Introduction

A paradox is a statement that seems self-contradictory or absurd, a situation that gives different answers if analyzed in different ways. According to the American physicist Richard Feynman, in Physics there aren’t real paradoxes because there is always one correct answer. Nature will act in only one way and it’s obviously the right way. So a paradox is a confusion caused by an insufficient understanding of the problem.

In the first chapter of this work Feynman’s disk paradox is presented. Even if it has been shown that qualitatively it seems not to be any paradox, Feynman’s one is considered a striking example of the existence of angular momentum of static electromagnetic fields. It demonstrates that motion is not required for a system to possess angular momentum. Resolution of the paradox necessarily relies upon the transfer of angular momentum from the fields (electric and magnetic) to the charges on the disk.

In chapter two is treated in detail a quantitative analysis of the situation from the point of view of two different physicists: Ernesto Corinaldesi [2.1] and Gabriel G. Lombardi [2.2]. They moved the paradoxical situation in systems similar to Feynman’ disk but simpler to analyze; preserving the main features and using little approximations Corinaldesi and Lombardi took different paths but reached the same conclusion, that is the presence of an angular momentum of the static electromagnetic field. It turned out to be a fundamental element in computing the total conserved quantity: the angular momentum, when associated to a static electromagnetic field, makes paradox disappear.
After the first two chapters the reader should ask himself spontaneously where does the angular momentum of the static electromagnetic field come from. Well, in the sections that constitute the third chapter this is shown step by step starting from the conservation of charge [3.1], finishing to the existence of a momentum flow that is necessary to maintain the conservation of angular momentum [3.4]. The idea of angular momentum is connected with that of motion about an axis, so it’s hard to accept that a static electromagnetic field may possess an angular momentum. In the third chapter the answer to this enigma is presented. Therefore it is shown the process which makes us conclude that when there are a magnetic field and some charges there will be an angular momentum in the field, put there when the field was built up.

The last chapter is dedicated to the concrete calculation of total angular momentum of the system in order to understand its storage in the field. Particularly interesting is the derivation of an expression for this quantity from the fields, electric and magnetic.
paradosso di Feynman mostra come non sia necessario che un sistema sia in moto per possedere un momento angolare. La sua soluzione si basa sul trasferimento del momento angolare dai campi (elettrico e magnetico) alle cariche presenti sul disco.

Nel secondo capitolo viene svolta in dettaglio un’analisi quantitativa del problema dal punto di vista di due fisici: Ernesto Corinaldesi [2.1] e Gabriel G. Lombardi [2.2]. Essi ricrearonno la paradossale situazione in sistemi simili al disco di Feynman ma più semplici da analizzare; preservando le caratteristiche fondamentali e servendosi di piccole approssimazioni, Corinaldesi e Lombardi giunsero alla medesima conclusione pur avendo seguito diverse linee di ragionamento, ovvero alla presenza di un momento angolare del campo elettromagnetico statico. Esso si rivela fondamentale nell’elaborazione della totale quantità conservata: il momento angolare, se associato ad un campo elettromagnetico statico, annulla il paradosso.

Dopo i primi due capitoli il lettore avrebbe dovuto chiedersi da dove proviene il momento angolare di un campo elettromagnetico statico. Bene, nei paragrafi che costituiscono il terzo capitolo viene riportata passo passo la derivazione di tale grandezza, iniziando dalla conservazione della carica [3.1], per finire al flusso dell’impulso posseduto dal campo, necessario per la conservazione del momento angolare [3.4]. Il concetto di momento angolare è associato a quello di moto rispetto ad un asse, dunque è difficile accettare che un campo elettromagnetico statico possegga un momento angolare. Nel terzo capitolo è esposta la soluzione di tale enigma. Viene mostrato, inoltre, il procedimento che permette di concludere che in presenza di un campo magnetico e di cariche, nel campo si ha anche un momento angolare postovi al momento della sua creazione.

L’ultimo capitolo è dedicato al calcolo effettivo del momento angolare totale del sistema con lo scopo di osservare nel dettaglio la conservazione di tale grandezza nel campo. Particolarmente interessante è la derivazione di un’espressione per la suddetta quantità dai campi, elettrico e magnetico.
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Chapter 1

Feynman’s paradox

The situation that Feynman analyzed [1] shows an apparent contradiction between mechanics and electromagnetism principles: the conservation of angular momentum and Maxwell’s equations. On the one hand there is one of the three fundamental properties of motion: angular momentum. Together with linear momentum and energy, it gave enormous contributions to classical mechanics, especially to astronomy and astrophysics. On the other hand there are the keys of modern physics; the introduction of Maxwell’s equations in the late 1860’s rebuilt the connection between electric and magnetic field unifying electricity and magnetism definitively, and linking together all the previous results concerning these spheres.

The following figure (Fig. 1.1) represents the situation Feynman focused on: a thin, plastic disc is free to rotate about a concentric frictionless axis. A rail of wire in the form of a short solenoid is mounted on the disc, concentric with its axis of rotation; also placed on the disc there is a small battery that provides a steady current $I$ to the superconducting solenoid so a magnetic field, whose flux goes through it more or less parallel to the axis of the disk (Fig. 1.2), is generated.
Uniformly spaced around the periphery of the disc there are small metal spheres, insulated from each other and from the solenoid by the plastic material of the disk. Each of these conducting spheres is charged with the same electrostatic charge $Q$. The disk is initially at rest. Everything is quite stationary until, without any intervention from the outside, the current in the solenoid is interrupted.

Qualitatively it seems not to be any paradox. It is said to arise since electromagnetic theory predicts that the disk would rotate whereas mechanics would not make such a prediction. With the system initially at rest there is no mechanical angular momentum ($L$). However, there are electrons moving in a circular path with their drift speed and an $L$ should be present. Moreover, the magnetic field appearing in all the equations would be proportional to the product of $L$ by the charge/mass ratio of the electrons. This $L$ could be transferred to the disk by the collision process responsible for resistance. Thus, at least qualitatively, there is no paradox.

A quantitative view of the problem make us conclude that with no more
current, the magnetic flux should drop toward zero and, according to Faraday’s law of electromagnetic induction, an electric field \( E_\phi(r) = -\frac{rB_z}{2c} \), where \( c \) is the speed of light, will be induced. It will start circulating around in circles centered at the axis of the disk and each charged sphere will experience an electric field tangential to its perimeter. This electric force has the same sense for all charges so it will result in a net torque on the disk and it will begin to rotate. Therefore, knowing the moment of inertia \( I \) of the disk, the current in the solenoid and the charges on the small spheres, it could be possible to compute the resulting angular velocity:

\[
\omega = \int_0^\infty \frac{d\omega}{dt} dt = -\frac{\lambda r^2}{2cI} \int_0^\infty \dot{B}_z dt.
\] (1.1)

However, the problem may be analyzed from a different point of view. While current was flowing in the wire the angular momentum of the disk with all its equipments was supposedly zero and the disc was at rest. Consequently, after the current is interrupted, the disk should remain at rest in order for angular momentum to be conserved.

Here is Feynman’s paradox.

Figure 1.2: Blue lines represent the magnetic flux that goes through the solenoid; red ones represent the current circulating in it.
Chapter 2

The solution: the static electromagnetic field

The resolution of the paradox depends upon the realization that static electromagnetic field has angular momentum which must be considered in computing the total conserved quantity. The gain in angular momentum of the disk compensates for the loss by the field. When an angular momentum is attached to the static electromagnetic field the paradox disappears; during the assembling process in the field there is storage of the angular momentum. In our situation it means that when the current is interrupted the angular momentum becomes zero and the plastic disk will begin to rotate so that the total angular momentum, mechanical and electromagnetic, may be conserved.

Both Ernesto Corinaldesi [2] and Gabriel G. Lombardi [3] examined Feynman’s disk paradox taking their stand on this concept. In particular they put the problem in mechanical systems similar to Feynman’s one in order to let the situation be simpler to analyze. Even taking different paths they reached the same goal.

In the following two sections the resolutions of the mentioned Physicists are treated in detail.
2.1 Corinaldesi’s resolution

Ernesto Corinaldesi (Locri, 1923), currently emeritus Professor at Boston University, proposed a simpler model in order to facilitates calculations of Feynman’s paradox and solved it considering the angular momentum of the static electromagnetic field existing at the outset

\[ M = \int r \times p \, dV = \epsilon_0 \int r \times (E \times B) \, dV. \]  \hspace{1cm} (2.1)

where \( p \) is the momentum density, which is proportional to the Poynting vector \( \star \). The model Corinaldesi analyzed is not faithful to Feynman’s disk, but it has some of its features and the advantage of allowing a simple calculation: it’s a mechanical system free to rotate about an axis, generating both an electric and a magnetic field by virtues of charges attached to its parts. The only consequence of Corinaldesi’s unfaithfulness is the approximation which consists in disregarding Maxwell’s displacement current, namely the time derivative of the electric field.

Two infinitely long coaxial cylinders may rotate without friction, in the same or in opposite direction, about their axis (\( z \) axis), that is supposed to be fixed. One cylinder has radius \( R \) and is uniformly charged, charge \( \lambda > 0 \) per unit length along the \( z \) axis. The other, also uniformly charged, has radius \( R' > R \) and charge \( -\lambda \) per unit length. Let \( I \) and \( I' \) be the moments of inertia per unit length of the cylinders, \( \phi \) and \( \phi' \) their angular velocities, not necessarily of the same sign. Suppose the mechanical torques per unit length \( \tau_{\text{mech}} \) and \( \tau'_{\text{mech}} \) (not necessarily of the same sign) are applied to the inner and outer cylinders, respectively. The sum of these torques cannot equal the rate of change of the mechanical angular momentum, because of the induced electric field arising from the time variation of the magnetic field produced by rotating charged cylinders. If \( \tau_{\text{elec}} \) and \( \tau'_{\text{elec}} \) are the torques per unit length exerted on the cylinders by the induced electric field, we can write

\[ \frac{d}{dt}(I \dot{\phi} + I' \dot{\phi}') = \tau_{\text{mech}} + \tau'_{\text{mech}} + \tau_{\text{elec}} + \tau'_{\text{elec}}. \]  \hspace{1cm} (2.2)

\(^*\text{Section 3.3 for details.}\)
Denoting by $r$ the distance from the $z$ axis, the only existing component of the magnetic field generated by the inner cylinder is

\[
B_z = \mu_0 \frac{\lambda \phi}{2\pi} \quad \text{for } r < R
\]
\[
B_z = 0 \quad \text{for } r > R
\]
as can be seen by regarding the rotating cylinder as equivalent to a solenoid with an arbitrary number of turns per unit length $n$, and current $i = \frac{\lambda}{2\pi R n} R \phi$.

Similarly the magnetic field due to the outer cylinder is

\[
B'_z = -\mu_0 \frac{\lambda \phi'}{2\pi} \quad \text{for } r < R'
\]
\[
B'_z = 0 \quad \text{for } r > R'.
\]

In cylindrical coordinates, the induced electric field has components

\[
E_r = 0,
\]
\[
E_z = 0,
\]
\[
E_\phi = -\frac{1}{2\pi R} \frac{d}{dt} \left[ \pi R^2 (B_z + B'_z) \right]
\]at the position of the inner cylinder,

\[
E'_\phi = -\frac{1}{2\pi R'} \frac{d}{dt} \left( \pi R'^2 B_z + \pi R'^2 B'_z \right)
\]at that of the outer cylinder. Therefore, the torque acting on a unit length of the inner cylinder is

\[
\tau_{\text{elec}} = R \lambda E_\phi = -\frac{\lambda R^2}{2} \frac{d}{dt} (B_z + B'_z)
\]
and that on the outer cylinder is

\[
\tau'_{\text{elec}} = -R' \lambda E'_\phi = \frac{\lambda}{2} \frac{d}{dt} (R^2 B_z + R'^2 B'_z).
\]
Hence

\[
\tau_{\text{elec}} + \tau'_{\text{elec}} = \frac{\lambda}{2} (R^2 - R'^2) \frac{dB'_z}{dt} = \frac{\mu_0 \lambda^2}{4\pi} (R^2 - R'^2) \phi'. \tag{2.3}
\]
2.1 Corinaldesi’s resolution

We now show that
\[ \tau_{\text{elec}} + \tau'_{\text{elec}} = -\frac{dM_z}{dt}, \]
(2.4)
where \( M_z \) is the \( z \) component of the angular momentum of the electromagnetic field, Eq. (2.1). In fact in cylindrical coordinates:
\[
\varepsilon_0 \mu_0 (E \times H)_\phi = -\varepsilon_0 E_r = \frac{\lambda}{2\pi \varepsilon_0 r} \quad \text{for } R < r < R',
\]
\[
\varepsilon_0 \mu_0 (E \times H)_\phi = -\varepsilon_0 E_r = 0 \quad \text{for } r < R \text{ and } r > R',
\]
where \( E_r = \frac{\lambda}{2\pi \varepsilon_0 r} \) is the static electric field between the cylinders, so
\[
M_z = \int_R^{R'} [r \varepsilon_0 \mu_0 (E \times H)_\phi] 2\pi rdr = \frac{\mu_0 \lambda^2}{4\pi} (R'^2 - R^2) \phi'.
\]
(2.5)
The time derivative of this coincides up to a sign with the sum of the electric torques. We can therefore cast Eq.(2.2) in the form
\[
\frac{d}{dt} (I \dot{\phi} + I' \dot{\phi}' + M_z) = \tau_{\text{mech}} + \tau'_{\text{mech}}.
\]
(2.6)
A situation similar to that in Feynman’s disk experiment would be the following. The inner cylinder is initially at rest, the outer is rotating with uniform angular velocity. There is an amount \( M_z^{(0)} \) of angular momentum of the static electromagnetic field stored between the two cylinders. A mechanical torque is applied to the outer cylinder so as to stop it completely. Thereby the angular momentum of the electromagnetic field becomes zero. The inner cylinder squires a mechanical angular momentum equal to \( M_z^{(0)} \). From the symmetry of the model it is easily seen, without solving Maxwell’s equations, that these can be satisfied by taking \( E_r = \frac{\lambda}{2\pi \varepsilon_0 r} \) (static value) \( E_z = 0, H_r = 0, H_\phi = 0, E_\phi \) and \( H_z \) functions of \( r \) and \( t \). With this ansatz, the equation for Faraday’s law reads
\[
\frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) = -\frac{\partial B_z}{\partial t}.
\]
(2.7)
Then the expression for the electromagnetic torques can be transformed
2.2 Lombardi’s resolution

as follows:

\[
\tau_{elec} + \tau'_{elec} = R\lambda E_\phi(R, t) - R'\lambda E_\phi(R', t)
\]

\[
= \lambda \int_{R'}^{R} \frac{\partial}{\partial r} (rE_\phi) dr
\]

\[
= -\lambda \int_{R'}^{R} \frac{\partial}{\partial t} (rB_z) dr
\]

\[
= \epsilon_0 \frac{d}{dt} \int_{R'}^{R} E_z B_z 2\pi r dr
\]

\[
= -\frac{dM_z}{dt}.
\]

This accounts for the simplicity of the model.

2.2 Lombardi’s resolution

As Corinaldesi did, Gabriel G. Lombardi [3] asserted that the initial static electromagnetic field should possess an angular momentum and proved the results for a general initial configuration of static charges and steady currents and a final configuration of slowly moving charges.

Adopting some little approximations Lombardi showed that the angular momentum stored in the static fields appears as mechanical angular momentum.

Let \(E\) and \(B\) be the initial electric and magnetic fields of the charges and currents, respectively. The electric field may be expressed in terms of density of charge \(\rho\) by Gauss’s law:

\[
\nabla \cdot E = \frac{\rho}{\epsilon_0}.
\]

(2.9)

The electric field \(E'\) induced by the interruption of the current is given by Faraday’s law:

\[
\nabla \times E' = -\frac{\partial B}{\partial t}.
\]

(2.10)

The torque \(\frac{dL_M}{dt}\) exerted on the charges is

\[
\frac{dL_M}{dt} = \int (r \times \rho E') d^3r,
\]

(2.11)
where \( L_M \) is the mechanical angular momentum of the system and \( r \) is the radius vector from an arbitrary origin. Using Eq. (2.9), this becomes

\[
\frac{dL_M}{dt} = \int (r \times E') (\epsilon_0 \nabla \cdot E) d^3r. \tag{2.12}
\]

Consequently, interruption of the current changes the angular momentum by

\[
\Delta L_M = \int \int (r \times E') \epsilon_0 \nabla \cdot E d^3r dt. \tag{2.13}
\]

The angular momentum of the static fields, before the interruption of the current, is

\[
L_F = \epsilon_0 \int r \times (E \times B) d^3r. \tag{2.14}
\]

The field \( B \) can be expressed as the same time integral of the induced electric field by using Eq. (2.10):

\[
\Delta L_F = -\epsilon_0 \int \int r \times [E \times (\nabla \times E')] d^3r dt. \tag{2.15}
\]

In Eqs. (2.13) and (2.15), the time dependence of \( E \) has been neglected. This is permissible as long as the motion of the charges is slow, which may be ensured by making them massive. Inclusion of the time variation of \( E \) would necessitate consideration of the radiation field of the accelerating charges, which has also been neglected. In order of the total angular momentum \( L_M + L_F \) to be conserved, the two changes must balance:

\[
\Delta L_M = -\Delta L_F. \tag{2.16}
\]

If it can be shown that

\[
\int (r \times E') (\nabla \cdot E) d^3r = \int r \times [E \times (\nabla \times E')] d^3r, \tag{2.17}
\]

then Eq. (2.16) follows. Since \( E \) is generated by static charges \( \nabla \times E = 0 \); likewise \( \nabla \cdot E' = 0 \) since \( E' \) is not. Consequently, both sides of Eq. (9) may be symmetrized in \( E \) and \( E' \):

\[
\int [(r \times E')(\nabla \cdot E) + (r \times E)(\nabla \cdot E')] d^3r = \int r \times [E \times (\nabla \times E') + E' \times (\nabla \times E)] d^3r. \tag{2.18}
\]
To show that Eq.(2.18) holds, only one Cartesian component needs to be considered, let it be $x$. Let indices 1, 2 and 3 denote coordinates $x, y$ and $z$, respectively. The terms in Eq.(2.18) may be combined and cast in the form

$$\int r_i \frac{\partial}{\partial r_j} (E_k E_i) d^3r. \tag{2.19}$$

For terms with $i \neq j$, the integral vanishes if $E_k E_i$ approaches zero sufficiently fast as $r_i$ approaches infinity. The remaining terms are

$$\int r_3 \frac{\partial}{\partial r_3} (E_2 E_3' + E_2' E_3) d^3r = \int r_2 \frac{\partial}{\partial r_2} (E_2 E_3' + E_2' E_3) d^3r. \tag{2.20}$$

As above, if $E_k E_i$ approaches zero sufficiently fast, an integration by parts reduces Eq.(2.18) to an identity. These mathematical manipulations are possible if the product $E_i E_j$ approaches zero at infinity faster than $r_k^{-3}$ for any $i, j$ and $k$. This condition is satisfied by fields in Feynman’s disk problem.

Two tacit approximations have been made: the displacement current and the magnetic field generated by the motion of the charges have been neglected. The first effect is simply included by stipulating that $B$ is the total magnetic field, not just that of the steady currents. As the current changed, $B$ would include terms arising from the time dependence of $E'$. The second effect may be made negligible by making the mass of the disk large, or simply by including the field generated by them in $B$.

Lombardi applied this principle to an infinite solenoid of radius $R$ with current $i$ and $n$ turns per meter, concentric with two cylindrical tubes of charge $Q$ and $-Q$ and radii $a$ and $b$, respectively. The charges is distributed uniformly over the cylindrical surfaces and both tubes have length $l$, where $l > b > R > a$. The coil and the cylindrical tubes are all stationary initially but free to rotate without friction about their common axis. According to Lombardi the angular momentum possessed by the initial electromagnetic field is transferred to the cylindrical tubes as the current in the solenoid drops to zero in such a way that angular momentum is conserved.

The apparent paradox arises when the current in the solenoid is interrupted. The changing magnetic flux causes a tangential electric field that
acts on charged tubes, giving them a mechanical angular momentum as follows:

\[ L_{ma} = \int aQE_a dt \]  
\[ L_{mb} = \int bQE_b dt \]  

where \( E_a \) and \( E_b \) are the tangential electric fields induced at the inner and outer tubes respectively. According to Faraday’s law and Ampere’s circuital law we get:

\[ E_a = \frac{d\phi_a}{dt} = \frac{\pi a^2}{2\pi a} \frac{dB}{dt} = a \frac{dB}{dt} \]  
\[ E_b = \frac{d\phi_b}{dt} = \frac{\pi R^2}{2\pi b} \frac{dB}{dt} = R^2 \frac{dB}{dt} \]  

tangential in the direction of the original current. Replacing these values in Eqs.(2.21) we get

\[ L_{ma} = Qa^2 \int \frac{dB}{2} = Qa^2 \frac{B}{2} \]  
(2.22)

in the direction of the solenoid axis (considering that the rotation is in the same direction as the original current), and

\[ L_{mb} = QR^2 \int \frac{dB}{2} = QR^2 \frac{B}{2} \]  
(2.23)

in opposite direction to that of \( L_{ma} \), where \( B \) is the initial magnetic field within the solenoid. The total final mechanical angular momentum is

\[ L_m = L_{mb} - L_{ma} = QB \frac{R^2 - a^2}{2} \]  
(2.24)

in the direction of \( L_{mb} \).

In order to check Lombardi’s thesis for the present example we note first that initially \( B \) is equal to zero everywhere except within the solenoid where it is uniform. The electric field is nearly zero everywhere except in the region between the cylindrical shells where it is radially outward and of a magnitude \( E = \frac{Q}{2\pi r_0 r} \) where \( r \) is the cylindrical radius. The field angular momentum is,
following the procedure given by Lombardi:

\[
L_F = \epsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3r
\]

(2.25)

\[
L_F = \epsilon_0 \int_{r=a}^{R} \frac{Q}{2\pi \epsilon_0 \ell} B 2\pi r l dr
\]

\[
= QB \int_{r=a}^{R} r dr = QB \frac{R^2 - a^2}{2}
\]

in the direction of \(L_m\). Hence the initial field angular momentum (2.25) is equal to the final mechanical angular momentum (2.24).

There is an alternative method to find the field angular momentum transferred to mechanical angular momentum that uses vector potential. A vector potential for the insinuate solenoid that works is

\[
A = B \times \frac{r}{2}, \quad r < R, \quad \nabla \times A = B,
\]

\[
A = R^2 \frac{B \times r}{2r^2}, \quad r > R, \quad \nabla \times A = 0.
\]

The angular momentum associated with \(A\) is

\[
L_{Aa} = \left| \int \mathbf{r} \times A dQ \right| = QB \frac{a^2}{2}
\]

in the direction of \(L_{ma}\),

\[
L_{Ab} = \left| \int \mathbf{r} \times A dQ \right| = QB \frac{R^2}{2}
\]

in the direction of \(L_{mb}\), for the inner and outer shells, respectively. The total died angular momentum transferred to the tubes as \(A\) drops to zero is the sum

\[
L_A = L_{Ab} - L_{Aa} = B^2 Q \frac{R^2 - a^2}{2}
\]

in the direction of \(L_m\), which is the same result given by Eq(3).

This second method to show angular momentum is conserved as the current in the solenoid drops to zero is sometimes much easier, particularly if \(\mathbf{E}\) and \(\mathbf{B}\) are not so neatly confined as they are in the setup described here. An example where this is the case is the setup described above with the inner tube omitted.
Chapter 3

Where does angular momentum come from?

After reading and analyzing Corinaldesi’s and Lombardi’s work it should be clear that the key of the paradox resolution is the presence of an angular momentum in a static electromagnetic field. This has always been a fascinating subject. The idea of angular momentum is connected with that of motion about an axis, so it seems plausible that a circularly polarized wave should possess an angular momentum, since both the electric and the magnetic field rotate about the direction of propagation; it is harder to accept that a static field may possess an angular momentum. The answer to this enigma is that if we have a magnetic field and some charges, there will be an angular momentum in the field. It must have been put there when the field was built up. When the field is turned off, the angular momentum is given back so the disk in the paradox would start rotating. This mystic circulating flow of energy, which at the beginning seemed so ridiculous, is absolutely necessary.

Let’s proceed now step by step, starting from the conservation of energy, finishing to prove the existence of the momentum flow necessary to maintain the conservation of angular momentum.
3.1 Local conservation

It is well known that energy of matter is not conserved; when an object wastes some energy, for example in the form of light, the lost energy is describable in some other form, let’s say in light. So the theory of the conservation of energy will be incomplete if energy associated to the electromagnetic field is not considered. We will see in the following how energy can be conserved considering that when it goes away from a region it’s because it flows away through the boundaries of that region. To catch the meaning of this statement, let’s see how the law of conservation of charge works. In this case we know that there is a charge density $\rho$ and a current density $j$; moreover, when the charge decreases, there must be a flow of charge going away from that place. We call it conservation of charge. The mathematical form of conservation law is

$$\nabla \cdot j = -\frac{\partial \rho}{\partial t}. \quad (3.1)$$

A consequence of this law is that in the world there is never any net gain or loss of charge, in other words the total charge in the world is always constant. However, it can be constant in another way. Suppose that there is some charge $Q_1$ near some point (1) while there is no charge near some point (2) some distance away (Fig......). Now suppose that, as time goes on, the charge $Q_1$ gradually fade away and that simultaneously with the decrease of $Q_1$ some charge $Q_2$ will appear near some point (2) in such a way that in every instant the sum of $Q_1$ and $Q_2$ was a constant. This means that at any intermediate state the amount state of charge lost by $Q_1$ would be added to $Q_2$. Then the total amount of charge in the world would be conserved.

But this statement cannot be considered a ”local” conservation because in order for the charge to get from (1) to (2), it didn’t have to appear anywhere in the space between point (1) and point (2). So locally the charge was just wasted. It is difficult to introduce a ”world-wide” conservation law in the theory of relativity; the concept of ”simultaneous moments” at distant points is not equivalent in different systems. For ”world-wide” conservation similar
3.1 Local conservation

Figure 3.1: Two ways to conserve charge: (a) $Q_1 + Q_2$ is constant; (b) $\frac{dQ_1}{dt} = \int j \cdot n \, da = -\frac{dQ_2}{dt}$

...to the one described, it is necessary that the charge lost from $Q_1$ should appear simultaneously in $Q_2$; otherwise there should be some moments when the charge was not conserved. It seems to be no way to make the law of charge conservation relativistically invariant without making it a local conservation law.

Local conservation involves another idea. It says that a charge can get from one place to another only if there is something happening in the space between. To describe the law we need not only the charge density $\rho$ but also another kind of quantity, namely $j$, a vector giving the rate of flow of charge across a surface. Then the flow is related to the rate of change of the density by Eq.(3.1). This is the most extreme kind of conservation law: it says that charge is conserved locally.

It turns out that energy conservation is also a local process. There is not only an energy density in a given region of space but also a vector to represent the rate of flow of the energy through a surface. For example, when a light source radiates, we can find the light energy moving out from the source; if we imagine a mathematical surface surrounding the light source, the energy lost from inside the surface is equal to the energy that flows out through the surface.
3.2 Energy conservation in electromagnetic fields

Let’s analyze now quantitatively the energy conservation for electromagnetism. To do that it’s necessary to describe how much energy there is in a volume element of space and the rate of energy flow. At first we think only of the electromagnetic field energy. Let \( u \) represent the energy density in the field (that is the amount of energy per unit volume in the space) and let the vector \( \mathbf{S} \) represent the energy flux of the field (that is the flow of energy per unit time across a unit area perpendicular to the flow). Then, in perfect analogy to the conservation of charge (Eq. 3.1), we can write the local law of energy conservation in the field as

\[
\frac{\partial u}{\partial t} = - \nabla \cdot \mathbf{S}.
\] (3.2)

This law is not true in general, obviously. Equation 3.2 is not the complete conservation law, because the field energy alone is not conserved, there is the energy of matter too. The field energy will change if there is some work done by matter on the field or by field on matter. What we are looking for is an equation which says that the total field energy in a known volume decreases either because field energy flows out of the volume or because field gives it energy to matter (or gains energy, which is just a negative loss). The field energy inside a volume \( V \) is

\[
\int_V u \, dV,
\]

and its rate of decrease is minus the time derivative of this integral. The flow of field energy out of the volume \( V \) is the integral of the normal component of \( \mathbf{S} \) over the surface \( \Sigma \) that encloses \( V \), \( \int_\Sigma \mathbf{S} \cdot \mathbf{n} \, da \). So

\[
-\frac{\partial}{\partial t} \int_V u \, dV = \int_\Sigma \mathbf{S} \cdot \mathbf{n} \, da + \text{(work done on matter inside } V).\]

It is well known that the field executes work on each unit volume of matter at the rate \( \mathbf{E} \cdot \mathbf{j} \) and this statement comes from the fact that the force on a particle is \( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) and the rate of execute work is \( \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} \);
if there are $N$ particles per unit volume, the rate of execute work per unit volume is $NqE \cdot v$, but $Nqv = j$. So the quantity $E \cdot j$ must be equal to the loss of energy per unit time and per unit volume by the field. Equation 3.2 becomes

$$-\frac{\partial}{\partial t} \int_V u dV = \int_S \mathbf{n} d\mathbf{a} + \int_V \mathbf{E} \cdot \mathbf{j} dV.$$  \hspace{1cm} (3.3)

This is the conservation law for energy in the field; it can be converted in a differential equation changing the second term to a volume integral that is easily to do with Gauss’ theorem. The surface integral of the normal component of $\mathbf{S}$ is the integral of its divergence over the volume inside. So Eq. 3.3 is equivalent to

$$-\int_V \frac{du}{dt} dV = \int_V \nabla \cdot \mathbf{S} dV + \int_V \mathbf{E} \cdot \mathbf{j} dV,$$  \hspace{1cm} (3.4)

where we put the time derivative of the first term inside the integral. Since this equation is true for any volume, we can take away the integrals and we have the energy equation for electromagnetic fields:

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{E} \cdot \mathbf{j}.$$  \hspace{1cm} (3.5)

### 3.3 Energy density and energy flow in electromagnetic fields

In this section the only dependence of the field energy density $u$ and the flux $\mathbf{S}$ upon the fields $\mathbf{E}$ and $\mathbf{B}$ will be proved. Let’s begin rewriting the quantity $\mathbf{E} \cdot \mathbf{j}$ in such a way that it becomes sum of two terms: one that is the time derivative of a quantity and another that is the divergence of a second quantity. The first quantity would then be $u$ and the second would be $\mathbf{S}$. Both of them must be written in terms of the fields only. So:

$$\mathbf{E} \cdot \mathbf{j} = -\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{S}.$$  \hspace{1cm} (3.6)

The left side could be written using Maxwell’s equations, in particular from the one for the curl of $\mathbf{B}$ we have:

$$\mathbf{j} = \epsilon_0 c^2 \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$  \hspace{1cm} (3.7)
Substituting this in (3.6) we will have only $E$’s and $B$’s:

$$E \cdot j = \epsilon_0 c^2 E \cdot (\nabla \times B) - \epsilon_0 E \cdot \frac{\partial E}{\partial t}.$$  \hfill (3.8)

The last term is a time derivative: it is $(\frac{\partial}{\partial t})(\frac{1}{2}\epsilon_0 E \cdot E)$, hence $\frac{1}{2}\epsilon_0 E \cdot E$ is at least part of $u$. What we have to do now is to make the other term into the divergence of something. Notice that, from vector algebra, the first term on the right side is the same as

$$(\nabla \times B) \cdot E$$  \hfill (3.9)

and

$$\nabla \cdot (B \times E)$$  \hfill (3.10)

so we have a divergence as we wanted. Unfortunately this result is wrong. We knew that $\nabla$ is ”like” a vector but not ”exactly” the same thing, in fact there is a convention from calculus saying that when a derivative operator is in front of a product, it works on everything to the right. In Eq. 3.8, $\nabla$ operates only on $B$, instead in (3.10) the normal convention would say that $\nabla$ operates both on $B$ and $E$. In this case it should be useful to adopt a trick that allows us to use all the rules of vector algebra on expressions with $\nabla$ operator without getting into trouble. The trick is to throw out the calculus rule and make a new rule that does not depend on the order in which terms are written down; then we can juggle terms around without worrying. Let’s indicate with a subscript what a differential operator works on, the order has no meaning. Let $D$ stand for $\frac{\partial}{\partial x}$. Then $D_j$ means only the derivative of the variable quantity $f$ is taken. So we have:

$$fg = gD_j f = fD_j g = fgD_j.$$  \hfill (3.11)

With this new convention a derivative of something like $fg$ will be $D_j(fg) + D_g(fg)$. Now it will be easier to work out an expression for $\nabla \cdot (B \times E)$:

$$\nabla \cdot (B \times E) = \nabla_B \cdot (B \times E) + \nabla_E \cdot (B \times E).$$  \hfill (3.12)

It’s not necessary to keep the order straight any more and $\nabla$ can be used as an ordinary vector. Even if it looks freakish, now we can interchange dots.
and crosses and make other kinds of rearrangements of the terms. If we try to go back to the ordinary convention, we have to arrange that $\nabla$ operates only on its "own" variable. Analyzing Eq. 3.12 we notice that we have just to rearrange the second term to put $\nabla$ in front of $E$, reversing the cross product and changing sign:

$$B \cdot (E \times \nabla E) = -B \cdot (\nabla E \times E).$$

(3.13)

Now it is in a conventional order, so we can return to the usual notation. Hence Eq. 3.13 is equivalent to

$$\nabla \cdot (B \times E) = E \times (\nabla \times B) - B \cdot (\nabla \times E).$$

Using our new result to transform the $\nabla \times B$ term of Eq. 3.8, the energy equation becomes

$$E \cdot j = \epsilon_0 c^2 \nabla \cdot (B \times E) + \epsilon_0 c^2 B \cdot (\nabla \times E) - \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon_0 E \cdot E \right).$$

Thank to Maxwell’s equation $\nabla \times E = -\frac{\partial B}{\partial t}$ we have an expression of energy equation very similar to Eq. 3.8; that is

$$E \cdot j = \nabla \cdot (\epsilon_0 c^2 B \times E) - \frac{\partial}{\partial t} \left( \frac{\epsilon_0 c^2}{2} B \cdot B + \frac{\epsilon_0}{2} E \cdot E \right).$$

In particular, this last equation coincides with Eq. 3.8 if we make de definitions

$$u = \frac{\epsilon_0}{2} E \cdot E + \frac{\epsilon_0 c^2}{2} B \cdot B \quad \text{and} \quad S = \epsilon_0 c^2 E \times B.$$

(3.14)

We obtained an expression for the energy density that is the sum of an "electric" energy density and a "magnetic" energy density. Moreover we found a formula for the energy flow vector of the electromagnetic field. This new vector, $S = \epsilon_0 c^2 E \times B$, is called "Poynting’s vector" after its discoverer. It tell us the rate at which the field energy moves around in space. The energy which flows through a small area $da$ per second is $S \cdot n da$, where $n$ is the unit vector perpendicular to $da$. 
3.4 Field momentum

Let’s proceed introducing the momentum of the electromagnetic field. Just as the field has energy, it will have a momentum per unit volume. We will call $g$ the momentum density. Obviously momentum has various possible directions, so $g$ is clearly a vector. Since each component of the momentum is conserved we should be able to write down a law similar to the following:

$$-\frac{\partial}{\partial t}(\text{momentum of matter})_x = \frac{\partial g_x}{\partial t} + (\text{momentum outflow})_x$$

where has been considered the $x$-component. The rate-of-change of the momentum of matter is just the force on it; for a particle it is $F = q(E + v \times B)$, instead for a distribution of charges the force per unit volume is $(\rho E + j \times B)$. The “momentum outflow” term, however, cannot be the divergence of a vector because it’s not a scalar; it could be the $x$-component of a vector. Anyway it should look something like $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$, because the $x$-momentum could be flowing in any of the three directions. In any case, whatever $a$, $b$ and $c$ are, the combination is supposed to equal the outflow of the $x$-momentum.

Now we should write $\rho E + j \times B$ only in terms of $E$ and $B$, eliminating $\rho$ and $j$ by using Maxwell’s equations, and make substitutions to get into a form that looks like $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$. Then we would have an expression for $g_x$, $a$, $b$ and $c$.

But an important theorem of Mechanics (treated in detail in sec. 3.4.1) helps us finding an expression for the momentum density $g$ in a simpler way; this theorem assures that whenever there is a flow of energy, the energy flowing through a unit area per unit time, when multiplied by $\frac{1}{c^2}$, is equal to the momentum per unit volume in the space. This fundamental result comes from an analysis of the magnetic field that is associated with light and its effects; in the special case of electrodynamics, it gives the result that $g$ is $\frac{1}{c^2}$ times the Poynting vector. In symbols:

$$g = \frac{1}{c^2}S$$

(3.15)

where $S$ is the energy flow vector.
There could be several examples justifying this thesis; let’s consider the one due to Einstein. Suppose that we have a railroad car on wheels (assumed frictionless) with a certain big mass $M$. At one end there is a device that shoots out some particles of light (or anything, it doesn’t make any difference what it is) which are stopped at the opposite end of the car. There was some energy originally at one end, say the energy $U$ indicated in the following Fig.3.1(a), and later it is at the opposite end as shown in Fig.3.1(c). The energy $U$ has been displaced the distance $L$, the length of the car.

Figure 3.2: The energy $U$ in motion at speed $c$ carries the momentum $\frac{U}{c}$. 
Now the energy $U$ has the mass $\frac{U}{c^2}$, so if the car stayed still, the centre of gravity of the car would be moved. Einstein didn’t accept the idea that the centre of gravity of an object could be moved by fooling around only on the inside, so he assumed that it is impossible to move the centre of gravity by doing anything inside. But if that is the case, when we moved the energy $U$ from one end to the other, the whole car must have recoiled some distance $x$ as shown in part (c) of the figure. It is possible to see, in fact, that the total mass of the car, times $x$, must equal the mass of the energy moved, $\frac{U}{c^2}$ times $L$ (assuming that $\frac{U}{c^2}$ is much less than $M$):

$$Mx = \frac{U}{c^2}L.$$  \hspace{1cm} (3.16)

Let’s consider now the case of the energy being carried by a light flash. When the light is emitted there must be a recoil, some unknown recoil with momentum $p$. It’s this recoil which makes the car roll backward. The recoil speed $v$ of the car will be:

$$v = \frac{p}{M}.$$  

The car moves with this speed until the light energy $U$ gets to the opposite end; then, when it hits, it gives back its momentum and stops the car. If $x$ is small, the time the car moves is nearly equal to $\frac{L}{c}$; so:

$$x = vt = v \frac{L}{c} = \frac{p}{M} \frac{L}{c}.$$  

Putting this $x$ in Eq.(3.16) we get

$$p = \frac{U}{c}.$$  

Again we have the relation of energy and momentum for light. Dividing by $c$ to get the momentum density $g = \frac{p}{c}$, we obtain:

$$g = \frac{U}{c^2}.$$  

The previous treatment is founded on the truth of the centre-of-gravity theorem; tanks to an example it’s possible to prove that if it would be wrong,
3.4 Field momentum

Figure 3.3: The energy $U$ must carry the momentum $\frac{U}{c}$ if the angular momentum about P is to be conserved.

The conservation of angular momentum will get lost. Suppose that the boxcar is moving along a track at some speed $v$ and that we shoot some light energy from the top to the bottom of the car (from A to B in Fig.3.2). Let’s consider now the angular momentum of the system about the point P. Before the energy $U$ leaves A it has the mass $m = \frac{U^2}{c^2}$ and the speed $v$, so it has angular momentum $mvr_A$. When it arrives in B it has the same mass and, if the linear momentum of the whole boxcar has not to change, it must still have speed $v$, so its angular momentum about $P$ is $mvr_B$. The angular momentum will be changed unless the right recoil momentum was given to the car when the light was emitted, that is unless the light carries the momentum $\frac{U}{c}$. It turns out that the conservation of angular momentum and the centre-of-gravity theorem are closely related in relativity theory. Hence it can be considered a general law, and in the case of electrodynamics it can be used to get the momentum in the field. What we obtained is the answer at the question ”Where does angular momentum come from?”

in fact knowing that energy flow and momentum are proportional we can conclude that there is momentum circulating in the space; this means that
there is angular momentum. Linking this thesis to Feynman’s disk paradox it’s possible to assert that if there are a magnetic field and some charges there will be an angular momentum, put in the field when it has built up. When the field is turned off the angular momentum is given back. So the disk in the paradox would start rotating and the momentum flow which Corinaldesi and Lombardi talk about really exists in order to maintain the conservation of angular momentum.

3.4.1 The momentum of the light

Let’s move for a while from the cornerstone of this chapter, focusing our attention on the magnetic field associated with light. Suppose that light is coming from a source and is acting on a charge and driving it up and down. We will suppose that the electric field is in the $x$-direction, so the motion of charge is also in $x$-direction: it has a position $x$ and a speed $v$. The electric field acts on the charge and moves it up and down; the magnetic field acts on the charge (say an electron) only when it is moving, that means always! So the two fields, electric and magnetic, work together. While the electron is going up and down it has a speed and there is a force on it, $F = Bvq$, in the direction of the propagation of light. Therefore, when light is shining on a charge and it is oscillating in response to that charge, there is a driving force in the direction of the light beam. This is called radiation pressure or light pressure. Let us determinate how strong the radiation pressure is. Evidently it is $F = qvB$ or, since everything is oscillating, it is the time average of this, $\langle F \rangle$. From $B = -e_{R'} \times \frac{E}{c}$ (where $e_{R'}$ is the unit vector which points in the apparent direction of the charge) we know that the strength of the magnetic field is the same as the strength of the electric field divided by $c$, so we need to find $\langle F \rangle = \frac{q\langle vE \rangle}{c}$. But the charge $q$ times the field $E$ is the electric force on a charge, and the force on the charge times the speed is the work $\frac{dW}{dt}$ done on the charge. Therefore the force, the ”pushing momentum”, that is delivered per second by the light, is equal to $\frac{1}{c}$ times the energy absorbed from the light per second. That is a general rule since
we did not say how strong the oscillator was, or whether some of the charges cancel out. So in any circumstance where light is being absorbed, there is a pressure. The momentum that the light delivers is always equal to the energy that is absorbed, divided by \( c \), in symbols:

\[
<F> = \frac{dW}{dt} \frac{1}{c}.
\]

We already knew that light carries energy; now it's clear that it also carries momentum, and further, that the momentum carried is always \( \frac{1}{c} \) times the energy.
Chapter 4

How to calculate angular momentum

A way to understand why angular momentum is stored in the fields is calculating it; it is possible, in fact, to obtain an expression for angular momentum from the fields. In this derivation the magnetic field due to the solenoid may be taken as that of a magnetic dipole \( \mathbf{u} \) at the centre of the disk:

\[
B(r) = \frac{3(\mathbf{u} \cdot r)r - r^2 \mathbf{u}}{r^5} = -\nabla \left( \frac{\mathbf{u} \cdot r}{r^3} \right) = \nabla \left( \nabla \cdot \frac{\mathbf{u}}{r} \right). \tag{4.1}
\]

The ring of charged particles at \( r = R, \theta = \frac{\pi}{2} \) produces a static electric field \( \mathbf{E} = -\nabla V \), where \( V \) is, in spherical coordinates system, the solution to Laplace’s equation

\[
V(r, \theta) = Q \sum_{l=0}^{\infty} \frac{r_<}{r_>^{l+1}} P_l(0) P_l(\cos \theta), \tag{4.2}
\]

where \( r_< (r_> \) is the smaller (greater) of \( r \) and \( R \). The field angular momentum is well known to be

\[
1 = \frac{1}{4\pi c} \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \frac{1}{4\pi c} [\mathbf{E} (r \cdot \mathbf{B}) - \mathbf{B} (r \cdot \mathbf{E})]. \tag{4.3}
\]

The symmetry of the problem suggest that the angular momentum has only \( z \) component if it is calculated for spherical shells. The radial field angular
momentum distribution function is given by
\[ l_z(r) = \int_{-1}^{1} d(cos\theta) \int_{0}^{2\pi} d\phi \cdot \hat{z}. \] (4.4)

Using Eqs. (4.1), (4.2), and (4.3) this can be calculated to be
\[ l_z(r) = \frac{2}{5} uQ \frac{r^2}{cr^3} - \frac{2}{3} uQ d \left( \frac{1}{r^3} \right) = -\frac{1}{15} uQ \frac{d}{dr} \left( \frac{r^2}{r^3} \right). \] (4.5)

It’s possible to obtain now the angular momentum \( L_\leq \) within the sphere of radius \( R \) doing the \( r \) integration of \( l_z(r) \) from 0 to \( R \):
\[ L_\leq = \int_{R}^{\infty} dr \left[ \frac{2}{5} uQ \frac{R^2}{cr^4} - \frac{2}{3} uQ d \left( \frac{R^2}{r^6} \right) \right] = \frac{2uQ}{15cR}. \] (4.6)

Outside of the sphere the field angular momentum \( L_\geq \) may also be found by integrating from \( R \) to \( \infty \):
\[ L_\geq = \int_{R}^{\infty} dr \left[ \frac{2}{5} uQ \frac{R^2}{cr^4} - \frac{2}{3} uQ d \left( \frac{R^2}{r^6} \right) \right] = \frac{13uQ}{15cR}. \] (4.7)

The total angular momentum is thus found to be \( \frac{uQ}{cR} \).

It may be surprising to find that there is a finite fraction of the total field angular momentum within a sphere of finite volume.
Appendix A

Feynman’s biography

Nobelist Physicist, teacher, storyteller and bongo player, Richard Feynman was born the 11 of May 1918 in the community of Far Rockaway, just on the southern skirt of Manhattan. Financially his family was neither rich nor poor. They were materially comfortable, but not wealthy. As a young man he had the opportunity to learn to work industriously, but without undergoing pressure to perform. He was a free man. But what to do with his freedom? In this direction his father, Melville, would be most influential. It was he who, as the birth of who would be Richard approached, determined that, if the child turned out to be a boy, then he would grow up to be a scientist. His son did arrive, and Melville dove into his plans with all sincerity. But never would he push Richard along too narrow a path. Instead his approach was much more intuitive and subtle. He encouraged young Richard to identify not what he knew, but rather what he did not know. This is the essence of Richard Feynman’s style of understanding. By absolutely asking what his ignorance consisted of, he freed himself from the tyranny of conventional wisdom. He learned that it’s entirely possible, and even likely, for a person to live not knowing the answers to important questions. What’s most important for knowledge is the well asked question. The answers will wait patiently for their discovery. But this is only half of the story of his formative experiences. His mother, Lucille, instilled into Richard a quality that, although less obvious,
was nevertheless of equal importance to his future success as an explorer. That critical quality was a powerful sense of humor. While Melville provided Richard with the tools to choose his own path, Lucille taught him how to laugh out loud at self-importance, giving him the important courage to "step onto the path". Both of his parents were a formidable combination to guide him his whole lifetime.

Once his mental ingredients were well mixed as a child, the next phase of his life began, that of becoming, first of all, a physicist, and second, a soul mate. Although placing "physicist" ahead of "soul mate" may seem appropriate in its way, it certainly doesn’t imply that one was of greater personal importance than the other. As a young man in school he found himself who would be perhaps the single most important person in Richard’s life, Arlene Greenbaum, his high school sweetheart. Arlene shared, as it turned out, his taste for life, and became his most trusted confidant in all things. But in the same years that Richard and Arlene became inseparable, he also crystallized in his own mind what it was that he was truly to do with his life. As a young man he was good at most things scientific, that is to say, topics of study that are conventionally associated with the word ‘science’: astronomy, physics, chemistry, biology, geology, etc. He was also exceptionally talented in mathematics. At one point he considered seriously becoming a mathematician, but it didn’t quite click. It was nature "herself" (as Feynman liked to put it) that goaded the very best questions from within him, and as a physicist he’d be able find out more than in any other field.

Eventually he attended college as a physics major. He finished his first four years at one of the best schools for physics, the MIT, Massachussets Institute of Technology in Boston, and then moved to Princeton as a graduate student for the Ph. D. During this time in his life he became engaged to marry Arline, which they’d do after completion of his Ph.D. However, Arline at one point started to display serious symptoms of some sort of illness. After some time she was positively diagnosed with tuberculosis, and was not expected to live too many more years. Richard figured that there was only one right
thing for him to do, and that was to marry her as soon as possible. He wanted to be responsible for her welfare as much as he could muster.

The turning point of Richard Feynman’s life was his participation to the well known Manhattan Project; thank to that he received the tag of youngest figure working on atomic bomb. It had been known for some time by scientists that there is a tremendous amount of energy trapped in the nucleus of every atom, just waiting to be liberated and put to work. In particular, it was estimated that at the very least extremely powerful explosives could be made from this principle, and work was being done in this direction both in Nazi Germany and, on some smaller scale, in the U.S. But when the U.S. eventually entered battle during the Second World War it was feared that Germany was very far along in the engineering of nuclear bombs. The United States then started its Manhattan Project with the purpose of perfecting nuclear bombs ahead of the axis powers so as to ensure victory before it was too late. Toward this end the U.S. Army established the cloistered research city of Los Alamos, well into the New Mexico desert. The best mathematicians, physicists and chemists were encouraged to join the project; it was the physicist Robert Wilson who gently prodded Richard Feynman, still at Princeton, to join what was considered one of the most vital wartime projects of all. At first Feynman’s reaction was that this wasn’t the sort of thing he’d be interested in, but the thought nagged at him that the Nazi’s might create their own nuclear device first and use it to disastrous ends. So he took the job, moving himself to Los Alamos and Arline to a hospital in Albuquerque for the care of her illness. During this time he acquired what was to become a definitive fascination with safecracking. Reading books by professionals and developing his own methods, he eventually became notorious for his ability to open safes.

In the months just near the end of the war Arline’s tuberculosis advanced to a desperate degree. In July, 1945, just before the very first test of the bomb, she finally passed away due to her illness. Richard was fortunate enough at least to be by her side at the moment. He made his way back to Los
Alamos and temporarily put it out of his mind by further immersing himself in his work. After the conclusion of the war Feynman moved on. He accepted a professorship with Cornell University, but fell into somewhat of a slump. He lost his inspiration and confidence as a physicist, and speculated that perhaps his better days were behind him. So it surprised him to no end that he would get solicitations from competing universities to more lucrative professorships at other schools. Finally he even received an invitation to the Institute for Advanced Study at Princeton. This was something for him to ponder, since the institute was one of the most prestigious academic institutions anywhere. Only the best minds were offered posts there, for instance Einstein, and to him this didn’t make sense since as far as he could fathom he himself was pretty well tapped out. It was with this that one of his personal revelations occurred to him. Suddenly his slump snapped and he regained all of his intellectual vigor. Just prior to the war he’d been working on an idea of his for his Ph.D. thesis, having to do with his own new method in quantum mechanics that recognized to him the epithet of ”quantum’s man” by the american astronomer, physicist and essayist Lawrence Krauss. The method, typical of Feynman’s approach, had to do with computing the probability of a transition of a quantum from one state to some other subsequent state. In principle, every possible path from one state to the other is considered equally likely, with the final path between being a kind of sum of all paths. This was an entirely new formalism in quantum mechanics, and he eventually adapted it directly to the physics of quantum electrodynamics, also called QED. For this work he was awarded the Nobel Prize in physics in 1965, which he shared with Schwinger and Tomonaga, who also independently found their own methods in the same problem.

Feynman also was able to make a breakthrough in the physics of the superfluidity of super cold liquid helium, wherein the liquid displays no frictional resistance whatsoever while flowing. He successfully applied the well known Schrodinger’s equation to the question, showing that the superfluid was displaying what amounted to quantum mechanical behavior at macro-
scopic scales. A very close relative to superfluidity is the phenomenon of superconductivity, wherein electric current moves without resistance in certain materials at extremely low temperatures. Feynman also attempted to solve this important problem in physics, but this came to be one of his most spectacular failures as a theorist. Another close race involved what is called “weak decay”, which shows itself most familiarly in the decay of a free neutron into an electron, a proton, and an anti-neutrino. Feynman worked intensely on this problem himself while fellow theorist Murray Gell-Mann did likewise. In due time the pair collaborated on a broad new theory of the weak process in a joint research paper which was published just days before a similar theory was presented by fellow physicists Robert Marshak and E.C.G. Sudarshan. The new theory was of no trivial importance, and the work of these four gentlemen constituted the revelation of a new law of nature. For the first time in Feynman’s life, he had been instrumental in changing the very course of humankind’s understanding of nature.

During the 1950s he married his second wife, Mary Lou, but this was not to last for too long a time. It turned out that his feelings for the lady were hasty at best, and they were a mismatch. They divorced and went their separate ways. However, in the early 1960s he happened to be at a professional conference in Europe and became acquainted with a charming lady by the name of Gweneth Howarth, a native of Great Britain. He’d long been in need of someone to fill the loneliness after Arline’s death, and after having surveyed the field extensively, he became certain that Gweneth would be the one for him. She was patient with his eccentricities, yet shared much of his taste for adventure. Gweneth became his third and last wife. The pair stayed together for the remainder of Richard’s life, during which they had one child of their own, Carl, and adopted a daughter, Michelle.

From the 1950s onward Feynman was a professor of physics with the California Institute of Technology, popularly called Caltech. In the early 1960s there was some consensus among the teachers there that the freshman physics curriculum was badly in need of renovation. Professors Robert Leighton and
Matthew Sands approached Feynman with the proposal that he might be just the ticket; it would mean devoting himself full time to the project without significant time for his beloved research. But Richard knew that this was a rare opportunity to make a difference to the emerging younger generation of physics students, so he accepted the job and dove into it without looking back. The new physics course occupied most of the next three years of his life; the entire series of lessons was turned into what has since become a classic set of three bound textbooks called "The Feynman Lectures on Physics". Even nearly four decades has not faded the vitality of these works, and they continue to be a staple among both students and experienced professors seeking the valuable insights that lurk within. His first and foremost intellectual interest was, of course, in physics, but there was also something further that brewed in his mind for several years. He was building a curiosity for the nature of art. Art appeared as something of a mystery to him as an adult, as there was nothing clearly necessary driving the appeal which works of art held for human beings. It was obvious, of course, that art did in fact have a satisfying effect upon the human spirit, but the connection itself eluded him. He eventually decided to study the problem from the inside, and took beginners’ courses in drawing and painting. At first he had no particular ability, but in due time he developed into a skilled artist, especially in portrait sketching. He found that the lure of art lay, for him, in the personal satisfaction that his works could bring to others. He continued to practice art along side with physics for the rest of his life.

Although in the late 1970s he was looking at his 60th birthday, his intellectual form barely showed any age whatsoever, and he continued to make noteworthy contributions to his chosen field. But also at about this time he entered into what has since become one of his most mythic adventures. By chance he had the opportunity to mention to his close friend Ralph Leighton, son of Robert Leighton that there was a "lost land" of sorts from which he had collected postage stamps as a youngster, but which was nowadays nowhere on the map. The lost land was the country of Tanmu Tuva, tucked between Mon-
golia and Russia, that during Richard’s boyhood had been annexed by the Soviet Union. Nevertheless, Tuva was still in existence in practice regardless of its political status, and it was uniquely isolated by its mountainous geography, making it a tempting object of adventure in Richard’s mind. Richard, Gweneth and Ralph then determined that they would find a way to journey to Tuva and see what few outsiders have seen. An even harder climb was in store for them as they petitioned the Soviet bureaucracy for permission to travel to Tuva.

In the mean time, Richard was to enter into an adventure of another sort entirely. It was discovered that he was harboring a rare form of cancer, growing in the form of a massive tumor in his abdomen. Surgery successfully removed the tumor at the time, but substantial damage had already been done to his internal organs, leaving him weakened. In particular one of his kidneys had been crushed beyond saving. The timeliness of the surgery added time onto what otherwise would surely have been an early death. Still, over the next decade he experienced recurrences of cancer, and undoubtedly the illness was destined to be a fact of life. Never being one to fret, however, he always tended to be cheerfully grateful for whatever time was his, however short it may be.

In early 1986 the NASA space shuttle Challenger was destroyed in a disastrous explosion of its large fuel tank, consequently killing the seven crewmembers aboard. A presidential commission was quickly assembled to investigate the disaster, and one of Feynman’s former students in a position with the government nominated Richard as a member of the commission. Richard, having been asked to join, at first tended to recoil from such a task. He was, of course, suffering from his continuing cancer, and he also preferred to stay out of the obvious political charade sure to be made of the whole affair. But Gweneth, being more objective as to his clear value to the investigation, reminded him of his better self. She encouraged him to accept the job and he did, even if it turned out to be Feynman the only commission member without a vested interest for or against the shuttle program. He discovered that,
for one thing, NASA and its major contractors had an unspoken tendency to
discourage their own people from constructive criticism of valid safety issues.
This was old hat to him, as he’d encountered a similar blind eye attitude
about the security of the bomb secrets back at Los Alamos. A number of
low ranking individuals in the space program had reason to expect certain
systems of the spacecraft to be a catastrophe, but NASA continued to launch
its shuttle fleet ignoring flight data that told the tale of the Challenger well
in advance of its eventual destruction. In particular, the spacecraft launch
was assisted by large solid rocket boosters built of cylindrical sections butted
together and sealed at the joints with rubber O-rings. There was accumu-
lating evidence from previous flights that the O-rings tended to be damaged
by the burning solid rocket fuel inside. It eventually became quite clear to
Feynman that the culprit was to be found in the elastic properties of the rings
at low temperatures. For the morning of the fatal launch the weather had
produced particularly low temperatures, the lowest launch day temperatures
ever during the shuttle program. Richard speculated that the rubber rings
quite possibly were unable at such a temperature to expand quickly enough
at launch to fully seal the joints, perhaps then allowing the extremely hot ex-
hauast gasses to leak past the joint and burn through the large fuel tank filled
with liquid hydrogen, thus causing the final destructive explosion. To test the
plausibility of his theory he managed to acquire a section of genuine O-ring
of the same type used in the rocket joints. He then found a nearby hardware
store and bought a small C-clamp. Back in his hotel room he compressed
the sample in the clamp and then dipped it in ice water for a time, dropping
its temperature to 32 degrees Fahrenheit. At that point he removed it from
the ice water and unclamped it. As he expected, the rubber remained highly
compressed, even after being released, for an unacceptably long time while
at low temperature. He now felt confident enough to present this finding to
the commission. But he felt that he needed to generate enough impact that
the theory couldn’t be ignored or buried, and decided that the best place to
reveal it would be on live television the next day, during the meeting of the
commission. At an opportune moment he prepared the clamped sample in the freezing water and then irretrievably demonstrated the behavior of the O-rings to anyone and everyone with their television tuned in. The demo did its job very well, as the number of ranking officials had been claiming that the explosion might never be solved. Feynman’s experiment showed not just the likely ”mechanical” cause of the accident, but also just as much revealed the political cause of the death of the shuttle crew members. Anyone might potentially have shown the rubber rings to be the cause of the disaster, but only an ”outsider” was free to actually do so. At an official level, insiders were systematically disempowered to pursue the truth.

Feynman was one of the highest caliber intellectually and morally as well. Over the years he acquired both scientific knowledge and wisdom. His continuing ill health made him decide to put, together with his close friend Ralph, his very best anecdotes in print. A volume of Feynman’s stories was published in 1985 as Surely You’re Joking, Mr. Feynman. Even better than he’d ever expected, the book became a national best seller, spreading his particular approach to life now among millions of readers. Later, posthumously, a second volume titled What Do You Care What Other People Think? also sold very well. In the autumn of 1987 doctors discovered yet another cancerous tumor, which was treated surgically, but Richard was left still extremely weak and in considerable physical pain. It was only a matter of time before he’d be in the hospital once again. In February 1988 he was admitted to the hospital, where doctors discovered a further complication in the form of a ruptured gastrointestinal ulcer. Before too long his condition worsened when his one remaining kidney finally failed. It was certainly possible to have added a few months to his life by way of dialysis, but he estimated that enough was enough, and renounced any further medical treatments. Death is a certainty, and Feynman had chosen to take his without undue indignity. He empowered himself even in the matter of his own passing. On February 15, 1988, he at last made a farewell.
Bibliography


[4] R. P. Feynman, "Surely you’re joking, Mr. Feynman!", W. W. Norton, USA (1985);