Asymmetric Dark Matter in Leptogenesis models

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L’essentiel est invisible pour les yeux.

Antoine de Saint Exupéry, *Le Petit Prince.*
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1.3 Weak/Strong</td>
<td>75</td>
</tr>
<tr>
<td>6.1.4 Weak/Weak</td>
<td>77</td>
</tr>
<tr>
<td>6.2 Weak $\beta$ dependence</td>
<td>77</td>
</tr>
<tr>
<td>6.3 Negative asymmetry</td>
<td>80</td>
</tr>
<tr>
<td>6.4 Conclusions and perspectives</td>
<td>83</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>87</td>
</tr>
<tr>
<td>Bibliography</td>
<td>90</td>
</tr>
</tbody>
</table>
Abstract

Experimental evidence has pointed out important cosmological problems that are still open and far from a solution. Here we shall take into consideration two of them: the presence of Dark Matter in the Universe and the asymmetry between baryons and anti-baryons, trying to provide a possible solution from the Quantum Field Theory point of view. We shall briefly describe them and give a brief review of a class of models that attempt to give an explanation. Among them, a particular kind of models, called two-sector models will be studied in depth. These models provide a joint explanation to both problems, tracing them back to a common origin due to the same mechanism already considered in standard leptogenesis models. Asymmetry is generated between leptons and anti-leptons, as in standard leptogenesis, but also between Dark Matter particles and antiparticles, whence the name Asymmetric Dark Matter models. In addition to that, the problem of massive neutrinos receives an answer too, thanks to the see-saw mechanism which is automatically embedded in leptogenesis mechanism. A detailed study of a minimal model will be carried out in order to acquire a full understanding of the processes that give rise to the asymmetries. The model allows to make predictions about the mass of the Dark Matter candidate. Different regimes will be considered, by varying the free parameters of the model and Dark Matter mass spectra will be obtained. Remarks about the obtained values and possible ways of Dark Matter detectability will be made in the Conclusions, though leaving these issues open to future works.
Chapter 1

Introduction

1.1 Open problems

The so-called Standard Cosmological Model (SCM) is surely one of the greatest result of modern Physics. It is able to explain the evolution of the Universe from its early moments up to today’s time and it can number many experimental confirmations. However, if the pure cosmological structure can be considered well established, the model lacks of side theories that complete the picture and, above all, can explain some vital problems still unsolved.

It is well known that the SCM rest on the assumption of a homogeneous and isotropic Universe at early stage, which is described by Friedman-Robertson-Walker’s metric (e.g. see [1,2]). From the metric, through Friedman’s equations, the evolution of the expansion parameter can be derived. This depends on the densities of the species present in the Universe, normalized to today’s value of the critical density

$$\rho_0 \equiv \frac{3H_0^2}{8\pi G}$$

(1.1)

where $H_0$ is today’s Hubble’s parameter. Thence, it is customary to define the just mentioned ratio as

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}$$

(1.2)

where the index $i$ stands for the considered species. $\rho_i$ is the energy density of the species, so $\Omega_i$ is the density of the species $i$ over the critical density $\rho_c$.

According to its dynamical behaviour, the content of the Universe can be divided into radiation ($\Omega_r$), matter ($\Omega_m$) and cosmological constant ($\Omega_\Lambda$). Different values of the $\Omega$’s result in different cosmological evolutions, therefore it is necessary to have precise experimental measurement for these parameters. To this aim, it is possible to cross data from various experiment projects based on different phenomena, such as Cosmic Microwave Background (CMB) measurements, weak lensing, baryon acoustic oscillation, Big
Bang Nucleosynthesis (BBN) and Type-Ia Supernovae. Taking into consideration the experimental data obtained, more insight is gained into the composition of our Universe. Referring to the recent observation carried out by WMAP-7 \[3\], we know that the relevant parameters for the Universe today are

\[
\begin{align*}
\Omega_{\text{TOT}} & = 1.099^{+0.100}_{-0.085}, \\
\Omega_m & = (0.258 \pm 0.004), \\
\Omega_r & \sim 10^{-5}, \\
\Omega_{\Lambda} & = (0.742 \pm 0.027).
\end{align*}
\]

Eq. (1.3) tells us that the geometry of our Universe is highly compatible with a flat one, while Eq. (1.6) indicates that the cosmological constant dominates over matter and radiation. This striking result has been confirmed by the study of high-redshift type I-a supernovae \[4, 5\] and leads to the conclusion that the Universe is accelerating (see Fig. 1.1). The cosmological constant

![Figure 1.1: Hubble plot of low and high redshift supernovae, compared to theoretical cosmological models \[5\]. On the y-axis we have the bolometric magnitude, while on the x-axis the redshift.](image)

content which in the SCM causes the accelerated expansion of the Universe is often called *Dark Energy* (DE). As the name suggest, the nature of DE is unknown. Many models try to give an explanation to this puzzle, invoking, for instance, General Relativity modification, scalar field theories \[6\] or QFT vacuum energy \[7\].

Also the matter contribution, Eq. (1.4), requires more reflection. Study of astronomical dynamics and, in particular, of the radial velocity of rotating galaxies \[8\] showed that a certain amount of mass is missing in the Universe
CHAPTER 1. INTRODUCTION

at the galactic scale. That is, the standardly interacting matter (also referred to as luminous matter) is not enough to justify galactic dynamics (see Fig. 1.2). The curves of galactic rotational velocity are incompatible with the theoretical prediction made on the basis on the only luminous matter content of the galaxy. This anomaly was first discovery by Vera Rubin [8]. The missing matter cannot be directly detected, because it should interact very weakly with ordinary matter, and therefore it is called Dark Matter (DM). Since it is able to form structures, it must be also non-relativistic at early times, and so it is called Cold Dark Matter (CDM). To similar conclusion we

Figure 1.2: Experimental and theoretical rotational velocity of galaxies. 1.2(a) is taken from [8] and 1.2(b) from [9]

are led by BBN which gives a value for the baryon density of the Universe by studying the formation of the lightest nuclei [10–12]

\[ \Omega_b \sim 0.045. \]  \hspace{1cm} (1.7)

Also CMB anisotropies raise a question about matter. Indeed, the size of these fluctuations is too small to allow the origin of structures as those we know. Ordinary matter can effectively aggregate only after recombination era, because of electrostatic forces, but density fluctuation at CMB release are too small for this purpose. Therefore, structure formation needs an electrically neutral form of matter that could start to segregate before recombination. From WMAP7 data [3] we have

\[ \Omega_b = 0.0441 \pm 0.003, \]  \hspace{1cm} (1.8)

\[ \Omega_c = 0.214 \pm 0.027, \]  \hspace{1cm} (1.9)
where $c$ index stands for CDM. Eq. (1.7) is in very good agreement with (1.8) and together with (1.9) show that the major part of matter content is represented by dark matter. The current results for Universe contents are well summarized in figure 1.3. The need for an unknown and undetectable kind of matter is highly stimulating for theorists that have been exploring different ways of explanation.

Besides modified gravity theory, particle physics is the most attempted path. The most common scenario refers to the DM feature of being a Weakly Interacting Massive Particle (WIMP). The reactions that keep the WIMP in equilibrium with the other species are very weak and as the Universe cools down they become inefficient, causing the exit from equilibrium status of the particle. Hence, the particle remains decoupled from the thermal bath and evolves independently. An obvious requirement is that the WIMP be stable, otherwise, after decoupling, the number of DM particles is totally depleted by decays. Therefore, the WIMP is said to “freeze-out” and it survives till today. Even if the paradigm is clear, it remains unknown what the WIMP is. Many models have been brought forward and a great part of them is represented by supersymmetric (SUSY) models. The models still wait for experimental confirmation or falsification which is carried out mainly via direct detection. However, supersymmetry on its own still waits for a direct test via detection of signals at colliders.

Dark matter and dark energy are not the only problems left open by of the standard cosmological model. According to the standard cosmological model and using particle Standard Model (SM) results, the Universe has its origin from a homogeneous, isotropic plasma made of all the elementary particles, still massless because of electroweak symmetry restoration at high temperature. The early Universe SM content is therefore made up of the three generations of quarks $q_i \in \{(u,d),(c,s),(t,b)\}$, the three generations of leptons $l_i \in \{(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau)\}$ and the gauge bosons $\{g,Z,W^\pm,\gamma\}$.
The original number density of particles and antiparticles should be the same, since there is not either cosmological or QFT trivial reason for this situation not to occur. Anyway, it is self evident that the number of particles is far greater than that of antiparticles, that is our Universe is evidently made of matter by the major part. Quantitatively, this problem can be addressed neglecting the small contribution of leptons and considering the difference between the number density of baryons and antibaryons. This must be zero in case of a symmetry, otherwise it is said that there is baryon asymmetry. It is customary to consider the ratio of the difference between baryon and antibaryon densities, \( n_B \) and \( n_{\bar{B}} \), to the number density of photon, so defining the \( \eta \) parameter as

\[
\eta \equiv \frac{n_B - n_{\bar{B}}}{n_\gamma}. \tag{1.10}
\]

This parameter plays a relevant role in determining the final abundances of the nuclei in the framework of BBN. Indeed, the relic abundances of primordial elements depends on \( \eta \) and so a precise measurement of these quantity can give a numerical value to this asymmetry. Estimations of \( \eta \) can be done by finding the value corresponding to the measured element abundance and comparing it to the one predicted by BBN theory \[10\]–\[12\]. The agreement very good and today’s value for \( \eta \) according to BBN is

\[
\eta = (6.21 \pm 0.16) \cdot 10^{-10}, \tag{1.11}
\]

which, evidently, is not compatible with zero. Therefore we must conclude that our Universe shows a marked baryon asymmetry, whose origin is not yet understood.

This asymmetry can be originated by the initial conditions of Universe evolution. In order to obtain the observed value, we must require roughly 6 million and one quark every 6 million anti-quark. Clearly this fine-tuned initial condition is not fully satisfactory and so dynamical mechanisms able to produce this asymmetry have been searched for. The general scheme for the generation of baryon asymmetry is called baryogenesis. There is a great number of baryogenesis models, exploiting different theories and processes. All of them must satisfy a common set of conditions first pointed out by Sakharov \[14\]. We shall deal with this subject more in detail in Section 2.2, considering another dynamic mechanism which traces back baryon asymmetry to lepton asymmetry and that is then called leptogenesis. Even if many models exist both for baryogenesis and for leptogenesis, the problem is still far from a solution. Despite the effort made to make every model predictive, it remains quite difficult to find effective tests and experiments able to choose among the models.

In addition to pure cosmological problems, experimental observations have pointed out a leak in the Standard Model proposed by Weinberg, Glashow and Salam \[15\]–\[17\], proving that neutrinos oscillate. This feature is
an unquestionable evidence that neutrinos have a mass, in spite of Weinberg-Glashow-Salam model which considers neutrinos as massless particles. As a consequence, the model must be expanded in order to include a term which can give mass to neutrinos without spoiling any symmetry of the model. Adding new terms to pure SM opens a vast field commonly called physics beyond SM, in which many theories can be considered and included. Therefore, different possibilities are available and among them the most interesting try to give a unitary explanation to all the problems discussed. In a single framework baryon asymmetry, dark matter and neutrino masses problems are addressed and (maybe) solved.

In chapter 3, we will give a brief review of some of these models, then we will study in detail a particular class of models.
Chapter 2

Theoretical framework

In this chapter we will introduce the theoretical background on which this work is based. In section 2.1 we will describe how to give mass to neutrinos, in particular we will deepen one interesting possibility: the so-called see-saw mechanism. In section 2.2 we will discuss the main characteristic of a process leading to an asymmetry in baryon or lepton number. Then we will turn our attention to leptogenesis process, just pointing out the main features and leaving a review of leptogenesis models to chapter 3. Eventually, in section 2.3 we will briefly consider the lepton/baryon number processes in SM that are essential to leptogenesis.

2.1 Neutrino oscillations and the see-saw mechanism

Neutrino oscillation was first proposed by Pontecorvo [18] as a quantum-mechanical phenomenon due to the interference of different massive neutrinos. Since then many experiments have been set up, involving solar (Homestake [19,20], Kamiokande [21], GALLEX/GNO [22], etc.), atmospheric (Kamiokande [23], IMB [24], etc.), reactor (KamLAND [25,26], CHOOZ [27], Daya Bay [28,29], etc.) and accelerator (OPERA) neutrinos. A first approach makes use of standard quantum mechanics and plane wave approximation [30]. It considers the basis of the interaction eigenstates (also called flavour eigenstates) $|\nu_\alpha\rangle$ and the basis of hamiltonian (mass) eigenstates $|\nu_i\rangle$, normalised so that

$$\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha \beta}, \quad \langle \nu_i | \nu_j \rangle = \delta_{ij}.$$  

A flavour eigenstate can be written in the mass basis by means of a unitary matrix $U$ called mixing matrix, and vice versa

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle, \quad |\nu_i\rangle = \sum_\alpha U^*_{\alpha i} |\nu_\alpha\rangle.$$  

11
Since $|\nu_i\rangle$ are eigenstates of the Hamiltonian, their time evolution operator allows to write

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} e^{-iE_i t} |\nu_i\rangle,$$

and passing to the flavour basis we get

$$|\nu_\alpha(t)\rangle = \sum_\beta \left( \sum_i U_{\alpha i} e^{-iE_i t} U^*_{\beta i} \right) |\nu_\beta\rangle.$$

The projection of $|\nu_\alpha(t)\rangle$ onto the flavour eigenstate $|\nu_\beta\rangle$ gives the amplitude of the transition probability from flavour $\alpha$ to flavour $\beta$

$$A_{\nu_\alpha \rightarrow \nu_\beta}(t) \equiv \langle \nu_\beta | \nu_\alpha \rangle = \sum_i U^*_{\beta i} U_{\alpha i} e^{-iE_i t}.$$

The transition probability is therefore

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = \sum_{ij} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-i(E_i - E_j)t}.$$

Eq. (2.1) can be approximated considering ultra-relativistic neutrinos, in which case the dispersion relation can be substituted with

$$E_i \simeq E + \frac{m^2_i}{2E},$$

where $E$ is the neutrino energy neglecting the mass contribution. Hence we have

$$E_i - E_j \simeq \frac{\Delta m^2_{ij}}{2E},$$

where $\Delta m^2_{ij} = m^2_i - m^2_j$ and which can be used in (2.1) in order to have

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = \sum_{ij} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \exp \left\{ -i \frac{\Delta m^2_{ij}}{2E} t \right\}.$$

We can conclude that there is a non-zero probability of transition from an interacting eigenstate to another, depending on the mixing matrix $U$ and on some experimental parameters such as the neutrino energy $E$ and the time of flight $t$, which is commonly approximated by the source-detector distance. The experiments have obtained two values for the square mass differences [31]

$$|\Delta m^2_{21}| = (7.50 \pm 0.20) \cdot 10^{-5} \text{ eV}^2,$$

$$|\Delta m^2_{32}| = 0.0023^{+0.00012}_{-0.00008} \text{ eV}^2.$$
Since (2.3) was obtained in experiments involving solar neutrinos, it is referred to as solar, while (2.4) as atmospheric, because it was measured while studying oscillation of atmospheric neutrinos.

As seen, in order to have oscillations it is necessary to have mass eigenstates with non-zero eigenvalue. The part of the Standard Model SU(2)_L ⊗ U(1)_Y lagrangian which contain lepton Yukawa coupling, is made of the charged leptons \( l_\alpha \), the neutrinos \( \nu_\alpha \), where the index \( \alpha \) counts the three flavour generations, and the Higgs scalars. We consider both left handed \( l_{L\alpha} \) and right handed \( l_{R\alpha} \) fields for the charged leptons, while we consider only left handed neutrinos \( \nu_{L\alpha} \). These fields are organized into representations of the gauge group as follows.

<table>
<thead>
<tr>
<th>Field</th>
<th>( I_{SU(2)} )</th>
<th>( Y_{U(1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left handed leptons</td>
<td>( l_{L\alpha} = (\nu_{L\alpha}, l_{L\alpha}) )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>Right handed leptons</td>
<td>( l_{R\alpha} )</td>
<td>0</td>
</tr>
<tr>
<td>Higgs scalars</td>
<td>( \Phi = (\phi^+, \phi^0) )</td>
<td>( \frac{1}{2} )</td>
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where \( I \) is the weak isospin and \( Y \) the hypercharge.

The original lagrangian gives mass only to charged leptons via Electroweak Symmetry Breaking (EWSB), while neutrinos remain massless. It is possible to give mass to them introducing a term similar to the one that gives mass to the up-like quarks. To do so it is necessary to consider the right-handed components \( \nu_{R\alpha} \) of the neutrino fields. They are singlets of \( SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \) and have \( Y = 0 \), so they do not interact but gravitationally. For this reason they are called sterile. The lepton mass sector of the lagrangian is then extended

\[
\mathcal{L}_{-H} = - [Y'_\nu]^{\alpha\beta} \bar{l}_\alpha \Phi^{\dagger}_\beta - [Y'_\nu]^{\alpha\beta} \bar{\nu}_{R\alpha} \tilde{\Phi} \bar{L}_\beta + h.c.,
\]

where \( \tilde{\Phi} \) is derived from Higgs doublet as

\[
\tilde{\Phi} = -i\Phi^i \sigma_2 = (\phi^0, -\phi^+).
\]

After EWSB Higgs field is shifted as usual by its v.e.v. \( v \) and we are left with the same charged lepton mass term, a new interaction term

\[
\mathcal{L}_{\nu}^{\text{int}} = [Y'_\nu]^{\alpha\beta} \bar{\nu}_{R\alpha} \nu'_{L\beta} \ h + h.c.,
\]

where \( h \) is the Higgs scalar, and a new mass term for neutrinos

\[
\mathcal{L}_{\nu}^{\text{mass}} = - \frac{v}{\sqrt{2}} [Y'_\nu]^{\alpha\beta} \bar{\nu}_{R\alpha} \nu'_{L\beta} + h.c.,
\]

appears in the lagrangian.
The matrix of neutrino Yukawa couplings $Y^\nu$ can be diagonalized through a bi-unitary transformation $V_L^\nu Y^\nu V_R^\nu = Y^{\nu}_{\text{diag}}$, where $Y^{\nu}_{ij} = y^{\nu}_{(i)} \delta_{ij}$ and the fields can be rotated by $V_L^\nu$ and $V_R^\nu$, obtaining

$$\nu_L^i = \left[ V_L^{\nu L} \right]_{i\alpha} \nu_L^{\nu \alpha}, \quad \nu_R^i = \left[ V_R^{\nu R} \right]_{i\alpha} \nu_R^{\nu \alpha},$$

where $\nu_i$ will be the mass eigenstates. This can be substituted into Eq. (2.7) so to have a symmetric mass term

$$\mathcal{L}_{\nu \text{mass}} = -\frac{\nu}{\sqrt{2}} y^{\nu}_{(i)} \delta_{ij} \bar{\nu}_i \nu_j.$$  

(2.8)

Also the weak interaction currents can be changed by field rotation. The charged weak current reads

$$j^\mu_W = \delta^{\alpha\beta} \bar{\nu}_L^{\nu \alpha} \gamma^\mu l_L^{\nu \beta} = \bar{\nu}_L^{\nu \alpha} \left[ V_L^{\nu L} \right]^{i\alpha}_{\nu} \gamma^\mu \left[ V_L^{\nu L} \right]_{i\alpha}^{L} l^L_j$$

$$= \bar{\nu}_L^{\nu \alpha} U^{L\nu}_{ij} \gamma^\mu l^L_j,$$

where $V_L^i$ is the charged lepton analogous of $V_L^\nu$ and $U^{L\nu} = V_L^\nu V_L^{\nu L}$ is the mixing matrix in lepton sector, analogous to the CKM mixing matrix of quarks \cite{32} \cite{33}. The weak neutral current remains unchanged. The introduction of a mass term for the neutrinos brings about the appearance of a mixing matrix $U$ that, as mentioned before, causes neutrinos oscillations.

The method just described results in a Dirac mass term for neutrinos but these particles can be considered also as Majorana fermions \cite{34}. The feature of such particles is to coincide with their charge-conjugated field

$$\psi = \psi^{C} \equiv C \bar{\psi}.$$  

Given a left-handed field, its charge-conjugated is right-handed and transform as the LH field under Lorentz transformation. Hence, it is possible to write an invariant mass term using only a left handed field,

$$\mathcal{L}_{M \text{mass}} = -\frac{1}{2} m \bar{\nu}_L \nu_L + h.c.$$  

(2.9)

where the factor 1/2 is needed to avoid double counting due to the dependence of $\nu_L^C$ and $\bar{\nu}_L$. Eq. (2.9) can be easily rewritten as

$$\mathcal{L}_{M \text{mass}} = \frac{1}{2} m \nu_L^C \bar{\nu}_L + h.c.$$  

(2.10)

Such a term is called Majorana mass term and causes lepton number violation, because the complete lagrangian is no more invariant under global $U(1)$ transformations. A term like (2.10), (2.9) must origin from some high-energy theories, because it cannot be generated by EWSB. Indeed, since $\nu_L$
has $I_3 = 1/2$ and $Y = -1$, the mass term $\nu_L^i C^\dagger \nu_L$ is an isospin triplet with $Y = -2$ and SM does not contain any triplet with $Y = 2$ to directly form an invariant term. For this reason, the only way by which a Majorana mass term can arise via EWSB is being generated by a dimension-5 operator

$$\mathcal{L}_5 = \frac{g}{\mathcal{M}} (L' \sigma_2 \Phi) C^\dagger (\Phi' \sigma_2 L) + h.c.,$$

where, however, $g$ is a dimensionless constant and $\mathcal{M}$ is a mass scale, usually deriving from GUT scale. Such a term is clearly non-renormalizable in SM, but it can be allowed if SM is considered as the low-energy effective theory of a high-energy unified theory.

Eventually, it is possible to consider a lagrangian with both Dirac and Majorana mass terms, if both left-handed and right-handed neutrinos exist. It is not necessary that the number of RH, i.e. sterile, neutrinos is equal to that of the LH, i.e. active, ones. We will consider three active neutrinos $\nu$ and $n_s$ sterile neutrinos $N_i$. The lagrangian can contain a Majorana mass term for the RH neutrinos

$$\mathcal{L}_M = \frac{1}{2} N_t^i C^\dagger [M_R]^{ij} N_j + h.c.,$$

where the mass matrix is assumed to be already diagonal $M_R = \text{diag}(M_{N_1}, \ldots, M_{n_s})$, and a Dirac mass term

$$\mathcal{L}_D = -\frac{v [Y'_\nu]^i_\alpha}{\sqrt{2}} \tilde{N}_i \nu_L^\dagger + h.c. = -\tilde{N}_i [M_D]^{i_\alpha} \nu_L^\dagger + h.c. \quad (2.11)$$

originating from the second term in (2.5), where we define

$$[M_D]^{i_\alpha} = \frac{v}{\sqrt{2}} [Y'_\nu]^i_\alpha. \quad (2.12)$$

This way, after EWSB we have

$$\mathcal{L}_{D+M} = \frac{1}{2} N_t^i C^\dagger [M_R]^{ij} N_j - \tilde{N}_i [M_D]^{i_\alpha} \nu_L^\dagger + h.c. \quad (2.13)$$

while before the breaking the “starting point” lagrangian reads

$$\mathcal{L}_{ss} = \frac{1}{2} N_t^i C^\dagger [M_R]^{ij} N_j - [Y'_\nu]^i_\alpha \tilde{N}_i \tilde{\Phi} L_\alpha + h.c. \quad (2.14)$$

or, exploiting matrix notation

$$\mathcal{L}_{ss} = \frac{1}{2} \mathbf{N}^\dagger C^\dagger M_R \mathbf{N} - \mathbf{\tilde{N}} Y'_\nu \tilde{\Phi} \mathbf{L}' + h.c., \quad (2.15)$$

where

$$\mathbf{N} = \begin{pmatrix} N_1 \\ \vdots \\ N_{n_s} \end{pmatrix}, \quad \text{and} \quad \mathbf{L}' = \begin{pmatrix} L'_1 \\ L'_2 \\ L'_3 \end{pmatrix}.$$
CHAPTER 2. THEORETICAL FRAMEWORK

A Majorana mass term for LH neutrinos is not allowed by the symmetries of the standard model. The two mass matrices are complex: $M_R$ is $n_s \times n_s$ symmetric, while $M_D$ is $n_s \times 3$ rectangular. (2.13) can be re-written using a vectorial notation by defining the $n = 3 + n_s$-dimensional vector

$$\mathcal{N}'_L \equiv \begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \\ N^C_1 \\ \vdots \\ N^C_{n_s} \end{pmatrix},$$

and the $n \times n$ mass matrix

$$M \equiv \begin{pmatrix} 0 & M_D' \\ M_D & M_R \end{pmatrix}.$$

The lagrangian now reads

$$\mathcal{L}_{D+M} = \frac{1}{2} \mathcal{N}'_L^t C^\dagger M \mathcal{N}'_L + h.c. \quad (2.16)$$

$M$ can be diagonalized through a bi-unitary transformation $(V_L^e)^t M V_L^e = D_m$ with $[D_m]_{ij} = m_{(i)} \delta_{ij}$ and it can be defined a new field vector

$$\mathcal{N}'_L = V_L^e \mathbf{n}_L, \quad \text{where} \quad \mathbf{n}_L = \begin{pmatrix} \nu_{L1} \\ \vdots \\ \nu_{Ln} \end{pmatrix}. \quad (2.17)$$

This way, Eq. (2.16) can be transformed into

$$\mathcal{L}_{D+M} = \frac{1}{2} m_{(p)} \delta^{pq} \nu_{Lp}^t C^\dagger \nu_{Lq}, \quad (2.18)$$

where $p, q = 1, \ldots, N$.

This way we are left with $N$ Majorana neutrinos $\nu_{Lp}$, and by defining $\nu_p = \nu_{Lp} + \nu_{Lp}^C$, we obtain

$$\mathcal{L}_{D+M} = \frac{1}{2} m_{(p)} \delta^{pq} \nu_p^t C^\dagger \nu_q. \quad (2.19)$$

The massive neutrinos we get are then Majorana fermions.

The diagonalization procedure deserves more attention, in particular in case that the elements of $M_R$ are much larger than the elements of $M_D$. If all the eigenvalues of $M_R$ are larger than all the elements of $M_D$, we can neglect the mixing with the heavy neutrinos $N_i$, thus considering an effective $3 \times 3$ mass matrix for the LH neutrinos. This procedure can be rigorously proven.
by integrating out $N_i$ from (2.14) or (2.15), once it has been established that $M_R$ eigenvalues are much larger than the others. In the static limit, the equation of motion for $N$ is

$$\frac{\partial \mathcal{L}_{ss}}{\partial N} = N^t C^\dagger M_R - \bar{\Phi} \Phi^\dagger Y_{\nu}^\dagger \sim 0,$$

whence we have, solving for $N$,

$$N^t \simeq \bar{\Phi} \Phi^\dagger Y_{\nu}^\dagger M_R^{-1} C,$$

(2.20)

$$N \simeq -C M_R^{-1} Y_{\nu}^\ast \bar{\Phi}^\ast (\bar{\Phi}^\ast)^\dagger.$$

(2.21)

Substituting (2.20) and (2.21) into (2.15), we get

$$\mathcal{L}_{ss} \simeq -\frac{1}{2} N^t C^\dagger M_R N + h.c.$$

$$= \frac{1}{2} \left( \bar{\Phi} \Phi^\dagger \right) C \left( Y_{\nu}^t M_R^{-1} Y_{\nu}^\ast \right) \left( \bar{\Phi}^\ast \bar{\Phi}^\dagger \right).$$

After EWSB, it becomes

$$\mathcal{L}_{ss} = \frac{1}{2} v^2 \bar{\nu}_L^t C \left( Y_{\nu}^t M_R^{-1} Y_{\nu}^\ast \right) (\bar{\nu}_L^t)^\dagger + h.c.$$

$$= \frac{1}{2} v^2 \left( \bar{\nu}_L^t \right)^\dagger \left( Y_{\nu}^t M_R^{-1} Y_{\nu}^\ast \right) C^\dagger \left( \bar{\nu}_L^t \right) + h.c.$$

$$= \frac{1}{2} v^2 \left( \bar{\nu}_L^t C^\dagger \left( Y_{\nu}^t M_R^{-1} Y_{\nu}^\ast \right) \nu^t_L + h.c. \right).$$

(2.22)

(2.22) shows that the active LH neutrinos can be considered Majorana particles with an effective mass given by

$$M_{\text{light}} \equiv \frac{v^2}{2} Y_{\nu}^t M_R^{-1} Y_{\nu}^\ast = M_D^t M_R^{-1} M_D,$$

(2.23)

where we recall def. (2.12). $M_{\text{light}}$ must now be diagonalized in order to find the mass eigenvalues. To this end, in this approximation we can use a square mixing matrix $U$

$$D_m = U^t M_{\text{light}} U = U^t M_D^t (M_R^{-1}) M_D U.$$

The mixing matrix is a $3 \times 3$ unitary matrix, thus requiring $N^2 = 3^2$ parameters. Since it appears in weak currents, the charged lepton fields can be re-phased so that 3 phases can be eliminated. The neutrino fields cannot be re-phased because of the Majorana mass term which is not invariant. $U$ is left with $N^2 - N = N(N - 1) = 6$ parameters, which can be divided into 3 mixing angles and 3 physical phases. The neutrino mixing matrix can be divided into a unitary square matrix $U_D$ with one phase, similar to CKM,
and a diagonal matrix $D_M$ that carries two more phases called Majorana phases

$$U = U_D P_M.$$ 

The Dirac matrix can be parametrized exactly as CKM matrix

$$U_D = \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{13}} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{13}} & c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_{13}} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_{13}} & c_{23} c_{13}
\end{pmatrix},$$  

(2.24)

where $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$ and $\theta_{ij}$ are the mixing angles. The Majorana matrix reads

$$P_M = \text{diag}(1, e^{i \alpha}, e^{i \beta}).$$  

(2.25)

Values for the parameters in $U_D$ can be found in [31,35,36], while Majorana phases are unknown.

$U$ can be related to the Yukawa couplings via the so-called Casas-Ibarra parametrization, which is very useful for calculations [37]

$$Y'_\nu = \frac{\sqrt{2}}{v} M_{\text{light}}^{\frac{1}{2}} R D_R^{\frac{1}{2}} U^\dagger,$$  

(2.26)

where $R$ is an orthogonal matrix.

What has been described so far is the so called see-saw mechanism [38,39]. It can give a clear explanation of the smallness of neutrino masses with respect to the other fermion masses, thanks to the masses of the sterile neutrinos which are supposed to be heavy. This is possible because Dirac masses $M_D$ are constrained by EWSB not to be much larger than the electroweak scale $v$, whereas $M_R$ can assume any value since it belongs to physics beyond SM. This term can be generated by some high-energy symmetry breaking, at great unification scale, for instance. This mechanism is very much celebrated because of the natural process by which light neutrinos are produced. Indeed, simpler methods such as pure Dirac masses require small Higgs-neutrino Yukawa couplings that are not explained but by fine-tuning boundary conditions.

To be more specific, this model is often referred to as type-I see-saw. At least other two kind of see-saw mechanism have been proposed so far. While type-I make use of singlet fermions ($N_i$), type-II considers $SU(2)$-triplet scalars [40,42] and type-III $SU(2)$-triplet fermions [43,45]. These models will not be described here. As we will see in the following section, type-I see-saw plays an essential role in leptogenesis.

## 2.2 Baryogenesis and Leptogenesis

As mentioned in the introduction, the baryonic asymmetry that is typical of our Universe can be generated through a dynamical process called baryogenesis. The initial condition is simply a baryon symmetric Universe, but the
CHAPTER 2. THEORETICAL FRAMEWORK

process, whatever it may be, must have precise characteristics first pointed out by Sakharov [14]: 

i) interactions violating baryon number \( B \),

ii) interactions violating \( C \) and \( CP \),

iii) interactions that take place out of equilibrium.

We shall briefly analyse these conditions.

i) At least one interaction must violate baryon number, since starting from a baryon symmetric universe we must reach today’s asymmetry values. It is self evident that \( B \) must not be globally conserved.

\[ \Gamma(a \rightarrow b + c) \neq \Gamma(\bar{a} \rightarrow \bar{b} + \bar{c}). \]

However, even though \( C \) is violated but \( CP \) is conserved, considering chiral processes, we have \( \Gamma(a \rightarrow b_L + c_L) = \Gamma(\bar{a} \rightarrow b_R + c_R) \) and so the global process is still symmetric in baryon number

\[ \Gamma(a \rightarrow b_L + c_L) + \Gamma(a \rightarrow b_R + c_R) = \Gamma(\bar{a} \rightarrow \bar{b}_L + \bar{c}_L) + \Gamma(\bar{a} \rightarrow \bar{b}_R + \bar{c}_R). \]

Therefore, in order to have a different number of baryon and antibaryon \( CP \) must be violated as well.

iii) It is still necessary to consider thermodynamic aspects. In order to establish an asymmetry, processes must take place outside thermal equilibrium. That is, a reaction and its \( T \)-conjugated must differ strongly due to statistical suppression of one of the two.

In principle all these conditions can be met in the SM, nonetheless it cannot generate large enough baryon asymmetry. Therefore, baryogenesis has required new physics in order to be explained. Some of the possible baryogenesis scenarios are GUT, electroweak or Affleck-Dine baryogenesis. These models will not be considered in this work, they can be found, for instance, in [46–49].

The first GUT baryogenesis models presented some problems hard to explain, such as unacceptable decay rates for the proton. A new path was then explored first by Fukugita and Yanagida in [50], where a new mechanism, called \textit{leptogenesis}, was proposed. The starting point of leptogenesis is type-I see-saw lagrangian (2.14), of which only Majorana and interaction with Higgs scalar \( h \) terms are considered:

\[ \mathcal{L} \supset \frac{1}{2} N^i_C \tilde{C}^\dagger [M_R]^{ij} N_j - [Y^\nu_L]^{i\alpha} N_i \nu^\alpha_C h + h.c., \]

This lagrangian provides vertices for the decay of the RH neutrinos \( N_i \) into Higgs scalar and active LH neutrinos. It must be kept in mind that RH
neutrinos are still sterile with respect to color and EW interactions. Since sterile neutrinos are Majorana fermions, \( N_i \) can decay both to \( \nu_\alpha h \) and to \( \bar{\nu}_\alpha h \), so no net asymmetry is produced if the decay rate into leptons equals the decay rate into anti-leptons. According to Sakharov second condition, to obtain an asymmetry it is necessary to violate CP, which can be done considering complex Yukawa couplings. Moreover, Yukawa couplings can be small enough to cause departure from equilibrium of the reactions involving \( N_i \), that, therefore, begin to decay out of thermal equilibrium. Eventually, the presence of Majorana term in Eq. (2.27) assures lepton number \( L \) violation. Such a mechanism naturally satisfies Sakharov conditions and it has as net result the generation of a lepton asymmetry. However, the asymmetry we measure is evidently baryonic, hence a way to transfer it from lepton to baryon sector is needed. It is found in the SM, thanks to non perturbative configurations that violates \( B+L \) called sphalerons at high temperature. We shall deal with this subject in the following Section.

The leptogenesis framework we shall adopt in this work is named thermal leptogenesis because \( N_1 \) population is produced through scattering in the thermal bath at high temperatures (\( T \sim 10^9 \) GeV). Baryon asymmetry can be estimated from the equilibrium number density by means of different parameters as follows

\[
Y_{\Delta B} \simeq Y_{N_1}^{eq} (T \gg M_{N_1}) \sum_\alpha \varepsilon_\alpha \eta_\alpha C.
\]

(2.28)

where

\[
Y_i(T) \equiv \frac{n_i(T)}{s(T)},
\]

(2.29)

and the superscript “eq” means that the equilibrium number density is considered. We shall consider only the lightest sterile neutrino \( N_1 \), assuming a strict hierarchy among sterile neutrinos masses. Therefore, thermal production of \( N_2 \) and \( N_3 \) will be assumed to be negligible.

The equilibrium abundance of \( N_1 \) is supposed to “transform” into lepton asymmetry abundance in flavour \( \alpha \), \( Y_{\Delta L_\alpha} \), thanks to CP-violating decays. CP-violation is estimated through the asymmetry parameter into flavour \( \alpha \), \( \varepsilon_\alpha \), defined as

\[
\varepsilon_\alpha \equiv \frac{\Gamma(N_1 \rightarrow \nu_\alpha h) - \Gamma(N_1 \rightarrow \bar{\nu}_\alpha h)}{\Gamma(N_1 \rightarrow \nu_\alpha h) + \Gamma(N_1 \rightarrow \bar{\nu}_\alpha h)},
\]

(2.30)

where \( \Gamma(N_1 \rightarrow \nu_\alpha/\bar{\nu}_\alpha h) \) is the \( N_1 \) decay rate into (anti-)neutrino and Higgs particle.

\( \varepsilon_\alpha \) is zero if the decay rates are computed at tree-level. On the contrary, interference between tree-level amplitude and loop corrections gives rise to a contribution proportional to the imaginary part of the involved Yukawa
Yukawa couplings. Thus, as we told in advance, asymmetry is generated only if Yukawa couplings are complex, i.e. if CP is violated. More in detail, the CP asymmetry $\varepsilon$ arises from the interference of tree-level and one-loop amplitudes, considering both vertex and self-energy corrections \[51,52\]. Considering the vertices arising from the lagrangian in Eqs. (2.27), and drawn in Fig. 1.1, the diagrams contributing to $\varepsilon$ are shown in figure 2.1. Tree level amplitude must considered together with loop corrections on the right. The first diagram in brackets is a vertex correction with a loop closed by a Higgs, a neutrino and a RH neutrino. The second diagram in brackets considers a self energy correction at one loop of the $N_i$ propagator.

![Diagram](image.png)

Figure 2.1: Contribution to asymmetry parameter. Tree-level - one loop interference. The first diagram on the left shows the $N_1$ decay into neutrino and Higgs at tree level, while diagrams in the brackets are one loop corrections of the same process. The first diagram in the brackets is a vertex correction with a loop closed by a Higgs, a neutrino and a RH neutrino. The second diagram in brackets considers a self energy correction at one loop of the $N_1$ propagator.

To obtain $\varepsilon$ at first order (not necessarily order one) in the expansion parameter, we consider the numerators in Def. (2.30) at one loop while the denominator at tree-level. The amplitude of the process at one loop can be written as $c_0 A_0 + c_1 A_1$, where $c_0$ and $c_1$ are the overall coupling constant at tree level and one loop respectively. The CP conjugated process is $c_0^* A_0 + c_1^* A_1$, and $|\bar{A}_i|^2 = |A_i|^2$. We can explicit $\varepsilon$: \[
\varepsilon = \frac{\int |c_0 A_0 + c_1 A_1|^2 \tilde{\delta} \, d\Phi_{\nu h} - \int |c_0^* A_0 + c_1^* A_1|^2 \tilde{\delta} \, d\Phi_{\nu h}}{\Gamma_D} = 2\text{Im}\{c_0 c_1^*\} \int \text{Im}\{A_0 A_1^*\} \tilde{\delta} \, d\Phi_{\nu h},
\]
where \[
\tilde{\delta} = (2\pi)^4 \delta^4(P_i - P_f), \quad d\Phi_{\nu h} = d\Phi_{\nu} d\Phi_{h} = \frac{d^3p_\nu}{2E_\nu(2\pi)^3} \frac{d^3p_h}{2E_h(2\pi)^3}.
\]
Imaginary parts arise from loop amplitudes when particles circulating in the loop are put on-shell. According to Cutkowski’s cutting rule \[53,54\] it is possible to write:

$$2 \text{Im}\{A_0 A_1^*\} = A_0(N \rightarrow h\nu) \int A_0^*(N \rightarrow \bar{\nu}'\bar{\nu}') A_0^*(\bar{\nu}'\bar{\nu}' \rightarrow h\nu) \delta' d\Phi_{\nu',\bar{\nu}'}$$

(2.32)

where $\nu'$ and $\phi'$ are the particles in the loop. The calculation of $c_1$ coefficient is made easier in the limit $M_2, M_3 \gg M_{N_1}$ since an effective dimension-5 operator can be used to shrink the $N_{2,3}$ propagator to a point.

Actually, asymmetry generation is not as simple as described before, because of other interactions that disturb the plain decay of $N_1$, such as inverse decay and other processes commonly called washout processes. The name is due to the fact that they tend to deplete and cancel the asymmetry generated by the decay. The effect of washout in flavour $\alpha$ is estimated through the efficiency parameter $\eta_\alpha$. $0 < \eta_\alpha < 1$ and there can be different possibilities.

If the total decay rate $\Gamma_D$ is larger than the Hubble’s parameter, $\Gamma_D > H(T = M_{N_1})$ at $T \sim M_{N_1}$ we are in the strong washout situation. This means that at that temperature a thermal population of $N_1$ is obtained and there is no lepton asymmetry due to equilibrium. As the Universe cools, the inverse decays can be still in equilibrium, thus washing out the asymmetry generated by direct decays. Therefore, the final asymmetry is related to the $N_1$ population at the temperature $T_\alpha^{\text{inv}}$ at which inverse decay for flavour $\alpha$ departs from equilibrium. However, at $T_\alpha^{\text{inv}}$, $N_1$ is already Boltzmann suppressed so the efficiency parameter for flavour $\alpha$ can be roughly estimated as

$$\eta_\alpha \sim \frac{n_1(T_\alpha^{\text{inv}})}{n_1(T \gg M_{N_1})} \sim \exp \left\{ -\frac{M_{N_1}}{T_\alpha^{\text{inv}}} \right\}.$$  

Considering the decay rate of $N_1$ into the flavour $\alpha$, $\Gamma_\alpha$, the scenario in which $\Gamma_\alpha < H(T = M_{N_1})$ but $\Gamma_D > H(T = M_{N_1})$ is called intermediate washout. It is important to distinguish if $N_1$ has already thermalised, i.e. $n_1(T_0) = n_1^{\text{eq}}(T_0)$, or we have zero population initial condition. The former case does not present any particular problem, while in the latter case $N_1$ reaches its thermal population thanks to all the interactions but those with flavour $\alpha$. Therefore, an anti-asymmetry is produced in this flavour but it is annihilated at lowest order by the asymmetry generated subsequently by $N_1$ decays. The asymmetry is not actually completely eliminated because part of the anti-asymmetry is already washed out during $N_1$ production. Hence, a rough estimation of the efficiency factor can be given by

$$\eta_\alpha \sim \frac{\Gamma_\alpha}{H(T = M_{N_1})}.$$ 

If $\Gamma_D < H(T = M_{N_1})$ we are in the weak washout scenario. The situation described for the intermediate case now applies to all flavours, and the
efficiency factor can be estimated as
\[ \eta_\alpha \sim \frac{\Gamma_\alpha \Gamma_D}{H^2(T = M_{N_1})}. \]

Finally, recalling Eq. (2.28), C factor describe further reductions of the asymmetry due to other effects such as sphalerons which transform lepton asymmetry into baryon asymmetry. In order to have the global asymmetry, the sum over all flavours is then performed.

The lepton asymmetry estimation made this way, i.e. through Eq. (2.28), can be very useful in order to understand the fundamental behaviour of the main interaction involved. Anyway, a more precise calculation must rely on the Boltzmann Equations (BEs). An introduction to the subject can be found in common texts such as [1,2]. In the framework of leptogenesis, BEs study the evolution of the abundances of the relevant species and yield the final lepton asymmetry. This must then be converted into a baryon asymmetry through the mentioned sphaleron processes. The set-up of these equation is quite general and it will be discussed in detail in Chapter 4.

2.3 Sphalerons

In the SM baryon and lepton number can be violated through non-perturbative field configurations [55]. This can happen because the electroweak theory has a non-trivial vacuum structure, according to which different vacua have different baryon and lepton numbers. The vacua are separated by an energy barrier and tunnelling (instantons) between the different vacua is a non-perturbative effect. Due to temperature corrections, these processes become more and more efficient, as the temperature increases, while today they are strongly suppressed. The thermally induced transitions between two adjacent vacua is called sphaleron transitions. A sphaleron changes \( B + L \), whereas the orthogonal combination \( B - L \) is left unchanged. This way, lepton asymmetry can be turned into baryon asymmetry at high temperatures thanks to these non-perturbative processes. Pure standard model lagrangian is invariant under global \( U(1) \) transformation acted on lepton fields \( U_L(1) \) and on quark fields \( U_B(1) \). These invariances correspond to Noether’s conserved currents

\[ \partial_\mu j_B^\mu = \partial_\mu \frac{1}{3} \sum_i \bar{q}_i \gamma^\mu q_i = 0, \]

\[ \partial_\mu j_L^\mu = \partial_\mu \sum_\alpha \bar{l}_\alpha \gamma^\mu q_\alpha = 0, \]

where \( q_i \in \{ u, d, c, s, t, b \} \) and \( \alpha \) is the lepton flavour index.

Eqs. (2.33) and (2.34) only hold in the classical theory, while at quantum level they are modified by loop corrections. These corrections give rise to
the so called chiral anomaly \[56,57\]. In a gauge theory on a simple gauge group, the divergence of the axial current \( j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \) is no more zero, but we have \[58\]

\[
\partial_\rho j_A^\rho = -k g^2 \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_\mu^a F_{a\rho\sigma}, \tag{2.35}
\]

where \( F_\mu^a \) is the gauge field strength, \( g \) is the coupling constant and \( k \) depends on the gauge group representation of the spinor \( \psi \). The divergence of the vectorial current \( j^\mu = \bar{\psi} \gamma^\mu \psi \) can be split into LH \( j_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L \) and RH \( j_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R \) pieces, and thanks to Eq. (2.35) we get

\[
\partial_\mu j_L^\mu = -k_L g^2 \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_\mu^a F_{a\rho\sigma}, \tag{2.36}
\]

\[
\partial_\mu j_R^\mu = k_R g^2 \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_\mu^a F_{a\rho\sigma}, \tag{2.37}
\]

where \( k_L \) and \( k_R \) are the coefficients depending on the gauge group representation respectively of the LH and RH spinors. A different value of these coefficient causes a non-zero divergence of the vectorial current.

In the framework of the SM, we have \( k_{SU(3)}^{C} = k_{SU(3)}^{C} \), because colour interaction is not chiral, whereas \( k_{SU(2)_L \otimes U(1)_Y}^{SU(2)_L \otimes U(1)_Y} \neq k_{SU(2)_L \otimes U(1)_Y}^{SU(2)_L \otimes U(1)_Y} \) because LH and RH fermions belong to different \( SU(2)_L \otimes U(1)_Y \) representations. In the end, we have

\[
\partial_\mu j_L^\mu = \partial_\mu j_B^\mu = \frac{n_G}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \left( -g^2 W_\mu^a W_{a\rho\sigma} + g^\prime 2 B_\mu B_{\rho\sigma} \right), \tag{2.38}
\]

where \( W_\mu^a, g \) and \( B_\mu, g^\prime \) respectively are the \( SU(2)_L \) and \( U(1)_Y \) field strength tensors and coupling constants, and \( n_G = 3 \) is the number of generations. It is clear from Eq. (2.38) that \( j_L^\mu + j_B^\mu \) is no more conserved, while \( j_L^\mu - j_B^\mu \) still is. This means that neither \( B \) nor \( L \) nor \( B + L \), but \( B - L \) quantum number is exactly conserved in the standard model. In order to investigate further \( B + \) violation we can write the right hand side of Eq. (2.38) as the divergence \( n_G \partial_\mu K^\mu \) of a current

\[
K^\mu = -\frac{2g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} W_\nu^a \left( \partial_\rho W_{a\sigma} + g^a_{\quad bc} W_b W_{c\sigma} \right) + \frac{g^\prime 2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} B_\nu B_{\rho\sigma}, \tag{2.39}
\]

where \( W_\nu^a \) and \( B_\nu \) are the gauge fields. This divergence can be integrated on the space-time obtaining

\[
\int d^4x \partial_\mu K^\mu = \frac{g^3}{96\pi^2} \int_{\partial V} d\nu^\mu \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} W_a^\nu W_b^\rho W_c^\sigma, \tag{2.40}
\]

where \( B_\mu \) does not appear since the result does not depend on the abelian part of the gauge group. Exploiting the gauge invariance of \( \partial_\mu K^\mu \) it is possible to choose a temporal gauge which makes calculation easier. Choosing
CHAPTER 2. THEORETICAL FRAMEWORK

also a suitable surface we get a result that depends only on the integrals over the time-constant surfaces at times \( t_i \) and \( t_f \). These contribution are integral numbers called Chern-Simons numbers \( N_{CS}(t) \) and in the temporal gauge we have

\[
N_{CS}(t) = \frac{g^3}{96\pi^2} \int d^3x \varepsilon^{ijk} \varepsilon_{abc} W_i^a W_j^b W_k^c,
\]

hence

\[
\int d^4x \partial_{\mu} R_{\mu} = N_{CS}(t_f) - N_{CS}(t_i) \equiv \Delta N_{CS},
\]

which is gauge invariant. Eq. (2.42) gives a topological characterization to the gauge fields \( W_{\mu}^a \), because the topological non-triviality of the gauge group arises from its non abelian part. This way the theory has an infinite number of ground state, whose vacuum gauge field configurations have different topological charges, given by \( \Delta N_{CS} = 0, \pm 1, \pm 2, \ldots \) The potential is a functional of the Higgs and the gauge fields \( V[\Phi, W^a] \).

Integrating also the left-hand side of Eq. (2.38) we get

\[
\Delta B = \Delta L = n_G \Delta N_{CS},
\]

which shows that \( B + L \) can be violated only by multiples of \( n_G \), i.e. in the SM it can change at least by three units. This means that if we consider small gauge field quantum fluctuations around the perturbative vacuum \( W_{\mu}^a = 0 \), the right-hand side of Eq. (2.43) is zero, and \( B \) and \( L \) number remain conserved. This is the usual case, since perturbative theory is commonly used. However, large gauge fields \( W_{\mu}^a \neq 0 \) with \( \Delta N_{CS} \) exist and they can induce transitions between fermion states with baryon and lepton numbers that differ according to Eq. (2.43). The dominant transition at \( T = 0 \) is the narrowest, with \( \Delta B = \Delta L = \pm 3 \), i.e. with \( \Delta N_{CS} = \pm 1 \), which is carried by the instanton configuration [59]. The probability of such a process to occur is exponentially suppressed and therefore it can be said that there are no \( B + L \) violating processes today. On the contrary, at higher temperatures such as in the early Universe these transition become possible. A Higgs and gauge field configuration with \( N_{CS} = n + 1/2 \) \((n \in \mathbb{Z})\) is called sphaleron. This solution is not stable because it correspond to a maximum of the potential \( V[\Phi, W^a] \), given by

\[
E_{sph}(T) \propto v(T),
\]

where \( v(T) \) is the v.e.v of the Higgs field at temperature \( T \). While the EW symmetry is unbroken, the rate of transition per unit volume is Boltzmann suppressed

\[
\Gamma_{sphaleron}/V \propto g^{-6} e^{-E_{sph}(T)/T}.
\]

EW symmetry is restored at \( T \gtrsim T_{EW} \) and so thermal excitation are greater than the potential barrier. Thus, transition processes rate is no more be
exponentially suppressed, and the rate per unit volume is given by $\Gamma_{\text{sphaleron}}/V \propto g^{10}T^4$.

This result can be used in order to calculate when $B + L$ violating processes are in thermal equilibrium, comparing the decay rate to the Hubble’s parameter. We get that sphaleron processes are in equilibrium for temperatures $T_{\text{EW}} \lesssim T \lesssim 10^{12} \text{GeV}$.

This means that at high temperature sphaleron transition are allowed and fast enough to transform lepton asymmetry into baryon asymmetry.
In this chapter we will consider some models that at once explain the problems described in the introduction. We will give a brief review of some of them and then we shall concentrate on the one that is the object of this work.

The search for a common explanation of DM and baryon asymmetry problem arises because of the size of their densities. The common paradigm that explains the origin of DM relies on weakly interactive massive particles (WIMP) whose mass is at GeV scale. Their weak interaction rate with the thermal bath provides their departure from equilibrium and its thermal freeze-out. This approach is supported by the widespread hunt for new physics at TeV scale and it gives correct predictions for DM relic abundances. However, it is completely unrelated to the baryon asymmetry and the fact that DM and baryon densities are very close is just a mere coincidence. For this reason, new models have been proposed aiming at a unification of the solution to the problems [61]. In these models baryon asymmetry can be generated before or at the same time of DM production, but in all models DM carries a quantum number which is violated. Therefore, DM is produced with different number density for particles and antiparticles and interaction are assumed to annihilate away the symmetric component. Thus, we are left with a net DM particle/antiparticle asymmetry which terms this class of models Asymmetric Dark Matter (ADM) models. A rough distinction can be made dividing those models which envisage baryon asymmetry production before ADM generation and those according to which baryon asymmetry and ADM production takes place at the same time in different sectors. We shall call the former class transfer models and the latter two sector models.
3.1 Transfer models

This class of models generally exploits standard baryogenesis/leptogenesis results. Through these mechanisms a net baryon/lepton asymmetry is first produced by means of $B - L$ violating processes. This asymmetry is then transferred to the DM via new kinds of interactions that exit from equilibrium causing ADM freeze-out. Eventually, particle-antiparticle annihilations eliminate the symmetric component, leaving an ADM relic density.

3.1.1 Extended symmetry

This kind of models assumes the existence of new symmetries beyond the usual SM ones in order assure asymmetry transfer from SM particles to DM particles [62]. DM is assumed to be composed of three generations Dirac fermions $\chi_\alpha \in \{\chi_1, \chi_2, \chi_3\}$, assumed to carry a DM quantum number $X$ which must be related somehow to $B$ and $L$. Lepton asymmetry is generated through standard leptogenesis (cfr. sec. 2.2) and baryon asymmetry is produced by means of usual $SU(2)_L$ sphalerons. This mechanism can in principle be exploited to transfer baryon/lepton asymmetry to DM as well. However, DM couplings to $SU(2)_L$ would result into too light $\chi$ which has been excluded by LEP experiments. Therefore, $SU(2)_L$ sphalerons are used in order to exchange lepton and baryon asymmetry as in standard leptogenesis, while non-perturbative processes associated to a new extension of the gauge group provide SM-DM transfer. To this end, it is necessary to extend SM gauge group so that it couples both to SM and do DM, but avoiding DM-neutrino mixing. This mechanism, exploiting an extended gauge group is often called aidnogenesis [62]. The model, then, has a new gauge group

$$G_{ADM} = \frac{SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes SU(2)_H \otimes SU(3)_{dc}}{SM}$$

where the added $SU(2)_H$ is a new horizontal chiral asymmetry and $SU(3)_{dc}$ provides a DM color interaction that prevents mixing with singlets. Indeed, RH $\chi_1 R$ and $\chi_2 R$ are put into $SU(2)_H$ doublets, as well as $e_R$ and $\mu_R$, while third generation RH fields $\chi_3 R$ and $\tau_R$ are $SU(2)_H$ singlets. $\chi_\alpha$ are put into $SU(3)_{dc}$ triplets. The general scheme of fields in this model is given in table 3.1 $SU(2)_H$ sphalerons satisfy $\Delta B = 2\Delta L = 2\Delta X$, therefore baryon asymmetry is converted into DM asymmetry while $B - L - X$ remains conserved. After their departure from equilibrium DM asymmetry is frozen-out and can be computed. DM mass is then extracted and $m_\chi \sim 5$ GeV.

This model has interesting phenomenology due to the dark colour interactions. Indeed, DM can be composed of dark baryon since $SU(3)_{dc}$ is confining. Therefore, blow $SU(3)_{dc}$ phase transition DM is made of dark-mesons that can decay into SM particles via $SU(2)_H$. This can put a lower bound on the coupling constant because decays are required to take place after BBN.
CHAPTER 3. ASYMMETRIC DARK MATTER MODELS

<table>
<thead>
<tr>
<th>Field</th>
</tr>
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<tbody>
<tr>
<td>$L_\alpha \left( \nu_{\alpha L} \frac{1}{2} \right)$</td>
</tr>
<tr>
<td>$\nu_{\alpha R} \left( \nu_{\alpha R} \right)$</td>
</tr>
<tr>
<td>$\tau_R \left( \tau_R \right)$</td>
</tr>
<tr>
<td>$Q_{L\alpha} \left( u_{\alpha L} \frac{1}{6} \right)$</td>
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<tr>
<td>$Q_{R} \left( u_{R} \frac{2}{3} \right)$</td>
</tr>
<tr>
<td>$b_{R} \left( b_{R} \frac{-1}{3} \right)$</td>
</tr>
<tr>
<td>$L_H \left( \chi_{1R} \frac{0}{1} \right)$</td>
</tr>
<tr>
<td>$X_H \left( \chi_{3R}, \chi_{3L} \right)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2)_L$</td>
</tr>
<tr>
<td>$SU(2)_C$</td>
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<tr>
<td>$SU(2)_H$</td>
</tr>
<tr>
<td>$SU(3)_{\text{dec}}$</td>
</tr>
</tbody>
</table>

Table 3.1: List of extended gauge group multiplets in aidnogenesis

Despite some artificial assumptions, such as a new gauge group horizontal in flavour space, this model can nicely explain the similarity between baryon and DM densities, tracing back their origin to a common process. In addition, its rich phenomenology provides interesting way of DM detection and model testing.

### 3.1.2 Supersymmetric ADM

A large part of the proposed model make use of supersymmetry. Even if it is not strictly necessary, it can provide a connection to other models of electroweak physics and interesting collider phenomenology.

DM is assumed to be a gauge singlet chiral superfield $X$, which can carry lepton or baryon number $[63]$. If it carries lepton number, e.g. $L = \pm 1/2$, it can provide $\Delta L = 2$ due to Majorana neutrino mass terms. Asymmetry in lepton number must be generated at high scales according to standard processes, then it is transferred to DM. It can happen thanks to effective interactions like

$$\frac{1}{M} \bar{X}^2 L H_u.$$  \hspace{1cm} (3.1)

Here $M$ is the high-energy scale, while $H_u$ is the Higgs field.

At early stages this interaction is in equilibrium and the asymmetry is transferred. As the temperature drops, $T \gtrsim 100 \text{ GeV}$ if $M \gtrsim 10^9 \text{ GeV}$, this inter-
action falls out of equilibrium and DM asymmetry is frozen. Eq. (3.1) can be unwrapped and the component interaction we get can contain at most two fermion fields. Thus, the only interactions that change $X$ quantum number involve sneutrinos and/or Higgs scalars. Comparing the interaction rate of Eq. (3.1) and the expansion rate of the Universe it is possible to predict the value of the $X$ asymmetry, which depends on the exit from equilibirum of interaction in Eq. (3.1). If this interaction departs from equilibrium at a temperature haigher than the EWSB, we have \[ \Delta X = -\frac{11}{79} \Delta (B - L). \]

Resorting to SUSY enables interesting interlaces between cosmology and collider physics. DM mass should arise from electroweak symmetry breaking due to its closeness to the EW scale. This can be achieved, for instance, in the next-to-minimal supersymmetric standard model. In addtion to that, in this framework the annihilation of the symmetric component can be obtained via the exchange interaction or Higgs coupling. Therefore, annihilation processes can have chances to be detected at colliders in SUSY hunt experiments.

The interaction introduced in Eq. (3.1) is just a low energy effective vertex, belonging to a complete high-energy theory. The building of this theory gives rise to further model possibilities.

### 3.2 Two-sector models

As in the extended symmetry model described in the previous section, in two-sector models lepton asymmetry is generated through standard leptogenesis (cfr. Sec. 2.2). The difference between this model and the previous one is that here lepton asymmetry and ADM are generated at the same time and, even if transfer processes between SM and DM are still present, they do not require non-perturbative mechanism such as sphalerons. The usual $SU(2)_L$ sphalerons are, obviously, required to transfer asymmetry from leptons to baryons.

This simultaneous generation of lepton and DM asymmetries can be achieved by exploiting the same process, i.e. the decay of the RH sterile
neutrinos, usually responsible only for leptogenesis. In this model the same neutrinos are coupled also to the DM candidate, that resides in the so called *hidden* or *DM sector*, via new Yukawa couplings. The Standard Model sector and the DM sector are then completely independent and they do not interact with each other but through the sterile neutrinos\(^1\). Thence, RH neutrinos can decay not only into SM particles (i.e. leptons) but also into the DM sector. As in leptogenesis, DM particles and antiparticles are not produced with the same number density, and so they do not fully annihilate, because of CP violation that is present also in the DM sector since Yukawa couplings can be complex.

We shall consider three sterile RH neutrinos \(N_i\) that couple to the three families Dirac neutrinos in the SM sector via Yukawa couplings \(y_{\alpha i}\). The interaction term, with the Higgs scalar \(h\), is the same as in standard leptogenesis and type-I seesaw, cfr. Eq.\(^{(2.14)}\) [65,66]:

\[
y_{\alpha i} \bar{N}_i \nu_L h,
\]

where \(\alpha\) counts the SM families and \(y_{\alpha i}\) are the Yukawa couplings for the SM sector. With respect to the see-saw lagrangian in Eq.\(^{(2.14)}\), as mentioned above, here we have considered only the interaction term with the Higgs scalar, i.e. we have chosen unitary gauge. Moreover, we have defined \(y_{\alpha i} \equiv [Y_\nu]_{i\alpha}\) and, for notation sake, we have abolished prime indices.

The DM sector is made of a candidate \(\chi\) which is a Dirac fermion and a complex scalar \(\phi\) so to have a term similar to the SM one

\[
\lambda_i \bar{N}_i \chi \phi,
\]

where \(\lambda_i\) are the Yukawa coupling of the hidden sector.

Putting it all together, we can write a type-I see-saw lagrangian extended by the DM term,

\[
\mathcal{L} \supset -\frac{1}{2} M_i N_i^T C^\dagger N_i + y_{\alpha i} \bar{N}_i \nu_L h + \lambda_i \bar{N}_i \chi \phi + h.c., \tag{3.2}
\]

where the first term gives the standard Majorana mass to \(N_i\). For diagrammatic sake, we can explicit the \(\gamma_5\)'s. Writing explicitly the hermitian conjugate as well, we obtain the interaction lagrangian

\[
\mathcal{L} \supset \frac{y_{\alpha i}}{2} \bar{N}_i (1 - \gamma_5) \nu_a h + \frac{y_{a i}^*}{2} \bar{\nu}_a (1 + \gamma_5) N_i h + \frac{\lambda_i}{2} \bar{N}_i (1 - \gamma_5) \chi \phi + \frac{\lambda_i^*}{2} \bar{\chi} (1 + \gamma_5) N_i \phi^\. \tag{3.3}
\]

The vertices given by Eq.\(^{(3.3)}\) are shown in Figure 3.2.

---

\(^1\)As mentioned in Section\(^{(2.2)}\) they are called sterile with respect to the colour and EW interactions.
Since $\lambda_i \in \mathbb{C}$, as well as $y_{i\alpha}$, there will be CP violation also in the hidden sector. Hence, the decay of the neutrinos into this sector will be CP-violating and so it will satisfy the second Sakharov’s condition. This decay will be ineffective until the neutrino becomes non relativistic, due to statistical suppression. Therefore, it will take place only after the exit from equilibrium condition. This situation of Out of Equilibrium (OoE) decay is perfectly complying with the third Sakharov condition. As a result, the process starting from an equilibrium population of $N_i$ at early times, will end not only in an different abundance for SM particles and antiparticles as in standard leptogenesis, but also in an asymmetry in the hidden sector. DM produced in this way is asymmetric and the hidden interactions in the DM sector are understood in order to allow the annihilation of the symmetric component. These interactions are assumed to be fast enough to thermalise the sector and to avoid any asymmetry carried by $\phi$.

In each sector the asymmetry is slightly smoothed by inverse decay, but it can also be strongly modified by additional processes that can smear the net value resulting from the plain neutrino direct/indirect decay. They are $2 \leftrightarrow 2$ washout processes, mediated by $N_i$. An example is given in Fig. 3.3.
CHAPTER 3. ASYMMETRIC DARK MATTER MODELS

In Fig. 3.3(a) a neutrino in the in-state is turned into an anti-neutrino by means of the exchanga of a RH neutrino in the s-channel. Higgs scalars are both in the in- and out-states. The Majorana fermion propagator allows a lepton number violation of two units $\Delta L = 2$. In Fig. 3.3(b) the analogous process takes place in the hidden sector: a DM particle is turned into a particle thanks to the RH neutrino in the s-channel. DM scalars $\phi$ and $\phi^\dagger$ are both in the in- and out-states.

Figure 3.3: $2 \leftrightarrow 2$ washout processes. All momenta are assumed to flow from left to right. In 3.3(a) a neutrino lepton number is violated by two units $\Delta L = 2$, due to the Majorana fermion in the s-channel propagator. Indeed, a neutrino $L = +1$ is turned into an anti-neutrino $L = -1$. In Fig. 3.3(b) the analogous process takes place in the hidden sector: a DM particle is turned into a particle thanks to the RH neutrino in the s-channel.

Moreover, since the two sectors are connected via $N_i$, there can also be $2 \leftrightarrow 2$ scattering processes that transfer the asymmetry from one sector to the other. These processes violate lepton number too and they are shown in Fig. 3.4. In these processes a particle in the in-state that belongs to one sector is turned into a particle belonging to the other sector. In Fig. 3.4(a) a DM particle $\chi$ and an hidden scalar $\phi$ are turned into a (anti-)neutrino and a Higgs scalar by the exchange of a RH neutrino in the s-channel. In Fig. 3.4(b) we have scattering of SM and DM particles. The first and second diagram show processes that eliminate SM and DM fermions in favour of scalars, while in the third and in the fourth fermion and scalar contents are mixed up.

In particular, in the following of our analysis, we will fix the sterile neutrino masses to be hierarchical $[64] M_1 : M_2 : M_3 = 1 : 10 : 100$, and in many applications it is possible to integrate out the contribution of $N_{2,3}$. The greatest contribution to the final asymmetries is due to the decay of the lightest neutrino $N_1$. The heavier states are therefore neglected in most treatments of the subject. Variations of standard thermal leptogenesis are
Figure 3.4: 2 ↔ 2 transfer processes. All momenta are assumed to flow from left to right. In Fig. 3.4(a) s-channel processes where DM particles are turned into SM neutrinos and Higgs scalar. In Fig. 3.4(b) t-channel processes where DM and SM particles can be found both in \( m \) and \( out \) states.

The choice of parameters can end up into highly variable situations. The asymmetry predicted by the model can vary very much in both sectors. Such a variability will cause DM mass to vary in a wide range, from keV up to TeV scale. As mentioned in the introduction, our aim is to study the spectrum of DM candidate and its dependence on the model parameters. However, before studying the spectrum of \( m_\chi \) it is necessary to verify that the model is compatible with reality. This is the case if it predicts the correct amount of lepton/baryon asymmetry and the right active neutrino masses and mixing. As mentioned in Section 2.1 the neutrino mixing matrix and Yukawa coupling can be conveniently related exploiting the Casas-Ibarra parametrization \[37\]. The orthogonal matrix \( R \) can be varied so to have different Yukawa couplings for each model, directly embedding the values for the neutrino mixing matrix.

Then, the final asymmetries can be found by solving the BEs and comparing the lepton one with the experimental data. If this is acceptable, a value for DM mass can be extracted from the DM asymmetry abundance.
CHAPTER 3. ASYMMETRIC DARK MATTER MODELS

The requirement of perturbativity in DM annihilation reactions puts an upper bound of order $\sim 10$ TeV on the value of $m_{\chi}$. Whereas, perturbative coupling in the DM sector requires $m_{\chi} \lesssim 10$ keV \[64\].

3.2.1 Minimal model

In order to study the available spectrum of the mass of DM candidate, it is sufficient to consider a straightforward simplification of the full model described in Section 3.2. Instead of the full three generation Standard Model, only one lepton generation and two sterile neutrinos $N_i, i = 1, 2$, are considered. Therefore, the lagrangian is reduced to the form:

$$
\mathcal{L} \supset -\frac{1}{2} M_i N_i^T C^\dagger N_i + y_i \bar{N}_i \nu_L h + \lambda_i \bar{N}_i \chi_L \phi + h.c. \tag{3.4}
$$

Yukawa couplings both in the SM sector and in the hidden sector are now vectors, since they just carry the sterile neutrinos index $i$. By redefining the fields it is possible to have $y_1$ and $\lambda_1$ real and positive, while $y_2$ and $\lambda_2$ remain complex. Therefore

$$
y_1, \lambda_1 > 0 \quad y_2 = |y_2| e^{i\phi_L}, \quad \lambda_2 = |\lambda_2| e^{i\phi_\chi}. \tag{3.5}
$$

Writing also explicitly the hermitian conjugate, we obtain the interaction lagrangian:

$$
\mathcal{L} \supset \frac{y_i}{2} \bar{N}_i (1 - \gamma_5) \nu h + \frac{y_i^*}{2} \bar{\nu}(1 + \gamma_5) \bar{N}_i h^\dagger + \\
\frac{\lambda_i}{2} \bar{N}_i (1 - \gamma_5) \chi \phi + \frac{\lambda_i^*}{2} \bar{\chi}(1 + \gamma_5) \bar{N}_i \phi^\dagger. \tag{3.6}
$$

From Eq. (3.6) we obtain the same vertices in Fig. 3.2 but with Yukawa coupling without the SM generation index $\alpha$.

The general features exposed for the full model are still valid for the minimal model. Clearly, we will not have to verify the match with the experimental active neutrino masses and mixing.

In the following this model will be studied in detail in order to get a mass spectrum for the DM candidate $\chi$. The main step towards this result is represented by the BEs describing the evolution of the abundances of the involved species. In Chapter 4 we will write down these equations and calculate all the relevant leptogenesis parameters. In Chapter 5 we will discuss the computational set-up and the numerical code used to solve the equations, while in Chapter 6 we will expose the main results obtained in the analysis.
Chapter 4

Boltzmann Equations

In this chapter we will briefly describe how the DM candidate mass, $m_\chi$, can be extracted from the model introduced in section 3.2.1. To this aim, we will not follow the estimation described in section 2.2, rather BEs for all the relevant abundances will be derived and in the last sections the needed coefficient will be calculated. A detailed treatment of the BEs can be found for instance in [1,2].

Before looking at the mass of the DM candidate, it is compulsory to verify if the model complies with the observed lepton asymmetry, commonly measured by the lepton asymmetry abundance, cfr. Def. 2.29

\[ Y_{\Delta l} \equiv Y_l - Y_\bar{l}. \]

An estimate of this asymmetry at present day can be obtained through a precise measure of the baryon-to-photon ratio $\eta$, introduced in the introduction. From recent data [3,10] we get the value for $\eta$ in Eq. (1.11) and thence a value for the baryon asymmetry abundance

\[ Y_{\Delta B} \equiv \left. \frac{n_B - n_\bar{B}}{s} \right|_0 = (8.75 \pm 0.23) \cdot 10^{-11}. \] (4.1)

From this value for the baryon asymmetry a value for $Y_{\Delta l}$ can be obtained, taking into consideration the EW sphaleron mechanism (cfr. Sec. 2.3). In equilibrium we have [70]

\[ Y_{\Delta l} \simeq \frac{37}{12} Y_{\Delta B} \sim 2.6 \cdot 10^{-10}. \] (4.2)

The value of the lepton asymmetry abundance we measure today will be referred to as asymptotic lepton asymmetry abundance, $Y_{\Delta l}^\infty$.

Once $Y_{\Delta l}^\infty$ is obtained and matches the observation, thanks to BE solution, we also have the value of the asymptotica DM asymmetry abundance $Y_{\Delta \chi}^\infty$. The value of the Dark Matter candidate is then obtained using the value of the current relic density of Cold Dark Matter (CDM). In a sense, this
mechanism force $\chi$ to be a CDM candidate. Experiment such as WMAP-7, \[3\] have measured with high precision the density ratio to critical density $\rho_c$ of the various component of the Universe, see Def. (1.2). For the CDM we have (WMAP-only)

$$\Omega_{CDM}h^2 = 0.1099 \pm 0.0062,$$

which is related to the present CDM abundance via $m_\chi$ and the entropy. $\chi$ will be non-relativistic, therefore

$$\rho(T) = m_\chi n(T) = m_\chi Y_\Delta(T) s(T),$$

and so

$$\Omega_{CDM} = \frac{m_\chi Y_\Delta(T_0)s(T_0)}{\rho_c}.$$ 

Inverting this relation and using (cfr. Refs. \[1,3\]) the following values for the critical density $\rho_c$ and the entropy $s$

$$\rho_c \equiv \frac{3H_0^2}{8\pi G} \simeq 1.878 \cdot 10^{-29} h^2 \text{ g cm}^{-3},$$

$$s(T_0) \simeq 2970 \text{ cm}^{-3},$$

we have

$$Y_\Delta^\infty \simeq 4 \cdot 10^{-10} \left( \frac{\text{GeV}}{m_\chi} \right).$$

Hence, inverting the equality, it is easy to obtain a value for $m_\chi$, which, therefore, directly depends on the solution of the BEs

$$m_\chi = 4 \cdot 10^{-10} \frac{Y_\Delta^\infty}{Y_\Delta} \text{ GeV.} \quad (4.3)$$

According to leptogenesis scenario, the lepton asymmetry origins from the decay of a heavy sterile neutrino. As we have seen, our model can explain the origin of Dark Matter assuming that the same decay give rise to a DM asymmetry. For this reason, the BEs will have to describe the evolution of the sterile neutrino abundance $Y_{\Delta N}$ and of those of the lepton and DM asymmetry $Y_{\Delta l}$ and $Y_{\Delta \chi}$.

### 4.1 Definitions and notation

In order to refer to the relevant quantities appearing in the BEs, we shall use the notation commonly adopted in leptogenesis literature (e.g. \[65\]). The thermally averaged rate for the $A \rightarrow B$ process is defined as

$$\gamma_{ij} \equiv \gamma(a_i \rightarrow b_j) = \int \prod_{i \in \text{in}} (d\Phi_i f_i^{eq}) \prod_{j \in \text{out}} (d\Phi_j) |\mathcal{M}(a_i \rightarrow b_j)|^2 (2\pi)^4 \delta^4,$$  

(4.4)
CHAPTER 4. BOLTZMANN EQUATIONS

where \( a_i \)'s are the species in the in-state, while \( b_j \)'s are those appearing in the out-state. \( d\Phi_x \) stands for the Lorentz-invariant phase space integral of the species \( x \) and \( \delta^4 \) is a short notation for the 4-momentum conservation delta function. \( f_i^{\text{eq}} \) is the equilibrium distribution of the species \( a_i \). Here we have neglected both Bose enhancement and Pauli blocking, and all the species are assumed to be in equilibrium.

The difference between a process and its CP-conjugated is then indicated by

\[
\Delta \gamma^A_B \equiv \gamma^A_B - \gamma^A_B. \quad (4.5)
\]

It is also useful to define the difference between a process and its T-reversed, all weighted by the ratios between the real abundance and the equilibrium abundance of the occurring initial species

\[
[A \leftrightarrow B] \equiv \left( \prod_{a \in \text{IN}} \frac{Y_a}{Y_{eq_a}} \right) \gamma^A_B - \left( \prod_{b \in \text{IN}} \frac{Y_b}{Y_{eq_b}} \right) \gamma^B_A \quad (4.6)
\]

The decay rates define also the branching ratios. For instance, referring to the SM final states, the total branching ratio is given by

\[
\text{Br}_l \equiv \frac{\Gamma(N_1 \rightarrow \nu h) + \Gamma(N_1 \rightarrow \bar{\nu} h)}{\Gamma_D}, \quad (4.7)
\]

where \( \Gamma_D \) also stand for the total decay rate of the RH neutrino.

Using definition (2.30) and Eq. (4.4), in this case we can also write

\[
\varepsilon_l \equiv \frac{\gamma(N_1 \rightarrow \nu h) - \gamma(N_1 \rightarrow \bar{\nu} h)}{\gamma_D} \quad (4.8)
\]

where \( \gamma_D \) is the thermally averaged total decay rate

\[
\gamma_D \equiv \gamma(N_1 \rightarrow \text{anything}) = \sum_{\text{out}} \gamma(N_1 \rightarrow \text{out}) \quad (4.9)
\]

In the following Sections we shall see these parameters at work and in Section 4.3 they will be fully calculated in the framework of the minimal model.

Eventually, the time derivative of the abundance can be related to its derivative with respect to \( z \) as follows

\[
Y'_i = \frac{dY_i}{dz} = \frac{z}{s(z)H_i} \dot{Y}_i, \quad (4.10)
\]

where \( H_i \equiv H(T = M_i) \).
For our purposes, it is sufficient to consider the contribution to the BEs up to the second order of the Yukawa coupling $O(y^2)$ for the $N_1$ equation, and up to $O(y^4)$ for the asymmetry equations. We can also safely neglect any scattering with heavy quarks or gauge bosons.

Eventually, we shall carry on using $Y_l$ to indicate lepton abundance and $Y_{\Delta l}$ to refer to lepton asymmetry abundances, however it is understood that in the model we are studying lepton abundances are due to neutrinos. In a sense it is as

$$Y_{l\equiv\nu}$$

### 4.2 $N_1$ decay

The abundance of the sterile neutrino is determined by its decay and inverse decay that may occur to both SM and DM particles. Considering the vertices shown in Sec. 3.2.1, we must consider

1. the SM decay process into a neutrino $N_1 \to \nu h$,
2. the T-reversed SM process (inverse SM process) $\nu h \to N_1$,
3. the SM decay process into an anti-neutrino $N_1 \to \bar{\nu} h$,
4. the T-reversed SM process (inverse SM process) $\bar{\nu} h \to N_1$,
5. the DM decay process into a DM particle $N_1 \to \chi \phi$,
6. the T-reversed DM process $\chi \phi \to N_1$,
7. the DM decay process into a DM anti-particle $N_1 \to \bar{\chi} \phi^{\dagger}$,
8. the T-reversed DM process $\bar{\chi} \phi^{\dagger} \to N_1$.

Processes 1 and 2 can be kept together using Def. (4.6), as well as 3 and 4, 5 and 6, and 7 and 8. Therefore, the variation of the sterile neutrino abundance is given by

$$\dot{Y}_{N_1} = [\nu h \leftrightarrow N_1] + [\bar{\nu} h \leftrightarrow N_1] + [\chi \phi \leftrightarrow N_1] + [\bar{\chi} \phi^{\dagger} \leftrightarrow N_1].$$

(4.11)

Working, as said, at order two in the Yukawa couplings in $N_1$ BE, we can approximate the asymmetry abundances to the equilibrium ones. Using Eq. (4.6), we have

$$\dot{Y}_{N_1} = \gamma(\nu h \to N_1) - y_{N_1}\gamma(N_1 \to \nu h) + \gamma(\bar{\nu} h \to N_1) - y_{N_1}\gamma(N_1 \to \bar{\nu} h)$$

$$+ \gamma(\chi \phi \to N_1) - y_{N_1}\gamma(N_1 \to \chi \phi) + \gamma(\bar{\chi} \phi^{\dagger} \to N_1) - y_{N_1}\gamma(N_1 \to \bar{\chi} \phi^{\dagger})$$

$$= -y_{N_1}\gamma(N_1 \to \text{anything}) + \gamma(\text{anything} \to N_1).$$
Recalling Def. (4.4) and thanks to CPT invariance [71], we finally have
\[ \dot{Y}_{N_1} = -\gamma_D (y_{N_1} - 1), \]
which we can transform into
\[ Y'_{N_1} = -\gamma_D \frac{z}{s(z) H_1} \left( \frac{Y_{N_1}}{Y_{N_1}^{eq}} - 1 \right) \]
(4.12)
by using Eq. (4.10)

### 4.3 Asymmetry abundances

Generally speaking, the asymmetry, both in SM and in DM sector, is due
to $1 \leftrightarrow 2$ processes such as the direct and indirect decays of $N_1$ and $2 \leftrightarrow 2$
processes like the scattering mediated by $N_1$ between the same species (i.e. in
the same sector, which we called washout) or between the two sector (which
we called transfer), cfr. Sec. 3.2. A complete treatment of the subject should
take into consideration all the processes, but we have chosen to devote our
study to the so called Narrow Width Approximation (NWA) [64,65,72]. In
this case we have
\[ \frac{\Gamma_D}{M_{N_1}} \ll 1, \quad \frac{\Gamma^2_D}{M_{N_1} H_1} \ll 1, \]
(4.13)
and the $2 \leftrightarrow 2$ processes are completely negligible. It is intended hereafter
that all our study will be carried out within this framework. Consequently,
in order to set up the BEs for the asymmetries we will consider only the
$1 \leftrightarrow 2$ contributions. Looking at the lepton asymmetry, we must consider

1) the SM decay process into a neutrino $N_1 \rightarrow \nu h$,
2) the T-reversed SM process (inverse SM process) $\nu h \rightarrow N_1$,
3) the SM decay process into an anti-neutrino $N_1 \rightarrow \bar{\nu} h$,
4) the T-reversed SM process (inverse SM process) $\bar{\nu} h \rightarrow N_1$.

That is, always using Def. (4.6)

\[
\begin{align*}
\dot{Y}_i &= [N_1 \leftrightarrow \nu h] \\
\dot{Y}_{\bar{h}} &= [N_1 \leftrightarrow \bar{\nu} h]
\end{align*}
\]

Subtracting the latter from the former, we get
\[ \dot{Y}_{\Delta l} = y_{N_1} \left( \gamma^{N_1}_{\nu h} - \gamma^{N_1}_{\bar{\nu} h} \right) - y_l y_{\nu h}^{\nu h} + y_{\bar{\nu} h}^{\nu h} \]
where we have taken into account that $h$ distribution is the equilibrium one. Now, recalling that we must work to order $O(y^4)$, we separate the terms like $y_l \gamma_{N_1}^{lh}$ into two pieces:

$$y_l \gamma_{N_1}^{lh} = \gamma_{N_1}^{lh} + \tilde{\gamma}_{N_1}^{lh} y_l + O(y^4),$$

where in the first part we take the 1-loop corrected rate $O(y^4)$ and $y_l = 1$, while in the second we take the corrected abundance $y_l = O(y^4)$ times the tree-level rate. Hence, making use of Def. (4.5) and CPT theorem,

$$\dot{Y}_{\Delta l} = y_{N_1} \Delta \gamma_{N_1}^{l} - \gamma_{N_1}^{l} + \tilde{\gamma}_{N_1}^{l} (y_l - y_l)$$

$$= \Delta \gamma_{N_1}^{l} (y_{N_1} - 1) - y_{\Delta l} \gamma_{N_1}^{l}$$

$$= \gamma_{D} \gamma_{N_1}^{l} (y_{N_1} - 1) - y_{\Delta l} \gamma_{D} \frac{\text{Br}_{l}}{2}$$

$$= \gamma_{D} \left[ \varepsilon_{l} \left( \frac{Y_{N_1}}{Y_{eq}} - 1 \right) - \frac{\text{Br}_{l} Y_{\Delta l}}{2 Y_{eq}} \right].$$

(4.14)

In the third line we have used Eq. (4.8) and Eq. (4.7), recalling that at tree level there is no difference between $\gamma_{N_1}^{l}$ and $\gamma_{N_1}^{l}$. The first term in Eq. (4.14) is related to the direct decay of $N_1$ and it appears with plus sign, while it is appears in Eq. (4.12) with a minus sign to show that it causes a depletion of the $N_1$ number density. The second term in Eq. (4.14) is related to the inverse decay of $N_1$. For this reason, it appears with a minus sign, because it tends to decrease the asymmetry abundance, turning leptons, and Higgs, back to RH neutrinos.

The same calculation can be carried out in the DM sector, obtaining

$$\dot{Y}_{\Delta \chi} = \gamma_{D} \left[ \varepsilon_{\chi} \left( \frac{Y_{N_1}}{Y_{eq}} - 1 \right) - \frac{\text{Br}_{l} Y_{\Delta \chi}}{2 Y_{eq}} \right].$$

(4.17)

to which the same consideration as those for Eq. (4.14) apply.

To sum up, the three BEs in the NWA read

$$Y_{N_1}' = -\gamma_{D} \frac{z}{s(z) H_{1}} \left( \frac{Y_{N_1}}{Y_{eq}} - 1 \right)$$

(4.15)

$$Y_{\Delta l}' = \frac{z}{s(z) H_{1}} \left[ \varepsilon_{l} \left( \frac{Y_{N_1}}{Y_{eq}} - 1 \right) - \frac{\text{Br}_{l} Y_{\Delta l}}{2 Y_{eq}} \right]$$

(4.16)

$$Y_{\Delta \chi}' = \frac{z}{s(z) H_{1}} \left[ \varepsilon_{\chi} \left( \frac{Y_{N_1}}{Y_{eq}} - 1 \right) - \frac{\text{Br}_{l} Y_{\Delta \chi}}{2 Y_{eq}} \right]$$

(4.17)

What we still need is to calculate in detail each coefficient appearing in the equation. That is, values for $\gamma_{D}$, Br$_l$, Br$_\chi$, $\varepsilon_l$ and $\varepsilon_\chi$ are needed and will be obtained in the following section.
4.4 Parameters of the BEs

We shall calculate the parameters needed by the BEs, i.e. the total decay rate $\Gamma_D$ and its thermal average $\gamma_D$, the branching ratios into SM Br$_l$ and DM Br$_\chi$ sectors, and the lepton and DM asymmetry parameters, $\varepsilon_l$ and $\varepsilon_\chi$.

4.4.1 Decay rate and branching ratios

One of the most important terms that appears in BE is the thermally averaged decay rate of the lightest sterile neutrino $N_1$. First, it is convenient to explicitly calculate the standard decay rate $\Gamma_D$ and then perform the thermal average.

The total decay rate is given by

$$\Gamma_D = \Gamma(N_1 \rightarrow \nu h) + \Gamma(N_1 \rightarrow \bar{\nu} h) + \Gamma(N_1 \rightarrow \chi \phi) + \Gamma(N_1 \rightarrow \bar{\chi} \phi)$$

(4.18)

We now calculate each of these terms.

As already shown in Fig. 3.2, the Standard Model part gives the following two vertices. We will assume the momentum of the incoming neutrino flows inwards the vertex, while the momenta of the outgoing particles flow outwards, as in Fig. 4.2. Therefore, Fig. 4.1 also shows the two diagram contributing to the decay of $N_1$ into the SM sector at tree level. We shall compute the square modulus of the S-matrix for the decay into neutrino and Higgs, $|M(N_1 \rightarrow \nu h)|^2$ and for the decay into anti-neutrino and Higgs $|M(N_1 \rightarrow \bar{\nu} h)|^2$.

- $|M(N_1 \rightarrow \nu h)|^2$

  Due to our assumptions on the momenta, this decay can proceed through the first diagram. Since the outgoing Dirac fermion is a particle ($\bar{u}_2$), the incoming Majorana neutrino will be a $u_1$ spinor. For brevity we shall use $\mathcal{M}(N_1 \rightarrow \nu h) \equiv \mathcal{M}_1$. We have

  $$\mathcal{M}_1 = \frac{y_1}{2} \bar{\nu}_2 (1 + \gamma_5) u_1 \quad \mathcal{M}_1^* = \frac{y_1}{2} \bar{u}_1 (1 - \gamma_5) u_2$$

  $$|\mathcal{M}_1|^2 = \frac{|y_1|^2}{4} \bar{\nu}_2 (1 + \gamma_5) u_1 \bar{u}_1 (1 - \gamma_5) u_2$$

  $$= \text{tr}\{(1 + \gamma_5) u_1 \bar{\nu}_1 (1 - \gamma_5) u_2 \bar{u}_2\}.$$
Summing over the spins and averaging over the incoming neutrino:

\[
\frac{1}{2} \sum_{\text{spin}} |\mathcal{M}_1|^2 = \frac{|y_1|^2}{8} \text{tr}\{(1 + \gamma_5)(p_1 + M_{N_1})(1 - \gamma_5)(p_2 + m_\nu)\}
\]

\[
= \frac{|y_1|^2}{8} \left\{ \text{tr} [(1 + \gamma_5)^2 p_1 (p_2 + m)] + M_{N_1} \text{tr} [(1 + \gamma_5)(1 - \gamma_5)(p_2 + m)] \right\}
\]

\[
= \frac{|y_1|^2}{8} \left\{ \text{tr} [p_1 (p_2 + m)] + \text{tr} [\gamma_5 p_1 (p_2 + m)] \right\}
\]

\[
= \frac{|y_1|^2}{4} \text{tr}\{p_1 p_2\} = |y_1|^2 p_1 \cdot p_2,
\]

where we have used the conventions that can be found, for instance, in [74].

- $|\mathcal{M}(N_1 \to \bar{\nu} h)|^2$
  Due to our assumptions on the momenta, this decay can proceed
through the second diagram. Always following Ref. [73], we have a $\bar{v}_1$ spinor for the incoming Majorana neutrino, since the outgoing Dirac fermion is an antiparticle ($\bar{\nu}$). For brevity, here $\mathcal{M}(N_1 \rightarrow \bar{\nu}h) \equiv \mathcal{M}_2$. We have

$$\mathcal{M}_2 = \frac{y_1}{2} \bar{v}_1(1 - \gamma_5)v_2 \quad \mathcal{M}_2^* = \frac{y_1^*}{2} \bar{v}_2(1 + \gamma_5)v_1$$

$$|\mathcal{M}_2|^2 = \text{tr}\{(1 - \gamma_5)v_2\bar{v}_2(1 + \gamma_5)v_1\bar{v}_1\}.$$  

Summing over the spins and averaging over the incoming neutrino and
performing the same calculations
\[
\frac{1}{2} \sum_{\text{spin}} |M_2|^2 = \frac{|y_1|^2}{8} \text{tr}\{ (1 - \gamma_5)(p_2 - m_\nu)(1 + \gamma_5)(p_1 - M_{N_1}) \}
\]
\[
= \frac{|y_1|^2}{4} \text{tr}\{ p_1 p_2 \} = |y_1|^2 p_1 \cdot p_2.
\]

The hidden sectors part of the lagrangian yields also two vertices that are identical to the previous ones. What changes is only the coupling constant, which now is \( \lambda_1 \) instead of \( y_1 \). Therefore, we just have to substitute \( \lambda_1 \) for \( y_1 \) in all the previous formulae so to have
\[
|M(N_1 \rightarrow \chi \phi)|^2 = |\lambda_1|^2 p_1 \cdot p_2,
\]
\[
|M(N_1 \rightarrow \bar{\chi} \phi)|^2 = |\lambda_1|^2 p_1 \cdot p_2.
\]

In order to obtain the decay rates, we must integrate over the 2-body phase space, which is common to all the previous processes. We have
\[
\int d\Phi_{2,3} p_1 \cdot p_2 \delta^4(\ldots) = \int d^4 p_2 d^4 p_3 \delta(p_2^2 - m^2) \theta(p_2^0) \delta(p_3^2 - M_{N_1}^2) \theta(p_3^0) \delta^4(p_1 - p_2 - p_3).
\]

After a straightforward calculation we obtain
\[
\int d\Phi_{2,3} p_1 \cdot p_2 = \frac{\pi}{4M_{N_1}} \left( M_{N_1}^2 - M_s^2 + m^2 \right) \left\{ \frac{(M_{N_1}^2 - M_s^2 + m^2)^2}{4M_{N_1}^2} - m^2 \right\}^{1/2},
\]
where \( M_s \) stands for \( M_h \) or \( M_\phi \) and \( m \) is either \( m_\nu \) or \( m_\chi \).

If we assume that both \( m \) and \( M_s \) are much smaller than \( M_{N_1} \), or that the processes take place before EWSB, which is indeed our case, we get a simpler result
\[
\int d\Phi_{2,3} p_1 \cdot p_2 \sim \frac{\pi}{4M_{N_1}} M_{N_1}^2 \left\{ \frac{M_{N_1}^4}{4M_{N_1}^2} \right\}^{1/2} = \frac{\pi}{4} M_{N_1}^2.
\]

Now we have every ingredient for a decay rate. It is defined as
\[
\Gamma = \frac{1}{(2\pi)^{3n-4}} \frac{1}{2M} \int d\Phi_{2,3,\ldots,n} |M(A \rightarrow \alpha_2, \ldots, \alpha_n)|^2,
\]
where \( 1/2M \) is the flux factor. In our case we have, for instance
\[
\Gamma(N_1 \rightarrow \nu h) = \Gamma(N_1 \rightarrow \bar{\nu} h) = \frac{1}{4\pi^2} \frac{1}{2M_{N_1}} |y_1|^2 \int d\Phi_{2,3} p_1 \cdot p_2
\]
\[
= \frac{1}{4\pi^2} \frac{1}{2M_{N_1}} |y_1|^2 \frac{\pi}{4} M_{N_1}^2
\]
\[
= \frac{1}{32\pi} M_{N_1} |y_1|^2.
\]
And similarly
\[ \Gamma(N_1 \rightarrow \chi \phi) = \Gamma(N_1 \rightarrow \bar{\chi} \phi) = \frac{1}{32\pi} M_{N_1} |\lambda_1|^2. \]

Summing up every term as in (4.18) we finally have
\[ \Gamma_D = \frac{1}{16\pi} M_{N_1} \left( |y_1|^2 + |\lambda_1|^2 \right). \tag{4.20} \]

Once we have calculated every decay rate and the total rate \( \Gamma_D \), it is easy to obtain the branching ratios into each sector. We have
\[ \text{Br}_l = \frac{|y_1|^2}{|y_1|^2 + |\lambda_1|^2}, \tag{4.21} \]
\[ \text{Br}_\chi = \frac{1}{2} \frac{|\lambda_1|^2}{|y_1|^2 + |\lambda_1|^2}. \tag{4.22} \]

What actually enters the BE is the thermally averaged total decay rate. According to Eq. (4.4) we have
\[ \gamma_D = \frac{1}{2E_1(2\pi)^3} e^{-E_1/T} \int d\Phi_{2,3}(2\pi)^4 \delta^4(\ldots) |\mathcal{M}(N_1 \rightarrow \text{anything})|^2 \]
\[ = \int \frac{d^4 p_1}{E_1(2\pi)^3} e^{-E_1/T} M_{N_1} \frac{1}{2M_{N_1}} \int d\Phi_{2,3}(2\pi)^4 \delta^4(\ldots) |\mathcal{M}(N_1 \rightarrow \text{anything})|^2 \]
\[ = M_{N_1} \int \frac{d^3 p_1}{E_1(2\pi)^3} e^{-E_1/T} \Gamma_D \]
\[ = \frac{M_{N_1} \Gamma_D}{8\pi^3} \int d^4 p_1 \frac{e^{-E_1/T}}{E_1} = \frac{M_{N_1} \Gamma_D}{2\pi^2} \int dE_1 p_1 e^{-E_1/T}, \]
where we have assumed a Maxwell-Boltzmann equilibrium distribution for \( N_1 \)
\[ f_{N_1}^{\text{eq}} \equiv e^{-E_1/T}. \]

Defining \( x = E/T \) and \( z = M_{N_1}/T \) we have
\[ \frac{M_{N_1} \Gamma_D}{2\pi^2} \int dE_1 p_1 e^{-E_1/T} = \frac{M_{N_1} T^2 \Gamma_D}{2\pi^2} \int dx e^{-x\sqrt{x^2 - z^2}}. \]

Due to the thermal bath the integral is over \([z, +\infty)\), so we can use \( K_1 \) Bessel function \( 75 \)
\[ zK_1(z) = \int_{z}^{+\infty} dx e^{-x\sqrt{x^2 - z^2}}. \tag{4.23} \]

Thence
\[ \frac{M_{N_1} T^2 \Gamma_D}{2\pi^2} \int dx e^{-x\sqrt{x^2 - z^2}} = \frac{M_{N_1} T^2 \Gamma_D}{2\pi^2} zK_1(z) = \frac{M_{N_1}^2 z^2}{2\pi^2} K_1(z) \Gamma_D. \tag{4.24} \]
In the end, the thermally averaged total decay rate is
\[ \gamma_D = \frac{M_{N_1}^2 K_1(z)}{2\pi^2 z} \Gamma_D. \] (4.25)

### 4.4.2 Asymmetry parameters

The asymmetry parameters describe the CP-violation in the decays of \( N_1 \) into particles or antiparticles. They are obtained as the difference between the decay rate into particles and into antiparticles over the total decay rate, see Def. (2.30). In the minimal model the lepton asymmetry parameter \( \varepsilon_l \) and the DM asymmetry parameter \( \varepsilon_\chi \) are given by
\[ \varepsilon_l = \frac{\Gamma (N_1 \to \nu \bar{h}) - \Gamma (N_1 \to \bar{\nu} h)}{\Gamma_D}, \]  
(4.26)
\[ \varepsilon_\chi = \frac{\Gamma (N_1 \to \chi \phi) - \Gamma (N_1 \to \bar{\chi} \phi^\dagger)}{\Gamma_D}, \]  
(4.27)

where \( \bar{N}_1 = N_1 \) since it is a Majorana fermion.

The diagram contributing to \( \varepsilon_l \) are shown in Figure 4.4. As described in Section 2.2, the asymmetry parameters arise from the interference between tree level and one-loop diagrams. The contributions to \( \varepsilon_l \) in the minimal model are the same occurring in standard leptogenesis, see Fig. 2.1, plus an additional \( N_1 \) self-energy correction involving DM particles in the loop. Following the same procedure described in Section 2.2 after a lengthy but straightforward calculation we get
\[ \varepsilon_\chi \simeq \frac{M_{N_1}}{M_{N_2}} \frac{1}{16\pi(y_1^2 + \lambda_1^2)} \left[ 2\lambda_1^2|\lambda_2|^2 \sin (2\phi_\chi) + y_1 y_2 \lambda_1 |\lambda_2| \sin (\phi_l + \phi_\chi) \right], \]
(4.28)
\[ \varepsilon_l \simeq \frac{M_{N_1}}{M_{N_2}} \frac{1}{16\pi(y_1^2 + \lambda_1^2)} \left[ 2y_1^2|y_2|^2 \sin (2\phi_l) + y_1 y_2 \lambda_1 |\lambda_2| \sin (\phi_l + \phi_\chi) \right], \]
(4.29)
where we have used the expression for $\Gamma_D$, Eq. (4.20).

Due to Defs. (4.26) and (4.27), $\varepsilon_l$ and $\varepsilon_\chi$ cannot be greater than 1,

$$|\varepsilon_l|, |\varepsilon_\chi| \leq 1.$$  \hspace{1cm} (4.30)

It can be noticed that the asymmetry parameters in Eqs. (4.28) and (4.29) differs only by the first term in square brackets. $\varepsilon_l$ depends on the SM Yukawa couplings $y_1$ and $y_2$, while $\varepsilon_\chi$ on the DM couplings $\lambda_1$ and $\lambda_2$.

Their expressions will be modified in form in the following Chapter, in order to make the numerical analysis simpler and clearer.
Chapter 5

Towards Dark Matter mass spectra

The BEs which have been written in Chapter 4 must be solved for different values of the involved parameters, i.e. in for different models. The aim of this chapter is to give an overview on the numerical and computational set-up required by this task. We will show how the BEs have been shaped in order to be numerically solved and how the original numerical code in C++ language has been developed.

Before addressing the details of the numerical solution, it is convenient to recall section 4.3. The solution will be carried out in the NWA, in which the main source of washout is represented by the inverse decays and the $2 \leftrightarrow 2$ contributions can be neglected. For this reason, we must assure that the model whose BEs are to be solved actually lies in the NWA. Thus, there will be a test in order to verify this approximation. Secondly, it must verified that the asymptotic lepton asymmetry $\Delta Y^l_\infty$ matches the observations (4.2). A minimal model that passes both these tests is acceptable and the value for $m_\chi$ can be extracted through (4.3). However, due to perturbativity bounds, $m_\chi$ must lie between 10 keV and 10 TeV, [64].

5.1 Parameters setup

The original parameter set of the model is made of seven parameters:

$$\{M_{N_1}, y_1, |y_2|, \varphi_l, \lambda_1, |\lambda_2|, \varphi_\chi \}.$$  

It can be convenient, here, to recall that $M_{N_1}$ is the mass of the decaying RH neutrino, $y_1$ and $y_2$ are the $N_1$ Yukawa coupling to the SM, while $\lambda_1$ and $\lambda_2$ are the $N_1$ couplings to the DM sector. As pointed out in Eq. (3.5), $y_1$ and $\lambda_1$ are real, while $y_2$ and $\lambda_2$ are complex, with phases $\varphi_l$ and $\varphi_\chi$ respectively.

This set can be modified in order to point out some feature of the model and better understand the abundances evolution and mass spectra.
From Eq. (4.7) we know that the branching ratios are completely determined by \( y_1 \) and \( \lambda_1 \) so that we can define a new parameter \( \omega \) as
\[
\omega \equiv \frac{\lambda_1}{y_1}.
\]
We have therefore
\[
\text{Br}_l = \frac{1}{1 + \omega^2} \quad \text{and} \quad \text{Br}_\chi = \frac{1}{1 + \omega^{-2}},
\]
while the decay width becomes
\[
\Gamma_D = \frac{M_{N_1}}{16\pi} y_2^2 (1 + \omega^2).
\]
With regard to the asymmetry parameters (4.28) and (4.29), we have
\[
\epsilon_\chi = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^2} y_2^2 \left[ 2 \frac{\lambda_2}{y_2} \sin(2\varphi_\chi) + \frac{y_1}{\lambda_1} \frac{\lambda_2}{y_2} \sin(\varphi_l + \varphi_\chi) \right]
= \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^2} y_2^2 \left[ 2 \beta \sin(2\varphi_\chi) + \omega^{-1} \beta \sin(\varphi_l + \varphi_\chi) \right], \quad (5.1)
\]
\[
\epsilon_l = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^2} y_2^2 \left[ 2 \sin(2\varphi_l) + \frac{\lambda_1}{y_1} + \frac{\lambda_2}{y_2} \sin(\varphi_l + \varphi_\chi) \right]
= \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^2} y_2^2 \left[ 2 \sin(2\varphi_l) + \beta \omega \sin(\varphi_l + \varphi_\chi) \right], \quad (5.2)
\]
where we have defined
\[
\beta \equiv \frac{\lambda_1}{|y_2|}.
\]
Thence the parameters set may now be rewritten as
\[
\{ M_{N_1}, y_1, |y_2|, \omega, \beta, \varphi_l, \varphi_\chi \}.
\]
We will carry out the solution of the BE fixing \( y_1 \) and \( |y_2| \), that is, studying the scenarios in Table 5.1.

| Scenario             | \( y_1 \) | \( |y_2| \) |
|----------------------|----------|----------|
| Strong/Strong (S/S)  | \( 10^{-3} \) | \( 10^{-3} \) |
| Strong/Weak (S/W)   | \( 10^{-3} \) | \( 10^{-5} \) |
| Weak/Strong (W/S)   | \( 10^{-5} \) | \( 10^{-5} \) |
| Weak/Weak (W/W)     | \( 10^{-5} \) | \( 10^{-5} \) |

Table 5.1: Scenarios originating from the combinations of \( y_1 = 10^{-3}, 10^{-5} \) and \( |y_2| = 10^{-3}, 10^{-5} \).
Furthermore, we have also chosen to fix the phases $\varphi_l$, $\varphi_\chi$ to $\pi/4$, so that the two contributions to $\varepsilon_\chi$ and $\varepsilon_l$ are both maximized. We call this setup \textit{Highest Asymmetry Case} (HAC):

$$
\varepsilon_\chi = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^{-2}} |y_2|^2 \left[ 2\beta^2 + \omega^{-1}\beta \right], \quad (5.3)
$$

$$
\varepsilon_l = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{1}{1 + \omega^2} |y_2|^2 \left[ 2 + \beta\omega \right]. \quad (5.4)
$$

Within HAC, we are left with only three varying parameters, $M_{N_1}$, $\omega$ and $\beta$, whose ranges are discussed below.

Other two cases are considered, in which the phases $\varphi_l$ and $\varphi_\chi$ are varied from the HAC values. In the first case we put $\varphi_\chi = 0$ and $\varphi_l = 1.4 \lesssim \pi/2$ so to eliminate the square dependence on $\beta$ in Eq. (4.29).

$$
\varepsilon_\chi \simeq \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{|y_2|^2}{1 + \omega^{-2}} \beta\omega^{-1}, \quad (5.5)
$$

$$
\varepsilon_l \simeq \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \frac{|y_2|^2}{1 + \omega^2} (0.66 + \beta\omega). \quad (5.6)
$$

This implies a weaker dependence on $\beta$ of $Y_{\Delta \chi}$ and therefore lower asymmetry values. We shall call this setup \textit{weak $\beta$ dependence}.

In the last case we put $\varphi_\chi = -\pi/12$ and $\varphi_l = \pi/4$ so to have negative values for $\varepsilon_\chi$. They will result in negative DM asymmetry abundances, which is allowed. Indeed, we do not distinguish between DM particles and antiparticles, therefore it is not compulsory to consider only $Y_{\Delta \chi} > 0$, because DM particle/antiparticle are just a matter of convention. We have

$$
\varepsilon_\chi = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \left( -\beta^2 + \frac{\beta}{2\omega} \right), \quad (5.7)
$$

$$
\varepsilon_l = \frac{1}{16\pi} \frac{M_{N_1}}{M_{N_2}} \left( 2 + \frac{1}{2}\beta\omega \right). \quad (5.8)
$$

From Eq. (5.7) we have negative values for $\varepsilon_\chi$, i.e. of $Y_{\Delta \chi}$, of $\omega > 1/2$. We shall call this setup \textit{negative asymmetry}.

## 5.2 Solution of the Boltzmann equations

The main task which the numerical code must do is solving the BEs for a given set of values for the parameters. The solution of BE requires a numerical integration routine and a routine to solve differential equation \cite{76}.

The former is needed in order to calculate the values of the number distribution of the sterile neutrino, $n_{N_1}^{eq}$, which appears in the definition of the abundance $Y_{\Delta \chi}^{eq}$. This is because it is not possible to approximate $n_{N_1}^{eq}$ either to its relativistic expression or to its non-relativistic one. Indeed, the crucial
events of thermal leptogenesis occur in the ballpark of $z \equiv M_{N_1}/T \sim 1$, that is when $N_1$ is in a “mixed” regime. On the other hand, $l$ and $\chi$ does not need any integration, because they are fully relativistic at $z \sim 1$.

Moreover, this routine is used to calculate $K_1(z)$ as it appears in Eq. (4.25). It is clear that a massive use of numerical integration in different parts of the code results in a slower programme, but it is not possible to approximate these functions.

In the calculation of the abundances as well as in the BEs (4.15)-(4.17), the entropy $s$ appears. It is defined as

$$s = \frac{2\pi^2}{45} g^{(s)} (T) T^3,$$  (5.9)

where $g^{(s)}$ accounts for the relativistic degrees of freedom:

$$g^{(s)} = \frac{7}{8} \sum_{\text{fermions}} g_f \left( \frac{T_f}{T} \right)^3 + \sum_{\text{bosons}} g_b \left( \frac{T_b}{T} \right)^3. $$  (5.10)

In our model none of the particles has already decoupled from the thermal bath, so we have no contribution from the temperature terms. In the minimal model we must take into account some differences from the full model. Since we have just one leptonic family, we have

$$g^{(s)} (T > 200 \text{ GeV}) = \frac{7}{8} \left( \nu + l + \chi + 2 + 12 \times 6 \right) + 2 \left( g + W, Z + 3 + 1 \right) = 92.25.$$

This is the only interesting value of $g^{(s)}$, since leptogenesis occurs at temperatures far higher than the electroweak symmetry breaking scale $\sim 200 \text{ GeV}$.

The set of partial differential equations is solved by a four points Runge-Kutta algorithm.

As initial conditions, at $z_{\text{start}}$ we have chosen thermal distribution for the sterile neutrino and zero abundances for lepton and DM asymmetries.

$$Y_{N_1}(z_{\text{start}}) = Y_{\text{eq}}^{N_1}(z_{\text{start}}),$$  (5.11)

$$Y_{\Delta l}(z_{\text{start}}) = 0,$$  (5.12)

$$Y_{\Delta \chi}(z_{\text{start}}) = 0.$$  (5.13)

This condition has been put very early in the history of the Universe, when $N_1$ is still relativistic. In terms of $z$, we set the starting point at $z_{\text{start}} = 0.001$, that is at a temperature that can vary from $10^{11} \text{ GeV}$ up to $10^{18} \text{ GeV}$ according to $M_{N_1}$.

Before considering the parameters range, it is convenient to study the solution of the BEs for some arbitrary values, just to get familiar with the behaviour of the abundances. Within the NWA the shape of the abundances of the species considered is generally depicted by Fig. 5.1.
Figure 5.1: Sterile neutrino, leptonic and DM abundances. In Fig. 5.2(a) the $N_1$ decay is plotted. The red line is the real $N_1$ abundance normalized to the initial one $y_{N_1}/y_{N_1}(z_{\text{start}})$, while the green one is the equilibrium abundance normalized to the initial one $y_{N_1}^e/y_{N_1}(z_{\text{start}})$. The RH neutrino evolves out of equilibrium and starts to decay until it reaches the equilibrium distribution. In Fig. 5.2(b) the red line corresponds to $y_{\Delta l}/y_{\Delta l}^\infty$, while the green one to $y_{\Delta \chi}/y_{\Delta l}^\infty$. The asymmetry abundances rise while $N_1$ decays and they then reach their asymptotic value when $N_1$ starts to decay.

shows the sterile neutrino abundance evolution (in red) in comparison with the equilibrium abundance (in green). The reactions involving the sterile neutrinos depart from equilibrium and $N_1$ abundance remains constant until decays are stronger than inverse decay and then it starts to decrease. Eventually the “real” abundance reaches the equilibrium one and they evolve together. The shape of $N_1$ abundance therefore shows a typical “drift and decay” behaviour [1].

Figure 5.2(b) shows the asymmetries abundances evolution. We have plotted their ratio to the asymptotic leptonic asymmetry abundance $y_{\Delta l}^\infty$. According to Eqs. (4.16) and (4.17), while $y_{N_1} \neq y_{N_1}^e$, that is during the departure from equilibrium $y_{\Delta l}$ and $y_{\Delta \chi}$ rise up to an asymptotic value that is reached when the neutrino is decaying and eventually reaches the equilibrium abundance so that the first term in Eqs. (4.16) and (4.17) vanishes. As we already expected, the behaviour of the asymmetries is fully controlled by the neutrino decay and we can tell when these are close to their asymptote by studying the departure from equilibrium of the reaction involving $N_1$. A rough estimate can be obtained comparing $\Gamma_D$ to the value of the Hubble’s parameter $H(z)$ [63]. For the expression of the Hubble’s parameter can adopt the one valid for a radiation dominated Universe

\[ H(z) \simeq 1.66 \sqrt{g_\star} \left( \frac{M_{N_1}}{z} \right)^2 \left( \frac{1}{M_{Pl}} \right) = 1.66 \sqrt{g_\star} \frac{M^2_{N_1}}{M_{Pl}} z^{-2}. \] (5.14)
Using Eq. (4.20) we have

\[ H(z) = \Gamma_D \Rightarrow z_{\text{eq}} = \left( \frac{M_{\text{Pl}}}{16\pi 1.66\sqrt{g_*}} \right)^{-\frac{1}{2}} \frac{1}{y_1} \sqrt{\frac{M_{N_1}}{1 + \omega^2}}, \] (5.15)

where \( M_{\text{Pl}} \) is the Planck Mass, \( M_{\text{Pl}} \sim 10^{19} \text{GeV} \). As we can see from Fig. 5.2, around \( z_{\text{eq}} \) \( N_1 \) starts to decay and the asymmetries reach their asymptotic value. As already mentioned (cfr. Sec. 4.3), the inverse decay is represented in Eqs. (4.16) and (4.17) by the terms involving the branching ratios and its effect is noticeable when \( \text{Br}_x \gtrsim 0.9 \).

In Fig. 5.3(a) the inverse decay affects the lepton sector and it can be noticed that, as in Fig. 5.1, \( Y_{\Delta l} \) stops to increase but it does not reach a limit value, rather it slightly decrease to the asymptotic value. In the meanwhile, the behaviour of \( Y_{\Delta \chi} \) is untouched by any inverse decay since \( \text{Br}_x \ll 1 \). The opposite situation is depicted in Fig. 5.3(b), where the inverse decay affects the DM abundances.

As we have seen in figure 5.2, in order to have an asymptotic value for the leptonic and DM abundances, it must be paid attention to the value of \( z_{\text{eq}} \). If this value is “great”, i.e. \( z_{\text{eq}} \gg 1 \), the asymptote is reached very late and a long time is needed. Moreover, for \( z_{\text{eq}} \gg 1 \) it can happen that the sterile neutrino abundance \( Y_{N_1} \) becomes too little, even below the computational limit allowed by the hardware and software (i.e. \( 10^{300} \)). For this reason we have chosen to consider in the following analysis only those case in which \( z_{\text{eq}} < 100 \). In addition to that, care must be taken also to \( z_{\text{eq}} \) lower limit. Indeed, \( z_{\text{eq}} \) must be greater than the initial value \( z_0 = 0.001 \), hence we have chosen \( z_{\text{eq}} > 0.1 \), so that the abundances have enough time to evolve. To sum up, a test is carried out on the value of \( z_{\text{eq}} \) and the model is accepted if

\[ 0.1 < z_{\text{eq}} < 100. \] (5.16)
Figure 5.3: Inverse decay in lepton (a) and DM abundances (b). In Fig. 5.3(a) inverse decay affects the lepton asymmetry. The lepton abundance reaches a maximum which is not its asymptotic value, because it slightly decrease to a lower value. This is due to a depletion of the lepton number density due to the inverse decay. In Fig. 5.3(b) inverse decay affects DM asymmetry in the same way.

5.2.1 Details of the numerical code and parameters ranges

The integration routine is the core of the numerical code. It is inserted into a programme whose output are the mass spectrum of DM mass with respect to $\omega$ and $\beta$ and the spectrum of $m_\chi$ vs $M_{N_1}$ too. The latter is straightforwardly obtained from the former by sorting the output file.

Since we are left with three varying parameters, the code structure is represented by three nested loop. The outer one is a for loop on the mass values, which increases logarithmically, inside there is a while loop on $\omega$ and the inner one is again a while loop on $\beta$. Immediately after entering the second loop, we can calculate the value of the decay width $\Gamma_D$ since we have a value for $y_1$ (fixed), $M_{N_1}$ (first loop) and $\omega$ (second loop). The earlier $\Gamma_D$ is obtained, the better it is, since it is possible to verify the NWA condition. If this is not satisfied, the code skips the inner loop and the integration routine too.

It is also possible to calculate $z_{eq}$ and verify if it lies within the boundary mentioned in the previous section. If so, the programme enters the $\beta$ loop and calculate $\varepsilon_l$, $\varepsilon_\chi$, otherwise it skips the loop and the integration routine.

In order to gain a better understanding of the spectra, we have chosen at first a quite narrow range for the sterile neutrino mass $M_{N_1} \in [10^9, 10^{11}]$, so that a finer logarithmic scan is allowed.

The range of $\omega$ parameter is generally $[10^{-4}, 10]$, but, as we will see, it is often cut during the runtime due to NWA or $z_{eq}$ boundaries. The step on $\omega$ is logarithmic as well. The upper bound to $\omega$ is fixed by the NWA condition.
Indeed, using \((4.20)\), we have from the first condition in Eq. \((4.13)\)
\[
\frac{1}{16\pi} (y_1^2 + \lambda_1^2) M_{N_1} \ll M_{N_1} \Rightarrow y_1, \lambda_1 \ll 1,
\]
which, considering \(y_1\)'s values, gives roughly \(\omega \lesssim 10^4\).

Eventually, \(\beta\)'s lower bound is \(10^{-7}\) while its upper bound is chosen each time so that \(\varepsilon_l\) and \(\varepsilon_\chi\) are smaller than \(1\) \((4.30)\), i.e. \(\beta\) starts from \(10^{-7}\) and grows up while \(\varepsilon_l, |\varepsilon_\chi| < 1\). The ranges chosen for the parameters are summarized in Table 5.2. The step of \(\beta\) is variable because the value of \(\beta\) is adjusted so that \(Y_\Delta^\infty\) (calculated as follows) lies between the experimental bounds. The programme starts form a low value, that usually yields \(Y_\Delta^\infty < Y_{\Delta l}^{\exp}\). If the starting \(Y_\Delta^\infty\) is already greater than the experimental value, \(\beta\) is decimated. Then, \(\beta\) is incremented logarithmically. If the asymptotic lepton asymmetry exceeds \(Y_{\Delta l}^{\exp}\) the previous value of \(\beta\) is recovered and the step is divided by ten. The process is then repeated, with the new step, until an acceptable value is found. In this way, for given values of \(M_{N_1}\) and \(\omega\), an acceptable \(Y_\Delta^\infty\) is obtained.

Before integrating the BEs, it is possible to avoid some useless calculations thanks to an estimate of the asymptotic values for \(Y_{\Delta l}\). At least an upper bound to the order of greatness of the asymptotic value can be given by the value of the abundance at \(z_{eq}\). This can be obtained by choosing \(dz = z_{eq}\), so that a single run of the integration routine can give an approximate value of \(Y_{\Delta l}(z_{eq})\). This is better approximate if \(z_{eq}\) is closer to \(z_0\). If \(Y_{\Delta l}(z_{eq}) > Y_{\Delta l}^{\exp}\) it is useless to integrate BEs with a fine step, because \(Y_{\Delta l}^{\infty} > Y_{\Delta l}^{\exp}\) and the model will not be accepted. Anyway, it cannot be a strict condition, since in some cases the inverse decay is considerable and so \(Y_{\Delta l}\) tends to decrease a bit after \(z_{eq}\). As mentioned, the condition can be on the order of greatness.

After this last test, the integration routine starts. Integration is carried out with a little step in order to minimise the integration error. Generally, we have \(dz = 0.005\), which is increased to \(dz = 0.1\) if \(z_{eq} > 10\). This because if \(z_{eq}\) is closer to \(z_0\) a finer mesh is needed to obtain a reliable value at \(z_{eq}\) and after.

The integration is performed until a constancy condition for both the asymmetries is reached. At each integration step the derivative of the abundances
is calculated and compared to a threshold value. If it is smaller than the latter the abundance can be considered as constant and the integration can be stopped. The choice of the threshold does not influence very much the result, and a value of $10^{-5}$ has been used. This condition avoids wasting of time since once the asymptotic value is reached the abundances remain constant.

The resulting lepton asymmetry value is then compared to the experimental one. The error is assumed to be on the first decimal digit, that is, considering (4.2), the model is accepted if

$$2.5 \cdot 10^{-10} \leq \frac{Y}{\Lambda} \leq 2.7 \cdot 10^{-10}. \quad (5.17)$$

In this case the value of $m_{\chi}$ can be calculated using Eq. (4.3).

The test on $m_{\chi}$ are not performed in the code. Any value obtained is then plotted in the graphs of the various spectra.

In order to speed up the programme, the code has been parallelised. The outer loop, the for loop on $M_{N_1}$, has been split on all the available CPU cores. Each iteration is assigned to a core so that the calculation is carried out in parallel, resulting in a smaller computation time. The parallelisation has been performed by means of OpenMP. This is an application programme interface that explicitly directs multi-threaded, shared-memory parallel programming [77]. It is directly specified for C/C++ and Fortran and most major platforms have been implemented, including Unix/Linux platforms and Windows NT.

Parallelisation is obtained through the use of compiler directives, that are already embedded in C/C++ codes, whose general structure is

```
#pragma omp <directive_name> [clause,...]
```

In our case a for loop has been parallelised, therefore the compiler directive looks like

```
#pragma omp parallel for shared(<v_name>,..) schedule(dynamic)
```

where shared and schedule are the optional clauses of the directive. The former specifies the variables to be shared among the threads, i.e. those that can be read and written by all threads. All the other variables are private, they are allocated independently by each thread. The last clause specifies how iterations of the loop are divided among the threads. dynamic indicates that when a thread finishes an iteration it is dynamically assigned another. The compiler directive has no effect if nothing is specified in the compiling command line. To enable the parallelisation, option -fopenmp must be added:

```
$ g++ -o <executable_name> -fopenmp <file_name> .
```

Parallel programming has great advantages, above all in codes that have many loops. The time gain is noticeable: roughly the computing time is divided by the number of available cores.
Chapter 6

Results and conclusions

In this chapter we shall review the results obtained by solving the BEs in the NWA. The aim of the simulations carried out are mass spectra for the DM candidate \( \chi \) plotted against the relevant parameters, i.e. \( \omega \) and \( \beta \), and the sterile neutrino mass \( M_{N_1} \). The spectrum of \( m_\chi \) with respect to \( M_{N_1} \) is important in order to show the range of value that DM mass can acquire depending on the mass of the heavy neutrino. As mentioned in the introduction, this range is in principle quite large, that is very light and very heavy DM candidate can be both predicted by the same model. However, we shall see that high mass values cannot fully reached within the NWA approximation.

In order to better understand the behaviour of the abundances with respect to the simulation parameters, it is convenient to first consider the other plot, against \( \omega \) and \( \beta \). This allows us to notice some features of the model and understand the spectrum characteristic.

We shall carry out this study in the Narrow Width Approximation (see Sec. 4.3). We shall consider first the Highest Asymmetry case and then we shall vary the phases into the weak \( \beta \) dependence and the negative asymmetry cases (see Sec. 5.1).

6.1 Highest asymmetry

In the Highest Asymmetry case we set both the phases to \( \pi/4 \) so that the value of the asymmetry parameters reaches its maximum with respect to \( \varphi_1 \) and \( \varphi_\chi \). The moduli of \( y_1 \) and \( y_2 \) can assume independently two values, thus generating the four scenarios shown in Table 5.1. In the HAC the asymmetry parameters are always positive.

We shall take into detailed consideration the Strong/Strong scenario, and then extend the study to the other situations.
6.1.1 Strong/Strong

The Strong/Strong scenario is named after the strong coupling of the sterile neutrinos $N_1$ and $N_2$ with the SM particles. The highest value of the SM coupling $y_1$ is defined by the NWA condition. Using (4.20) and (5.14) with $z = 1$ we have

$$ \frac{1}{(16\pi)^2 M_{N_1}} \frac{(y_1^2 + \lambda_1^2)^2 M_{N_1}^2}{M_p^2} \ll 1. $$

Considering $1.66(16\pi)^2 \sqrt{g_*} \sim 10^4$ and $M_{N_1} \sim 10^{10}$ we get

$$ \frac{10^{10}}{10^4} (y_1^2 + \lambda_2)^2 \ll 1, $$

and so, choosing

$$ \frac{10^{10}}{10^4} (y_1^2 + \lambda_2)^2 \lesssim 10^{-2}, $$

we have

$$ (y_1^2 + \lambda_1^2)^2 \lesssim 10^{-8}. $$

Thence we can put an upper bound to $N_1$ SM coupling

$$ y_1 \lesssim 10^2. \quad (6.1) $$

However, $y_1 = 10^{-2}$ results into too low values of $z_{eq}$, which becomes smaller than the limit in Eq. (5.16). Therefore, we choose $y_1 = 10^{-3}$ as the value for the strong SM coupling. In the Strong/Strong scenario we also put $y_2 = 10^{-3}$. The mass spectrum of DM candidate with respect to the simulation parameters is plotted in Fig. 6.1 and 6.2. The dots are divided into four different classes corresponding to four different ranges for $M_{N_1}$. Two figure are shown in order to make clear the shape of the surface described by the DM mass values. The borders of this surface are determined by comparing the asymptotic lepton asymmetry with the experimental data, according to Eq. (5.17).

The surface obtained is quite complex and it shows two distinct regimes. For low values of $\omega$ it can be approximated by a plane. The blank spaces between the stripes corresponding to a fixed $\omega$ value are due to the logarithmic scale on all the axes. At $\omega \sim 1$ the plane starts to wrap showing a curve behaviour. This can be seen clearly by considering sections of the surface at different $\omega$. In Fig. 6.3(a), 6.3(b) and 6.3(c) sections at $\omega = 0.001$, $\omega = 3$ and $\omega = 7$ are plotted respectively. Thence we can notice that the surface gets a curve shape for high $\omega$ values, gaining a maximum for $M_{N_1} \sim 10^{10}$. In the last section ($\omega = 7$) $m_\chi$ spectrum is empty for light RH neutrinos.

In order to understand how this surface is generated by the BEs, it is more convenient to consider the two regimes separately, that is, we shall study the case with $\omega < 1$ firstly, and then the other with $\omega > 1$. 
Figure 6.1: DM mass candidate spectrum plotted with respect to simulation parameters $\omega$ and $\beta$. Front view.

$\omega < 1$ case physically correspond to weak $N_1$ DM coupling. Recalling Def. (1.2), we have

$$\frac{\lambda_1}{y_1} \ll 1 \implies \lambda_1 \ll y_1,$$

that is RH neutrino $N_1$ is very weakly coupled to the DM candidate $\chi$. This clearly explains why $Br_\chi$ is so small in this regime. The very low branching ratio in the DM and the strong statistical suppression make the inverse decay highly ineffective, while the direct decay is less suppressed. Therefore, $N_1$ decays mainly in the SM sector, but the small fraction that decays into the DM sector yields an enhanced asymmetry thanks to $\varepsilon_\chi$. The asymmetry parameters depend on the values of $N_2$ couplings to SM and DM sector, respectively given by $y_2$ and $\lambda_2 = |\beta| |y_2|$. So, the effects of the heavier RH neutrino $N_2$ are important in defining the size of the asymmetries, by virtue of its presence in the loop corrections of the process amplitude, see Fig. 4.4.

As it can be seen in Fig. 6.1 and 6.2 for $\omega < 1$ the surface, in logarithmic scale, is well approximated by a plane. The highest values of the DM candidate are obtained for heavy RH neutrinos, but light neutrino cases occupy a larger part in parameter space. Different $M_{N_1}$ ranges describe neat hori-
Figure 6.2: DM mass candidate spectrum plotted with respect to simulation parameters $\omega$ and $\beta$. Back view.

horizontal stripes and they overlap on a small slice. This means that somehow, given a value for the RH neutrino mass, the model finds the right values for $\omega$ and $\beta$ that yields the same final DM asymmetry. However, the only test on the abundances which the model must pass is about the final lepton asymmetry, while $Y_{\Delta\chi}$ follows. Therefore, it must be that the accepted models all have the same asymptotic lepton asymmetry which corresponds to the same asymptotic DM asymmetry and hence to the same value of $m_\chi$.

Let us see how it can happen.

For $\omega < 1$, according to Eq. (4.20), we have

$$\Gamma_D = \frac{1}{16\pi} y_1^2 (1 + \omega^2) M_{N_1} \sim \frac{1}{16\pi} y_1^2,$$

(6.2)

that is the total decay rate $\Gamma_D$ does not depend on $\omega$ anymore. Hence, following Def. (4.25) and (4.4), $\gamma_D$ does not depend on $\omega$ as well. This implies that, fixed a value of $M_{N_1}$, the BE describing the RH neutrino abundance in Eq. (4.15) gives the same evolution for any value of $\omega$ and $\beta$. A careful examination of the terms appearing in the BEs for $Y_{\Delta\chi}$, Eqs. (4.16) and (4.17) shows that for $\omega < 1$ the second term in each equation is constant. They depend on the branching ratio of $N_1$ and, recalling Def. (4.21) and
CHAPTER 6. RESULTS AND CONCLUSIONS

(a) DM spectrum surface section at $\omega = 0.001$.

(b) DM spectrum surface section at $\omega = 3$.

(c) DM spectrum surface section at $\omega = 7$.

Figure 6.3: DM mass spectrum sections at different value of $\omega$. 
CHAPTER 6. RESULTS AND CONCLUSIONS

(4.22), we have

\[ \text{Br}_l = \frac{1}{1 + \omega^2} \rightarrow 1, \quad \text{as } \omega \to 0 \]  
(6.3)

\[ \text{Br}_\chi = \frac{1}{1 + \omega^{-2}} = 1 - \text{Br}_l \rightarrow 0. \quad \text{as } \omega \to 0 \]  
(6.4)

The behaviour of \( \text{Br}_l \) and \( \text{Br}_\chi \) is shown in Fig. 6.4.

Figure 6.4: Behaviour of the branching ratios \( \text{Br}_l \) and \( \text{Br}_\chi \) with respect to \( \omega \). Logarithmic scale on both axes is assumed.

The first term in Eq. (4.16) and in Eq. (4.17) depends on the behaviour of the RH neutrino abundance through the factor \( Y_{N_1}/Y_{N_1}^{eq} - 1 \), which actually is constant throughout the parameter space due to the independence of Eq. (4.15) from \( \omega \) and \( \beta \). Therefore, the only values that change with the parameters are the asymmetry parameters \( \varepsilon_l \) and \( \varepsilon_\chi \). The asymptotic lepton asymmetry generated in a given value with \( \omega < 1 \) depends only on the mass of the RH neutrino and the asymmetry parameter \( \varepsilon_l \). Thus, given a value of \( M_{N_1} \) we have to find \( \omega \) and \( \beta \) so that the model is acceptable. That is, we can calculate the trajectory in the parameter space that a model with a given \( M_{N_1} \) must follow in order to have always suitable values for \( \varepsilon_l \).

To be more precise, given a value for \( \gamma_D \varepsilon_l \) that yields an acceptable lepton asymmetry value, it is possible to compute the path \( \Upsilon \) in the plane \( \omega, \beta \) that must be followed in order to have other accepted models with different \( \omega \) and \( \beta \). That is

\[ \Upsilon \equiv \{ (\omega, \beta) \mid \varepsilon_l \gamma_D = \text{const.} \}. \]  
(6.5)

\( \Upsilon \) trajectory is shown in Figure 6.5. Along the red line, then, models with that given RH mass are accepted. To see how DM asymmetry behaves we must consider \( \gamma_D \varepsilon_\chi \) and the values it assumes along the trajectory just
determined. This is plotted in Figure 6.6. It can be noticed that along $\Upsilon$ also $\gamma_D \varepsilon_l$ takes constant values thus originating the same DM asymmetry. Therefore, we can conclude that for $\omega < 1$, given $N_1$ mass, every model accepted not only has the same value for $\gamma_D \varepsilon_l$, but also the same value for $\gamma_D \varepsilon_\chi$, which yields the same value for $Y_\infty^\Delta \chi$. Thence, for $\omega < 1$ dots in Fig. 6.1 and 6.2 corresponding to a fixed value of $M_{N_1}$ are plotted along $\Upsilon$-like trajectories, as can be seen in Fig. 6.7. This clearly explains how horizontal stripes are formed for $\omega < 1$. In Figure 6.7, $m_\chi$ values corresponding to $\Upsilon$ trajectory with $\omega \gtrsim 0.1$ are also plotted to show that they do not match with the simulation ones, because the latter bend away from $\Upsilon$. This is because Eq. (6.2) does not hold anymore and thence the reasoning carried out so far is no more valid.

The ordering of $M_{N_1}$ horizontal stripes and the bend of $\omega = \text{const.}$ vertical stripes are due to the behaviour of the asymmetry parameters. For low $M_{N_1}$ values, in order to reach a suitable value for $\gamma_D \varepsilon_l$ so that the model can be accepted, higher values of $\beta$ are needed. Therefore, models with low RH neutrino masses can be found for relatively high value of $\beta$. High values of $\beta$
causes, according to Eq. (5.3), high values of \( \varepsilon_\chi \), which depends on \( \beta^2 \), and so high DM asymmetries are generated. By means of Eq. (4.3) these values correspond to low \( m_\chi \). Increasing \( M_{N_1} \) lower values of \( \beta \) are enough to accept the model, therefore lower DM asymmetries, i.e. higher DM masses, are generated. This process can be seen even better by considering the evolution of the DM asymmetry abundance \( Y_\Delta \chi \) for various \( M_{N_1} \) values, at fixed \( \omega \). It is plotted in Fig. 6.8. It can be noticed that higher asymmetry values correspond to higher RH neutrino masses. The abundances reach their maximum value and remain constant. Indeed, due to \( Br_\chi \sim 0 \), the inverse decay term in Eq. (4.17) is completely negligible. Hierarchy in \( M_{N_1} \) is inversely reflected by \( Y_\Delta \chi \): low \( N_1 \) mass models yields higher DM abundances than those obtained in heavy RH neutrino models. This is due to the combination of the factor multiplying the square parenthesis in Eqs. (4.16), (4.17) and the value of \( z_{eq} \). Considering first the multiplying factor, we have

\[
\mathcal{F} \equiv \gamma_D \frac{z}{s(z)H_1}.
\]

We can expand the terms in \( \mathcal{F} \) pointing out their dependence on \( M_{N_1} \). Re-
Figure 6.7: DM mass spectrum plotted with $\Upsilon$ trajectory corresponding to $M_{N1} = 5 \cdot 10^{10}$. $m_\chi$ values given by models accepted along $\Upsilon$, with $M_{N1} = 5 \cdot 10^{10}$ and $\omega \lesssim 0.1$, are aligned along the blue line. Curve dotted line is made of $m_\chi$ values corresponding to $\Upsilon$ trajectory with $\omega \gtrsim 0.1$. The departure from the actual simulation values shows that this reasoning is no more valid for $\omega \gtrsim 0.1$.

calling Defs. (4.25), (4.20), (5.9) and (5.14) we have

$$\mathcal{F} \propto \frac{M_{N1}^3}{M_{N1}^2 M_{N1}^3} = \frac{1}{M_{N1}}, \quad (6.7)$$

which tells us that the slope of the $Y_{\Delta \chi}$ curve is inversely proportional to the mass of the RH neutrino. So, though both asymmetries start from zero value, due to the chosen initial conditions Eqs. (5.11)-(5.13), the light $N_1$ models reach immediately a higher asymmetry and then evolve with a slightly steeper slope. This result must be kept together with a $z_{\text{eq}}$ analysis. Recalling Def. (5.15) we can see that heavy $N_1$ yields high $z_{\text{eq}}$ values, that is the asymptotic value of the asymmetries is reached later. This also implies that also the peak in the asymmetry derivative, corresponding to the maximum value reached by $Y_{N1}/Y_{\text{eq}}^{N1} - 1$, is reached later, see Fig. 6.9. Therefore, we can understand why low $M_{N1}$ models get high $Y_{\Delta \chi}$ values before than high
CHAPTER 6. RESULTS AND CONCLUSIONS

Figure 6.8: Evolution of DM asymmetry abundances for different values of RH neutrino mass, at fixed $\omega = 0.001$. It can be noticed that $Y_{\Delta \chi}$ hierarchy plainly reflects $M_{N_1}$ hierarchy. No inverse decay effects are present, due to $Br_{\chi} \sim 0$.

The colours of the lines try to recall the $M_{N_1}$ classes in the previous plots.

This implies that SM branching ratio is smaller than the DM one and so the total decay rate is due to the decay into DM sector. According to Eq. (4.20), we have

$$\Gamma_D = \frac{1}{16\pi} y_1^2 (1 + \omega^2) M_{N_1} \sim \frac{1}{16\pi} y_1^2 \omega^2 = \frac{1}{16\pi} \lambda_1^2,$$

therefore BEs are now highly sensible to $\omega$ and the final asymmetries strongly depends on this parameter. From Fig. 6.1 and 6.2 it can be noticed, as already mentioned, that the $m_\chi$ surface completely changes behaviour. The horizontal stripes described above are lost and DM mass shows a maximum which is not coincident with one of the two boundaries of $M_{N_1}$ range as it happens for $\omega < 1$. The most interesting feature is that high and low RH neutrino masses, i.e. $M_{N_1} \lesssim 5 \cdot 10^9$ GeV and $5 \cdot 10^{10}$ GeV $< M_{N_1} < 10^{11}$ GeV, gives similar low values of $m_\chi$, while the maxima are reached for $M_{N_1} \sim 10^{10}$ GeV. This is naturally reflected by the asymmetry abundances evolution.
Figure 6.9: Evolution of the DM asymmetry abundance derivative $Y'_{\Delta \chi}$ with respect to $z$ for $\omega = 0.001$, $M_{N_1} = 3 \cdot 10^9$ GeV and $M_{N_1} = 10^{11}$ GeV. It is clear the difference between the peaks in the derivative for different $N_1$ mass values. Vertical lines show the values of $z_{eq}$ in the two cases, showing that the peak is reached well before in light $N_1$ models.

as in Fig. 6.10. It can be noticed that the lowest DM asymmetry values are obtained for intermediate RH neutrino masses $M_{N_1} \sim 7 \cdot 10^9$ GeV, while light and heavy $N_1$ gives almost the same $Y_{\Delta \chi}$ values. The ordering of the asymmetry abundances is the same as in Fig. 6.8 for $z < 1$, while for $z \gtrsim 1$ the asymmetries begin to interweave. This means that until $z \sim 1$ we have the same behaviour described for the $\omega \ll 1$ case, i.e. light $N_1$ reach their pronounced peak before heavy $N_1$ arrive at their low one. For $z > 1$, however, things changes because of inverse decay. As already mentioned, inverse decay (ID) is related to the second term in the asymmetry BEs. To be significative it must be greater than the direct decay (DD) term (corresponding to the first term in the BEs), causing a decrease of the abundance. In Figs. 6.11(a) and 6.11(b) DD and ID terms are plotted for light and heavy RH neutrino. DD term for $M_{N_1} = 3 \cdot 10^9$ GeV grows up almost immediately, reaching its peak. On the contrary, for $M_{N_1} = 10^{11}$ GeV takes little more time to arrive at its maximum which, anyway, is two order of greatness smaller than in light $N_1$ case. This is perfect agreement with the previous explanation that holds for $\omega = 0.001$. In Figs. 6.11(a) and 6.11(b) it can be noticed that the ID decay term get values which are similar to the DD ones. More precisely, for $M_{N_1} = 3 \cdot 10^9$ GeV DD rises before ID but it is then overtook by ID whose maximum is higher than DD one. After they reach their peaks,
they get closer and closer while evolving to zero. This explains why for \( M_{N_1} = 3 \cdot 10^9 \text{ GeV} \) DM abundance shows an evident inverse decay effect: after the usual abundance rise, \( Y'_{\Delta \chi} \) takes negative values causing a decrease in the abundance until its asymptotic values. The overall derivative is plotted in Fig. 6.12. On the contrary, in Fig. 6.11(b) it can be noticed that for heavy \( N_1 \) the ID does not manage to overtake DD, but its values are smaller in the peak region and, then, similar in the high \( z \) tail. This is reflected in Fig. 6.12 by the fact that the curve corresponding to \( M_{N_1} = 10^{11} \text{ GeV} \) reaches a large, low peak and then goes to zero, causing no decrease in the DM abundances. The different behaviour of light and heavy RH neutrino is due to the first part of abundances evolution in which \( Y_{\Delta \chi} \) rise to their maxima (rise part). Indeed, in this phase, as we have seen, light \( N_1 \) models show higher asymmetry values than heavy \( N_1 \) ones. The ID term in Eq. (4.17) is given by

\[
\gamma_D Br_{\chi} \frac{Y_{\Delta \chi}}{2Y_{\chi}^{eq}},
\]

that is, it is directly proportional to the DM asymmetry abundance value. Therefore, if \( Y_{\Delta \chi} \) rise to high values, the ID term gets also high values that can be larger than the DD term in the following evolution. This generate the light \( N_1 \) “bouncing” behaviour shown in Fig. 6.12 considering the total

Figure 6.10: Evolution of DM asymmetry abundances for different values of RH neutrino mass, at fixed \( \omega = 3.01 \). \( Y_{\Delta \chi} \) corresponding to different \( N_1 \) masses overlap, spoiling the hierarchy found before. Low \( N_1 \) masses are affected by large inverse decay effects, while high \( N_1 \) masses are unchanged. The colours of the lines try to recall the \( M_{N_1} \) classes in the previous plots.
CHAPTER 6. RESULTS AND CONCLUSIONS

Figure 6.11: DD and ID term plots for $\omega = 3$ and different $M_{N_1}$ values. For $M_{N_1} = 3 \cdot 10^9$ GeV, Fig. (a), ID reaches a higher peak than DD, so causing negative $Y'\Delta\chi$ values. In the high $z$ tail both terms take the same values and go to zero together. For $M_{N_1} = 10^{11}$ GeV, Fig. (b), ID takes values that are always lower than DD ones, thus $Y'\Delta\chi$ is always positive. In the high $z$ tail the two terms take the same values and go to zero together.
CHAPTER 6. RESULTS AND CONCLUSIONS

Figure 6.12: Evolution of the DM asymmetry abundance derivative $Y'_{\Delta \chi}$ with respect to $z$ for $\omega = 3$, $M_{N_1} = 3 \cdot 10^9$ GeV and $M_{N_1} = 10^{11}$ GeV. The analysis for the rise part is the same as that for $\omega = 0.001$. After the peak, the red curve gets negative values and then goes to zero, showing a characteristic “bouncing” behaviour. This behaviour of the magenta curve is much different, because it is always positive. The “bounce” for heavy $N_1$ is damped. This explains why light $N_1$ are affected by large inverse decay effects, while heavy $N_1$ are not.

DM derivative: the immediate rise of $Y'_{\Delta \chi}$ causes high values of $Y_{\Delta \chi}$, so enhancing the ID term, which causes a drop of the derivative to negative values. Clearly, this bounce do not take place for heavy $N_1$ because of the lower rise phase, which is not enough to trigger high inverse decay effects.

Going back to Fig. 6.9, we can notice the absence of any bounce, even for light RH neutrino masses. This is obviously due to the strong suppression of ID term owing to the branching ratio, as it can be seen in Figs. 6.13(a) and 6.13(b).

The absence of ID effects in simulations of models with heavy RH neutrino is in perfect agreement with physics. As seen for instance in Fig. 6.11(a), ID becomes relevant for $z \gtrsim 1$, i.e. when $N_1$ is already non-relativistic. In this situation, the processes $\chi \phi \rightarrow N_1$ and its conjugated are suppressed by Boltzmann factor.

A final consideration must be done about the $\omega > 1$ case. Comparing Fig. 6.3(b) and 6.3(c) we can notice that increasing $\omega$ value we lose the models with light $N_1$ masses. Indeed, in Fig. 6.3(c) red dots corresponding to $M_{N_1} < 5 \cdot 10^9$ GeV are completely absent. The disappearance of light $N_1$
CHAPTER 6. RESULTS AND CONCLUSIONS

(a) DD and ID for $M_{N_1} = 3 \cdot 10^9$ GeV and $\omega = 0.001$

Figure 6.13: DD and ID term plots for $\omega = 0.001$ and different $M_{N_1}$ values. Here logarithmic scale is adopted on both axes, in order to show the difference in the order of greatness between DD and ID terms. It is clear that for $\omega = 0.001$ for both light and heavy $N_1$ no inverse decay effects can take place due to the highly suppressive value of the branching ratio.
models for high $\omega$ values is due to the condition put on $z_{eq}$ in Eq. (5.16). In Fig. 6.14(a) we plot $z_{eq}$ with respect to $M_{N_1}$ and $\omega$. It can be seen that for high $\omega$ values and light $N_1$, $z_{eq}$ acquires values which are below the lower bound $z_{eq} \geq 0.1$. Therefore, increasing $\omega$, we are left with only the higher $M_{N_1}$ models. It is well depicted by Fig. 6.14(b), where the higher $\omega$, the narrower allowed $M_{N_1}$ range becomes. For $\omega \gtrsim 100$, only models with $M_{N_1} > 10^{11}$ GeV are acceptable.

Besides $m_\chi$ spectrum against the simulation parameters $\omega$ and $\beta$, it is interesting to take into consideration the plot of DM mass versus RH neutrino masses, Fig. 6.15. In this case, dots corresponding to $m_\chi$ are divided into classes with respect to the branching ratio into the DM sector. It is possible to explain the structures in Fig. 6.15 thanks to the analysis carried out so far about the other spectrum (Fig. 6.1). It can be convenient to write down the synopsis between $\text{Br}_\chi$ and $\omega$ intervals (see table 6.1). Low $\omega$'s correspond to

\begin{table}[h]
\centering
\begin{tabular}{ |c|c| }
\hline
$\text{Br}_\chi$ & $\omega$ \\
\hline
$\text{Br}_\chi < 0.1$ & $\omega < 0.33$ \\
$0.1 < \text{Br}_\chi < 0.9$ & $0.33 < \omega < 3$ \\
$0.9 < \text{Br}_\chi < 0.99$ & $3 < \omega < 9.95$ \\
$\text{Br}_\chi > 0.99$ & $\omega > 9.95$ \\
\hline
\end{tabular}
\caption{Relation between $\text{Br}_\chi$ and $\omega$ intervals.}
\end{table}

green dots in Fig. 6.15 and part of the blue ones. It is evident the continue rise of $m_\chi$ with respect to $M_{N_1}$, reflected by the inclination of the plane in Fig. 6.1. Magenta dots correspond to high $\omega$ values and they reflect the curvature of the surface in Fig. 6.1 showing a maximum for intermediate $N_1$ masses, i.e. $M_{N_1} \sim 10^{10}$ GeV. Red dots correspond to very high $\omega$'s and, as described before and as shown in Figs. 6.14(a) and 6.14(b), they can be found only for high $N_1$ masses, due to $z_{eq}$ lower cut.

As a final remark on $m_\chi$ spectrum for the Strong/Strong case in the HAC scenario, we can notice that DM mass is well comprised within the perturbativity bound mentioned in Sec. 3.2

$$10 \text{keV} \lesssim m_\chi \lesssim 10 \text{TeV}.$$ (6.9)

Actually, $m_\chi$ values fall into a rather narrow interval:

$$10 \text{MeV} < m_\chi < 100 \text{MeV}.$$ (6.10)

In order to widen the DM mass range we must consider the other scenarios exposed at the beginning of this chapter. We must also take into account that our analysis is carried out in the NWA, Eq. (4.13), and therefore we cannot expect a full coverage of the entire range allowed by perturbativity constraint, Eq. (6.9).
(a) $z_{eq}$ values with respect to $M_{N_1}$ and $\omega$. The surface is cut at $z_{eq} = 0.1$.

(b) $M_{N_1}$ and $\omega$ values corresponding to $z_{eq} > 0.1$. The green region is made of models allowed by $z_{eq}$ constraint Eq. (5.16).

Figure 6.14: Models with $z_{eq} \geq 0.1$. 
6.1.2 Strong/Weak

In the Strong/Weak setup $N_1$ Yukawa coupling to SM is still $y_1 = 10^{-3}$, while $N_2$ Yukawa coupling is reduced to $y_2 = 10^{-5}$. The DM mass spectrum against $\omega$ and $\beta$ we get in this case is shown in Fig. 6.16. The general shape of the surface is similar to that in the previous case, and the analysis carried out before holds here as well. However, it can be noticed that $m_\chi$ values obtained are far smaller than those obtained in the Strong/Strong case. This is because in order to obtain acceptable $Y_{\Delta l}$, due to $y_2$ suppression, high $\beta$ values are required. From Eq. (5.3) we know that $\varepsilon_\chi$ is proportional to $\beta^2$ which causes high DM asymmetry values. We get DM mass values that lies in

$$10 \text{ eV} < m_\chi < 100 \text{ MeV}. \quad (6.11)$$

Thence, we must operate a cut, rejecting $m_\chi$ values which are below the lower bound in Eq. (6.9).

6.1.3 Weak/Strong

In the Weak/Strong setup we have $y_1 = 10^{-5}$ and $y_2 = 10^{-5}$. This results in lower $N_1$ decay rate $\Gamma_D$ and so in a longer OoE drift of the RH neutrino. Since, as we have seen, the OoE part of the neutrino evolution (i.e. from its
Figure 6.16: DM mass candidate spectrum plotted with respect to simulation parameters $\omega$ and $\beta$.

departure to its re-attachment to the equilibrium distribution) correspond to the rising phase of the asymmetry evolutions, this also result into a longer rise phase. This is often “too” long, because of the limit on the simulation due to the computing precision. Indeed, while asymmetries grow towards their asymptote, $Y_{N_1}$ decreases to smaller and smaller values, often exceeding the lower precision limit of both the software and the hardware. This is the reason for the upper bound on $z_{eq}$, Eq. (5.16). Therefore, in Fig. 6.17 we can find very few dots corresponding to heavy $N_1$, since they tend to yield too high $z_{eq}$ values.

The total number of points in Fig. 6.17 is generally small due to $z_{eq}$ rejection but also because the step on $\omega$ is bigger in order to reduce the number of models examined. High values of $z_{eq}$, as said, cause longer rising time and so longer computing time. In order to keep simulation time within a reasonable range, a reduction of analysed models is necessary. In this case $m_\chi$ values are slightly smaller than those in the Strong/Strong setup. They lie within

$$10 \text{ keV} \lesssim m_\chi \lesssim 100 \text{ MeV},$$

and therefore are acceptable.
6.1.4 Weak/Weak

In the Weak/Weak setup both $N_1$ and $N_2$ Yukawa coupling to the SM take their lowest value: $y_1 = 10^{-5}$, $y_2 = 10^{-5}$. This case is the joining of the two previous cases, i.e. Strong/Weak and Weak/Strong, thus it inherits the feature of both of them. As we can see in Fig. 6.18, we have no points but those corresponding to the light $N_1$. This is due to the $z_{\text{eq}}$ cut we mentioned before, while we can see that the $\beta$ values are the same order of those for the Strong/Weak case. $m_\chi$ range is similar to Eq. (6.11).

Thus we can conclude that within the NWA, in the HAC it is not possible to obtain high DM mass values, i.e. we have at most $m_\chi \sim 100$ MeV. In order to reach higher values for $m_\chi$, without leaving NWA, we shall explore other two configuration, varying the complex phases of $y_2$ and $\lambda_2$.

6.2 Weak $\beta$ dependence

As mentioned, it is possible to change the values of the phases $\varphi_l$ and $\varphi_\chi$ that in the HAC are set to the same value, $\varphi_l = \varphi_\chi = \pi/4$, see Section...
Figure 6.18: DM mass candidate spectrum plotted with respect to simulation parameters $\omega$ and $\beta$. The number of points is small due to computing time. Only light $N_1$ models are present, due to $z_{\text{eq}}$ cut, see Eq. (5.16).

Interesting cases are those in which one of the two terms in Eq. (5.1) or (5.2) is negligible with respect to the other, i.e. it is interesting to make comparison between the sine factors in the expressions for the asymmetry parameters $\varepsilon_l$ and $\varepsilon_\chi$. The sine functions which appear in their definitions, Eqs. (4.28), (4.29), are

$$\sin(2\varphi_l), \quad \sin(2\varphi_\chi), \quad \sin(\varphi_l + \varphi_\chi).$$

Their plots are shown in Fig. 6.19, which is useful to understand their relative behaviour and determine the most interesting case.

In order to get higher values for $m_\chi$, it is possible to consider a weaker dependence on $\beta$ of $\varepsilon_\chi$. This can be achieved by cancelling the term proportional to $\beta^2$ in Eq. (??), choosing $\varphi_\chi = 0$. $\varphi_l$ appears in the expression for $\varepsilon_l$ in the common $\sin(\varepsilon_l + \varepsilon_\chi)$ term and in a constant term, which can be made arbitrarily small. Therefore, we can choose

$$\varphi_l = 1.4 \sim \frac{\pi}{2}, \quad \varphi_\chi = 0,$$

(6.13)
Figure 6.19: Plot of sine functions involved in the asymmetry parameters definitions. \( \sin(2\varphi_l) \) is plotted in red, \( \sin(2\varphi_\chi) \) is plotted in blue, while \( \sin(\varphi_l + \varphi_\chi) \) in orange.

thus obtaining the results anticipated in Eqs. (5.6), (6.14):

\[
\varepsilon_\chi \simeq \frac{1}{16\pi} \frac{M_1}{M_2} \frac{|y_2|^2}{1 + \omega^{-2} \beta \omega^{-1}}, \tag{6.14}
\]

\[
\varepsilon_l \simeq \frac{1}{16\pi} \frac{M_1}{M_2} \frac{|y_2|^2}{1 + \omega^2 (0.66 + \beta \omega)}. \tag{6.15}
\]

The absence of square dependence on \( \beta \) in Eq. (6.14) yields lower \( \varepsilon_\chi \) values at given \( \beta \), and so lower \( Y_{\Delta \chi}^\infty \). By virtue of Eq. (4.3), this results into higher values for \( m_\chi \). Higher \( m_\chi \) will be found for \( \omega > 1 \), i.e. for those models whose dependence on \( \omega \) is relevant. The behaviour for \( \omega < 1 \) must be similar to the one described in the Strong/Strong case. In Fig. 6.20 \( m_\chi \) spectrum with respect to the simulation parameters is plotted. We can notice a relevant rise in \( m_\chi \) for \( \omega > 1 \). In this case the high \( \beta \) values chosen by the models so that \( Y_{\Delta \chi}^\infty \) be acceptable do not cause a strong increase in \( \varepsilon_\chi \), as in the pure S/S case, rather, due to the \( \omega^{-1} \) suppression in Eq. (6.14) \( \beta \) dependence is considerably milder. Fig. 6.20 also shows that the aim of higher \( m_\chi \) values can be achieved by suitably varying \( \varphi_l \) and \( \varphi_\chi \). Indeed, in the weak \( \beta \) dependence case DM masses lie in an interval which is shifted towards higher values

\[
1 \text{ GeV} \lesssim m_\chi \lesssim 100 \text{ GeV}, \tag{6.16}
\]

as it can be seen considering also Fig. 6.21 To analyse it, Table 6.1 is still
Figure 6.20: Low $\beta$ dependence case. Yukawa couplings are set as in the Strong/Strong case: $y_1 = y_2 = 10^{-3}$, while phases are $\varphi_\chi = 0$ and $\varphi_l = \pi/2$. The $\omega < 1$ regime is the same as in S/S case, whereas $\omega > 1$ case shows a pronounced rise of $m_\chi$.

It can be worthy to cast a glance also to a particular regime which cannot be contemplated within the HAC. Both in the four HAC regimes and in the weak $\beta$ dependence it is assumed that the asymptotic DM asymmetry is positive $Y_{\Delta \chi}^\infty > 0$, that is the asymmetric DM component which survives until today

6.3 Negative asymmetry
Figure 6.21: $m_\chi$ spectrum vs. $M_{N_1}$ in the weak $\beta$ dependence case. DM masses lie within $\sim 1\,\text{GeV}$ and $\sim 100\,\text{GeV}$. Green and part of the blue dots correspond roughly to low $\omega$’s, while magenta and red dots to high $\omega$’s is made of DM particles $\chi$. However, while lepton asymmetry is strictly constrained by experiments to be positive, i.e. there are more leptons than anti-leptons, it is just a matter of convention to consider the DM content of the Universe composed of particle $\chi$ or anti-particle $\bar{\chi}$. Therefore, there is no reason to take into consideration only models that yields $Y_{\Delta\chi}^{\infty} > 0$, but it is interesting to look also at models whose final result is $Y_{\Delta\chi}^{\infty} < 0$. Due to the initial conditions, Eqs. (5.11), (5.12) and (5.13), this can be achieved only by means of negative asymmetry parameter $\varepsilon_\chi < 0$. Within the HAC, the phase choice completely avoids this possibility, as it can be noticed from Eqs. (5.3), as well as in the weak $\beta$ dependence, see Eq. (6.14). In order to keep naturally $\varepsilon_l > 0$ while allowing negative values for $\varepsilon_\chi$, phases can be chosen as follows

$$\varphi_l = \frac{\pi}{4}, \quad -\frac{\pi}{12},$$

so to have the asymmetry parameters already shown in Eqs. (5.7), (5.8)

$$\varepsilon_\chi = \frac{1}{16\pi} \frac{M_1}{M_2} \left( -\beta^2 + \frac{\beta}{2\omega} \right), \quad (6.17)$$

$$\varepsilon_l = \frac{1}{16\pi} \frac{M_1}{M_2} \left( 2 + \frac{1}{2} \beta \omega \right). \quad (6.18)$$
As in the HAC case, $\varepsilon_\chi$ still maintain a square dependence on $\beta$, but it can get negative values for

$$\beta > 1/2\omega$$

(6.19)

The $m_\chi$ spectrum with respect to the simulation parameters $\omega$ and $\beta$ is plotted in Fig. 6.22. As it can be seen, the simulation dots are all into the $\varepsilon_\chi < 0$ region, therefore they correspond to $Y_{\Delta_\chi}^\infty < 0$. Obviously, Eq. (4.3) is suitably modified with the absolute value of the DM asymptotic asymmetry.

The shape of the surface in Fig. 6.22 is rather similar to those in the S/S case and we do not find points in the $\varepsilon_\chi > 0$ region. This means that the same reasoning carried out in the S/S case are valid here with the only change in the sign of the DM asymmetry abundances. Indeed, in $\varepsilon_\chi > 0$ region, $\varepsilon_l$ is too low to yield acceptable models, exactly as in Fig. 6.1. Therefore, this phase choice results into negative DM asymmetries, but the shape of $m_\chi$ with respect to $\omega$ and $\beta$ does not change from the first case examined because of the test on $Y_{\Delta l}^\infty$. It is still interesting to notice that the largest CP violations in favour of DM particle or anti-particle sector yields $m_\chi$ spectra
which are comparable.

6.4 Conclusions and perspectives

In this work we have made a brief review of some models addressing the open problems of the baryon (through lepton) asymmetry and the presence of DM in our Universe, devoting ourselves to a particular class of models proposed in Ref. [64]. The main characteristic of this model is to give a joint and rather simple explanation to both problems while also solving the problem of the neutrino masses. Asymmetry is generated into the lepton sector via thermal leptogenesis, thus automatically embedding the celebrated standard see-saw mechanism, and also into the so-called hidden sector, in which resides the DM candidate $\chi$, via new coupling of the decaying RH neutrinos.

To better understand the asymmetry generation mechanism a minimal model has been considered, in which only one SM family is present. Such models has been proven to be very flexible, though being always able to give DM relic abundances in agreement with the observed one. Thus, a wide range of DM masses can be accommodated, spanning from few eV up to hundreds of GeV. However, considering perturbativity constraint, the lighter predicted candidates must be rejected, leaving a DM mass spectrum whose lower bound is of the order of tens of keV. The upper part of $m_\chi$ spectrum is, thus, also in agreement with the mass requirement of the DM WIMP theories $\sim 10 \div 100$ GeV.

The model, thus, is able to predict a DM content of the Universe made of cold particles with a mass that can be in perfect agreement with the WIMP models and observations [78,79]. However, commonly the adjective weak is referred to weak interactions, while in this model weak should be referred more properly to the weakness of the interaction itself. The model assumes the DM candidate $\chi$ to be in a completely separate sector, which communicate with the SM one just via RH neutrinos. Therefore, $\chi$ are not coupled to any SM particle through $SU(2)_L \otimes U(1)_Y$ interactions, but only by the Yukawa term shown in Eq. (3.4). The adjective weak still suitably depict the situation because it can be realized that interaction mediated by RH neutrinos are always strongly suppressed. We can conclude that the DM candidate of this model lies within the allowed WIMP mass ranges, but it belongs to a different kind of DM, since it is completely blind to SM weak interaction.

The calculation of the cross section of $\chi$ with SM particles is one of the tasks that could be worth address in a future development of the subject. It would give a clearer understanding of the framework and would cast more light on the behaviour of the DM candidate of which, here, we have just calculated the asymptotic asymmetry abundance. In addition to that, the full
model should be considered in its complexity, as well as the other dynamics beyond the NWA. This would surely modify the DM mass spectra seen so far, by widening their range, thanks to the relaxation of the condition imposed in the NWA framework, but at the same time imposing new condition on the active neutrino spectrum and mixing.

Eventually, any model can be appreciated by itself, but physically it must provide viable ways of testing and disproving. It is certainly worth exploring the possible ways of detection of the DM candidate of this model. Any future work on the subject would take into consideration this aspect that so far has been almost neglected. The model as it is considered here does not give any clues on how the symmetric component of the DM annihilated in the early Universe. As said in Section 3.2 the model generally assumes interactions suitable for the task, but they are not considered neither in Eq (3.4), nor in Eq. (3.2). It is clear that interactions that involves only particles in the hidden sector are difficult, if not impossible, to discover or identify. It would be even a subject at the boundaries of physical approach. However, contribution to the $\chi \bar{\chi}$ annihilation due to interactions involving vertices in Fig. 3.2 with SM particles in the out-state should be considered in any case. Examples of this kind of $\chi \bar{\chi}$ annihilation are shown in Figs. 6.23 and 6.24.

In Fig. 6.23 $\chi \bar{\chi}$ annihilate into two RH neutrinos in the t-channel mediated by the hidden scalar $\phi$. Even though $N_i$ are not directly detected in experiment since they are sterile with reference to electroweak interactions, they can decay into a Higgs scalar $h$ and a neutrino. Therefore, the diagram in Fig. 6.23 potentially can yields from two SM particles, $h$ and $\nu$ if only one RH neutrino decays into the SM sector, up to four SM particles if both decay into SM sector. $\chi \bar{\chi}$ annihilation receives contribution also from one loop processes, whose some examples are shown in Figs. 6.24. The box diagram in Fig. 6.24(a) contributes at one loop to the cross section of $\chi \bar{\chi} \rightarrow N_i N_i$ whose tree level is in Fig. 6.23. In Figs. 6.24(b), 6.24(c), 6.24(d) and 6.24(e) boxes with at least two active neutrinos in the final state are shown. The box is closed by two RH neutrinos, a hidden scalar $\phi$ and a Higgs scalar $h$ whose propagator can be split into two by inserting a SM vertex which can be found in common text such as Ref. [80]. Therefore, beyond the $2 \leftrightarrow 2$ process, we can have $2 \leftrightarrow 3$ by inserting a $hhh$ vertex, or $2 \leftrightarrow 4$ processes by considering $hhZZ$ or $hhW^+W^-$ vertices. The cases with vector boson in the out state can be interesting for detectability purposes, since can be clearly handled better than those processes with elusive particles in the out state such as $\nu$ or $h$. In Figs. 6.24(f) and 6.24(g) there are at least two Higgs scalar in the out-state, while the box is closed by a neutrino propagator. This can be divided if SM neutrino annihilation vertex is inserted, thus adding a $Z$ boson in the out-state. Cross section of these processes can be worth calculating, especially differential cross sections with respect, for instance, to the energy of the outgoing bosons. As it is well known, however, box diagrams are problematic because of UV divergences. Therefore, even if
these processes can be the most interesting, additional difficulties are added because of regularization (e.g. Ref. \[81\]) and renormalization issues.

The hope is to find any possible signature of the DM annihilation whose signal can be detected by experiments. This certainly would result in a strong confirmation or disproof of this class of models, which still needs a tighter link to reality by means of testable predictions.

Figure 6.23: Tree level $\chi \bar{\chi}$ annihilation into a couple of RH neutrinos. $N_i$ cannot be directly detected, but they can subsequently decay into SM particles ($h$ and $\nu$). The hidden scalar $\phi$ is exchanged in the t-channel. Momentum are assumed to flow from left to right.
Figure 6.24: Examples of $\chi\bar{\chi}$ annihilation into SM particles at one loop. Momenta are assumed to flow from left to right, while loop momenta are assumed to flow anti-clockwise. In Fig. 6.24(a) and 6.24(b) processes with two particles in out-state are considered. In the others, three or four particles are in out-state thanks to weak SM vertices involving three Higgs scalars (Fig. 6.24(c)), two Higgs and two vector bosons (Figs. 6.24(d) and 6.24(e)), or two neutrinos and Z-boson (Fig. 6.24(g)). See for instance Ref. [80].
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### List of Figures

1.1 Type Ia supernovae .................................................. 6  
1.2 Rotational velocity of galaxies .................................... 7  
1.3 Universe content .................................................... 8  
1.4 Big-Bang Nucleosynthesis .......................................... 10  

2.1 Tree-level - one loop interference ................................. 21  

3.1 Two sector leptogenesis scheme .................................... 30  
3.2 Interaction vertices with RH neutrinos ............................ 32  
3.3 $2 \leftrightarrow 2$ washout processes ............................... 33  
3.4 $2 \leftrightarrow 2$ transfer processes ................................. 34  

4.1 Standard model vertices ............................................ 43  
4.2 $N_1 \rightarrow \nu h$ ................................................... 44  
4.3 $N_1 \rightarrow \bar{\nu} h$ ............................................... 44  
4.4 Tree-level - one loop interference ................................. 47  

5.1 Abundances evolution ............................................... 53  
5.2 Abundances evolution with $z_{eq}$ ............................... 54  
5.3 Inverse decay examples ............................................ 55  

6.1 DM mass vs simulation parameters. S/S case, front view ....... 60  
6.2 DM mass vs simulation parameters. S/S case, back view ....... 61  
6.3 DM mass spectrum sections ........................................ 62  
6.4 Branching ratios $Br_I(\omega)$, $Br_\chi(\omega)$ .................... 63  
6.5 Constant $\gamma_D \varepsilon_l$ trajectory ................................ 64  
6.6 Values of $\gamma_D \varepsilon_\chi$ along $\gamma_D \varepsilon_l = const.$ trajectory ... 65  
6.7 DM mass spectrum and $Y$ trajectory ............................ 66  
6.8 $Y_{\Delta \chi}$ evolution for different $M_{N_1}$ at $\omega = 0.001$ ...... 67  
6.9 $Y_{\Delta \chi}$ evolution for $\omega = 0.001$ and different $M_{N_1}$ ...... 68  
6.10 $Y_{\Delta \chi}$ evolution for different $M_{N_1}$ at $\omega = 3.01$ .... 69  
6.11 Direct and indirect decay terms for $\omega = 3$ and different $M_{N_1}$ 70  
6.12 $Y'_{\Delta \chi}$ evolution for $\omega = 3$ and different $M_{N_1}$ .......... 71
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.13</td>
<td>Direct and indirect decay terms for $\omega = 0.001$ and different $M_{N_i}$</td>
<td>72</td>
</tr>
<tr>
<td>6.14</td>
<td>Models with $z_{eq} \geq 0.1$</td>
<td>74</td>
</tr>
<tr>
<td>6.15</td>
<td>$m_\chi$ spectrum vs. $M_{N_i}$</td>
<td>75</td>
</tr>
<tr>
<td>6.16</td>
<td>DM mass vs. simulation parameters. S/W case.</td>
<td>76</td>
</tr>
<tr>
<td>6.17</td>
<td>DM mass vs. simulation parameters. W/S case.</td>
<td>77</td>
</tr>
<tr>
<td>6.18</td>
<td>DM mass vs. simulation parameters. W/W case.</td>
<td>78</td>
</tr>
<tr>
<td>6.19</td>
<td>Plot of sine functions</td>
<td>79</td>
</tr>
<tr>
<td>6.20</td>
<td>Low $\beta$ dependence case.</td>
<td>80</td>
</tr>
<tr>
<td>6.21</td>
<td>$m_\chi$ spectrum vs. $M_{N_i}$</td>
<td>81</td>
</tr>
<tr>
<td>6.22</td>
<td>Negative asymmetry.</td>
<td>82</td>
</tr>
<tr>
<td>6.23</td>
<td>Example of tree level $\chi \bar{\chi}$ annihilation</td>
<td>85</td>
</tr>
<tr>
<td>6.24</td>
<td>Examples of one loop $\chi \bar{\chi}$ annihilation</td>
<td>86</td>
</tr>
</tbody>
</table>
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