabla - ieA)^2\psi = 0, \quad (3.15)$$

$$\nabla \times B = 4\pi J \quad \text{with} \quad J = -\frac{ie}{2m}\left(\psi^*(\nabla - ieA)\psi - ((\nabla - ieA)\psi)^*\psi\right). \quad (3.16)$$

where the last equation is the Maxwell equation with a defined supercurrent.

It is important to remark that there is a strong limit of validity of this theory. We assumed $|\psi|$ to be small and took all the expansions about $T_c$, therefore the theory is strictly valid only around the critical temperature $T_c$.

Moreover $e$ and $m$ should be thought of as effective parameters. In fact experimental data turned to fit better with the values $e \sim 2e$ and $m \sim 2m$. From microscopic pairing theory, which will be illustrated in the following paragraph, one finds that the effective degrees of freedom responsible for the supercurrent are Cooper pairs, hence two electrons leading to $n_s \sim n_s^2$.

### 3.1.3 BCS theory

The microscopic theory of superconductivity was developed in 1957 by Bardeen, Cooper and Schrieffer [70]. The basic idea was presented by Cooper a year before in 1956 [71].
3.1. AN OVERVIEW OF SUPERCONDUCTIVITY

He showed that a weak attraction can bind a pair of electrons into a bound state called Cooper pair near the Fermi surface in momentum space. The electrons of this pair have equal opposite momenta and antiparallel spins. The noteworthy fact is that the instability of the Fermi surface against the formation of at least one bound pair occurs no matter how weak is the interaction between the particles. The attractive interaction is given by the electron-lattice interaction. Since there is no limit in formation of the Cooper pairs, at low enough temperature a considerable macroscopic part of the electrons have been turned to bound pairs. The excitations above this ground state are given by fermionic quasiparticles, whose dispersion relation contains an energy gap, which can be thought of as the energy needed to break a Cooper pair.

Formation of pairs

Let us first see how this binding comes about. Consider a simple model of two electrons added to the Fermi sea at \( T = 0 \), where all the other electrons feel the states up to the Fermi energy \( E_F \). Say that the two electrons interact with one another but not with the electrons of the Fermi sea, except via the Pauli exclusion principle. The orbital wave function of two electrons writes

\[
\psi_0(x_1, x_2) = \sum_{k > k_F} g_k e^{i \mathbf{k} \cdot \mathbf{x}_1} e^{-i \mathbf{k} \cdot \mathbf{x}_2},
\]

where \( g_k \) is an arbitrary weight factor and we have taken only the momenta above the Fermi surface \( \epsilon_k = E_F \) in order to respect the Pauli exclusion Principle. The Fermi surface realizes the Fermi sea ground state in momentum space: states with \( \epsilon_k < E_F \) are filled and states with \( \epsilon_k > E_F \) are empty. Let us introduce now the spin wave function and antisymmetrize the whole wave function. Two possibilities arise, the singlet and the triplet. Anticipating attractive interaction we expect the singlet (s-wave) to have the lower energy. The orbital wavefunction is then symmetric with \( g_k = g_{-k} \). Thus the total wave function writes

\[
\psi_0(x_1, x_2) = \sum_k g_k \cos(\mathbf{k} \cdot (x_1 - x_2))(\alpha_1 \beta_2 - \beta_1 \alpha_2)
\]

where \( \alpha_i \) and \( \beta_i \) are respectively the spin up and down functions. Hence, the wavefunction reproduces a couple of electrons with opposite momenta and antiparallel spins. Using the Schrödinger equation for the above wave function

\[
\left( -\frac{1}{2m} \left( \nabla_1^2 + \nabla_2^2 \right) + V(x_1 - x_2) \right) \psi_0(x_1, x_2) = E \psi_0(x_1, x_2),
\]

we find the relation

\[
(E - 2\epsilon_k)g_k = \sum_{k > k_F'} V_{kk'} g_{k'},
\]
where $E$ is the two-electron energy, $\epsilon_k$ are the single electrons energies, and $V_{kk'}$ are the matrix elements of the interaction potential for the scattering $(k, -k) \rightarrow (k', -k')$

$$V_{kk'} = \frac{1}{L^3} \int V(x) e^{i(k'-k) \cdot x} dx$$  \hspace{1cm} (3.21)

where $x = x_1 - x_2$ is the distance between the two electrons. If the summation in equation (3.20) was over all allowed values of $k'$, it would have no negative energy solutions, in fact the two-electron problem by itself does not give a bound state. However, in the presence of a Fermi sphere of additional electrons, a restricted set of momenta is allowed $k > k_F$. Thus, requiring that the interaction potential is negative $V_{kk'} < 0$, there exists a state with total energy lower than $2E_F$, leading to the formation of a bound pair in momentum space.

It is hard to analyze the solution for general matrix elements $V_{kk'}$. Cooper introduced the following assumption on the potential

$$V_{kk'} = \begin{cases} 
-V & E_F \leq \epsilon_k \leq E_F + \omega_c \\
0 & \text{otherwise},
\end{cases}$$  \hspace{1cm} (3.22)

with $V > 0$, and $\omega_c$ a cutoff energy away from $E_F$. The potential is constant up to the cutoff energy $\omega_c$, and it is vanishing above $\omega_c$. Thus equation (3.20) simplifies to

$$1/V = \sum_{k > k_F} \frac{1}{2\epsilon_k - E}.$$  \hspace{1cm} (3.23)

When we replace the summation with an integration, we find

$$\frac{1}{V} = \rho_F \int_{E_F}^{E_F + \omega_c} \frac{d\epsilon}{2\epsilon - E} = \frac{1}{2} \rho_F \ln \frac{2E_F - E + 2\omega_c}{2E_F - E},$$  \hspace{1cm} (3.24)

where $\rho_F$ is the electron density at the Fermi level

$$\rho_F = \frac{1}{2\pi^2} k_F^2 v_F,$$  \hspace{1cm} (3.25)

with $v_F$ the Fermi velocity. We can use the weak-coupling approximation $\rho_F V \ll 1$, satisfied for the most part of classic superconductors. Then the solution to equation (3.24) is given by

$$E \approx 2E_F - 2\omega_c e^{-\frac{2}{v_F^2}}.$$  \hspace{1cm} (3.26)

The energy of the two-electron state $E$ is lower than the energy of two electrons at the Fermi surface $2E_F$. Thus it results energetically favorable for the two electrons to form a bound state in momentum space, no matter how weak is the negative potential.

Roughly speaking we can show how bound pairs occur independently of the weakness of the interaction. Recall the fermi distribution

$$f(E, T) = \frac{1}{\left( e^{(E - \mu)/T} + 1 \right)}.$$  \hspace{1cm} (3.27)
It becomes a theta-function \( \theta(\mu - E) \) when \( T = 0 \). This means that all the states are occupied up to the Fermi energy \( E_F = \mu \), where \( \mu \) is the chemical potential. Adding or subtracting a fermion at the Fermi surface (i.e. with \( E = E_F = \mu \)) has no cost in free energy, since \( \Omega = E - \mu N \). However adding a bound pair (with binding energy \( E_B \)) produces a shift \( \Delta \Omega = -E_B \). Therefore the phase with bounded pairs at the Fermi surface is more stable.

Such bound state is the already mentioned Cooper pair. However, one must pay attention and not take the concept of bound state literally. It is more precise to speak about correlated particles in \( k \) space.

**Origin of the attractive interaction**

How can this attractive force arise inside a metal? Conducting electrons give a positive contribution to the potential, i.e. a net screening repulsive term. Negative terms in the interaction potential come only when one takes the motion of the ion cores into account. The idea is that one electron polarizes the medium by attracting positive lattice ions. The excess of positive ions in turn attracts the second electron, giving an effective attractive interaction between the electrons. If this potential is strong enough to override the repulsive screened Coulomb interaction, it gives rise to a net attractive interaction, and superconductivity results, see [7] for the details.

Although the phonon mediated attraction is the basis of superconductivity in the classic superconductors, it is important to note that the BCS pairing model requires only an attractive interaction giving a matrix element that can be approximated by \( -V \) over a range of energies near \( E_F \). Different pairing interactions, involving the exchange of other particles rather than phonons, may be responsible for superconductivity in some more exotic organic, heavy fermions, high-\( T_c \) superconductors. In this case the electron pairing may have a ground state with non zero angular momentum, thus it may have a \( p \)-wave or \( d \)-wave character rather an \( s \)-wave assumed here.

**The BCS ground state**

Now that we have seen the possibility of Cooper pair formation and where the attractive force can arise, we are ready to perform the next step. Since Cooper pairs are spontaneously created at low temperatures above the Fermi surface, there should be a critical temperature at which such bosonic bound states condense until an equilibrium is reached. The ground state is a Bose-Einstein condensate of Cooper pairs. In [70] the authors attempted to the theoretical construction of such BCS ground state.
CHAPTER 3. IMBALANCED SUPERCONDUCTORS

Define the operator $c^\dagger_{k\sigma}$, which creates an electron of momentum $k$ and spin $\sigma$, and the corresponding annihilation operator $c_{k\sigma}$, so that on the vacuum its action is given by

$$c_{k\sigma}|0> = 0.$$  (3.28)

These operators obey the standard commutation relations between fermions

$$\{c_{k\sigma}, c^\dagger_{k'\sigma'}\} = \delta_{kk'} \delta_{\sigma\sigma'}, \quad \{c_{k\sigma}, c_{k'\sigma'}\} = \{c^\dagger_{k\sigma}, c^\dagger_{k'\sigma'}\} = 0,$$  (3.29)

and particle number operator is defined as follows

$$n_{k\sigma} = c^\dagger_{k\sigma} c_{k\sigma}.$$  (3.30)

Out of the Fermi sea state $|F>$ containing all the electrons with momentum up to $k_F$

$$|F> = \prod_{k<k_F} c^\dagger_k c_k |0>,$$  (3.31)

one may construct the Cooper pair state applying two creation operators of two electrons with opposite momenta and anti-parallel spins

$$|\psi> = \sum_k g_k c^\dagger_k c^\dagger_{-k} |F>.$$  (3.32)

A naive definition for the BCS ground state constructed out of $N$ electrons paired into $N/2$ Cooper pairs could be

$$|\Psi_N> = \left( \sum_k g_k c^\dagger_k c^\dagger_{-k} \right)^{N/2} |F>.$$  (3.33)

However, this simple ansatz has technical problems, because there are too many possibilities in choosing pair occupancy of $N/2$ states. Furthermore $N$ results macroscopically too large, and it cannot be fixed to a single value in experiments. For this reason it is useful to work in the grand canonical ensemble with variable $N$. Write then the BCS ground state which doesn’t conserve the number of particles $N$ in the following way

$$|\Psi> = \exp\left( \sum_k g_k c^\dagger_k c^\dagger_{-k} \right) |F> = \prod_k \exp\left( g_k c^\dagger_k c^\dagger_{-k} \right) |F> = \prod_k \left( 1 + g_k c^\dagger_k c^\dagger_{-k} \right) |F>,$$  (3.34)

where in the last equivalence we used the fact that $(c^\dagger_k c^\dagger_{-k})^2 = 0$. By choosing another parametrization we finally find the BCS ground state wave function in the grand canonical ensemble

$$|\Psi_{BCS}> = \prod_k \left( u_k + v_k c^\dagger_k c^\dagger_{-k} \right) |F>.$$  (3.35)

where $|u_k|^2 + |v_k|^2 = 1$. The ground state $|F>$ can be reinterpreted as the state of zero Cooper pairs of any momentum $k$. 
3.1. AN OVERVIEW OF SUPERCONDUCTIVITY

The model Hamiltonian

Let us now see how to handle excitations above the BCS ground state. Start with the so-called BCS pairing Hamiltonian or reduced Hamiltonian in the grand canonical ensemble

\[ H_{\text{red}} = \sum_{k} \xi_k n_{k \sigma} + \sum_{kq} V_{kq} c^\dagger_k (c_{-k\downarrow}^\dagger c_{q\downarrow} + c_{-q\downarrow} c_{k\downarrow}^\dagger), \quad \text{with} \quad \xi_k = \epsilon_k + \mu. \] (3.36)

Here the first term is the kinetic energy of the particles as the sum over the product of the single particle energy \( \xi_k \) with momentum \( k \), and the number of particles operator given by (3.30). The single particle energy is defined with respect to \( \mu \), i.e. the chemical potential added to the Hamiltonian in order to account for the variable number of particles. The second term contains pair interactions and keeps only the terms which will be crucial for superconductivity. In fact it omits terms which involve electrons not paired as \((k\uparrow,-k\downarrow)\). Such terms, as one can see in detail in [7], have zero expectation value on the BCS ground state.

It is a good approximation to use the mean field theory approach (see [39] for a review) in which each Cooper pair is influenced only by the average value of the other Cooper pairs neglecting the fluctuations, which are small because of the large number of particles involved. Define a new operator as the average expectation value

\[ b_k = <c_{-k\downarrow} c_k\uparrow>_{av}. \] (3.37)

The model Hamiltonian out of (3.36) writes

\[ H_M = \sum_{k\sigma} \xi_k n_{k\sigma} + \sum_{kq} V_{kq} \left( c_k^\dagger c_{-k\downarrow}^\dagger b_q + b_k^\dagger b_{-q\downarrow} c_{q\downarrow} + b_{-k\downarrow} b_{-q\downarrow}^* \right). \] (3.38)

With this approximation we gained in simplicity because we eliminated the quartic terms in the \( c_k \) from the Hamiltonian. Defining the gap parameter

\[ \Delta_k = -\sum_q V_{kq} b_q = -\sum_q V_{kq} <c_{-k\downarrow} c_k\uparrow>. \] (3.39)

The model Hamiltonian becomes

\[ H_M = \sum_{k\sigma} \xi_k n_{k\sigma} + \sum_k \left( \Delta_k c_k^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_k\uparrow - \Delta_k^* b_k^* \right). \] (3.40)

This Hamiltonian can be diagonalized by a suitable linear transformation using the Fermi operators \( \gamma_k \). The appropriate transformation was found by Bogoliubov and Valatin

\[ c_k\downarrow = u_k^\dagger \gamma_k 0 + u_k \gamma_k\downarrow \quad (3.41) \]
\[ c_{-k\downarrow}^\dagger = -v_k^\dagger \gamma_k 0 + u_k \gamma_k\downarrow. \]
The numerical coefficients \( u_k \) and \( v_k \) satisfy \(|u_k|^2 + |v_k|^2 = 1\). The \( \gamma_k^\dagger \) create quasiparticle excitations from the superconducting ground state in terms of the electron creation operators \( c_k^\dagger \). The BCS ground state is now the vacuum state of \( \gamma \) particles
\[
\gamma_{k0}|\Psi_{BCS}>= \gamma_{k1}|\Psi_{BCS}>= 0.
\] (3.42)

Instead the excited states are constructed by applying the quasiparticle creation operator \( \gamma^\dagger \), for example
\[
\gamma_{k0}^\dagger |\Psi_{BCS}>= \gamma_{k1}^\dagger |\Psi_{BCS}>= 0.
\] (3.43)

The excited states correspond to put one single electron in one of the states of the pair \((k |, k |)\), while leaving the other state of the pair empty. This effectively blocks that Cooper pair in participating to the total ground state wavefunction, and increases the energy of the system.

Substituting these new operators given in (3.41) into the model Hamiltonian (3.38) and choosing the coefficients \( u_k \) and \( v_k \) so that the resulting non diagonal terms vanish, see [7] for details, one obtains the diagonalized form of the Hamiltonian
\[
H_M = \sum_k (\xi_k - E_k + \Delta_k b_k^\dagger) + \sum_k E_k (\gamma_{k0}^\dagger \gamma_{k0} + \gamma_{k1}^\dagger \gamma_{k1}).
\] (3.45)

with the coefficients defined
\[
|v_k|^2 = 1 - |u_k|^2 = \frac{1}{2} \left( \frac{\xi_k}{E_k} \right) \quad \text{and} \quad E_k = \sqrt{\Delta_k^2 + \xi_k^2}.
\] (3.46)

The first term in (3.45) is a constant. The second sum gives the increase in energy above the BCS ground state (3.35), condensate of Cooper pairs, in terms of the number operators \( \gamma_k^\dagger \gamma_k \). Thus \( \gamma_k \) describe the elementary fermionic quasi-particle excitations of the system, whose energy \( E_k \) is given by the second term in (3.46). \( \Delta_k \) plays the role of energy gap or minimum excitation energy. The quantity \( 2\Delta_k \) may be regarded as the binding energy of the Cooper pair, which would have to be expended in order to break it up.

**The gap parameter**

Let us now proceed to find the dependence of the gap parameter on the other parameters of the theory. Using (3.1), let us write an expression in terms of the collective excitations \( \gamma_k \) of the sistem
\[
\Delta_k = -\sum_q V_{kq} c_{-q|q|} c_{q|q|} = -\sum_q V_{kq} u_q^* v_q <1 - \gamma_{q0}^\dagger \gamma_{q0} - \gamma_{q1}^\dagger \gamma_{q1}> =
\] (3.47)
\[
= -\sum_q V_{kq} u_q^* v_q (1 - n_q - n_{-q}),
\]
3.1. AN OVERVIEW OF SUPERCONDUCTIVITY

where \( n_q \) are now the number operators for the \( \gamma \) particles. Using (3.46) the previous equation leads to the self-consistent condition for the gap parameter

\[
\Delta_k = -\frac{1}{2} \sum_q V_{qk} \frac{\Delta_q}{\sqrt{\xi_q^2 + \Delta_q}} (1 - n_{q\uparrow} - n_{q\downarrow}).
\]  

(3.48)

Using the simplified potential (3.22) and going to the integral representation, the self-consistent condition (3.48) writes

\[
\frac{1}{\rho_F V} = \int_{0}^{\omega_c} d\xi \frac{1 - n_{k\uparrow} - n_{k\downarrow}}{E_k},
\]  

(3.49)

where \( \rho_F \) again is the density of the electrons at the Fermi surface. The gap parameter acquires then a simplified form

\[
\Delta_k = \begin{cases} 
\Delta & |\xi_k| < \omega_c \\
0 & |\xi_k| > \omega_c.
\end{cases}
\]  

(3.50)

For the ground state at \( T = 0 \), where there are no quasi-particle excitations, (3.49) reads

\[
\frac{1}{\rho_F V} = \int_{0}^{\omega_c} \frac{d\xi}{\sqrt{\xi^2 + \Delta(0)^2}} = \sinh^{-1} \left( \frac{\omega_c}{\Delta(0)} \right),
\]  

(3.51)

where \( \Delta(0) \) is the gap parameter at zero temperature. Using the weak coupling limit \( \rho_F V \ll 1 \), one obtains the gap parameter at zero temperature

\[
\Delta(0) = \frac{\omega_c}{\sinh \left( \frac{1}{\rho_F V} \right)} = 2\omega_c e^{-\frac{1}{\rho_F V}}.
\]  

(3.52)

At finite temperature \( T > 0 \), we can use the Fermi distribution

\[
n_k = \frac{1}{e^{\beta E_k} + 1}, \quad \text{with} \quad \beta = \frac{1}{k_B T},
\]  

(3.53)

inside the self-consistency condition (3.49), leading to

\[
\frac{1}{\rho_F V} = \frac{1}{2} \int_{0}^{\omega_c} \frac{\tanh \left( \frac{\beta E_k}{2} \right)}{E_k}. \quad \text{(3.54)}
\]

The critical temperature is the temperature at which the gap parameter continuously goes to zero value. Furthermore the excitation spectrum becomes the same as that in the normal phase with \( E_k \to |\xi_k| \). Thus placing this requirement in equation (3.54) and using a dimensionless variable of integration we find

\[
\frac{1}{\rho_F V} = \int_{0}^{\frac{\beta \omega_c}{2}} \frac{\tanh x}{x} dx = \ln(A\beta \omega_c),
\]  

(3.55)
which yields to the result
\[
k_B T_c = 1.13 \omega_c e^{-\frac{1}{\tau_F}}. \tag{3.56}
\]
This equation together with (3.52) leads to the important result
\[
\frac{\Delta(0)}{k_B T_c} = 1.76, \tag{3.57}
\]
independent of the phenomenological parameters. This result holds for a large number of superconductors. However, there is an increasing number of unconventional superconductors which exhibit a larger constant value of (3.57), for example high-\(T_c\) superconductors, whose features will be clarified in the following.

BCS theory also predicts the dependence of the value of the energy gap \(\Delta\) at the temperature \(T\) on the critical temperature \(T_c\). Near the critical temperature the relation asymptotes to (see e.g. [7])
\[
\frac{\Delta(T)}{\Delta(0)} \approx 1.74 \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}} \text{ at } T \approx T_c. \tag{3.58}
\]
A typical shape of \(\Delta(T)\) is shown in figure 3.1.

A useful quantity is the condensation energy \(\delta U\) defined as the difference between the internal energy of the superconducting phase and the normal phase. In order to find its value it is sufficient to compute the expectation values of the reduced Hamiltonian (3.36) with respect to the BCS ground state (3.35) and with respect to the Fermi sea ground state (3.31). The result at zero temperature is
\[
\delta U = <\Psi_{BCS}|\mathcal{H}_{red}|\Psi_{BCS}> - <F|\mathcal{H}_{red}|F> = -\frac{1}{2}\rho_F \Delta(0)^2. \tag{3.59}
\]
3.2. INHOMOGENEOUS SUPERCONDUCTORS

Relation to the Ginzburg-Landau theory

In 1959 it was realized by Gorkov [72] that the BCS theory was equivalent to the Ginzburg-Landau (GL) theory. In particular it was proved that the BCS gap parameter $\Delta$ and the GL wave function were related by a proportionality constant and $\psi$ can be thought of as the Cooper pair wave function. Since all Cooper pairs are in the same two-electron state, a single function suffices. The order parameter does not refer to the relative coordinate of the electron inside a Cooper pair, hence the description of a superconductor by means of $\psi(x)$ is valid only for phenomena that vary slowly on the scale of the dimensions of the Cooper pair $\xi_0$.

3.2 Inhomogeneous superconductors

In 1964 a new feature of superconductors was theoretically found. A strong magnetic field coupled to the spins of the conduction electrons could give rise to a separation of the Fermi surfaces corresponding to the two fermions of opposite spins. If this separation is too high then the pairing is destroyed and the system finds the normal phase energetically favorable. This is the so called Chandrasekhar-Clogston bound [19] which determines a first order phase transition from the superconducting to the normal phase at zero temperature. Close to this first order phase transition a new state can be formed as it has been shown by Larkin and Ovchinnikov, and in a separate paper by Fulde and Ferrel [18]. The new LOFF phase exhibits an order parameter, or a gap parameter, periodically varying in space. The modulation arises because of the non zero total momentum of the Cooper pairs. Good reviews on this topic are [17, 60].

3.2.1 The Chandrasekhar-Clogston bound

When an external magnetic field is applied to the superconductor it interacts with the orbital angular momentum of the electrons. The interaction with the spin momentum through the Zeeman effect $H_I \sim \Psi_\gamma \sigma_3 \Psi H$ (where $\sigma_3 = \text{diag}(1, -1)$) is also present, but usually completely negligible. However, it happens that there are geometries which can reduce the coupling to the orbital degrees of freedom. Take for example a system made up of two dimensional layers. An external field parallel to the layers would produce currents in the perpendicular direction, but a very small coupling between the planes will prohibit the existence of such currents. In this situation the orbital coupling is strongly suppressed and the Zeeman effect dominates. The resulting critical magnetic field, above which the

\[\text{The Zeeman effect occurs also in the presence of paramagnetic impurities}\]
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The superconductive phase is destroyed, will be much higher than the ordinary one. The above geometry is realized in non standard superconductors as high-$T_c$ superconductors or quasi-two dimensional organic superconductors.

As the magnetic field is coupled to the electronic magnetic moments, the Zeeman effect effectively produces an imbalance between the spin up and down chemical potentials $\mu_\uparrow$ and $\mu_\downarrow$, or, analogously, between the corresponding number densities $n_\uparrow$ and $n_\downarrow$. The standard BCS state is unfavorable to this situation since the formation of Cooper pairs naturally implies $n_\uparrow = n_\downarrow$. Only when the applied field is such that the Zeeman energy is strong enough to flip one spin of the Cooper pair, BCS superconductivity can be destroyed.

It will be useful to rearrange the parameters to define the chemical potential mismatch and the averaged chemical potential

$$\delta \mu = \frac{\mu_\uparrow - \mu_\downarrow}{2},$$

$$\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2},$$

so that the effective chemical potentials of the two different species are

$$\mu_\uparrow = \mu + \delta \mu, \quad \mu_\downarrow = \mu - \delta \mu.$$  

Hence, the effective interaction is $H_z \sim -\delta \mu \Psi^\dagger \sigma_3 \Psi$. In the following we will consider implementing $\delta \mu$ in this general form without necessarily referring to a magnetic field-driven effect. Anyway, if we think of $\delta \mu$ as an effective magnetic field it is natural to deduce that there will be some critical $\delta \mu_c$ beyond which the superconducting phase is destroyed.

Take the self-consistency equation for the BCS gap parameter (3.49) where $n_{k\uparrow}$ are the number operators for the fermionic quasiparticles of spin up and down. These are related to the Fermi distributions (3.53) with energies $E_{k\uparrow, \downarrow} = E_k \pm \delta \mu$. At zero temperature $T = 0$ solving the equation (3.48), one finds that the gap parameter is independent of $\delta \mu$. Looking at the zeros of the gap equation one finds a curve $T_c(\delta \mu)$ ending at $\delta \mu^* = \frac{\Delta(0)}{2}$, where $\Delta(0)$ is the gap parameter at zero temperature, and $\delta \mu = 0$, see e.g. [17]. This is the continuation of the second order phase transition with $\delta \mu = 0$ towards the plane with non vanishing chemical potential mismatch. However the result displays a very strange reentrant behavior as we can see in figure 3.2.

It was realized in 1962 independently by Chandrasekhar and Clogston [19] that at $T = 0$ there is a first order phase transition from the superconducting to the normal phase at a higher value of the chemical potential mismatch $\delta \mu$ with respect to the one found before $\delta \mu^*$. One can find this value by looking at the free energies of the superconducting and the normal phase. The free energy $\Omega(\delta \mu)$ at zero temperature can be expanded when $\delta \mu \ll \mu$ up to second order in $\frac{\delta \mu}{\mu}$

$$\Omega(\delta \mu) = \Omega(0) + \Omega(0)’\delta \mu + \frac{1}{2} \Omega(0)’’\delta \mu^2 + O(\delta \mu^3) \simeq \Omega(0) - \frac{1}{2} \delta \mu (n_\uparrow - n_\downarrow),$$  

(3.63)
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Figure 3.2: Critical temperature of the phase transition between the normal and the superconducting BCS phase. The dashed line corresponds to the first order phase transition starting from the tricritical point (TCP) and ending at the Chandrasekhar-Clogston bound \( \delta \mu_1 = \frac{\Delta(0)}{\sqrt{2}} \). Above the tricritical point there is the second order phase transition. The other line is the one where the normal state is unstable with respect to the transition to the BCS state. At \( T = 0 \) it ends at \( \delta \mu^* = \frac{\Delta(0)}{2} \). Figure taken from [60], where \( \mu^* \) corresponds to \( \delta \mu \) in our notation.

where we have used the thermodynamical relation \( \delta n = -\frac{\partial \Omega}{\partial (\delta \mu)} \), and \( \delta n \approx 2\rho_F \delta \mu \) when \( \frac{\delta \mu}{\mu} \ll 1 \), where again \( \rho_F \) is the electron density at the Fermi surface. The normal phase admits a difference between the spin population and its free energy writes

\[
\Omega_n(\delta \mu) \simeq \Omega_n(0) - \rho_F \delta \mu^2. \tag{3.64}
\]

The superconducting phase, thanks to the presence of Cooper pairs, naturally has an equal number of spin up and spin down particle species, thus the free energy results

\[
\Omega_s(\delta \mu) \simeq \Omega_s(0). \tag{3.65}
\]

The difference between the two free energies is given by

\[
\Omega_n(\delta \mu) - \Omega_s(\delta \mu) \simeq \Omega_n(0) - \Omega_s(0) - \rho_F \delta \mu^2. \tag{3.66}
\]

Notice that at zero temperature \( \Omega(0) = E(0) \) and the difference between the energies of the superconducting and the normal phase is given by the so-called condensation energy (3.59). It follows that

\[
\Omega_n(\delta \mu) - \Omega_s(\delta \mu) \simeq \frac{1}{2} \rho_F \Delta(0)^2 - \rho_F \delta \mu^2. \tag{3.67}
\]
Thus at $T = 0$ the superconducting phase is favorable if its free energy is less than the free energy of the normal phase, i.e. only if the Chandrasekhar-Clogston bound

$$\delta \mu < \frac{\Delta(0)}{\sqrt{2}} = \delta \mu_1$$

(3.68)

is satisfied. At $\delta \mu = \delta \mu_1$ there is a first order phase transition since the gap jumps discontinuously from zero to $\Delta(0)$ going from the normal to the superconducting phase. The first order phase transition meets at non zero temperature the second order phase transition. The meeting point is a tricritical point (TCP) as shown in figure 3.2. The reentrant curve going from the TCP to the point $\delta \mu^* = \frac{\Delta(0)}{2}$ is a curve of instable points which does not play any role now.

### 3.2.2 The LOFF phase

In 1964 Larkin and Ovchinnikov and independently Fulde and Ferrel [18] showed that the situation is more complicated. Not only is there a second order phase transition from the normal to the superconducting phase in the $T - \delta \mu$ plane, which turns to a first order transition at lower temperatures below the tricritical point, but also the system may experience a new state: the LOFF phase.

Standard Cooper pairs have zero total momentum $q = 0$, because this situation is energetically favorable, see [7]. However, at zero temperature, there is a possibility for a formation of a new inhomogeneous phase at high enough chemical potential mismatch, where Cooper pairs have non zero total momentum $q$. Roughly speaking we can show how a non zero total momentum can arise. In the presence of a chemical potential mismatch there is a separation of the Fermi surface in spin up and spin down electron populations. To form a Cooper pair one must take the two electrons close to their own Fermi surface.
As showed in figure 3.3, the two electrons may have non opposite momenta \( k \downarrow + k' \uparrow = q \), due to the displacement \( q \) of the Fermi surface.

However, this is not a proof that Cooper pairs with non zero total momentum are energetically favorable globally. In fact the previous argument doesn’t apply for electrons situated on opposite sides of the Fermi surface. In this case the difference between the two momenta is not given simply by the displacement of the Fermi surface. Thus an explicit quantitative calculation is needed. The method is to minimize the free energy of the Ginzburg-Landau theory by letting the order parameter to vary in space in a determined manner. Since the total momentum of the pair is non zero, the order parameter has periodic spatial variations with wavelength of order of the size of the Cooper pair. The simplest ansatz for the order parameter, as in the work of Fulde and Ferrel, is

\[
\Delta(x) \sim e^{iq \cdot x}. \tag{3.69}
\]

However, as noted in the subsequent paper of Larkin and Ovchinnikov, a minimization of the energy would fix the modulus of \( q \) but not its direction. Thus in principle one must take a more general ansatz for the gap

\[
\Delta(x) = \sum_q \Delta_q e^{iq \cdot x}. \tag{3.70}
\]
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The Ginzburg-Landau free energy up to the fourth term writes

\[ F = \sum_{q} \alpha_0 |\Delta_{q}|^2 + \sum_{q_1+q_3=q_2+q_4} J(q_1, q_2, q_3, q_4) \Delta_{q_1} \Delta_{q_2}^* \Delta_{q_3} \Delta_{q_4}^*. \]  

Here \( J \) depends on the various relative orientations between the vectors. In principle one can compute both \( \alpha_0 \) and \( J \) explicitly by means of BCS theory. However, it is clear that such a computation is extremely complicated, because one has to perform it over a huge range of momenta. Larkin and Ovchinnikov [18] made a particular ansatz selecting some momenta \( q_i \) and showed that the best structure is obtained by taking the superposition of two opposite momenta with a resulting order parameter

\[ \Delta(x) \sim \cos(q \cdot x). \]  

The result is that at \( T = 0 \) the superconducting to normal phase transition occurs for a critical mismatch \( \delta \mu_2 = 0.754 \Delta(0) > \delta \mu_1 \). The transition is second order since the Ginzburg-Landau approach has been used. For \( \delta \mu_1 < \delta \mu < \delta \mu_2 \) the LOFF phase with varying gap parameter shows up. The LOFF region shrinks as one increases the temperature and disappears at the TCP as shown in figure 3.4.

Despite many experimental efforts, these phases have not yet been observed in standard superconductors. The difficulties arise when one wants to apply an external field to create a chemical potential mismatch. It will inevitably couple to the orbital angular momentum of the electrons, with a negligible contribution due to the coupling with the magnetic moments of the single electrons. An increase of the external field will destroy the superconductive phase before appreciably seeing the occurrence of the inhomogeneous superconductive phase. As we have already mentioned, layered geometries are more suitable for the experimental research of the LOFF phase. In fact the latter has been investigated within unconventional superconductors such as layered organic superconductors and heavy fermion superconductors, but all the experimental results are however inconclusive and the evidence of LOFF phase is still unclear. From the theoretical point of view the same limitation holds, since unconventional superconductors are non-BCS.

Inhomogeneous superconductivity is not peculiar to condensed matter superconductors. Color superconductivity (see e.g. [5, 73]) arises in QCD at small temperatures and very high quark densities where the color interaction favors the formation of a quark-quark condensate. The mechanism describing the formation of the condensate is similar to the one arising in BCS superconductivity. Thanks to the asymptotic freedom the precise structure of quark pairs is well established and the arising phase at high densities in the presence of three quarks is the Color-Flavor-Locked (CFL) one [74]. What happens at non-asymptotic densities is still unclear because perturbative QCD computations are no more reliable. One can get some insights by means of e.g. Nambu-Jona-Lasinio (NJL)-like
effective field theories, while lattice computations suffering from the so-called sign problem are not well suited for theories at finite density. Interestingly the LOFF phase in color superconductivity can be induced not only by a difference in the chemical potentials of the quarks forming the condensate, but it can be naturally implemented by mass differences among the quarks. In particular the large strange quark mass with respect to the up and down quark’s masses plays an important role in the formation of a LOFF phase in QCD, see [17] for a review. Again the occurrence of such phases at intermediate densities, where the coupling between quarks is strong, is not well established.

There is another area of physics where inhomogeneous phases can be experimentally investigated: ultracold trapped atoms of two fermionic species. The general feature of such systems is the presence of a crossover between a superconducting weakly coupled BCS phase to a strongly coupled Bose-Einstein condensate (BEC) by tuning a suitable coupling parameter. When the populations are unequal, hence in the presence of a chemical potential mismatch, recent experiments [75] indicate the presence of a bound (Chandrasekhar-Clogston like) above which the system turns to the normal unpaired state. It seems that beyond this limit, for a wide range of $\delta \mu$, superfluid atoms remain in the core of the trap, while the normal atoms in excess are expelled forming a surrounding shell, supporting the Phase Separation (PS) scenario [76] instead of the LOFF phase. However, a clear evidence is still lacking, and a possibility for the existence of the LOFF phase in some range of the coupling is unlikely to be discarded in trapped cold atoms experiments.

The LOFF phase of trapped cold atoms at $T = 0$ has been theoretically investigated by e.g. Mannarelli, Nardulli and Ruggieri [77] in a large range of the coupling parameter by means of effective theories with four-fermion interactions. It turns out that at weak coupling standard BCS results hold: a small chemical potential mismatch cannot destroy the BCS phase, while above the Chandrasekhar-Clogston bound a LOFF phase emerges. When the coupling is tuned from weak to strong coupling, it turns out that the LOFF window shrinks to a point at a certain critical value, beyond which LOFF phases cannot be realized and the homogeneous phase has a crossover transition directly to the normal phase.

The upshot of the above discussion is that the presence of the LOFF phase has been predicted within the standard BCS theory of weakly coupled electrons. However, the occurrence of the window of inhomogeneous superconductivity may be too narrow to experimental evidence within conventional superconductors. Unconventional superconductors are a more suitable playground for experimental investigation of the LOFF phase, but, as we will see in the following, they are not described by BCS theory and the mechanism driving superconductivity in such materials is still not well understood. Thus, also the presence of a LOFF window in their phase diagram is theoretically unknown. The results
of Mannarelli et al. suggest, at least for trapped cold atoms systems, that the LOFF phase could disappear in the strongly coupled regime.

3.3 Unconventional superconductors

Conventional superconductors such as mercury are those well explained by BCS theory. However, it is now clear that standard microscopic BCS theory does not well describe all the superconductors. Since 1979 [78] many “non-BCS”, unconventional, superconductors have been experimentally discovered. The ways of being “non-BCS” could be basically two. One is when the nature of the attractive interactions between couples of dressed electrons near the Fermi surface is not due to phonons. Examples are spin-spin interactions mediated by paramagnons, see e.g. [40] for references. The other departure from BCS theory is more radical: it may be that the normal state at temperatures just above the superconductive phase doesn’t admit a weakly coupled quasiparticle description at all. Thus the whole apparatus of BCS theory cannot be applied. This may happen when the onset of superconductivity occurs in the vicinity of a quantum critical point, as it has been suggested by Sachdev, see [6] for a review. The main classes of unconventional superconductors are the heavy fermion, high-$T_c$ and the layered organic superconductors. Their effective geometry is layered thus they are more likely to be described by effectively $2 + 1$ dimensional models, see e.g. [7]. The occurrence of quantum critical points within the phase diagram of heavy fermion metals is under experimental investigation, but many signals seem to support such hypothesis, see [57] for a review. Also the other unconventional superconductors are believed to display quantum critical points in the superconducting region of their phase diagram as suggested by Sachdev, see [61] for a review. A complete theoretical modeling of unconventional superconductors is still under research. The main difficulty is the lack of knowledge of standard methods in describing strongly coupled systems in the vicinity of their quantum critical points. One then needs to develop novel tools to model strongly coupled quantum criticality in $2 + 1$ dimensions. One tool can be $AdS/CFT$ using which analytic results for processes such as transport can be obtained.

In this section, which is mainly based on [6, 40, 51, 61, 62], we will introduce the notion of a quantum critical points first, and see their connection to unconventional superconductors in the following.
3.3. UNCONVENTIONAL SUPERCONDUCTORS

3.3.1 Quantum criticality

Quantum critical points are not peculiar of unconventional superconductors but a general feature of condensed matter systems. To understand their main properties let us compare them with the more usual classical critical points. A classical critical point is a point in the phase diagram where the system exhibits scale invariance, see e.g. [39]. In particular, thermal fluctuations become very strong with an infinite coherence length $\xi$, which characterizes the length scale over which the correlations between the fluctuations are lost. The free energy is a non-analytic function driving the system to a phase transition generally at a temperature $T = T_c$.

A quantum critical point is still a point at which the system exhibits scale invariance, but the phase transition is driven by quantum fluctuations of the fundamental state at $T = 0$ rather than thermal ones, see [6] for a review. When an external parameter $g$ (pressure, magnetic field, doping, . . .) is tuned, the fundamental state can undergo a quantum phase transition at a critical value $g_c$ as sketched in figure 3.5. At low temperatures the system sits in one of the two phases of the phase diagram. As one turns on the temperature the system may encounter an ordinary thermal phase transition at a certain critical point $(g_c, T_c)$.

At zero temperature, but away from the critical point, a system usually has an energy scale $\Delta$ associated with the energy difference between the ground and the first excited state [62]. At the quantum critical point we expect $\Delta$ to vanish and $\xi$ to diverge according

Figure 3.5: Prototypical phase diagram for a system that undergoes a quantum phase transition at the quantum critical point $g = g_c$ and $T = 0$. The solid lines could be classical thermal phase transitions. The region between the dashed lines is the quantum critical region (QCR).
to the scaling relation
\[ \Delta \sim (g - g_c)^{\nu z}, \quad \xi \sim (g - g_c)^{-\nu}. \] (3.73)

The quantity \( z \) is the dynamical scaling exponent, and the quantity \( \nu \) is the correlation critical exponent. The scaling invariance at the quantum critical point is in general the same as (2.87) which we write again for clearness
\[ t \to a^z t, \quad \vec{x} \to a\vec{x}. \] (3.74)

Different \( z \) occur in different condensed matter systems. \( z = 1 \) is common for spin systems, see e.g. [40], and it is a special case since other than scaling invariance, the system also exhibits Poincaré symmetry. The symmetry group is then enhanced to the conformal group. Thus, quantum critical points can be possibly described by scale (conformal) invariant field theories.

The crucial point is that the system away from the critical point can still be influenced by it because of the divergent coherence length. Provided that the temperature is increased more than the dimensionally appropriate power of \((g - g_c)\), the system goes in the quantum critical region (QCR) delimited by the dashed lines in figure 3.5. It seems that the effective scale invariant theory valid at the critical point can be generalized to nonzero temperature \( T \).

### 3.3.2 An example: high-\( T_c \) superconductors

BCS theory predicts an upper value for the critical temperature of around \( T_c \approx 30K \), see e.g. [7]. However, in 1986 Bednorz and Müller [79] discovered in “LBCO“, a mixed oxide of lanthanum, barium and copper, a transition to a superconducting phase at \( T_c \approx 35K \). This discovery enlarged the class of superconducting materials, including those with a higher temperature than predicted by BCS theory. These materials were for this reason called as high-\( T_c \) superconductors, and the highest achieved critical temperature is by now \( T_c \approx 130K \).

The common feature to these systems is the layered structure of the copper oxide planes (CO), in which the supercurrent is believed to flow. Neighbouring layers contain ions such as lanthanum, barium, strontium, and are used to dope the compound with additional electrons or holes onto the copper-oxyde layers in order to increase or reduce the number of conducting electrons per Cu atom. The doping is done for example by substitution \( \text{La}_2\text{CuO}_4 \rightarrow \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \), see e.g. [62]. Once the doping \( x \) is sufficiently large the compound superconducts at low temperature. The doping which yields the highest \( T_c \) is the optimal doping \( x_0 \). When \( x > x_0 \) the system is said to be over doped and when \( x < x_0 \) the compound is referred to as under doped. The schematic phase diagram for a
3.3. UNCONVENTIONAL SUPERCONDUCTORS

Figure 3.6: Schematic figure of a temperature versus doping phase diagram in a cuprate high-$T_c$ superconductor. The parent compound is in an antiferromagnetic ordered phase at low temperatures. By increasing the doping at low $T$ we find a pseudogap region, a strange metal (with a resistivity $\rho \sim T$ instead of $\rho \sim T^5$ as in standard metal), a superconducting phase and a normal phase. The optimal doping value is in the center of the superconducting “dome”, where the critical temperature has its highest value. Figure taken from [40].

Over doped high-$T_c$ superconductors are better understood because for $T > T_c$ the system behaves as a normal Fermi liquid, see e.g. [62], with weakly interacting quasiparticles, and the normal to superconducting phase transition can be described by standard BCS theory.

In contrast, in the under doped region $x < x_0$ the effective degrees of freedom are believed to be strongly interacting. How can we account here for the phase transition from the superconducting to the normal strongly coupled phase? One possible scenario is that the phase of the condensate gets disordered instead of breaking Cooper pairs. The electrons are allowed to remain in bound states in the normal state, thus this part of the phase diagram is called the pseudogap region. How does the pairing mechanism arise? The effective interaction between two electronic quasiparticles could be mediated by spin waves, i.e. paramagnons, see [40] for references. Cooper pairs can be formed but with a $d$-wave symmetry since in cuprate high-$T_c$ superconductors the orbital state of the copper ions is a $d$-wave. However, even if some indications have been given, there is still no clear
understanding in what is the mechanism driving superconductivity in these materials.

Recently it has been suggested for these materials the existence of quantum critical points beneath their "superconducting domes", see e.g. [6]. This feature makes these materials to be inherently strongly interacting. When \( z = 1 \), and this is the case of unconventional superconductors as suggested in e.g. [40], such quantum critical points are conformally invariant.

### 3.3.3 The role of gauge/gravity duality

As already mentioned many unconventional superconductors are layered and could be described by a 2 + 1 dimensional quantum field theory. The conjectured occurrence of quantum critical points within their superconducting domes makes these systems difficult to be described by means of standard condensed matter tools, i.e. as effective weakly coupled field theories. AdS/CFT comes in help as a possible approach to investigate the properties of these strongly coupled systems. Thanks to the scale (conformal) invariance of the quantum critical points, and to the extension of this feature to non zero temperatures in the quantum critical region, one may provide a gravitational dual to the unknown scale (conformal) invariant field theory. The approach is still at the phenomenological level in the sense that one relies on the scales and broken symmetries entering in the game and constructs a dual gravity theory with the minimal ingredients to realize those features. In this way it is not clear whether the correspondence describes real world materials, because the detailed microscopic description of the scale invariant quantum field theory is lacking. Moreover the gauge/gravity duality provides by itself limitations in its applicability. In fact from (1.3) the quantum field theory admits a gravitational classical dual only when the large \( N \) limit is performed. But what does this limit stand for in condensed matter systems is still unclear, hence one must take carefully these applications. The guide line is to construct phenomenologically minimal models to shed new light into universal properties of a class of strongly interacting field theories. The advantage is that AdS/CFT correspondence emerges as a unique theoretical tool to analyze real time properties, such as transport, in the vicinity of 2+1 dimensional quantum critical points.

It is important to stress on the fact that the gravitational description is not to be intended as an effective low energy description of the quantum critical point. It is instead an equivalent description of the same underlying theory, but done by other means.
3.4 Summary

Applications of the gauge/gravity duality to condensed matter systems rely on the existence of quantum critical points in the phase diagrams of strongly coupled models. The quantum critical region is also influenced by the quantum critical point leading to a larger domain where the theory should be inherently strongly coupled. Unconventional superconductors, the ones not described by standard BCS theory, may exhibit quantum critical points in their superconducting phase. A gravitational description may provide a unique approach for modeling such strongly coupled systems.
Chapter 4

Imbalanced holographic superconductors

In this chapter we present our model which tries to describe imbalanced strongly coupled superconductors using the AdS/CFT correspondence. Our basic assumption is that the superconducting phase emerges around a quantum critical point exhibiting full conformal symmetry in $2+1$ dimensions. The dual $3+1$ dimensional gravity model is then constructed following the approach of phenomenological effective theories: one tries to just implement the (broken) symmetries and the scales in the game, without taking into account the microscopic details of the quantum field theory side. Having in mind “non-BCS” (e.g. high-$T_c$) superconductors, this is a natural approach because the modeling quantum field theory is unknown. Moreover we don’t know how to construct a stringy embedding for condensed matter systems, which, in principle, would give us the precise form of the lagrangian from which to start. We are thus guided by the aim of searching for a minimal model describing a class of strongly coupled field theories exhibiting some requested features such as a spontaneous symmetry breaking below a critical temperature with a consequent formation of a charged condensate. Our simple setup is an improvement of the one studied in [15, 16]. We will construct a $(3+1)$-dimensional gravitational dual to a $(2+1)$-dimensional quantum field theory, which undergoes a phase transition below a critical temperature by adding a complex scalar and a $U(1)$ gauge field to an asymptotically Anti-de Sitter (AdS) black hole background of radius $L$. In our model, the scalar field has just a quadratic potential with mass parameter $m^2$. The chemical potential mismatch in the field theory side is accounted in the gravity setup by turning on the temporal component of another Maxwell field under which the scalar field is uncharged. The result is that the critical temperature below which a superconducting homogeneous phase develops decreases with the chemical potential mismatch, as is expected in weakly coupled setups. However, at least in the special case of $m^2L^2 = -2$, there is
no sign of a Chandrasekhar-Clogston bound at zero temperature and the phase transition is always second order. We believe that this feature does not allow for the presence of a Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase. A different situation can emerge for other values of \( m^2 L^2 \) as we will argue in the following.

The outline of this chapter is as follows. In section 4.1 we will introduce the setup, i.e. our “imbalanced holographic superconductor”. We will find the normal phase equilibrium solution and see under which conditions this phase can become unstable at \( T = 0 \). In section 4.2 we will present an approximate analytic solution for \( m^2 L^2 = -2 \) dual to the superconducting phase in a probe regime, whose meaning will become clear in the following. We will see that a non zero condensate arises below \( T_c \). Close to such critical temperature its shape is that of an order parameter of a second order phase transition. In section 4.3 we will present the results of numerical computations on the full model. In section 4.4 we will see an example of a transport quantity, namely the electric conductivity.

### 4.0.1 Minimal ingredients

How do we go about constructing a holographic dual for an imbalanced high-\( T_c \) superconductor? First of all remind that the geometry of such superconductors is layered in copper oxide planes, therefore the quantum field theory beneath the quantum critical point, around which the superconducting phase is believed to develop, is essentially (2+1)-dimensional. Thus, this three dimensional scale invariant field theory will be mapped to a four dimensional classical gravity theory and the correspondence will be of the type \( \text{AdS}_4/CFT_3 \). The field theory will admit a conserved stress-energy tensor \( T_{\mu\nu} \). Using the \( \text{AdS}/CFT \) dictionary summarized in table 1.1, its dual field will be a metric \( g_{ab} \) in the bulk. A superconductor must have a supercurrent \( J_{\mu} \), whose dynamics is captured by the classical dynamics of the bulk photon field \( A_a \). In the presence of two fermionic species with two different chemical potentials \( \mu_\uparrow \) and \( \mu_\downarrow \) the bulk should actually contain two Maxwell fields: a \( U_A(1) \)-field \( A_a \) to account for the total chemical potential \( 2\mu = \mu_\uparrow + \mu_\downarrow \) and a \( U_B(1) \)-field \( B_a \) for the chemical potential mismatch \( 2\delta\mu = \mu_\uparrow - \mu_\down\). From the Ginzburg-Landau point of view, superconductivity is a theory of spontaneous symmetry breaking with a charged bosonic order parameter. We will consider for simplicity the case of an \( s \)-wave condensate \( \mathcal{O} \), i.e. the one which does not carry angular momentum. Within the contest of the \( \text{AdS}/CFT \) correspondence such charged bosonic \( s \)-wave condensate is mapped to a scalar field \( \psi \) charged under the \( U_A(1) \)-field, but uncharged under the additional \( U_B(1) \)-field.

The latter condition is inspired by e.g. the coupling of an external magnetic field \( H_z \) with the spin up and down electrons. The Zeeman interaction term is \( \mathcal{H}_I = \bar{\psi} \gamma^0 \mu_B \sigma_3 \psi H_z \),
\[ CFT_3 \quad \text{AdS}_4 \]

| conserved \( T_{\mu\nu} \) | \( g_{ab} \) |
| finite \( T \) | black hole |
| \( \mu \) | \( U_A(1) \)-charge |
| \( \delta \mu \) | \( U_B(1) \)-charge |
| \( \mathcal{O} \) charged under global \( U_{em}(1) \) | \( \psi \) charged under local \( U_A(1) \) |

Table 4.1: Minimal ingredients

where \( \mu_B \) is the Bohr magneton and \( \sigma_3 = \text{diag}(1, -1) \). The effective chemical potential mismatch is given by \( H_z \mu_B \). The two fermionic particles have opposite “charges” with respect to the effective gauge field \( V_0 = H_z \mu_B \sigma_3 \), hence the condensate formed by antiparallel spins is uncharged with respect to \( V_0 \). In more general contexts, where a chemical potential mismatch \( \delta \mu \) is implemented also in the absence of an external magnetic field (e.g. in finite density QCD or in polarized cold atoms) the same reasoning holds. We will thus trade \( \delta \mu \) as the time component of a vector field. All these minimal ingredients of our setup are summarized in table 4.1.

It should be emphasized that the \( U(1)_{em} \) gauge symmetry, which undergoes a spontaneous symmetry breaking, is actually local and not global in the field theory side. However, photons in some condensed matter physics contexts can be treated as non-dynamical, in the sense that their interactions with the electrons have been integrated out and only contribute to the dressing of the quasiparticles. In particular, BCS theory only includes the electrons and phonons. The resulting symmetry is a weakly coupled one, in which the parameter \( e \) in the covariant derivative of the scalar operator \( (\partial_\mu - ieA_\mu)\mathcal{O} \) is very small. For our purposes we will always consider a weakly gauged symmetry as an almost global symmetry.

Remarkably on the quantum field theory side there is a puzzle. As it is well known the Mermin-Wagner theorem forbids continuous symmetry breaking in 2+1 dimensions because of large fluctuations in low dimensions. Nevertheless holographic superconductors are found in 2+1 dimensions [15, 16]. The reason is that holography concerns the limit (1.3) where the field theory side is considered at large \( N \). In this limit fluctuations are suppressed. An argument to support this feature has been given by Gregory et al. in [80]. They studied higher curvature corrections to 3+1 holographic superconductors and found that condensation becomes harder. A valuable check would be to construct a setup of 2+1 holographic superconductors with higher curvature corrections. At the moment such setups do not give any particular insight into what said before.
CHAPTER 4. IMBALANCED HOLOGRAPHIC SUPERCONDUCTORS

Moreover holographic superconductors concern a theory of gravity where one field is the metric. By the AdS/CFT prescription (see table 1.1) there should be a conserved stress-energy tensor in the dual field theory, hence the quantum field theory should be translationally invariant. This goes a bit against the condensed matter models of superconductivity where a lattice of ions has to be taken into account to have the formation of Cooper pairs. It is then believed that such holographic models of superconductivity describe somehow a set of electrons without taking into account the medium in which they propagate. Some interesting ideas of how to include a lattice have been discussed by Karchu et al. in [81].

The minimal bulk lagrangian writes

$$\mathcal{L} = \sqrt{-g}(\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{matter}}) = \frac{1}{2k^2_4} \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{1}{4} F_{ab} F^{ab} - \frac{1}{4} \mathcal{Y}_{ab} \mathcal{Y}^{ab} - V(|\psi|) - |\partial \psi - iqA\psi|^2 \right). \quad (4.1)$$

The first two terms are the Einstein-Hilbert lagrangian plus a negative cosmological constant, which means that the simplest solution to the equations of motion is an Anti-de Sitter space. The remaining terms contain the scalar and the vector fields. Note that we use the convention in which all the fields have been taken to be dimensionless and the gauge couplings have been reabsorbed in the overall gravitational constant $k^2_4$. Hence the charge $q$ of the scalar field has dimension of an energy. Field strengths are as usual defined as

$$F = dA, \quad Y = dB. \quad (4.2)$$

We will discard non-minimal couplings for simplicity. The potential for the scalar field can in principle contain all powers of $\psi$. Particular forms of these potentials can be guessed by requiring the phase diagram to mimic real world physics. This has been done by Gubser et al. [82] in the realm of high energy physics. For simplicity we will consider a potential containing only the mass term

$$V(|\psi|) = m^2 \psi \psi^\dagger. \quad (4.3)$$

The AdS/CFT correspondence provides a relationship between the mass of all fields in the bulk and the conformal dimension of the dual fields in the gauge theory side. In the case of scalar fields in $d = 3$ the mass/dimension relation (1.85) becomes

$$\Delta(\Delta - 3) = m^2 L^2. \quad (4.4)$$

Aimed by the wish of describing a fermionic Cooper pair-like condensate, $O_\Delta = \Psi \Psi^\dagger$ with canonical dimension $\Delta = 2$ (the fermionic field $\Psi$ in $d = 3$ has mass dimension 1), we choose the mass for the scalar field to be $m^2 = -\frac{2}{L^2}$. Even if it is negative it is above the Breitelhoner-Freedman bound (1.89), which in our case writes $m^2 \geq -\frac{9}{4L^2}$. Notice that the unitarity bound (1.91) is $\Delta \geq \frac{1}{2}$. 
With the choice (4.3) on the potential, the bulk lagrangian (4.1) depends only on two external parameters: the mass of the scalar field $m$ and its charge $q$. Thus, changing these parameters is like changing the class of universality of the dual bulk field theory we are trying to describe. In fact, the main features of superconductivity, such as the occurrence of the Chandrasekhar-Clogston bound, will depend on such parameters.

### 4.0.2 Equations of motion

From the lagrangian (4.1) one obtains the following equations of motion. The Einstein’s equations read

$$G_{ab} + \frac{1}{2}g_{ab}\Lambda = -\frac{1}{2}T_{ab}, \quad \text{with} \quad \Lambda = -\frac{6}{L^2},$$

with the stress-energy tensor

$$T_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta L_{\text{matter}}}{\delta g^{ab}} = -g_{ab}L_{\text{matter}} + 2\frac{\delta L_{\text{matter}}}{\delta g^{ab}} =$$

$$= \frac{1}{4}g_{ab}F_{cd}F^{cd} + g_{ab}V(|\psi|) +$$

$$+ \frac{1}{2}g_{ac}g^{cd}[(\partial_c\psi - iqA_c\psi)(\partial_d\psi^\dagger + iqA_d\psi^\dagger) + (c \leftrightarrow d)] +$$

$$+ \frac{1}{4}g_{abc}Y_{cd}Y^{cd} - F_{ac}F_{bd}g^{cd} - Y_{ac}Y_{bd}g^{cd} +$$

$$- [(\partial_a\psi - iqA_a\psi)(\partial_b\psi^\dagger + iqA_b\psi^\dagger) + (a \leftrightarrow b)],$$

Maxwell’s equations for the $A_a$ field read

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}g^{ce}F_{bc}) = iqg^{ce}[\psi^\dagger(\partial_c\psi - iqA_c\psi) - \psi(\partial_c\psi^\dagger + iqA_c\psi^\dagger)],$$

the scalar equation reads

$$- \frac{1}{\sqrt{-g}}\partial_a[\sqrt{-g}(\partial_b\psi - iqA_b\psi)g^{ab}] + ig^{ab}A_b(\partial_a\psi - iqA_a\psi) + \frac{1}{2}\frac{\psi}{|\psi|}V'(|\psi|) = 0,$$

and Maxwell’s equations for the $B_a$ field are

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}g^{ce}Y_{bc}) = 0.$$

Most of the work will revolve around solving these equations of motion. The first step will be finding the static black hole solutions describing the equilibrium phases of the theory.

For our purposes the most general ansatz for the spacetime metric is

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{r^2}{L^2}(dx^2 + dy^2) + \frac{dr^2}{g(r)},$$
together with an homogeneous ansatz for the fields

\[ \psi = \psi(r), \quad A_a dx^a = \phi(r) dt, \quad B_a dx^a = v(r) dt. \] (4.12)

We will focus on black hole solutions, with an horizon at \( r = r_H \) where \( g(r_H) = 0 \). The temperature of such black holes is found analogously to that in (2.16)

\[ T = \frac{g'(r_H) e^{-\chi(r_H)/2}}{4\pi}. \] (4.13)

Implementing the ansatz above into the equations (4.5-4.10) it is immediately seen that a component of the Maxwell’s equations implies that the phase of \( \psi \) must be constant. Without loss of generality we can then take \( \psi \) to be real. The scalar equation becomes

\[ \psi'' + \psi' \left( \frac{g'}{g} + \frac{2}{r} - \frac{\chi'}{2} \right) - \frac{V'(\psi)}{2g} + \frac{e^\chi q^2 \phi^2 \psi}{g^2} = 0, \] (4.14)

Maxwell’s equations for the \( \phi \) field become

\[ \phi'' + \phi' \left( \frac{2}{r} + \frac{\chi'}{2} \right) - \frac{2q^2 \psi^2}{g} \phi = 0, \] (4.15)

the independent component of the Einstein’s equations yield

\[ \frac{1}{2} \psi'^2 + \frac{e^\chi (\phi'^2 + v'^2)}{4g} + \frac{g'}{gr} + \frac{1}{r^2} - \frac{3}{gL^2} + \frac{V(\psi)}{2g} + \frac{e^\chi q^2 \phi^2 \psi^2}{2g^2} = 0, \] (4.16)

\[ \chi' + r \psi'^2 + r \frac{e^\chi q^2 \phi^2 \psi^2}{g^2} = 0, \] (4.17)

Maxwell’s equations for the \( v \) field become

\[ v'' + v' \left( \frac{2}{r} + \frac{\chi'}{2} \right) = 0. \] (4.18)

As already mentioned we’ll specialize to the case where the scalar potential only contains the mass term. The equations have been left in a more general form where the scalar potential is not specified for future development. In appendix A we have reported the same equations of motion for a general dimension \( d \) of the spacetime. In the following we will work in units \( L = 1, 2 \kappa_4^2 = 1 \).

**4.0.3 Boundary conditions**

To find the solution to these equations one must impose two suitable boundary conditions: one in the interior of the spacetime at \( r = r_H \) and one at the conformal boundary \( r = \infty \), where we require \( AdS \) asymptotics. At the horizon, as already discussed in section 1.4.2, we must impose regularity of the fields. \( g(r) \) should be vanishing, and the gauge fields
should be also zero in $r_H$ since they would have, as already mentioned, an otherwise infinite norm. Hence we must set

$$\phi(r_H) = v(r_H) = g(r_H) = 0, \quad \text{and} \quad \psi(r_H), \chi(r_H) \text{ constants.} \quad (4.19)$$

The series expansions of the fields out of the horizon $r_H$, implementing the above boundary conditions, are the following

$$\phi_H(r) = \phi_{H1}(r - r_H) + \phi_{H2}(r - r_H)^2 + \ldots, \quad (4.20)$$
$$\psi_H(r) = \psi_{H0} + \psi_{H1}(r - r_H) + \psi_{H2}(r - r_H)^2 + \ldots, \quad (4.21)$$
$$\chi_H(r) = \chi_{H0} + \chi_{H1}(r - r_H) + \chi_{H2}(r - r_H)^2 + \ldots, \quad (4.22)$$
$$g_H(r) = g_{H1}(r - r_H) + g_{H2}(r - r_H)^2 + \ldots, \quad (4.23)$$
$$v_H(r) = v_{H1}(r - r_H) + v_{H2}(r - r_H)^2 + \ldots. \quad (4.24)$$

At the conformal boundary we must impose a leading behavior according to the corresponding dual boundary operators as in (1.95). From (1.88), taking $\Delta = 2$, the scalar field should approach the boundary in the following way

$$\psi(r) = \frac{C_1}{r} + \frac{C_2}{r^2} + \ldots, \quad \text{as} \quad r \to \infty. \quad (4.25)$$

With an homogeneous ansatz (A.2) $C_1$ and $C_2$ are constants, independent on the field theory coordinates $x_\mu$. Our choice of mass does not lead to non normalizable modes ($-\frac{9}{4} < m^2 L^2 < -\frac{5}{4}$), hence we can in principle choose whether the leading or the subleading behavior in (4.25) should be the source of the dual operator $O$. Since we don’t want the condensate to be sourced but arising spontaneously we shall require either one or the other independent parameter in (4.25) to vanish. Specifically

$$C_1 = 0, \quad < O_2 > = \sqrt{2}C_2 \quad (4.26)$$

or

$$C_2 = 0, \quad < O_1 > = \sqrt{2}C_1. \quad (4.27)$$

The factor $\sqrt{2}$ in defining the condensate is a convenient normalization as in [16]. For definiteness we shall consider standard boundary conditions where the leading coefficient must vanish (4.26) motivated by the fact that the dual field $O_2$ has mass dimension 2, corresponding to the dimensions of a fermionic bilinear.

Vector fields at the boundary, analogously to (2.71), are given by

$$\phi(r) = \mu - \frac{\rho}{r} + \ldots, \quad \text{as} \quad r \to \infty \quad (4.28)$$
$$v(r) = \delta \mu - \frac{\delta \rho}{r} + \ldots \quad \text{as} \quad r \to \infty. \quad (4.29)$$
where $\rho$ and $\delta \rho$ are the charge density and the charge density mismatch of the dual field theory. We shall repeat here for completeness what said in section 2.2.1. The chemical potential and chemical potential mismatch $\mu$ and $\delta \mu$ are fixed boundary values in the grand canonical ensemble, namely the sources for conserved currents in the field theory side. Since only the time components of the Maxwell’s bulk fields are turned on, the dual fields are the time components of conserved currents, whose expectation value gives the charge density and the charge density mismatch $\langle J^t_A \rangle = \rho$ and $\langle J^t_B \rangle = \delta \rho$. Notice that the $AdS$ radius has been set to $L = 1$. In this case the metric acquires a dimension $l^{-2}$, and the gravitational coupling constant $k^2$ is dimensionless. For instance bulk fields $A_a$, $B_a$ and the parameters $\mu$, $\delta \mu$ have mass dimension 1, $\psi$ instead remains dimensionless; $\rho$ and $\delta \rho$ are charges per unit volume in the $(2+1)$-dimensional field theory, hence have dimension $l^{-2}$; the radial coordinate $r$ has thus dimension 1 in mass.

The other fields should be vanishing at the boundary according to the requirement of having an asymptotically $AdS_4$ spacetime metric

$$g(r) = r^2 + \ldots \quad \text{as} \quad r \to \infty \quad (4.30)$$
$$\chi(r) = 0 + \ldots \quad \text{as} \quad r \to \infty. \quad (4.31)$$

### 4.0.4 The Normal Phase

A possible solution to the equations of motion (4.14-4.18) is given by the normal phase, characterized by a vanishing vacuum expectation value of the condensate $O$, hence a vanishing scalar field $\psi = 0$ in the bulk. The equations of motion for the Maxwell field $A_a$ (4.8) simplify to

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} g^{ce} F_{bc}) = 0, \quad (4.32)$$

which, together to the Einstein’s equations (4.5) and the other gauge field equations (4.10), lead to a $U(1)^2$ charged Reissner-Nordstrom-$AdS$ black hole solution, with metric of the form of (2.66-2.68)

$$ds^2 = -f(r) dt^2 + r^2 (dx^2 + dy^2) + \frac{dr^2}{f(r)}, \quad (4.33)$$
$$f(r) = r^2 \left(1 - \frac{r_H^2}{r^2}\right) + \frac{\mu^2 r_H^2}{4 r^2} \left(1 - \frac{r}{r_H}\right) + \frac{\delta \mu^2 r_H^2}{4 r^2} \left(1 - \frac{r}{r_H}\right). \quad (4.34)$$

Here $r_H$ is the coordinate of the black hole outer horizon. Thus the solution for the gauge fields keeps the standard form as shown in chapter 2

$$\phi(r) = \mu \left(1 - \frac{r_H}{r}\right) = \mu - \frac{\rho}{r}, \quad (4.35)$$
$$v(r) = \delta \mu \left(1 - \frac{r_H}{r}\right) = \delta \mu - \frac{\delta \rho}{r}. \quad (4.36)$$
We can find the temperature of the doubly charged RN-AdS black hole as a straightforward generalization of the temperature of the RN-AdS black hole in (2.16)

$$T = \frac{r_H}{16\pi} \left( 12 - \frac{(\mu^2 + \delta \mu^2)}{r_H^2} \right).$$  \hfill (4.37)

We can solve (4.37) with respect to $r_H$

$$r_H = \frac{2}{3} \pi T + \frac{1}{6} \sqrt{16\pi^2 T^2 + 3(\mu^2 + \delta \mu^2)}.$$  \hfill (4.38)

and, generalizing the result (2.77) to a double charged RN-AdS black hole in $d = 3$, and setting again $L = 2k_4^2 = 1$, we find the Gibbs free energy density of the normal phase

$$\omega_n = -r_H^3 \left( 1 + \frac{(\mu^2 + \delta \mu^2)}{4r_H^2} \right).$$  \hfill (4.39)

Notice that, due to formula (4.38), this is a function of $T$ and $\mu$.

Let us now ask whether there are conditions under which, lowering the temperature, a superconducting phase ($\psi \neq 0$) might arise with a formation of a charged condensate below a certain critical temperature $T_c$. The instability of the normal phase must occur spontaneously, hence the bulk field $\psi$ at $T < T_c$, dual to the condensate operator $O$, should be non vanishing without being sourced.

### 4.0.5 Criterion for Instability

Suppose $\psi$ to be too small to significantly back-react on the geometry. So, we can consider such field as a fluctuation above the fixed double charged RN-AdS background (4.33). Taking again the simple homogeneous ansatz

$$\psi = \psi(r),$$  \hfill (4.40)

the equation of motion (4.9) writes

$$\psi'' + \left( \frac{f'}{f} + \frac{2}{r} \right) \psi' + \frac{q^2 \phi^2}{f^2} \psi - \frac{m^2}{f} \psi = 0,$$  \hfill (4.41)

where $f(r)$ is given by (4.34). What is the effective potential experienced by this scalar field? The electrically charged black hole gives an extra contribution to the scalar squared mass: $\frac{q^2 \phi^2}{f}$. This term comes with the opposite sign with respect to the mass term, hence it makes the field more likely to be unstable. The effective squared mass writes

$$m_{\text{eff}}^2 = m^2 - \frac{q^2 \phi^2}{f}.$$  \hfill (4.42)
Near the horizon \( f(r) \simeq f(r_H)'(r - r_H) = 4\pi T(r - r_H) \), where \( T \) is the temperature of the black hole, thus

\[
m^2_{\text{eff}} \approx m^2 - \frac{q^2\phi^2}{4\pi T(r - r_H)}.
\] (4.43)

It seems inevitable that the effective squared mass can turn negative near the horizon by lowering the temperature at fixed charge \( q \) provided that \( \phi \) is non zero just outside the horizon.

The instability of the scalar field to form a charged condensate near the horizon of a \( U(1) \)-charged RN-\( AdS \) black hole has been first studied by Gubser in [14]. The actual computation concerns finding marginally stable perturbations around the solution which does not break the \( U(1) \) symmetry. The result is that the Hilbert-Einstein gravity coupled to a vector field and to a charged scalar exhibits spontaneous symmetry breaking near the horizon of a RN-\( AdS \) black hole, provided the charge of the scalar is large enough.

Heuristically this mechanism has a simple quantum mechanical interpretation. The scalar bulk field can be seen as a gas of charged particles formed in pairs outside the horizon by Schwinger mechanism. Particles with opposite charge with respect to the one of the black hole will fall inside the horizon lowering the total charge of the black hole. The other particles will be repelled from the black hole when they are highly charged to overcome the gravitational attraction. In asymptotically flat spacetime, these particles escape to infinity, so the final result is a standard Reissner-Nordstrom black hole with a lower final charge. At this point \( AdS \) boundary conditions are crucial because the negative cosmological constant acts as a confining box which does not lead the scalar particles to escape outside the near-horizon region. They have no choice but condense and form a superconducting thin film just outside the horizon.

One expects that when \( q \to 0 \) the instability turns off, but it was noted in [16] that this is not the case, and a RN-\( AdS \) black hole remains unstable to forming neutral scalar hair. This means that there is a new source of instability. The best explanation [16] seems to be the following. At \( q = 0 \) the critical temperature \( T_c \) is nonzero but small. Therefore such unstable black holes are near extremal. The near horizon geometry of an extremal RN-\( AdS_4 \) black hole, as showed in section 2.2.3, is that of \( AdS_2 \times R^2 \). The squared radius of \( AdS_2 \) is \( L_2^2 = \frac{L^2}{6} \) where \( L \) is the \( AdS_4 \) radius. The BF-bound governing stability of scalar fields in \( AdS_2 \) is higher then the one in \( AdS_4 \). This means that scalars which are slightly above the BF-bound in \( AdS_4 \), can be below the corresponding bound for \( AdS_2 \).

In our model at \( T = 0 \) we can find a simple condition on the external parameters in order for the normal phase to become unstable. At \( T = 0 \) the doubly charged RN-\( AdS \) black hole becomes extremal and (see equation 4.37) the horizon radius reads

\[
r^2_H = \frac{1}{12}(\delta\mu^2 + \mu^2) \quad \text{at} \quad T = 0.
\] (4.44)
Analyzing the scalar equation (4.41) in these limits one gets the equation for a scalar field of mass (in units $L = 1$)

$$m_{\text{eff}(2)}^2 = \frac{m^2}{6} - \frac{q^2 \phi^2}{36 \tilde{r}^2},$$

(4.45)
on an $AdS_2$ background. From (4.33) and taking $r \simeq r_H + \tilde{r}$ with $\tilde{r} \to 0$, we have

$$\phi^2 = \frac{12\tilde{r}^2}{\left(1 + \frac{\delta \mu^2}{\mu^2}\right)},$$

(4.46)

so that

$$m_{\text{eff}(2)}^2 = \frac{1}{6} \left( m^2 - \frac{2q^2}{\left(1 + \frac{\delta \mu^2}{\mu^2}\right)} \right).$$

(4.47)

We have seen previously that in order to have an instability we must require this mass to be below the $AdS_2$ BF-bound

$$m_{\text{eff}(2)}^2 < -\frac{1}{4},$$

(4.48)

which leads to

$$\left(1 + \frac{\delta \mu^2}{\mu^2}\right) \left(m^2 + \frac{3}{2}\right) < 2q^2.$$  

(4.49)

When $(m^2 + \frac{3}{2}) < 0$, i.e. $m^2 < -\frac{3}{2}$, the instability occurs for every value of $\frac{\delta \mu^2}{\mu^2}$. This is indeed our case since we mostly consider $m^2 = -2$. Instead the case when $(m^2 + \frac{3}{2}) > 0$ is peculiar because instability occurs only when

$$\frac{\delta \mu^2}{\mu^2} < 2q^2 \frac{1}{(m^2 + \frac{3}{2})} - 1.$$  

(4.50)

This condition resembles the case of weakly interacting superconductors which exhibit a Chandrasekhar-Clogston bound.

To summarize the story so far there are apparently two distinct mechanisms rendering a RN-$AdS$ black hole unstable near its horizon: at very large charges for the scalar fields it is the coupling to the electric field in the covariant derivative that enhance the effective negative mass. On the other hand at very small charges, the black hole turns near extremal and the geometry produces a throat in which even neutral scalar with sufficiently negative mass squared becomes unstable. The instability also depends on the value of the external parameters of the theory $q$ and $m$. When $m^2 < -\frac{3}{2}$ a normal phase is unstable for every $\frac{\delta \mu^2}{\mu^2}$; when instead $m^2 > -\frac{3}{2}$ the normal phase is unstable provided the condition (4.50) is satisfied, i.e. below a Chandrasekhar-Clogston-like bound.

### 4.0.6 The Superconducting Phase

If the normal phase becomes unstable at low $T$, we must search for another static solution to the equations of motion (4.14-4.18) where the scalar field is non zero, and the black
hole is said to develop scalar hair. In the dual field theory this corresponds to turning on a vacuum expectation value of the condensate leading to a spontaneous symmetry breaking of an electromagnetic symmetry and the consequent emergence of a superconducting phase.

Before solving the full equations of motion numerically, we will consider in the following section a particular limit where things simplify. In this limit we will do analytic approximate computations obtaining results qualitatively comparable to the numerical ones in certain regimes.

### 4.1 The Probe Approximation

In order to start dealing with the whole set of differential equations (4.14-4.18), let us consider first the so-called probe regime, where the Maxwell field $A_a$ and the scalar field $\psi$ do not backreact on the metric. This can be obtained through the redefinition of the fields $\tilde{A}_a = \frac{4a}{q}$ and $\tilde{\psi} = \frac{\psi}{q}$. By taking $q \to \infty$ one obtains a decoupled sector where $\tilde{A}_a$ and $\tilde{\psi}$ can be threaten as fluctuations above a fixed metric background determined by the remaining field $B_a$. This limit is consistent as long as the fields are small in dimensionless units. The latter condition is not satisfied at small temperatures, where the fields are considerably big. As we shall see, however, this simplified model captures the physics of interest near the critical temperature $T_c$.

The fixed background in the $q \to \infty$ limit is a Reissner-Nordstrom-AdS black hole charged under $U_B(1)$

$$ ds^2 = -f(r)dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{f(r)}, $$

$$ f(r) = r^2(1 - \frac{r^3}{r^3_H}) + \frac{\delta \mu^2 r_H^2}{4r^2} (1 - \frac{r}{r_H}), $$

$$ v_t = \delta \mu (1 - \frac{r_H}{r}) = \frac{\delta \mu - \delta \rho}{r}, $$

The black hole temperature is given by (2.73)

$$ T = \frac{r_H}{4\pi} \left( 3 - \frac{\delta \mu^2}{4r_H^2} \right), $$

from which one can read the bound on the allowed charge of the black hole $\delta \mu \leq 12r_H^2$. When the bound is saturated one gets the extremal solution $AdS_2 \times R^2$. Using the dimensionless parameter

$$ c^2 = \frac{\delta \mu^2}{4r_H^2}, $$
4.1. THE PROBE APPROXIMATION

we may also write

\[ T = \frac{r_H}{4\pi} (3 - c^2). \] (4.56)

4.1.1 Fluctuations

Implementing the homogeneous ansatz (A.2) into the equations (4.8-4.9) on the \( U(1) \)-charged RN-AdS black hole background written above, we get the following coupled nonlinear, ordinary differential equations for the scalar field \( \psi \) and the Maxwell’s field \( \phi \)

\[
\begin{align*}
\psi'' + \left( \frac{f'}{f} + \frac{2}{r} \right) \psi' + \frac{\phi^2}{f^2} \psi - \frac{m^2}{f} \psi &= 0, \\
\phi'' + \frac{2}{r} \phi' - \frac{2\psi^2}{f} \phi &= 0.
\end{align*}
\] (4.57, 4.58)

These equations of motion are second order differential equations and need as usual two boundary conditions.

To find a solution to these equations we shall take the IR expansions (4.20) and (4.21) at the horizon. By multiplying the equation (4.57) by \( f(r) \) and evaluating it in \( r = r_H \) we find that \( f'(r_H)\psi'(r_H) = m^2\psi(r_H) \) so that the two coefficients \( \psi'(r_H) = \psi_{H1} \) and \( \psi(r_H) = \psi_{H0} \) are not independent. As a result we end up with an IR expansion which only depends on \( \psi_{H0} \) and \( \phi_{H1} \)

\[
\begin{align*}
\psi_H(r) &= \psi_{H0} + \psi_{H1}(\psi_{H0}, \phi_{H1})(r - r_H) + \ldots, \\
\phi_H(r) &= \phi_{H1}(r - r_H) + \ldots.
\end{align*}
\] (4.59, 4.60)

The solution at the conformal boundary is instead guessed using the \( AdS/CFT \) correspondence. Asymptotic behaviors of the fields are given by (4.25) and (4.28), which we rewrite here for completeness using (4.26)

\[
\begin{align*}
\psi &= \frac{C_2}{r^2} + \ldots \quad \text{as} \quad r \to \infty, \\
\phi &= \mu - \frac{\rho}{r} + \ldots \quad \text{as} \quad r \to \infty,
\end{align*}
\] (4.61, 4.62)

and the condensate is given by

\[ <O_2> = \sqrt{2}C_2. \] (4.63)

After imposing both boundary conditions at the horizon and at the conformal boundary we are led to a one parameter family of solutions.

4.1.2 Analytic solution

We should now proceed to numerically solve these equations. However we can get some qualitative information using an approximate analytical method suggested in [80]. Try
first to solve the differential equations close to \( r = \infty \) and to \( r = r_H \) in terms of Taylor expansions up to some order. The free parameters will be fixed by requiring the UV and the IR solutions, and their derivatives to match at some arbitrary intermediate point. The method gives closer results to the numerical ones as the order in the Taylor expansion is increased.

To this purpose it is useful to define another set of compact coordinates. Define \( z = \frac{r_H}{r} \) and rewrite the RN-AdS hole background (4.51) as

\[
ds^2 = r_H^2 f(z) dt^2 + \frac{r_H^2}{z^2} (dx^2 + dy^2) + \frac{dz^2}{z^4 f(z)},
\]

where we extracted a factor out of the blackening factor \( f(z) \rightarrow r_H^2 f(z) \), which writes (see (4.52))

\[
f(z) = \frac{1}{z^2} - (1 + c^2) z + c^2 z^2,
\]

where \( c \) is the dimensionless parameter defined in (4.55). The equations (4.57) and (4.58) take then the form

\[
\phi'' - \frac{2\psi^2}{z^4 f(z)} \phi = 0, \tag{4.66}
\]

\[
\psi'' + \psi' \frac{f'}{f} + \psi \left( \frac{\phi^2}{r_H^2 z^4 f^2} - \frac{m^2}{z^4 f} \right) = 0. \tag{4.67}
\]

The prime \( ' \) shall now denote \( \frac{d}{dz} \). The near horizon \( z = 1 \) expansions (4.59) and (4.60) up to second order read

\[
\phi_H(z) = \phi_H(z - 1) + \phi_H(z - 1)^2 + \ldots, \tag{4.68}
\]

\[
\psi_H(z) = \psi_H(z - 1) + \psi_H(z - 1)^2 + \ldots. \tag{4.69}
\]

Plugging these expressions into equations (4.66) and (4.67) we get the independent infrared (IR) parameters

\[
\phi_H^2 = -\frac{\phi_H^2}{3 - c^2}, \tag{4.70}
\]

\[
\psi_H^1 = \frac{2\psi_H}{3 - c^2}, \tag{4.71}
\]

\[
\psi_H^2 = \frac{\psi_H}{4} \left( \frac{28}{(3 - c^2)^2} - \frac{12}{3 - c^2} - \frac{\phi_H^2}{(3 - c^2)^2 r_H^2} \right). \tag{4.72}
\]

The IR expansions (4.68) and (4.69) read now

\[
\phi_H(z) = \phi_H(z - 1) - \frac{\phi_H^2}{3 - c^2} (z - 1)^2, \tag{4.73}
\]

\[
\psi_H(z) = \frac{2\psi_H}{3 - c^2} (z - 1) + \frac{\psi_H}{4} \left( \frac{28}{(3 - c^2)^2} - \frac{12}{3 - c^2} - \frac{\phi_H^2}{(3 - c^2)^2 r_H^2} \right) (z - 1)^2. \tag{4.74}
\]
4.1. THE PROBE APPROXIMATION

The ultraviolet (UV) solution (4.61) and (4.62) in \( z \to 0 \) up to second order is

\[
\phi_{UV}(z) = \mu - \frac{\rho}{r_H} z + Az^2 + \ldots, \tag{4.75}
\]

\[
\psi_{UV}(z) = Cz^2 + \ldots, \tag{4.76}
\]

where we have redefined \( C = \frac{C_2}{r_H^2} \). Implementing this ansatz into (4.67) we obtain that \( A = 0 \) and the UV solution gets the form

\[
\phi_{UV}(z) \sim \mu - \frac{\rho}{r_H} z, \tag{4.77}
\]

\[
\psi_{UV}(z) \sim Cz^2. \tag{4.78}
\]

Next step is to find the whole solution to our differential equations by imposing the IR solutions (4.73-4.74) to match the UV ones (4.77-4.78) at an intermediate point say \( z = \frac{1}{2} \): \( \phi_{UV}(\frac{1}{2}) = \phi_H(\frac{1}{2}), \phi'_{UV}(\frac{1}{2}) = \phi'_H(\frac{1}{2}) \) and analogous ones for \( \psi \). We obtain then the following conditions

\[
\frac{b}{2} + \frac{Q}{2} - \mu + \frac{1}{4(3 - c^2)} a^2 b = 0, \tag{4.79}
\]

\[
Q - b + \frac{1}{c^2 - 3} a^2 b = 0, \tag{4.80}
\]

\[
\frac{a}{4(3 - c^2)^2} (22 - 17c^2 + 4c^4) - \frac{C}{4} - \frac{1}{16(3 - c^2)^2 r_H^2} ab^2 = 0, \tag{4.81}
\]

\[
\frac{a}{(3 - c^2)^2} (8 - 5c^2) - C + \frac{1}{4(3 - c^2)^2 r_H^2} ab^2 = 0, \tag{4.82}
\]

where \( b = -\phi_H \), \( a = \psi_H \) and \( Q = \frac{C}{r_H} \), with \( a, b > 0 \). \( b \) and \( Q \) have dimension 1 in mass, while \( a \) is dimensionless. Eliminating the term containing \( a^2 b \) from (4.79) and (4.80) gives

\[
Q = \frac{4}{3} \mu - \frac{b}{3}, \tag{4.83}
\]

and assuming the non trivial solution with \( a^2 > 0 \)

\[
a^2 = (3 - c^2) \frac{Q}{b} \left(1 - \frac{b}{Q}\right). \tag{4.84}
\]

Eliminating the term containing \( ab^2 \) from (4.81) and (4.82) leads to

\[
C = \frac{a(5 - 2c^2)}{3 - c^2}, \tag{4.85}
\]

and

\[
\frac{a(-b^2 + 4(7 - 6c^2 + 2c^4) r_H^2)}{16(3 - c^2)^2 r_H^2} = 0, \tag{4.86}
\]

choosing the solution with \( b > 0 \)

\[
b = 2\sqrt{7 - 6c^2 + 2c^4} r_H. \tag{4.87}
\]
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The final result, restoring the original parameters in (4.83), (4.84), (4.85) and in (4.87) is

$$\psi_{H0} = \sqrt{3 - c^2} \sqrt{-\frac{\rho}{r_H \phi_{H1}}} \sqrt{1 + \frac{r_H \phi_{H1}}{\rho}}, \tag{4.88}$$

$$C = \frac{(5 - 2c^2)}{(3 - c^2)} \psi_{H0}, \tag{4.89}$$

$$\phi_{H1} = -2r_H \sqrt{7 - 6c^2 + 2c^4}, \tag{4.90}$$

$$\rho = +\frac{\phi_{H1}}{3} r_H + \frac{4}{3} \mu r_H. \tag{4.91}$$

We started with two IR parameters ($\psi_{H0}, \phi_{H1}$) and three UV parameters ($\mu, q, C$). The four conditions (4.79) leave us with one independent parameter which we can choose between $\mu$ and $\rho$ depending on which kind of ensemble we want to deal with: canonical or grand canonical. Let us consider the grand canonical ensemble in which the chemical potentials $\mu$ and $\delta \mu$ are fixed.

From the above relations (4.88-4.91) we may learn some additional informations. Note that $\psi_{H0}$, and thus $C$, is real only for $|\phi_{H1}| < Q$. This is precisely translated in the condition $T < T_c$ for the superconductive phase to develop. To find out the critical temperature $T_c$ one must set the vacuum expectation value of the condensate $< O_2 > = \sqrt{2}C_2 = \sqrt{2}C r_H^2$ to zero. Remembering the relationship between the horizon radius and the temperature given in (4.56) one finds the critical value of the horizon radius to be

$$r_{H0} = \frac{\mu x^2}{\sqrt{K(x)}}, \tag{4.92}$$

where

$$x = \frac{\delta \mu}{\mu} \quad \text{and} \quad K(x) = 1 + 6x^2 - \sqrt{1 + 2x^2 - 20x^4}. \tag{4.93}$$

The critical temperature $T_c$ is then

$$T_c = \frac{3}{4\pi} \mu \frac{x^2}{\sqrt{K(x)}} \left(1 - \frac{K(x)}{12x^2}\right), \tag{4.94}$$

which is reliable if

$$x^2 < \frac{1}{\left(\sqrt{56} - 6\right)} \quad \text{i.e.} \quad x < 0.81. \tag{4.95}$$

At $x = 0$ we get

$$T_c^{(0)} = \frac{3}{4\pi} \frac{\mu}{\sqrt{28}}, \tag{4.96}$$

a result which is in qualitative agreement with that found in [15]: at $\delta \mu = 0$ the critical temperature is set by the only other scale in the game, i.e. $\mu$. Defining

$$c_0 = \frac{\delta \mu}{2r_{HO}}, \tag{4.97}$$
4.2. THE FULLY BACKREACTIONED MODEL

the condensate can be written in terms of $T$, $T_c$ as

$$\langle \mathcal{O} \rangle = \frac{9\sqrt{3}}{5} \frac{(5-2c_0^2)}{(3-c_0^2)^{3/2}} \langle \mathcal{O}^{(0)} \rangle,$$  

(4.98)

where $\langle \mathcal{O}^{(0)} \rangle$ is as in [80]

$$\langle \mathcal{O}^{(0)} \rangle = \frac{80\pi^2}{9} \sqrt{2TT_c} \sqrt{1 + \frac{T}{T_c}} \sqrt{1 - \frac{T}{T_c}}.$$

(4.99)

Condensation occurs at $T < T_c$. The mean field theory result $\langle \mathcal{O} \rangle \sim (1 - \frac{T}{T_c})^{1/2}$ as the temperature approaches the critical one, typical of the second order phase transitions, is also recovered. Notice that in the allowed range of $x$, $T_c(x)$ in (4.94) never goes to zero. Thus there is no sign of a Chandrasekhar-Clogston bound. $T_c$ is not monotonic and decreases with $x$ for $x > 0.4$. We will see whether this persists also numerically.

Notice that we are in the probe regime and all the approach is sensible only if $\psi_{H0}$ and $C$ are very small in dimensionless units. Furthermore at extremely low temperatures we will eventually be outside the region of validity of our approximation. As we will show in the following section, a numerical analysis of the fully backreacted system will slightly correct the above rough findings where the approximation is applicable.

4.2 The fully backreacted model

At this point we have to solve the full set of equations (4.14-4.18) numerically. We use the shooting method as in [16], namely we try to match the numerical solutions found at the horizon and at the boundary.

Numerical calculations are more suited in a compact domain of the variables. Hence, as in the previous section, it is convenient to use the dimensionless coordinate $z = \frac{rH}{r}$. In this case the horizon is set to $z_H = 1$ and the boundary to $z = 0$. It is also convenient to redefine the blackening factor as

$$g(z) = \frac{r_H^2}{z^2} + h(z),$$

(4.100)

so that the asymptotically AdS part of it results evident ($g(z) \rightarrow \frac{r_H^2}{z^2}$ as $z \rightarrow 0$). With these conventions the metric ansatz (4.11) becomes

$$ds^2 = -e^{-\chi(z)} \left( \frac{r_H^2}{z^2} + h(z) \right) dt^2 + \frac{r_H^2}{z^2} (dx^2 + dy^2) + \frac{r_H^2}{z^4} \left( \frac{r_H^2}{z^2} + h(z) \right) dz^2.$$

(4.101)
Let us now rescale the scalar field by $z$: $\psi(z) = z \tilde{\psi}(z)$ and recall $\tilde{\psi}$ as $\psi$. The scalar equation (4.14) becomes

$$\psi'' + \psi'\left(\frac{2}{z} - \frac{2r_H^2}{z(r_H^2 + z^2 h)} - \frac{\chi'}{2} + \frac{h'z^2}{(r_H^2 + z^2 h)}\right) + \psi(\frac{2r_H^2}{z^2(r_H^2 + z^2 h)} + \frac{e^x q^2 r_H^2 \psi^2}{(r_H^2 + z^2 h)^2} + \frac{\chi'}{2z} + \frac{h'z}{(r_H^2 + z^2 h)}) - \frac{r_H^2 m^2 \psi}{2z^2(r_H^2 + z^2 h)} = 0, \quad (4.102)$$

Maxwell’s equations for the $\phi$ and $v$ fields (4.15-4.18) become

$$\frac{1}{r_H^2} \phi'' + \frac{1}{2r_H^2} \phi' \chi' - \frac{2q^2 \phi \psi^2}{(r_H^2 + z^2 h)} = 0, \quad (4.103)$$

$$\frac{1}{r_H^2} v'' + \frac{1}{2r_H^2} v' \chi' = 0, \quad (4.104)$$

and Einstein’s equations (4.16-4.17) become

$$\frac{1}{2} \psi'' + \frac{\psi'}{z} + \frac{\psi^2}{2z^2} + \frac{e^x (\phi^2 + v^2)}{4(r_H^2 + z^2 h)} + \frac{m^2 \psi^2 r_H^2}{2z^2(r_H^2 + z^2 h)} - \frac{h'}{z(r_H^2 + z^2 h)} + \frac{r_H^2}{z^4(r_H^2 + z^2 h)} + \frac{e^x q^2 r_H^2 \psi^2}{2(r_H^2 + z^2 h)^2} = 0, \quad (4.105)$$

$$\chi' - \psi^2 z - \frac{e^x q^2 r_H^2 \psi}{(r_H^2 + z^2 h)^2} \psi^2 - 2z^2 \psi' - 3 \psi^2 = 0. \quad (4.106)$$

### 4.2.1 Details of the numerical method

To solve such equations consider first the Taylor series expansion at the horizon $z_H = 1$

$$\phi_H(z) = \phi_{H1}(1 - z) + \phi_{H2}(1 - z)^2 + \ldots,$$

$$\psi_H(z) = \psi_{H0} + \psi_{H1}(1 - z) + \psi_{H2}(1 - z)^2 + \ldots,$$

$$\chi_H(z) = \chi_{H0} + \chi_{H1}(1 - z) + \chi_{H2}(1 - z)^2 + \ldots,$$

$$h_H(z) = -r_H^2 + h_{H1}(1 - z) + h_{H2}(1 - z)^2 + \ldots,$$

$$v_H(z) = v_{H1}(1 - z) + v_{H2}(1 - z)^2 + \ldots. \quad (4.107)$$

Boundary conditions (4.19) imply that the gauge fields $\phi$ and $v$ should vanish at the horizon, for this reason we have set $\phi_{H0} = v_{H0} = 0$. Also the blackening factor $g(z)$ should be zero at the horizon, hence we have set $h_{H0} = -r_H^2$. Implementing the IR expansions (4.107) into the equations of motion (4.102-4.106) the resulting independent IR parameters are

$$(r_H, \; \psi_{H0}, \; \phi_{H1}, \; \chi_{H1}, \; v_{H1}). \quad (4.108)$$
while the others are related to the previous ones by the following relations

\[\psi_{H1} = \psi_{H0} - \frac{8\psi_{H0}r_H^2}{(4(3 + \psi_{H0}^2)r_H^2 - e^{\chi_{H0}}(\phi_{H1}^2 + v_{H1}^2))},\]  
(4.109)

\[\chi_{H1} = -\frac{16\psi_{H0}r_H^2(e^{\chi_{H0}}\phi_{H1}^2 + 4r_H^2)}{(-4(3 + \psi_{H0}^2)r_H^2 + e^{\chi_{H0}}(\phi_{H1}^2 + v_{H1}^2))^2},\]  
(4.110)

\[h_{H1} = (1 + \psi_{H0}^2)r_H^2 - \frac{1}{4}e^{\chi_{H0}}(\phi_{H1}^2 + v_{H1}^2),\]  
(4.111)

and similar but rather more complicated ones for the second order coefficients \(\psi_{H2}, \phi_{H2}, \chi_{H2}, h_{H2}, v_{H2}\). Implementing the series expansion (4.107) where all the coefficients are expressed in function of the independent parameters (4.108) into the equations of motion (4.102-4.106) one is ready to perform a numerical derivation of the solutions via the command NDSolve of Mathematica. The solutions result to be functions of the independent IR parameters (4.108) and the external parameter \(q\)

\[\hat{\psi}(\phi_{H1}, \psi_{H0}, \chi_{H0}, v_{H1}, r_H, q),\]  
(4.112)

\[\hat{\psi}(\phi_{H1}, \psi_{H0}, \chi_{H0}, v_{H1}, r_H, q),\]  
(4.113)

\[\hat{\chi}(\phi_{H1}, \psi_{H0}, \chi_{H0}, v_{H1}, r_H, q),\]  
(4.114)

\[\hat{h}(\phi_{H1}, \psi_{H0}, \chi_{H0}, v_{H1}, r_H, q),\]  
(4.115)

\[\hat{v}(\phi_{H1}, \psi_{H0}, \chi_{H0}, v_{H1}, r_H, q).\]  
(4.116)

To obtain the whole solution one must impose the UV conditions (4.25), (4.28), (4.29), (4.30) and (4.31). In terms of the radial coordinate \(z\), and the rescaled scalar field they read

\[\psi(z) = \frac{C_1}{r_H} + \frac{C_2}{r_H}, \quad \text{as} \quad z \to 0,\]  
(4.117)

\[\phi(z) = \mu - \frac{\rho r_H}{z}, \quad \text{as} \quad z \to 0,\]  
(4.118)

\[v(z) = \delta \mu - \frac{\delta \rho r_H}{z}, \quad \text{as} \quad z \to 0.\]  
(4.119)

\[h(z) = -\frac{\epsilon}{2r_H}z, \quad z \to 0,\]  
(4.120)

\[\chi = -\log(1 + a), \quad z \to 0,\]  
(4.121)

where \(\epsilon\) is the mass of the black hole, while \(a\) is a constant required by numerics, which will be set to zero during the calculations.

Now that we have the UV behaviors of the fields (4.117-4.119), we can solve these equations in terms of the UV independent parameters \(\mu, \rho, C_1, C_2, \epsilon, a, \delta \rho, \delta \mu\). Hence we obtain equations depending on the fields and their derivatives \((z \to 0)\)

\[\mu = \phi - \phi'z,\]  
(4.122)
Figure 4.1: The value of the condensate as a function of the temperature with the chemical potential held at fixed value \( \mu = 1.87 \) and \( \delta \mu = 0.01 \). From top to bottom the various curves correspond to the values \( q = 3, 6, \) and \( 12 \).

\[
\sqrt{\frac{|<O_2>| q}{T_c}}
\]

\[
\begin{align*}
T & \quad T_c \\
0.0 & \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0
\end{align*}
\]

By imposing the boundary conditions \( C_1 = a = 0 \) and considering \( \mu \) and \( \delta \mu \) as external parameters, since we are dealing with a grand canonical ensemble, we can find the suitable values of the IR parameters which will bring us to define the solutions (4.112-4.116). In particular we use the command FindRoot of Mathematica, which finds the right values for the IR parameters near some suitable values which we put in by hand, namely the seeds. In particular since we have four constraining equations and five independent IR parameters we find the values for \( \phi_{H0}, \chi_{H0}, v_{H1}, r_H \) while \( \psi_{H0} \) is defined by hand as stated by the shooting method.

\[
\begin{align*}
\rho &= -\phi' r_H, \\
C_1 &= \psi r_H - \psi' r_H z, \\
C_2 &= \psi' r_H^2, \\
\epsilon &= -2\hbar' r_H, \\
a &= -e^{-\chi}(-1 + e^\chi), \\
\delta \rho &= -r_H v', \\
\delta \mu &= v - v' z.
\end{align*}
\]
4.2.2 The condensate

Let us now concentrate on what we can learn from these solutions. First of all let us find an expression for the temperature as a function of the IR parameters. Take the expression for the temperature (4.13), use the $z$ coordinate and (4.100), and use the expansions (4.107) with (4.111). The final result is

$$T = \frac{r_H}{16\pi} \left( (12 + 4\psi^2 H_0) e^{-\frac{\chi H_0}{2}} - \frac{1}{r_H^2} e^{\frac{\chi H_0}{2}} (\phi^2 + \psi^2) \right). \tag{4.131}$$

The critical temperature is found by setting $<O> \sim C_2 = 0$, hence by taking a really small value for $\psi H_0$. The dimensionless quantity is $\frac{T}{(\mu^2 + \delta\mu^2)^{1/2}}$.

Now to find a picture of the condensate one must try to plot $\sqrt{\frac{q <O>}{T_c}}$ as a function of $\frac{T}{T_c}$. The set of the data is given by performing an iteration varying the value of the input $\psi H_0$ from a small initial value to a higher value. Step by step one computes the right values of the IR parameters which will be the seeds for the following step. In each step one computes the value of $T$ given by (4.131) and the value of the condensate given by (4.26).

First we see that for small values of the chemical potential mismatch $\delta\mu = 0.01$ and for different values of the external parameter $q$ we obtain results similar to [16], see figure 4.1. A condensate arises below a certain critical temperature $T_c$ signalizing a phase transition from a normal to a superconducting phase. The general form of these curves is similar to the ones we find in BCS theory, where the behavior of the condensate (gap parameter) is given by (3.58), typical of mean field theories and second order phase transitions. The value of the condensate depends on the charge of the bulk field $q$. However, as in [16], it is difficult to get the numerics reliably down to very low temperatures.

Now, allowing for non zero values of the chemical potential mismatch $\delta\mu$, we obtain analogous figures for the condensate. Increasing the value of $\delta\mu$ (see figure 4.2) we obtain a decreasing value of the critical temperature. The phase transition is always second order.

The most interesting result is the plot of the critical temperature normalized to $T_c^0$, the critical temperature at zero chemical potential mismatch, against $\frac{\delta\mu}{\mu}$. The second order phase transition at zero chemical potential mismatch develops inside the $T_c - \delta\mu$ phase diagram. As it is shown in figure 4.3, the critical temperature decreases with $\frac{\delta\mu}{\mu}$, a qualitative feature which we have seen also in the weakly coupled case, see section 3.2, but differently from our approximate analytic approach in the probe limit (2.20). However, differently from the weakly coupled case, there is no finite value of $\frac{\delta\mu}{\mu}$ for which $T_c = 0$. Hence, there is no sign of a Chandrasekhar-Clogston bound. This result matches with
Figure 4.2: The value of the condensate as a function of the temperature at $\mu = 1$, $q = 2$. From right to left we have $\delta \mu = 0, 0.5, 1$, and the critical temperature is decreasing. Both $\sqrt{\Omega q}$ and $T$ are to be intended as divided by $T_c^0$, the critical temperature at $\delta \mu = 0$. At $T = 0$ we have just extrapolated.

the expectations coming from the formula (4.49), which actually suggested the absence of a Chandrasekhar-Clogston bound at $m^2 = -2$. However, it should be desirable to refine our numerics around $T = 0$ as done in [83] to definitely confirm this conclusion. In any case, we believe that it is unlikely that the curve in figure 4.3 will suddenly drop to zero with another flex. The phase transition we find is always second order. Together with the absence of a Chandrasekhar-Clogston bound, this leads us to conclude that LOFF phase is unlikely to develop.

4.2.3 The Gibbs free energy

Even if there is an instability of the RN black hole to formation of scalar hair, we must check that the superconducting phase is actually energetically favorable with respect to the normal phase. The superconducting phase is preferred when its Gibbs free energy is lower then the one of the normal phase (2.77). In order to compute the Gibbs free energy (2.76) of the hairy black hole we must compute the Euclidean continuation of the action (4.1) on the hairy black hole solution (4.11)

$$S_E = - \int d^4 x \sqrt{g} (\mathcal{R} + \mathcal{L}_{\text{matter}}) = - \int d^4 x \sqrt{g} \mathcal{L}_{\text{tot}},$$

where we set as above $2k_4^2 = L = 1$. Instead of computing directly such action, we may use a trick as in [16]. Notice that there is a relationship between the lagrangian of matter
4.2. THE FULLY BACKREACTED MODEL

There are always values of $T_c$ below which a superconducting phase arises.

and the stress energy tensor (4.6)

$$T_{ab} = -g_{ab}L_{\text{matter}}.$$  \hspace{1cm} (4.133)

Then Einstein’s equations (4.5) can be written in the following fashion

$$G_{ab} = \frac{1}{2}g_{ab}(L_{\text{matter}} + 6) = \frac{1}{2}g_{ab}(L_{\text{tot}} - \mathcal{R}).$$  \hspace{1cm} (4.134)

Taking the $xx$ and $yy$ component we find

$$L_{\text{tot}} - \mathcal{R} = G^x_x + G^y_y.$$  \hspace{1cm} (4.135)

The Ricci scalar is related to the Einstein tensor

$$-\mathcal{R} = G^a_a.$$  \hspace{1cm} (4.136)

Plugging here equation (4.135), one can rewrite the total lagrangian in terms of the components of the Einstein tensor

$$L_{\text{tot}} = -G^x_{t} - G^r_{r}.$$  \hspace{1cm} (4.137)

Plugging the hairy black hole ansatz (4.11) into the Einstein’s equations (4.5) we find the components of the Einstein tensor to be

$$G^r_{r} = g^{rr}G_{rr} = \frac{g - rg\chi' + rg'}{r^2}.$$  \hspace{1cm} (4.138)
\[ G^u_l = g^{\mu\nu} G_{\mu\nu} = \frac{g + rg'}{r^2}. \]  
(4.139)

The Euclidean action (4.132), using the expression for the total lagrangian (4.137) and plugging the Einstein tensor’s components (4.138-4.139), is a total derivative

\[ S_E = \int d^3x \int dr (2r ge^{-\frac{r}{2}})' . \]  
(4.140)

The surface term at the horizon vanishes since \( g(r_H) = 0 \). We are then led with the surface term at \( r = \infty \)

\[ S_E = \int d^3x (2rge^{-\frac{r}{2}})_{r=\infty} . \]  
(4.141)

As discussed in chapter 2 we have to add to this action the Gibbons-Hawking boundary term

\[ S_{GH} = \int d^3x \sqrt{\gamma}(-2K)|_{r=\infty}. \]  
(4.142)

The resulting action diverges at \( r = \infty \) and must be regulated. Hence we must add to the action a local counterterm of the form

\[ S_\Lambda = \int d^3x \sqrt{\gamma} 4 |_{r=\infty}, \]  
(4.143)

where again \( \gamma \) is the induced metric on the boundary at \( r = \infty \) (2.24) and \( K \) is the trace of the extrinsic curvature (2.22). Moreover we must add a counterterm quadratic in the scalar field as in [16] depending upon the boundary condition one chooses for \( \psi \sim \frac{C_1}{r} + \frac{C_2}{r^2} \)

which gives a contribution of the form

\[ S_{ct} = -\int d^3x (\alpha C_1 C_2)|_{r=\infty}. \]  
(4.144)

with \( \alpha = \frac{2}{3} \) if we choose the value \( C_1 \) at the boundary and \( \alpha = -\frac{4}{3} \) if we fix the other value at the boundary. For our purposes at least one of the \( C_i = 0 \), hence this term will not contribute in the computation of the free energy. The result is

\[ S_{GH} = \int d^3x e^{-\frac{r}{2}}r(-4g - r g' + r\chi'g), \]  
(4.145)

\[ S_\Lambda = \int d^3x 4e^{-\frac{r}{2}} \sqrt{g} r^2. \]  
(4.146)

Summing all the terms and taking the asymptotic behavior

\[ e^{-\chi} g \sim r^2 - \frac{\epsilon}{2r}, \quad \text{as} \quad r \to \infty, \]  
(4.147)

\[ \chi \sim 0, \quad \text{as} \quad r \to \infty, \]  
(4.148)

where \( \epsilon \) is again the energy density of the black hole, one finds

\[ S_E = S_{HE} + S_{GH} + S_\Lambda = -\frac{\epsilon}{2} V_2 \beta, \]  
(4.149)
4.3. **THE CONDUCTIVITY**

![Graph showing the difference between the superconducting and the normal Gibbs free energies](image)

Figure 4.4: Difference between the superconducting and the normal Gibbs free energies in dimensionless units against the temperature \( T \) (in units of \( \sqrt{\mu^2 + \delta \mu^2} \)) for \( \delta \mu = 0.1 \), \( \mu = 1.87 \) and \( q = 1 \). Such function is negative for temperatures below the critical temperature \( T_c = 0.2 \), making the superconducting phase to be favorable here; when \( T \to T_c \) the Gibbs free energy goes continuously to zero.

where \( V_2 = \int d^2x \). Using (2.76) we obtain the Gibbs free energy for the hairy black hole

\[
\omega_s = -\frac{\epsilon}{2}. \tag{4.150}
\]

Now we can numerically compute the Gibbs free energy density as a function of the temperature extracting the values of \( \epsilon \) as a function of the temperature \( T \) from our code. Subtracting to it the Gibbs free energy of the normal phase (4.39) and fixing \( \mu, \delta \mu, q \) we get the plot in figure 4.4 as a function of \( T \). We see that the superconducting phase is favored below the critical temperature. When the temperature approaches the critical temperature \( T_c \), the difference between the free energies vanishes continuously. This confirms that the phase transition is second order.

### 4.3 The conductivity

Let us here see an interesting computation of a transport coefficient, namely the optical conductivity, i.e. the electrical conductivity as a function of frequency. Thanks to the rotational invariance of the field theory in the \( x - y \) plane, it is sufficient to consider the conductivity in the \( x \) direction. According to the \( AdS/CFT \) prescription resumed in table 1.1, a conserved current is dual to a Maxwell field in the bulk. Hence to study the spatial component \( J_x \) one must turn on the Maxwell field in the \( x \) direction \( A_x \). Conductivity is a transport phenomenon, hence it requires a real time description. We must switch to
a Minkowskian prescription of the AdS/CFT correspondence. As already mentioned in chapter 1, in this case we must require in-going boundary conditions on the field \( A_x \) at the horizon. For this reason let us take the simple time dependence \( e^{-i\omega t} \) of the field, hence the following ansatz

\[
A = \phi(r) dt + A_x(r) e^{-i\omega t} dx,
\]

where \( \omega \) is the frequency of. In the probe approximation the field \( A_x \) fluctuates above the fixed metric background (4.51-4.52). The equation of motion (4.8) for the \( A_x(r) \) field writes

\[
A''_x + \frac{f'}{f} A'_x + \left( \frac{\omega^2}{g^2} - \frac{2\psi^2}{g} \right) A_x = 0.
\]

(4.152)

The fluctuations (4.152) are solved by imposing boundary conditions. Asymptotically we have

\[
A_x(r,t) = A_{x1}(t) + \frac{A_{x2}(t)}{r} + \ldots \quad \text{as} \quad r \to \infty.
\]

(4.153)

From the AdS/CFT correspondence, the value of this bulk field at the boundary should be the source of the conserved current \( J_x \), namely a gauge field \( A_{x1} = \tilde{A}_x \). The vacuum expectation value of the current is the subleading term \( \langle J_x \rangle = A_{x2} \) giving

\[
A_x(r,t) = \tilde{A}_x + \frac{\langle J_x \rangle}{r}.
\]

(4.154)

From Ohm’s law we get

\[
\sigma(\omega) = J_x / E_x,
\]

(4.155)

where \( E_x \) is the electric field on the boundary. It is related to the gauge field through \( E_x = \tilde{A}_x \). Hence we can write the conductivity as

\[
\sigma(\omega) = -\frac{J_x}{A_{x1}} = -\frac{iA_{x2}}{\omega A_{x1}}.
\]

(4.156)

The numerical results for the real part of the conductivity are given in figure 4.5 for the setup of [16] (\( \delta \mu = 0 \)). Above the critical temperature the conductivity is constant. As we start to lower the temperature below \( T_c \) a gap opens up at low frequency. There is also a delta function at \( \omega = 0 \) at \( T < T_c \), hence an infinite direct conductivity (DC). This cannot be seen from numerical solution because of its infinitesimal width, but by looking for a pole in \( Im(\sigma) \) [16]. A finite conductivity would imply dissipation. Preliminary results for our setup show that the same qualitative features persist at \( \delta \mu \neq 0 \) An interesting feature in the setup in [16] arises: the ratio between the gap frequency and the critical temperature is

\[
\frac{\omega_g}{T_c} \approx 8.
\]

(4.157)

Notice that this has an higher value than that found at weak coupling (3.57). Qualitatively the same happens for real-world high-\( T_c \) superconductors.
Figure 4.5: Real part of the optical conductivity at $\delta\mu = 0$. The straight line is at $T = T_c$. From top to bottom the critical temperature is lowered. There is a delta function at the origin in all cases. Figure taken from [15].
Conclusions and future developments

In this thesis we studied a model of imbalanced superconductors by means of gauge/gravity duality. This approach gives some limitations. As seen from (1.3), two limits are necessary to have a classical gravity dual to a conformal quantum field theory, namely the strong coupling limit and the large $N$ limit. Unconventional superconductors possibly contain within their “superconducting dome” quantum critical points, hence they are inherently strongly coupled. However, the interpretation of the large $N$ limit within the condensed matter realm has not been well understood, yet. Hence, let us stress here that our model is only a toy model, constructed to investigate some universal properties of a certain class of strongly coupled theories.

In this thesis we studied in particular thermodynamic properties of the imbalanced holographic superconductors. We saw first, through a rough approximate analytic and then through a numeric approach, that (in the parameter regime we have considered) a condensate arises below a critical temperature $T_c$, with a shape typical to second order phase transitions, as seen in figure 4.1. Observing the condensates for increasing values of $\delta \mu$, we see from figure 4.2 that the critical temperature decreases. This qualitative feature is also confirmed in figure 4.3 where we plot the ratio $\frac{T_c}{T_{c,0}}$ against $\frac{\delta \mu}{\mu}$. From figure 4.3 we also learn that there is no sign of a Chandrasekhar-Clogston bound for $m^2L^2 = -2$. This is consistent with our formula (4.49) where a Chandrasekhar-Clogston-like bound seems to exist for values of the mass parameter $m^2L^2 > -\frac{3}{2}$. Also LOFF phase is likely to be ruled out, because there is no first order phase transition above which it could develop. Numerically we also confirmed that the superconducting phase is actually energetically favorable by plotting the difference between the normal and the superconducting free energies as a function of the temperature in figure 4.4. The curve approaches continuously the zero line as the temperature goes to the critical one.

Our immediate developments are concerned with the study of the $T - \delta \mu$ phase diagram for different values of $m^2L^2$ to see whether a Chandrasekhar-Clogston bound occurs consistently with the formula (4.49). Following the line of [83], we want also to refine our zero temperature limit by studying different IR asymptotics for the bulk fields. Moreover, to confirm that there is no sign of a LOFF phase at least for $m^2L^2 = -2$, we can study
fluctuations of a scalar field but with an inhomogeneous ansatz, e.g. $\psi(r, x) = \Psi(r)e^{-ixk}$. Such request allows the presence of a spatial component of the Maxwell field $A_x(r, x)$. The peculiar request will be that the dual current should not be sourced. It will be interesting to study the behavior of other thermodynamical quantities such as the heat capacity, and see how thus it changes at $T = T_c$. Figure 4.2 seems to suggest that at zero temperature the values of the condensates, for different values of $\delta\mu$, converge (notice however that these curves have been extrapolated down to zero temperature where numerics is not reliable). It is then interesting to verify whether this property, resembling the weak-coupling property of the gap parameter (i.e. its independence on the chemical potential mismatch at zero temperature) is actually satisfied. A possibility is to study the optical conductivity for a wide range of $\delta\mu$ seeing whether the gap is constant.

Future developments are mainly addressed to a generalization of our setup to $4+1$ dimensions relevant for QCD-like models of color superconductors. Moreover it is challenging to find an explicit stringy embedding of our simple gravity model in a higher dimensional spacetime. This could give us more microscopic details of the underlying quantum field theory, as well as informations on the scalar potential.
Appendix A

Equations of motion in $d + 1$ bulk spacetime dimensions

Since superconductivity is not peculiar to condensed matter systems, holographic tools have been also applied to study color superconductivity in the realm of high energy physics. Such superconductors live in a 4 dimensional spacetime and holographic duals must be searched within 5 dimensional classical gravity theories. We will report here for completeness the equations of motion for the bulk fields of our model in a generic $d + 1$ spacetime for future applications.

The general ansatz for the spacetime metric is

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{r^2}{L^2}d\vec{x}^2 + \frac{dr^2}{g(r)},$$  \hspace{1cm} (A.1)

together with an homogeneous ansatz for the fields

$$\psi = \psi(r), \quad A_a dx^a = \phi(r)dt, \quad B_a dx^a = v(r)dt.$$ \hspace{1cm} (A.2)

The equation of motion for the scalar field reads

$$\psi'' + \psi'\left(\frac{g'}{g} + \frac{(d-1)}{r} - \frac{\chi'}{2}\right) - \frac{1}{2}V'(\psi) + \frac{e^\chi q^2\phi^2\psi}{g^2} = 0,$$ \hspace{1cm} (A.3)

Maxwell’s equations for the $\phi$ field are

$$\phi'' + \phi'\left(\frac{(d-1)}{r} + \frac{\chi'}{2}\right) - 2\frac{q^2\phi\psi^2}{g} = 0,$$ \hspace{1cm} (A.4)

the independent Einsteins’ equations are

$$\frac{1}{2}\psi'^2 + \frac{e^\chi(\phi'^2 + \psi'^2)}{4g} + \frac{(d-1)g'}{2gr} + \frac{1}{2}\frac{(d-1)(d-2)}{r^2} - \frac{d(d-1)}{2gL^2} + \frac{V(\psi)}{2g} + \frac{q^2\psi^2\phi^2e^\chi}{2g^2} = 0,$$ \hspace{1cm} (A.5)

$$\chi' + \frac{2}{(d-1)}r\psi^2 + \frac{2}{(d-1)}r\frac{q^2\phi^2\psi^2e^\chi}{g^2} = 0,$$ \hspace{1cm} (A.6)
finally Maxwell’s equations for the additional field are
\[ v'' + v' \left( \frac{d - 1}{r} + \frac{\chi'}{2} \right) = 0. \tag{A.7} \]

Taking \( z = \frac{r u}{r} \), rescaling the scalar field \( \psi = z \tilde{\psi} \), renaming \( \tilde{\psi} \rightarrow \psi \) and taking the ansatz for the metric
\[ ds^2 = -e^{-\chi(z)} \left( \frac{r_H^2}{z^2} + h(z) \right) dt^2 + \frac{r_H^2}{z^2} dx^2 + \frac{r_H^2}{z^4 \left( \frac{r_H^2}{z^2} + h(z) \right)} dz^2, \tag{A.8} \]
the scalar equation writes
\[ \psi'' + \psi' \left( \frac{5 - d}{2} - \frac{2r_H^2}{z(r_H^2 + z^2 h)} - \frac{\chi' + h' z^2}{2(r_H^2 + z^2 h)} \right) + \psi \left( 3 - d \right) \frac{2r_H^2}{z^2 (r_H^2 + z^2 h)} + \frac{e^\chi q^2 r_H^2 \phi^2}{(r_H^2 + z^2 h)^2} - \frac{\chi' + h' z}{2z (r_H^2 + z^2 h)} - \frac{r_H^2 V'(\psi)}{2z^2 (r_H^2 + z^2 h)} = 0. \tag{A.9} \]
The equation for the \( U_A(1) \)-Maxwell field writes
\[ \frac{1}{r_H^2} \phi'' - \frac{(d - 3)}{r_H^2 z} \phi' + \frac{1}{2r_H^2} \phi' \chi' - \frac{2q^2 \phi \psi^2}{(r_H^2 + z^2 h)} = 0. \tag{A.10} \]
The equation for the \( U_B(1) \)-Maxwell field
\[ \frac{1}{r_H^2} \phi'' - \frac{(d - 3)}{r_H^2 z} \phi' + \frac{1}{2r_H^2} \phi' \chi' = 0. \tag{A.11} \]
The Einstein’s equations are given by
\[ \frac{1}{2} \psi'^2 + \frac{\psi \psi'}{z} + \frac{\psi^2}{2z^2} + \frac{e^\chi (\phi^2 + \psi^2)}{4(r_H^2 + z^2 h)} + \frac{V(\psi) r_H^2}{2z^2 (r_H^2 + z^2 h)} - \frac{(d - 1)}{z} \frac{h'}{2 (r_H^2 + z^2 h)} + \frac{(d - 1)(d - 2)}{2z^4} - \frac{(d - 2)(d - 1) r_H^2}{2z^4 (r_H^2 + z^2 h)} + \frac{r_H^2 e^\chi q^2 \phi^2 \psi^2}{2 (r_H^2 + z^2 h)^2} = 0, \tag{A.12} \]
and
\[ \chi' - \frac{2}{(d - 1)} \psi^2 z - \frac{2}{(d - 1)} \frac{z^3 e^\chi \phi^2 q^2 r_H^2 \psi^2}{(r_H^2 + z^2 h)^2} - \frac{4}{(d - 1)} z^2 \psi \psi' - \frac{2}{(d - 1)} z^3 \psi'^2 = 0. \tag{A.13} \]
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APPENDIX A. EQUATIONS OF MOTION IN D+1 BULK SPACETIME DIMENSIONS
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