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Doctoral Degree Thesis in
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IMPACT OF INHOMOGENEITIES ON
COSMOLOGICAL OBSERVABLES

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“In the days of my youth, I was told what it means to be a man,
Now I reached that age, I’ve tried to do all those things the best I can.”
(Led Zeppelin)
Introduction

Understanding origin, history and composition of our Universe is one of the oldest challenges that the mankind has tried to face since its beginning.

Before the birth of modern science, the main attempts of answering questions about the Universe were based on philosophical or religious approaches. Since the ancient Greek civilization until the society of the XVII century, the dominant paradigm to describe the cosmos was based on the ideas of Aristotle and Ptolemy: it was believed that the Earth was at the center of the entire Universe, that planets, Moon and Sun were circularly orbiting around it, and that stars were static.

This picture describes a Universe which is stationary and unchanging forever. The natural laws governing the cosmos not only are regarded as perfect and unalterable laws, but also are different from the laws governing the more “prosaic” motion of bodies on the Earth surface, where changes may continuously happen. According to the Greek dictionary, the word “cosmos” itself refers to the idea of a perfect order, just the opposite of “chaos”.

This duality between a perfect cosmos and an imperfect Earth’s physics was the dominant one until the birth of modern science. Thanks to the work of eminent scientists like Copernicus, Galileo, Kepler, Newton and others, the basic ideas of the Aristotelian and Ptolemaic cosmology were finally confuted and discarded. In particular, after improving his telescope, Galileo for the first time pointed it to the sky and started to collect a series of data about planets. He discovered the appearing of dark spots on the Sun surface, and the presence of craters on the Moon. These irregularities proved that also far from the Earth the same physical laws are operating, and may be source of changes and imperfections.

On this grounds, Kepler studied the orbits of planets and formulated his three famous laws: the planetary orbits are elliptical trajectories not around the Earth but around the Sun (located in one of two ellipse’s focus). Also, the planetary velocities are not always constant in time, but are larger when the planets are closer to the Sun. He also discovered a precise relation between the orbital period $T$ and the semi-major axis of the orbit $R$, implying that
$T^2 / R^3$ is constant for all the planets in the solar system.

These evidences were crucial to reject the Aristotelian model and to accept the Copernican ideas that the Earth is not the center of the entire Universe. The definitive confirmation of this was due to Newton. He was able to explain why Kepler’s laws hold, but his contribution was even greater. Indeed, he separately proposed the universal law of gravitational attraction for describing the planetary motions, and also the dynamic laws for the motion of inertial observers on the Earth. By combining these different contexts, he provided an unique explanation for the motion of planets and of all falling bodies: this was the definitive prove that the physical laws are unique, and general enough to describe motion on the Earth and everywhere in the Universe.

Aside from the importance of this conceptual revolution in itself, the contribution of Newton, Galileo, Leibniz and others was relevant enough to make science, in general, and cosmology, in particular, quantitative, and then numerically predictive.

The approach to cosmology in terms of mathematically well-formulated laws is still nowadays highly successful. In particular, the introduction of the relativistic ideas due to Poincaré, Einstein, Lorentz, and the observations made by Eddington, Hubble etc. at the beginning of the XX century have intimately modified our understanding of nature. Einstein postulated that all observers, not only the inertial ones, must agree on the description and on the physical interpretation of physical phenomena. He also postulated an equivalence principle between inertial forces and gravity: according to this principle, they are locally indistinguishable. These ideas led him to formulate the theory of general relativity.

Among the most revolutionary consequences of this theory we can mention, in particular, the deflection of light trajectories due to gravity, and the different flowing of time in the presence of gravitational fields. In 1919, Eddington was able to provide the first experimental confirmation of these predictions: he observed that the deflection of light rays grazing the Sun during a solar eclipse was in perfect agreement with Einstein’s theory. A few years later, Hubble collected a lot of data about galaxies close to the Milky Way, and he observed that the wavelength of light-like signals from these galaxies was redshifted. An independent measure of distances led him to interpret this shift as the proof of a cosmological expansion. The last Aristotelian idea about stationarity of the cosmos, in this way, was definitively rejected. Hubble’s data too were explained by using the theory of general relativity, and this marked the beginning of modern relativistic cosmology.

Through the years, since Hubble’s work until now, the relativistic description of the Universe has been considerably improved. Important contributions
have been given by Penrose and Hawking during the 70’s: they have theoretically proved the need for an initial singular state of our expanding Universe. This initial singularity is one of the essential features of the Big Bang cosmological scenario. Such a scenario has greatly improved our understanding of the present state of the Universe, and its prediction about the existence of a relic Cosmic Microwave Background (CMB) has been confirmed in the 60’s by Penzias and Wilson. More recently, the inflationary completion of this scenario has led to the celebrated $\Lambda$CDM model, whose parameters are determined with ever increasing accuracy by present observations (like those provided by the WMAP and Planck missions).

Nevertheless, this standard scenario is based on the assumption that a typical scale exists above which the background cosmological geometry becomes exactly homogeneous and isotropic. Even if this idea allows a lot of mathematical simplifications, its validity is still waiting for a definitive confirmation. For instance, beside the deviations – of inflationary origin – from perfect isotropy and homogeneity that must be taken into account to describe the fluctuations of the CMB temperature as well as the dynamics of structures formation, an intrinsic background anisotropy might to be required to explain observed anomalies.

In addition, the conventional picture of homogeneous expansion is associated to basic questions in fundamental theoretical physics. Indeed, since 1998, observations of the Supernovae Ia due by Perlmutter, Schmidt and Riess have revealed a really surprising feature: a homogeneous and isotropic Universe is not only expanding, but its expansion has to be accelerated. This result is hard to be understood from a classical point of view. Indeed, usual matter feels an attractive gravitational force, so that the expansion should be decelerated by gravity. In order to explain such a surprising behaviour one must invoke a new cosmic component filling our present Universe: the so-called “dark energy”. However, our present knowledge about this particular source of gravity is not satisfactory at all. Standard particle physics seems to be unable to provide a good enough and unambiguous answer the question: what is dark energy? It is clear that such a discrepancy is one of the crucial problems of modern theoretical physics.

Motivated by these considerations, recent attempts have been made for a (possible) better understanding of the nature and of the origin of the large scale acceleration. In particular, it has been stressed that the presence of intrinsic background inhomogeneities could be responsible for biases in the determination of the cosmic distances and – as a consequence – of the dynamical evolution of the Universe. Hence, using a consistent theoretical scheme to correctly take into account possible deviations from exact homogeneity and isotropy, has become a crucial task in modern cosmology.
This work of thesis is then motivated by the purpose of providing answers to the questions: how much the homogeneity hypothesis is relevant for the theoretical interpretation of present cosmological observations? Do we have a consistent framework to test such hypothesis, and to take into account possible (even if small) deviations from a perfectly homogeneous and isotropic backgrounds geometry?
Chapter 1

Homogeneity and isotropy: the Friedmann-Lemaître-Robertson-Walker-metric

1.1 General Relativity and Cosmology

Modern cosmology aims at treating the Universe as a subject of our scientific knowledge. Of course, supporting the experimental investigation with a proper mathematical description is a crucial ingredient of this project. However, there are non-trivial problems to be solved. For instance, differently from other research fields in physics, the laws describing the global – past, present and future – evolution of the Universe must correctly take into account phenomena concerning a single system (our Universe as a whole), with no hope (at least at present) of reproducibility and direct experimental testing.

Having this in mind, it is clear that testing cosmological models and, more generally, gravitational models on cosmic scales may be harder than other physical models. Theoretical biases may occur in the description and analysis of experimental data, and this can be due to several reasons: for instance, on the assumption that standard General Relativity is valid without modifications also on very large scales of distance, or on the assumption of homogeneity and isotropy on such scales for the energy and matter distribution. Therefore, the models to be used should be the more general as possible in order to get a less biased description.

It will be enough, for our purposes, to assume that the Universe is well described by Einstein’s equations of General Relativity (GR). Recently, several and very interesting modifications of this theory have been proposed in the literature [1, 2, 3, 4, 5]. However, the need for such modifications is not always well motivated, also in view of the excellent agreement between
1.2. THE COSMOLOGICAL PRINCIPLE

GR and present observations on smaller scales. The Einstein theory, based on general covariance and on the equivalence principle, leads to field equations connecting the distribution of energy-momentum to the space-time curvature. Gravity is then represented, geometrically, by this curvature.

The successes of Einstein’s GR are numerous. First of all, from a theoretical point of view, it relates gravity, geometry and matter sources in a very elegant and formally consistent way. In addition, from the experimental side, a lot of independent confirmations of this theory have been accumulated during the 20-th century: let me mention only, for instance, the study of the propagation of light rays and electromagnetic signals in the presence of an external gravitational field. GR predicts that light has to feel the gravitational force because of its energy content. This implies that matter structures can deflect and delay the trajectories of light-like signals even if their are massless. The deflection effect was measured for the first time by Sir Arthur Eddington in 1919, and it provided the first great experimental confirmation of GR.

On the other hand, maybe this theory is still far from being definitive. Indeed, gravity is treated as a non-quantum field. All attempts of quantizing the theory have failed to solved the renormalization problem, and this is a big obstacle in order to get a coherent description of gravity in the realm of quantum phenomena. In addition, GR allows singular solutions, where the space-time curvature and the energy-density content of matter diverge. What really happens to the space-time when approaching these singularities is still unclear, and matter of scientific debate. Once again, the missing of a quantum gravitational theory and of appropriate quantum corrections to GR is probably responsible for such a divergent behavior.

Nevertheless, we believe that GR still represents the best available theory to describe gravity (at least at the classical level), and to formulate models for the late-time cosmological evolution. For such a reason we will always adopt its equations, whenever needed, for all the problems discussed in this work.

1.2 The cosmological principle

Our galaxy contains about $10^{11}$ stars, and has a spatial extension of about $10^4$ parsec (pc). Such a system is located itself within a larger cluster of galaxies, usually called the “Local Group”, containing more than 30 galaxies. The extension of such a group is of order of the megaparsec (Mpc). Once
again, the Local Group is part of a bigger cluster, the “Local Super Cluster”, called Virgo, with radius of about 50 Mpc.

Such groups can frequently appear everywhere in the present Universe, and are conventionally called “Super Clusters”. By studying them we can obtain interesting information on the large-scale matter distribution. We find, in particular, that the Universe seems to display a sponge-like (or “swiss cheese”) structure over length scales ranging from 50 to 100 Mpc, where such Super Clusters are located at the boundaries of really large under-dense regions \([6]\). It is thus clear that gravitational interactions too cannot be very homogeneous over such distance scales. However, one can assume that spatial homogeneity and isotropy are respected over length scales larger than the above ones. This assumption is well known under the name of “cosmological principle”, which postulates that Universe appears to be spatially homogeneous and isotropic over distances larger than 100 Mpc. Such an idea was also suggested by Einstein because of epistemological motivations, related to Mach’s principle.

Nevertheless, it should be stressed once again that the validity of the cosmological principle crucially depends on the scales we are considering. On small enough distances significant deviations from perfect homogeneity must be present, in order to describe structures formations due to gravitational instability. On the opposite side, assuming the property of isotropy and homogeneity on large enough distances is equivalent to assume that our position in the Universe is not preferred at all, and is statistically indistinguishable from that of any other observer. (By the way, the extrapolation of this idea to its maximum extension led cosmologists, in the past, to formulate the so-called “perfect cosmological principle”: according to it, the Universe should be the same not only at every point and along any direction, but also at all times).

We should recall, at this point, that the notion of isotropy is also relative, in principle, to the given observer. A moving observer (in particular, an accelerated observer) will unavoidably detect a preferred direction intrinsically related to its spatial motion, breaking the possible isotropy of the cosmological background. For this reason the cosmological principle refers in particular to static (or, better, “comoving”) observers which, in the very special case of a perfectly homogeneous and isotropic geometry, can exist at all points and are also automatically “free-falling” (i.e. following a geodesic trajectory).

Another point to be stressed is that, imposing on the spatial geometry to be isotropic with respect to any given static observer, automatically implies that there are no preferred positions, and this means that the geometry is also homogeneous. From this argument we learn that we may consider cosmological scenarios which are homogeneous but anisotropic; however, we are not allowed to consider scenarios which are inhomogeneous and simul-
1.3. THE FLRW METRIC

...taneously isotropic around all static observers. What we can consider, at most, is a model of inhomogeneous geometry which is isotropic around a given observer located at a special position, but no longer isotropic away from that position. We will largely discuss this situation during this work.

The last point we want to mention concerns the presence of (particle) horizons in the context of homogeneous and isotropic cosmological models. The horizon separates causally connected zones from unconnected ones: in spite of this, all spatial parts of a perfectly homogeneous Universe are characterized by the same physical and geometric properties, even if their distance is so large to escape the causal contact produced by a phase of primordial inflationary evolution. How could this be possible? Answering to this question is highly not trivial and, for sure, it suggests the possibility that perfect homogeneity cannot be true at all.

During this work we will discuss different approaches to the treatment of the cosmological inhomogeneities. This will be done after a short presentation of the standard cosmological model.

1.3 The FLRW metric

Following our previous discussion, we will start by implementing a homogeneous and isotropic model of cosmological space-time in the context of the GR gravitational equations.

We shall assume that our metric tensor $g_{\mu\nu}$ is invariant under the action of the $SO(3)$ group (namely, that the spatial rotations are isometries of the given metric, because of the assumed isotropy), and also invariant under the action of the group of spatial translations (because of the assumed homogeneity). From a mathematical point of view, this implies that the generators of the rotation and translation groups are Killing’s vectors for the given metric. Since the total number of generators of these two groups is 6, and since an $N$-dimensional maximally symmetric manifold is characterized by $\frac{N}{2}(N + 1)$ Killing vectors, by definition, we get:

$$\frac{N(N + 1)}{2} = 6 \Rightarrow N = 3. \quad (1.1)$$

This means that every spatially homogeneous and isotropic four-dimensional geometry must be characterized by (spatial) sections corresponding to max-
1.3. THE FLRW METRIC

Imally symmetric three-dimensional manifolds.

By adopting polar coordinates \( \{ r, \theta, \phi \} \) for the spatial sections, we can then write the cosmological line-element as follows [7, 8, 9]:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -b^2(t) dt^2 + \frac{a^2(t)}{1 - Kr^2} dr^2 + a^2(t) r^2 d\Omega^2 \tag{1.2}
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), where the functions \( a \) and \( b \) depend only on the time-like coordinate \( t \), and where \( K \) is a constant, determining the (inverse) curvature radius of the spatial geometry. Such a metric has still a gauge freedom to be fixed. In particular:

- We can impose \( b = a \). This is called “conformal gauge”. In this case, the time parameter is usually denoted by \( \eta \), and line-element becomes:

\[
ds^2 = a^2(\eta) \left[ -d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]. \tag{1.3}
\]

- Otherwise, we can choose \( b = 1 \). This is the so-called “synchronous gauge”, and the metric, expressed in terms of the cosmic time \( t \), assumes the standard Friedman-Lemaître-Robertson-Walker (FLRW) form:

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]. \tag{1.4}
\]

The two choices are obviously related by:

\[
a(\eta) d\eta = dt \quad \text{or equivalently} \quad \eta = \int \frac{dt}{a(t)}. \tag{1.5}
\]

For the moment we will adopt the second type of gauge fixing. Working in that gauge we can easily check that, for the given geometry, any static observer is also a geodesic observer. Indeed, from the explicit form of the geodesic equation:

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \tag{1.6}
\]

where:

\[
\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}), \tag{1.7}
\]

we obtain that static observers, defined by the velocity field \( \frac{dx^\mu}{d\tau} = \{ 1, \vec{0} \} \), automatically satisfy the equation:

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{tt} = 0 \tag{1.8}
\]

because, from the definition (1.7), \( \Gamma^\mu_{tt} = 0 \) since \( g_{tt} = -1 \). According to our previous discussion, this introduces a well-defined class of observers,
1.4 PERFECT BAROTROPIC FLUIDS

providing the natural reference frame for the description of homogeneous and isotropic backgrounds.

In order to obtain the dynamical evolution of the scale-factor $a(t)$ we have now to solve the corresponding Einstein’s equations, which in general are given by:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.9)$$

Here $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein’s tensor, $T_{\mu\nu}$ is the energy-momentum tensor, depending by the particular model of source we are adopting, $R_{\mu\nu}$ is the Ricci tensor, given by:

$$R_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu} = \partial_{\alpha} \Gamma^{\alpha}_{\mu\nu} - \partial_{\mu} \Gamma^{\alpha}_{\alpha\nu} + \Gamma^{\alpha}_{\alpha\beta} \Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\alpha\nu} \quad (1.10)$$

and $R = R_{\mu}^{\mu}$ is the scalar curvature.

By applying these general definitions to our FLRW metric we obtain:

$$R_{0}^{0} = 3 \frac{\ddot{a}}{a}$$

$$R_{1}^{1} = R_{2}^{2} = R_{3}^{3} = \frac{\dot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^{2} + \frac{K}{a^{2}} \quad (1.11)$$

where $\dot{a} = \frac{\partial a}{\partial t}$ and $\ddot{a} = \frac{\partial^{2} a}{\partial t^{2}}$. The scalar curvature is then given by:

$$R = 6 \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^{2} + \frac{K}{a^{2}} \right]. \quad (1.12)$$

As we can see from Eqs. (1.9), choosing a particular geometry is not enough to completely specify a cosmological model. We also need the explicit form of the matter-energy distribution filling the cosmological space-time, and appearing on the right side of the GR equations. Of course, different choices of $T_{\mu\nu}$ will produce different dynamical evolutions. In the following section we will discuss the most common one, which is also the relevant one for the models of late-time cosmological evolution in which we are interested.

1.4 Perfect barotropic fluids

In order to specify the gravitational sources we recall that standard cosmology usually assumes a mixture of perfect, barotropic, non-interacting fluids.
1.4. PERFECT BAROTROPIC FLUIDS

Denoting with $p$ the pressure of a single component of the mixture, and with $\rho$ the energy density, we have that the corresponding energy-momentum tensor can be written as:

$$T_{\mu}^{\nu} = (\rho + p) u_{\mu} u^{\nu} + p \delta_{\mu}^{\nu}. \quad (1.13)$$

where $u^{\nu}$ is the velocity field of the fluid element (assumed “comoving” with the static and geodesic observers of the background geometry). It follows that:

$$T_{\mu}^{\nu} = \text{diag} (-\rho, p, p, p). \quad (1.14)$$

More generally, we can take into account the presence of a mixture with more than one fluid component. In that case, the total pressure and total energy density of the mixture will be given by the discrete sums:

$$p = \sum_n p_n, \quad \rho = \sum_n \rho_n \quad (1.15)$$

where $\rho_n$ e $p_n$ refers to the $n$-th fluid component. These variables, because of the assumed homogeneity and isotropy of the background geometry, may be function only of time, namely $\rho = \rho(t)$ and $p = p(t)$; their evolution is governed by the Bianchi identity:

$$\nabla_{\nu} T_{\mu}^{\nu} = \partial_{\nu} T_{\mu}^{\nu} + \Gamma_{\nu\alpha}^{\nu} T_{\mu}^{\alpha} - \Gamma_{\nu\mu}^{\alpha} T_{\alpha}^{\nu} = 0 \quad (1.16)$$

which reduces to the condition:

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0. \quad (1.17)$$

Strictly speaking, Eq. (1.17) holds for the total energy and pressure of the mixture. Nevertheless, because we are assuming that the fluids don’t interact among themselves, we are allowed to write such equation for each component of the mixture. On the other hand, the assumptions that all components of the mixture are barotropic implies that pressure and density are not independent, but can be related by a simple proportionality equation, i.e.

$$p_n = \gamma_n \rho_n, \quad (1.18)$$

where $\gamma_n$ are specific constants describing the particular equation of state of the given fluid. From a physical point of view, the above equation implies that on every constant $\rho$ hypersurface there are no pressure differences which would modify the spatial distribution of energy, and break spatial homogeneity.

By combining Eqs. (1.17) and (1.18), we then easily obtain:

$$\dot{\rho}_n + 3 \frac{\dot{a}}{a} (1 + \gamma_n) \rho_n = 0 \implies \rho_n = \rho_{n0} \left( \frac{a_0}{a} \right)^{3(1+\gamma_n)} \quad (1.19)$$
1.5. EQUATIONS

where $\rho_{n0}$ and $a_0$ refer to the today values ($t = t_0$) of these variables.

1.5 Equations

We have now the explicit form of both sides of the cosmological equations, and are in the position of finding the required solutions. We obtain, in particular, two non-trivial equations corresponding to the $tt$ and $rr$ components of Einstein’s equations:

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \rho
\]

and

\[
2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = -8\pi G p.
\]  

(1.20)

With the addition of the Bianchi identity (equivalent to the fluid conservation equation) we have a total of three equations for two time-dependent functions, $\rho(t)$ and $a(t)$. As is well known, however, only two equations are independent, because it is always possible to derive one of them from the other two equations. It is also possible to combine these equations, to recast them in a more convenient form.

For instance, a combinations of the two Eqs. (1.20) gives the so called acceleration equation:

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) = -\frac{4\pi G}{3} \sum_n (1 + 3\gamma_n) \rho_n,
\]  

(1.21)

which has very important consequences. Indeed, it directly relates the cosmic acceleration to the total energy and pressure content of the cosmological sources. Since $\rho_n$ is positive it implies that we need $\gamma_n < -1/3$ (i.e. a source with a negative enough pressure) in order to get a positive acceleration. The role of the negative pressure is that of overcoming the usual gravitational attraction of ordinary matter.

Another interesting combination of the cosmological equations is the following one:

\[
H^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \sum_n \rho_{n0} \left(\frac{a_0}{a}\right)^{3(1+\gamma_n)},
\]  

(1.22)
1.5. EQUATIONS

where \( H(t) \equiv \frac{\dot{a}}{a} \). This is usually known as the Friedmann equation, and can be solved to obtain the time-dependence of the scale factor \( a(t) \). This equation can also be rewritten in the so-called “critical form”:

\[
\sum_n \Omega_n + \Omega_k = 1, \tag{1.23}
\]

where we have defined \( \Omega_n = \frac{8\pi G \rho_n}{3H^2} \) and \( \Omega_k = -\frac{k}{(aH)^2} \).

Given the constant spatial curvature \( K \), and given the explicit values of the constant coefficients \( \gamma_n \) of the fluid components, we can now analytically solve the equations of our cosmological model.

An illustrative example – useful for our subsequent discussion – is given by the case of a single pressureless fluid (\( \gamma = 0 \)), and vanishing spatial curvature (\( K = 0 \)). In this case, the second of Eqs. (1.20) becomes:

\[
2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = 0 \tag{1.24}
\]

Looking for solutions in the form \( a(t) = (a_o + \frac{H_0}{\alpha} t)^{\alpha} \), Eq. (1.24) gives:

\[
2\alpha (\alpha - 1) + \alpha^2 = 0 \tag{1.25}
\]

or, equivalently:

\[
\alpha (3\alpha - 2) = 0 \tag{1.26}
\]

which is satisfied for \( \alpha = 2/3 \) in the non-trivial case. The integration constant \( a_o \) thus corresponds to \( a(0) \). If we fix the time coordinate in such a way that \( t = 0 \) corresponds to the present time, then \( a_o \) is the today value of the scale factor. We can fix it to 1 without loss of generality. Also, the constant \( H_0 \) corresponds to the present value of the Hubble parameter \( \dot{a}/a \).

The above solution can also be conveniently expressed in terms of the conformal time \( \eta \), instead of the synchronous time \( t \). To this purpose let us recall that:

\[
\eta = \int \frac{dt}{a(t)} = \int \left( 1 + \frac{3}{2} H_0 t \right)^{-2/3} dt = \frac{2}{H_0} \left( 1 + \frac{3}{2} H_0 t \right)^{1/3} \tag{1.27}
\]

so that:

\[
t = \left( \eta + \frac{H_0}{2} \eta^2 + \frac{H_0^2}{12} \eta^3 \right). \tag{1.28}
\]

We then obtain:

\[
a(\eta) = a(t(\eta)) = \left( 1 + \frac{H_0}{2} \eta \right)^2. \tag{1.29}
\]
1.6 Physical observables

What is required, now, is a clear link between the cosmological variables used in the previous equations and the physical observables measured by astronomers and astrophysicists. There are many quantities of this type, of course, but, for the purposes of this work, we are mainly interested in clarifying what can be measured in order to describe the large scale evolution of our present Universe.

The sought target cosmologists would like to achieve is a direct measure of the distance, performed in a model independent way. Unfortunately this is not possible so that, in order to get precise experimental information on the cosmic evolution, they usually refer to two physical observables.

- **The redshift.** Let us consider a source which is emitting electromagnetic signals. The redshift is defined as the ratio \( z \equiv \frac{\lambda_o}{\lambda_e} \), where \( \lambda_o \) is the wavelength of the signal as measured by the observer and \( \lambda_e \) is the wavelength emitted by the source. It is well known, since the pioneer work by Hubble and Humason [10, 11], that the signals received from sources located at cosmological distances are characterized by wavelengths larger than expected, and this effect is universally interpreted as a proof of the expansion of our Universe. Of course, there are various possible contributions to the total observed redshift (the relative motion between source and observer, deviations from perfect homogeneity, and so on) but, for our purposes, we are mainly interested in the redshift due to the cosmic dynamics. Having this in mind, we recall that the redshift can be defined as follows:

\[
1 + z = \left( \frac{u_\mu k^\mu}{u_\mu k^\mu} \right)_o,
\]

where subscripts refer to emission (\( e \)) and observation (\( o \)) variables, \( u_\mu \) is the four-velocity of source or observer and \( k^\mu \) is the four-momentum of the travelling signal (in particular, of the transmitted photon). Thanks to this definition we can directly relate \( z \) to the given background geometry, and we can also express the cosmological equations and the geodesic equation of motion in terms of the redshift variable.

- **The luminosity distance.** According to the previous discussion, it follows that the redshift \( z \) can be identified as an appropriate time coordinate, useful to parametrize the large-scale evolution. We need now an independent good quantity to measure distances, in order to provide experimental plots describing the kinematics of the cosmological expansion. Unfortunately, the distance of the sources cannot be measured directly. However, we can measure several observables which are related to the distance. For instance, if we know the intrinsic size of a
source, we can measure its apparent size and obtain, in this way, what is usually called the “angular diameter distance” \( d_A \). In addition, if we exactly know the radiation power emitted by the source, we can deduce its distance by comparing the emitted power with the flux \( \mathcal{F} \) of the received radiation: this defines the so-called “luminosity distance” \( d_L \), related to the received flux by \( \mathcal{F} \sim d_L^{-2} \). Taking into account the cosmological expansion, it follows that \( d_L \) and \( d_A \) are not exactly the same, being related by Etherington’s reciprocity relation [12]:

\[
d_L = (1 + z)^2 d_A. \tag{1.31}
\]

This is a very powerful equality, as it is expected to hold in any space-time manifold if the number of photons is conserved while the radiated photons travel from the source to the observer. For the exact theoretical definition and computation of the luminosity distance we need introducing and solving the so-called equation of geodesic deviation, and this will be done in the last chapter. For the moment we will simply keep in mind that it is always possible to introduce good observable quantities, suitable to obtain a precise measure of the cosmological distances.

Using the two observables defined above we can easily produce experimental plots showing the luminosity distance of the sources as a function their measured redshift, as done for instance in Fig. 1.1. In that figure, in particular, the plotted variable is not exactly the luminosity distance \( d_L \), but the (more convenient for experimental reasons) variable \( \mu \), called the “distance modulus”, and related to \( d_L \) by:

\[
\mu = 5 \log_{10} \left( \frac{d_L(z)}{1 \text{ Mpc}} \right) + 25. \tag{1.32}
\]

The experimental results for the plots \( \mu(z) \) can be theoretically interpreted through the set of equations associated to a given cosmological model.

Let us briefly introduce an explicit example of this procedure for the case of the FLRW geometry, described by the metric (1.4), and for the case in which source and observer are both characterized by a static velocity field \( u_\mu = \{1, \vec{0}\} \). Considering the propagation of light-like signals of proper frequency \( \omega \), associated to the null vector \( k^\mu k_\mu = 0 \), one finds that Eq. (1.30) implies

\[
1 + z = \frac{\omega_\epsilon}{\omega_o} = \frac{a_o}{a_e}, \tag{1.33}
\]

where \( a_o = a(t_o) \) and \( a_e = a(t_e) \). To obtain the last equality we have imposed on \( k^\mu \) the condition of geodesic propagation along the null geodesic connecting source and observer.
Let us suppose that the static observer is located at the origin of our coordinate system and that the source, at a radial distance $r_e$, is emitting light-like radiation with a power (or intrinsic luminosity) $L_e$. The energy flux $F_o$ received by the observer at the time $t_o$ in the context of the FLRW geometry, is then given by

$$F_o = \frac{L_e}{4\pi(1+z)^2a_o^2r_e^2(t_o)},$$

where the factor $(1+z)^{-2}$ is due to the combined redshift and time-dilatation effects distorting the emitted power because of the cosmological expansion.

The effective distance $r_e(t_o)$ appearing in the above equation can be expressed in terms of the emission and observation times by integrating the null radial geodesic equation of the expanding FLRW geometry, namely:

$$\int_0^{r_e(t_o)} \frac{dr}{\sqrt{1-Kr^2}} = \int_{t_e}^{t_o} \frac{dt}{a(t)}.$$

The luminosity distance $D_L$, on the other hand, is defined so as to depend on the emitted power and the received flux only, in such a way that:

$$F_o = \frac{L_e}{4\pi d_L^2}.$$

It follows that

$$d_L = (1+z)a_o r_e(t_o) = (1+z)^2a_e r_e(t_o),$$

Figure 1.1: Plot for experimental point of 557 Supernovae Ia. Catalog: UNION2.
and the corresponding angular distance, according to Eq. (1.31), is given by \( d_A = a_e r_e(t_o) \).

To complete our program we have to integrate Eq. (1.35), shifting from the cosmic time \( t \) to the redshift parameter \( z \) so as to compute \( r_e(z) \) and \( d_L(z) \). Starting from the general relation \( 1 + z(t) = a_o/a(t) \), and differentiating with respect to \( t \), we have:

\[
dz = -\frac{\dot{a}(t)}{a^2(t)} a_o dt = -a_o H(t) \frac{d}{a(t)} \Rightarrow -\frac{dt}{a(t)} = \frac{dz}{a_o H(t(z))}.
\] (1.38)

Inserting this result into Eq. (1.35) and considering, for simplicity, the case \( K = 0 \), we immediately obtain from Eq. (1.37)

\[
d_L = (1 + z) \int_0^z \frac{dz'}{H(z')}.
\] (1.39)

The Hubble parameter \( H \) can be easily expressed in terms of \( z \) using the Friedman equation (1.22). By dividing such equations by \( H_0^2 = H^2(t_o) \), and defining \( \Omega_{n0} \equiv (8\pi G/3H_0^2)\rho_{n0} \), we finally obtain

\[
H(z) = H_0 \left[ \sum \Omega_{n0}(1 + z)^{3(1+\gamma_n)} \right]^{1/2}
\] (1.40)

where, again, \( 1 + z = a_o/a \). Hence, luminosity distance is given by:

\[
d_L = \frac{(1 + z)}{H_0} \int_0^z dz' \left[ \sum \Omega_{n0}(1 + z')^{3(1+\gamma_n)} \right]^{-1/2}.
\] (1.41)

Eq. (1.41) gives the theoretical prediction for all the FLRW models and it can be directly compared with experimental data in Fig. 1.1 by its logarithm. This comparison leads to the introduction of the cosmological constant. Indeed, a matter dominated model with only \( \gamma_{\text{matter}} = 0 \) fails this comparison. The introduction of a new gravitational source is necessary to this end. In particular, the simplest model needs the addition of a cosmological constant \( \Lambda \) which can be described equivalently as a perfect fluid with \( \gamma_\Lambda = -1 \). This component gives rise to an accelerated expansion of the universe. Nevertheless, because its nature is unknown at all, more general models have been considered in order to explain such acceleration. In general, we speak about dark energy.
1.7 Perturbation theory in a FLRW background

If inhomogeneities and anisotropies are small, they can be studied in a perturbative way. This is a very useful approach from a theoretical point of view (see [13, 14] for complete reports), as it gives us a method to correctly explain how structures can appear and evolve on small enough scales during the cosmological evolution. Also, within the perturbative approach we can properly take into account the deviations from perfect isotropy which are currently detected in the temperature of the Cosmic Microwave Background (CMB) radiation.

Let us start by adding a small perturbation $\delta g_{\mu\nu}$ to the FLRW metric $g_{\mu\nu}$:

$$g_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \vec{x}),$$

(1.42)

and use the conformal gauge to parametrize the time evolution of our variables. For the moment, we shall limit our study to first-order perturbations of the background dynamics. If no symmetry condition is imposed, then $\delta g_{\mu\nu}$ must contain 10 independent components, depending in principle on all coordinates. For a geometric interpretation such components can be conveniently classified as irreducible representations of a given symmetry group: because of the symmetries of the unperturbed background we can choose, in particular, the group of spatial rotations at fixed $\eta$, i.e. $SO(3)$. The perturbed line-element can then be written as:

$$ds^2 = (g_{\mu\nu} + \delta g_{\mu\nu}) dx^\mu dx^\nu$$

$$= a^2 \left\{ (1 + 2\phi) d\eta^2 + 2 (V_i + \partial_i B) d\eta d\eta$$

$$+ \left[ (1 - 2\psi) \delta_{ij} + 2\partial_i \partial_j E + 2\partial_i F_j + h_{ij} \right] dx^i dx^j \right\}. \quad (1.43)$$

Here $\{\phi, \psi, E, B\}$ are 4 scalar fields (with respect to the group of spatial rotations), $\{V_i, F_i\}$ are two transverse vectors, and $h_{ij}$ is a traceless transverse symmetric tensor. By counting the degrees of freedom we find a total number of 10, just as expected for the independent components of $\delta g_{\mu\nu}$.

Of course, the dynamic evolution of these fields is determined by the type of sources described by the energy-momentum tensor, which is to be completed by adding its own perturbations. For the late-time cosmological evolution we can concentrate, in first approximation, on the case of perfect barotropic fluids which are completely represented by two scalar fields $\{\rho, p\}$ and a by vector field $u^\mu$. The perturbations for this source can then be written as:

$$T^\nu_\mu(\eta) + \delta T^\nu_\mu(\eta, \vec{x}),$$

(1.44)

where

$$\delta T^\nu_\mu = (\delta \rho + \delta p) u_\mu u^\nu + (\rho + p) (\delta u_\mu u^\nu + u_\mu \delta u^\nu) + \delta p \delta^\nu_\mu.$$ 

(1.45)
While $\delta \rho$ and $\delta p$ are obviously scalar perturbations, $\delta u^\mu$ may also contain vector degrees of freedom. However, let us recall that for a comoving fluid source, at rest with a static observer, the velocity field in the conformal gauge is given by $u^\mu = \left\{ a^{-1}, \vec{0} \right\}$, and we have:

\[ \delta (g_{\mu\nu} u^{\mu} u^{\nu}) = \delta g_{\eta\eta} (u^\eta)^2 + 2 g_{\eta\eta} u^\eta \delta u^\eta = -2 \phi - 2a \delta u^\eta. \]  

(1.46)

Because the condition $u_\mu u^\mu = 1$ holds exactly, its perturbation is vanishing, hence we find:

\[ \delta u^\eta = -\frac{\phi}{a}, \quad \text{and} \quad \delta u_\eta = a \phi. \]  

(1.47)

This means that the perturbation of the time-like component of the velocity is of pure scalar type, and it doesn’t add any degree of freedom being completely fixed by the component $\phi$ of scalar metric perturbations.

On the other hand, the spatial component $\delta u_i$ contains three degrees of freedom which can be classified with respect to the $SO(3)$ group as done for the metric. In this way we find a pure scalar mode $w$ and a transverse vector $v_i$, and we can set:

\[ \delta u_i = - \left( a \partial_i w + v_i \right), \]  

(1.48)

where $\partial_i v^i = 0$.

In general, we have thus to consider a set of equations for scalar, vector and tensor perturbation variables. However, at the first perturbative order, such equations are linear and the scalar, vector and tensor variables evolve independently. In the absence of explicit vector sources it can be shown, in particular, that the contribution of vector perturbations is negligible [14]. In the rest of this chapter we will also neglect tensor perturbations, concentrating our illustrative discussion on the scalar case only.

Therefore, the variables we have to study can be written as follows:

\[ g_{\mu\nu} + \delta g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2\phi & \partial_i B \\ \partial_i B & (1 - 2\psi) \delta_{ij} + 2 \partial_i \partial_j E \end{pmatrix} \]

\[ T^\nu_{\mu} + \delta T^\nu_{\mu} = \begin{pmatrix} -\rho - \delta \rho & (\rho + p) \partial_i w \\ -(\rho + p) \partial_i w & (p + \delta p) \delta_i^j \end{pmatrix}. \]  

(1.49)

On these variables, we can still impose gauge conditions, by performing coordinate transformations and exploiting the intrinsic general covariance of the Einstein equations. In particular, the perturbation variables used in the above equation are not gauge-invariant, but we can always find an equivalent set of full gauge-invariant perturbations.

Let us recall, first of all, that given a (possibly perturbed) metric in a system of coordinates $x^\alpha$,

\[ g_{\mu\nu}(x^\alpha) + \delta g_{\mu\nu}(x^\alpha) \]  

(1.50)
and given the coordinate transformation (or diffeomorphism) \( x^\alpha \rightarrow \tilde{x}^\alpha \), we will call “gauge transformation” the local reparameterization (also called “functional reparametrization”) \( \delta g_{\mu\nu}(x) \rightarrow \delta \tilde{g}_{\mu\nu}(x) \), in which we compute the transformation of the metric at fixed space-time position, namely we compute \( \delta \tilde{g}_{\mu\nu}(x) \) instead of \( \delta g_{\mu\nu}(\tilde{x}) \).

Since we are interested in linear (first-order) perturbations, we can consider infinitesimal diffeomorphisms which can be expanded around the identity transformation as follows,

\[
x^\mu \rightarrow \tilde{x}^\mu(x) = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2),
\]

and limit ourselves to the first order in the vector generator \( \epsilon^\mu \). The inverse transformation is then given by:

\[
x^\mu(\tilde{x}) = \tilde{x}^\mu - \epsilon^\mu(\tilde{x}) + \mathcal{O}(\epsilon^2),
\]

Consider now the corresponding transformation of the metric tensor:

\[
\tilde{g}_{\mu\nu}(\tilde{x}^\rho) = \tilde{g}_{\mu\nu}(x^\rho + \epsilon^\rho) = \partial_x^\alpha \partial_{\tilde{x}^\mu} g_{\alpha\beta}(x^\rho).
\]

This relation can be evaluated at the translated point \( x^\rho - \epsilon^\rho \) and gives, in the limit \( \epsilon \rightarrow 0 \),

\[
\tilde{g}_{\mu\nu}(x^\rho) \approx \left( \delta^\alpha_\mu - \partial_\mu \epsilon^\alpha \right) \left( \delta^\beta_\nu - \partial_\nu \epsilon^\beta \right) [g_{\alpha\beta}(x^\rho) - \epsilon^\rho \partial_\rho g_{\alpha\beta}(x^\rho)]
\]

\[
\approx \left( g_{\mu\nu}(x^\rho) - g_{\mu\alpha} \partial_\nu \epsilon^\alpha - g_{\nu\beta} \partial_\mu \epsilon^\beta - \epsilon^\rho \partial_\rho g_{\mu\nu} \right). 
\]

Let us finally introducing our perturbations, by setting \( g \rightarrow g + \delta g \) and \( \tilde{g} \rightarrow \tilde{g} + \delta \tilde{g} \). Separating the perturbations from the background components, and considering only the terms of the first order in \( \epsilon \) and \( \delta g \), we obtain:

\[
\delta \tilde{g}_{\mu\nu}(x^\rho) = \delta g_{\mu\nu}(x^\rho) - g_{\mu\alpha} \partial_\nu \epsilon^\alpha - g_{\nu\beta} \partial_\mu \epsilon^\beta - \epsilon^\rho \partial_\rho g_{\mu\nu}. 
\]

Since we are considering the case of pure scalar perturbations, we must restrict ourselves to the class of infinitesimal coordinate transformations parametrized by two \( SO(3) \) scalars \( \epsilon^\eta, \epsilon \), such that:

\[
\epsilon^\mu = \{\epsilon^\eta, \partial^\eta \epsilon\}.
\]

Using the definition (1.49), the component \( \eta \eta \) of Eq. (1.55) gives the transformation of the perturbation \( \phi \):

\[
-2a^2 \phi = -2a^2 \phi + 2a^2 \partial_\eta \epsilon^\eta + 2\epsilon^\eta a \partial_\eta a \quad \Rightarrow \quad \tilde{\phi} = \phi - (\epsilon^\eta)' - \epsilon^\eta H
\]

where we have defined \( ' \equiv \partial_\eta \) and \( H \equiv a'/a \). In the same way, the component \( i \eta \) gives the transformation of \( B \):

\[
\tilde{B} = B + \epsilon^\eta - \epsilon'
\]
and components $ij$ provide the transformation of both $\psi$ and $E$, i.e.

$$2a^2\left(-\tilde{\psi}\delta_{ij} + \partial_i \partial_j \tilde{E}\right) = 2a^2\left(-\psi\delta_{ij} + \partial_i \partial_j E\right) - 2a^2\partial_i \partial_j \epsilon - 2\epsilon^a a' \delta_{ij}$$

(1.59)

from which:

$$\tilde{\psi} = \psi + \epsilon \eta H$$
$$\tilde{E} = E - \epsilon.$$  

(1.60)

Following the same procedure, we can now obtain the gauge transformations for the matter perturbations $\delta T_{\mu}^\nu$. By applying the general rule we start from

$$\tilde{T}_{\mu}^\nu (x^\rho + \epsilon^\rho) = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} T^\beta_{\alpha} (x^\rho),$$

(1.61)

and we end up, to first order in $\epsilon$ and $\delta T$, with the following result:

$$\delta \tilde{T}_{\mu}^\nu (x) = \delta T_{\mu}^\nu (x) - T_{\alpha}^\nu \partial_{\mu} \epsilon^\alpha + T_{\mu}^\beta \partial_{\beta} \epsilon^\nu - \epsilon^\rho \partial_{\rho} T_{\mu}^\nu.$$  

(1.62)

Inserting the perturbed components we get, from the $\eta \eta$ component:

$$\delta \tilde{\rho} = \delta \rho - \epsilon \eta \rho'$$

(1.63)

from the $ij$ components:

$$\delta \tilde{p} = \delta p - \epsilon \eta p'$$

(1.64)

and from $i \eta$ components:

$$\tilde{w} = w - \epsilon \eta.$$  

(1.65)

According to the above results for $\delta g$ and $\delta T$ it follows (as already stressed) that the above scalar variables are not gauge invariant. Nevertheless, it is always possible to define special combinations of these variables which represent gauge invariant perturbations. First of all, let us notice that:

$$\tilde{B} - \tilde{E}' = B - E' + \epsilon \eta.$$  

(1.66)

We can then easily check that the following variables:

$$\Phi = \phi + \mathcal{H} \left(B - E'\right) + (B - E')'$$
$$\Psi = \psi - \mathcal{H} \left(B - E'\right)$$
$$\mathcal{E} = \delta \rho + \rho' \left(B - E'\right)$$
$$\Pi = \delta p + p' \left(B - E'\right)$$
$$W = w + B - E'$$

(1.67)

are automatically gauge invariant, i.e. $\tilde{\Phi} = \Phi$, $\tilde{\Psi} = \Psi$, $\tilde{\mathcal{E}} = \mathcal{E}$, $\tilde{\Pi} = \Pi$ and $\tilde{W} = W$.  

29
Consider, for instance, the case of the scalar variable $\Phi$:

$$
\tilde{\Phi} = \tilde{\phi} + \mathcal{H} \left( \tilde{B} - \tilde{E}' \right) + \left( \tilde{B} - \tilde{E}' \right)'
= \phi - (\epsilon \eta)' - \epsilon \eta \mathcal{H} + \mathcal{H} \left( B - E' + \epsilon \eta \right) + \left( B - E' + \epsilon \eta \right)'
= \phi + \mathcal{H} \left( B - E' \right) + \left( B - E' \right)' = \Phi.
$$

(1.68)

The variables $\Phi$ and $\Psi$ are usually known as the “Bardeen potentials”. Clearly, any linear combination of these variables is still gauge invariant, provided the coefficients of the combination are homogeneous.

Working with gauge invariant variables allows us to obtain results which are not depending on the chosen system of coordinates. But, using the fact that the coordinate transformations (1.56) contains two scalar parameters ($\epsilon \eta$ and $\epsilon$), we can always impose two conditions on our perturbation variables, thus fixing a particular gauge, whenever this choice may be convenient to simplify our perturbation equations. The most frequently adopted choices of gauge are the following ones.

- **The longitudinal gauge**, defined by the conditions $E = B = 0$. Results within this gauge can easily find a physical interpretation. Also, the metric perturbations $\phi$ and $\psi$ coincide with the Bardeen potentials, and all the other scalar perturbations coincide with their gauge invariant version of Eq. (1.67).

- **The synchronous gauge**, defined by the conditions $\phi = B = 0$. With this choice we have $a \Phi = - (aE)'$, while the other variables have a more complicated expression. However, static observers in this gauge are still geodesic. It is important to stress that this gauge doesn’t completely fix the parameters $\epsilon \eta$ and $\eta$. There is a residual gauge freedom which will be discussed later.

- **The uniform curvature gauge**, corresponding to $\psi = E = 0$. Here we have the opposite of the synchronous gauge, in the sense that there are no perturbations in the pure spatial part of the metric, but only in the time-like and mixed one. This explain why, in such a case, the spatial hypersurfaces have no intrinsic curvature perturbations.

- **The comoving gauge**, which can be defined in two different ways: $w = B = 0$ or $w = E = 0$. Because $w$ is vanishing, we have that the spatial fluid velocity keeps unperturbed, and the fluid is comoving in the chosen coordinate.

To conclude this chapter let us report here the evolution equations satisfied by the gauge invariant variables, and obtained by perturbing the co-
1.7. PERTURBATION THEORY IN A FLRW BACKGROUND

responding cosmological equations:

$$
\begin{align*}
\partial_i \partial^j (\Phi - \Psi) &= 0 \\
\Psi' + \mathcal{H}\Phi &= 4\pi G a^2 (\rho + p) W \\
\nabla^2 \Psi - 3\mathcal{H}\Psi' - 3\mathcal{H}^2 \Phi &= 4\pi G a^2 \mathcal{E} \\
\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' - \frac{1}{3} \nabla^2 (\Psi - \Phi) + (2\mathcal{H}' + \mathcal{H}^2) \Phi &= 4\pi G a^2 \Pi^2.
\end{align*}
$$

(1.69)

We have 4 independent coupled equations for 5 variables and, in order to get a solution, we need one more condition. In particular, we can specify the properties of the fluid we are considering by giving $\Pi = \Pi(\mathcal{E})$, just as it happens for the background variables $p$ and $\rho$. Let us stress, also, that the first of the above equations is satisfied by $\Phi = \Psi$. This implies, in particular, the equality of the two metric perturbations in the longitudinal gauge:

$$
\phi = \psi.
$$

(1.70)

This holds, however, only for sources with $\delta T^i_j = 0$ for $i \neq j$.

As an illustrative example, let us recall here that the system of Eqs. (1.69) can be exactly solved for the perturbed Cold Dark Matter model with $\Pi = 0$. In that case, the equation for the Bardeen potential, in the longitudinal gauge, reduces to

$$
\psi'' + 3\mathcal{H}\psi' + (2\mathcal{H}' + \mathcal{H}^2) \psi = 0.
$$

(1.71)

In the matter dominated phase the scale factor satisfies $a \sim \eta^2$, so that $\mathcal{H} \equiv \frac{a'}{a} \sim \frac{2}{\eta}$ and $\mathcal{H}' \sim -\frac{2}{\eta^2}$. The previous equation thus becomes:

$$
\psi'' + \frac{6}{\eta} \psi' = 0
$$

(1.72)

and is solved by $\psi \sim A_1 \eta^{-5} + A_2$, where $A_1$, $A_2$ are two integration constants.
Chapter 2

The Lemaître-Tolman-Bondi metric

In this chapter we want to discuss a possible model of inhomogeneous large scale geometry [15, 16, 17, 18]. Let us choose, in particular, a system of spherical polar coordinates \((r, \theta, \phi)\) in order to write the spatial portion of the metric. This choice allows us to write the line element of an isotropic cosmological geometry in the following simple form,

\[
ds^2 = -dt^2 + X^2(t,r) dr^2 + A^2(t,r) d\Omega^2,
\]

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\), and where the functions \(X\) and \(A\) are independent of the angular directions. The metric (2.1) is written in terms of the synchronous time coordinate \(t\), and it is usually known as the Lemaître-Tolman-Bondi (LTB) metric [19, 20, 21]. Different choices of coordinates are also possible, in order to get analytical solutions for particular classes of geometries (this possibility will be discussed later). But let us first motivate the choice leading to the particular form (2.1).

Polar coordinates are very useful for the physical interpretation; indeed, thanks to a residual gauge freedom, we can always fix \(A(t,r)\) in such a way that, at a given time \(t_0\), \(A\) coincides the radial coordinate, i.e. \(A(t_0,r) = r\). When this condition is satisfied, as we shall see later, the function \(A\) is directly related to the time dependence of the measured distances. In our case, in particular, it will be convenient to choose \(t_0\) equal to the observer time \(t_o\).

In that case, the angular coordinates \(\theta\) and \(\phi\) can be easily related to the angular direction measured by the observer, even if he/she is placed far away from the origin of the polar coordinate system. In fact, differently from the case of the FLRW metric, it should be stressed that the LTB metric describes a geometry which is isotropic only when referred to a central observer. Different observers will be associated to an anisotropy, aligned
along the direction connecting the observer position to the center of the coordinates. The mathematical aspects and the consequences of such an anisotropy will be discussed in details in the last chapter of this work.

In this chapter we will discuss various properties of the LTB metric. First we shall derive the Einstein equations, and show how to find simple but physically interesting solutions. We will then introduce the physical observables, and we will provide the exact relations connecting the redshift with the distances of the cosmological sources.

### 2.1 Solving Einstein’s equations

As already stressed, in this work we discuss the dynamical properties of inhomogeneous geometries in the context of the theory of General Relativity. Discussing deviations from homogeneity within more general contexts (for instance, using the so-called $f(R)$ gravity theories) is certainly another interesting topic. However, it would be outside the main purpose of this research project.

Let us thus concentrate on the Einstein equations for the LTB metric. To this aim, we first report the nonvanishing components of the Ricci tensor:

\[
\begin{align*}
R_{rr} & = 2 \left( \frac{A' \dot{X}}{A X} - \frac{\dot{A}'}{A} \right) \\
R_{tt} & = \ddot{X} \frac{X}{X} + 2 \ddot{A} \frac{A}{A} \\
R_{rr} & = \ddot{X} \frac{X}{X} - 2 \frac{\dot{X}^2}{X^2} A + 2 \frac{\dot{A} \dot{X}}{A X} + 2 \frac{A' X'}{A X} \\
R_{\theta \theta} = R_{\phi \phi} & = \left( \frac{\dot{A}}{A} \right)^2 + \frac{\ddot{A}}{A} + \frac{\dot{X} \dot{A}}{X A} - \frac{1}{X^2} \left[ \left( \frac{A'}{A} \right)^2 + \frac{A''}{A} - \frac{A' X'}{A X} \right] + \frac{1}{A^2} \end{align*}
\]

(2.2)

where we have used the notation $\dot{f} \equiv \partial_t f$ and $f' \equiv \partial_r f$. From this we easily obtain the scalar curvature $R \equiv R_{\mu \nu}^\gamma$:

\[
R = 2 \frac{\ddot{X}}{X} + 4 \frac{\ddot{A}}{A} + 4 \frac{\dot{A} \dot{X}}{A X} + 2 \left( \frac{\dot{A}}{A} \right)^2 + \frac{2}{A^2} - \frac{1}{X^2} \left[ 4 \frac{A''}{A} - 4 \frac{A' X'}{A X} + 2 \left( \frac{A'}{A} \right)^2 \right].
\]

(2.3)
2.1. SOLVING EINSTEIN’S EQUATIONS

We are interested in gravitational sources characterized by a diagonal energy momentum tensors. According to the Einstein equations, this implies the condition $R_{tt} = 0$, i.e.

$$\frac{A' X}{A} = \frac{A'}{A} \Rightarrow \frac{X}{A} = \frac{A'}{A} \Rightarrow X(t, r) = F(r) A'(t, r), \quad (2.4)$$

where $F(r)$ is a free function, depending only on $r$. By defining $F(r) \equiv [1 - k(r)]^{-1/2}$ we can immediately identify this term as the contribution of a inhomogeneous spatial curvature. In the homogeneous limit, corresponding to the FLRW metric, we have in fact $A(t, r) \rightarrow r a(t)$ and $k(r) \rightarrow K r^2$.

This correspondence will be used later, when computing the equation of geodesic deviation in the presence of a nonvanishing spatial curvature. We now concentrate on the Einstein equations $G_{\nu \mu} = 8\pi G T_{\nu \mu}$, considering the two independent components $tt$ and $rr$ which can be written as:

$$-\left(\frac{A'}{A}\right)^2 + 2\frac{\dot{A}}{A} \frac{\dot{A'}}{A'} - \frac{k'}{AA'} = 8\pi G T^t_t$$

$$-\left(\frac{A}{A}\right)^2 - 2\frac{\ddot{A}}{A} - \frac{k}{A^2} = 8\pi G T^r_r. \quad (2.5)$$

We will use below several types of energy momentum tensor, corresponding to different physical configurations. The simplest case we may consider is that of a mixture of perfect, non-interacting, barotropic fluids. As we have seen in the previous chapter, the equation of state for the $n$-th component of this mixture, relating the pressure $p_n$ to the energy density $\rho_n$, can be written as:

$$p_n = w_n \rho_n, \quad (2.6)$$

where $w_n$ is constant. As the fluids are non-interacting, their evolution is separately constrained by the contracted Bianchi identity written explicitly for each component, i.e. by the equations $\nabla_{\mu} T_{n \mu}^\nu = 0$, where $T_{n \mu \nu} = (\rho_n + p_n) u_{\mu} u_{\nu} + p_n g_{\mu \nu}$ and $u_{\mu}$ is the static velocity field of a comoving observer, $u^\mu = (1, \vec{0})$.

Using the equations of state we have $T_{n t} = -\rho_n$ and $T_{n r} = p_n = w_n \rho_n$, and the equations (2.5) become:

$$\left(\frac{A'}{A}\right)^2 + 2\frac{\dot{A}}{A} \frac{\dot{A'}}{A'} + \frac{k'}{AA'} = 8\pi G \sum_n \rho_n, \quad (2.7)$$

$$\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\ddot{A}}{A} + \frac{k}{A^2} = -8\pi G \sum_n w_n \rho_n. \quad (2.8)$$
2.1. SOLVING EINSTEIN’S EQUATIONS

The energy conservation equations, obtained from the time-like component of the contracted Bianchi identity, gives:

\[ \dot{\rho}_n + \frac{\sqrt{-g}}{\sqrt{-g}} (1 + w_n) \rho_n = 0 \]  

(2.9)

from which, noting that \( k(r) \) doesn’t depend on \( t \):

\[ \rho_n(t, r) = \rho_{n0}(r) \left( \frac{A'A^2}{A_0'A_0^2} \right)^{-(1+w_n)} \equiv \tilde{\rho}_{n0}(r) (A'A^2)^{-1(1+w_n)}. \]  

(2.10)

Using the above equation, Eq. (2.8) can be rewritten as:

\[ \partial_t \left( A\dot{A}^2 \right) = -8\pi G \frac{\dot{A}}{A} \sum_n w_n \tilde{\rho}_{n0} \left( A'A^2 \right)^{-1(1+w_n)} - k\dot{A} \]  

(2.11)

It should be noted that the radial component of Bianchi identity, \( \nabla_\mu T^\mu_{rn} = 0 \), is nontrivially satisfied, in general, and provides an additional constraint which, for the case of a perfect barotropic fluid component, can be written as

\[ p'_n = w_n \rho'_n = 0. \]  

(2.12)

This implies that the pressure has to be homogeneous, and that, when \( w_n \neq 0 \), the energy density cannot depend on \( r \). Hence, radial inhomogeneities are allowed only for a perfect barotropic fluid of dust (i.e. pressureless) matter.

We are now in the position of discussing two interesting (inhomogeneous) analytical solutions for two cases: a spatially flat geometry with \( \Lambda \)CDM sources, and a spatially curved geometry with CDM sources.

2.1.1 An inhomogeneous \( \Lambda \)CDM model

Working in the geometric context illustrated in the previous section we shall now consider two fluid sources describing, respectively, cosmic dark energy in the form of a cosmological constant \( \Lambda \) (with \( w_\Lambda = -1 \)) and pressureless dark matter (with \( w_m = 0 \)). In this case Eq. (2.11) becomes:

\[ \partial_t \left( A\dot{A}^2 \right) = 8\pi G A^2 \dot{A}\Lambda - k\dot{A}. \]  

(2.13)

and a first integration with respect to \( t \) gives:

\[ \left( \frac{\dot{A}}{A} \right)^2 = -\frac{k(r)}{A^2} + 8\pi G \Lambda + \frac{\alpha(r)}{A^2}. \]  

(2.14)
where \( \alpha(r) \) is an arbitrary function of the radial coordinate.

We can interpret Eq. (2.14) by considering Eq. (2.7), which can be rewritten as

\[
\left( A \dot{A}^2 \right)' = 8\pi G \tilde{\rho}_m(r) + 8\pi G A^2 A' - (kA)',
\]

and which gives, after integration with respect to \( r \):

\[
\left( \frac{\dot{A}}{A} \right)^2 = -\frac{k(r)}{A^2} + 8\pi G \Lambda + 2G \frac{M(r)}{A^3},
\]

where we defined the quantity \( M(r) \equiv 4\pi \int \tilde{\rho}_m(r) dr \), representing the total mass enclosed within a sphere of radius \( r \) centered at the origin of our coordinate system. This interpretation is consistent because we have used the definition of \( \tilde{\rho} \) of Eq. (2.11), with initial condition \( A(t_0, r) = r \), which gives \( \tilde{\rho}_m = \rho_m r^2 \). By comparing Eqs. (2.14) and (2.16), we can thus conclude that \( \alpha(r) \) is closely related to the mass distribution of our cosmological model. We will analyze the consequences of this property in the next section, when considering a non-trivial matter distribution.

In this section we will concentrate on Eq. (2.14). In order to put the equation in a more familiar form, we can define

\[
H \equiv \frac{\dot{A}}{A}, \quad k(r) \equiv -H_0^2(r)\Omega_{k0}(r)A^2_0, \quad 8\pi G \Lambda \equiv H_0^2(r)\Omega_{\Lambda0}(r)A^2_0, \quad \alpha(r) \equiv H_0^2(r)\Omega_{m0}(r)A^2_0.
\]

In this way we obtain a form similar to the Friedmann equation:

\[
H^2 = H_0^2 \left[ \Omega_{m0} \left( \frac{A_0}{A} \right)^3 + \Omega_{\Lambda0} + \Omega_{k0} \left( \frac{A_0}{A} \right)^2 \right],
\]

where \( H \) is the corresponding inhomogeneous Hubble function, \( H_0 \) represents its present value at the observation time \( t = t_o \), and \( A_0(r) = A(t_o, r) \); finally, \( \{\Omega_{m0}, \Omega_{\Lambda0}, \Omega_{k0}\} \) are the current values of the critical fractions of matter, dark energy and curvature contributions to the Einstein equations. Just as in the FLRW case, and because \( H(t_o, r) = H_0(r) \), these fractions are constrained by the condition \( 1 = \Omega_{m0} + \Omega_{\Lambda0} + \Omega_{k0} \).

By integrating Eq. (2.17) we obtain

\[
H_0(r) (t_o - t) = \int_{\frac{A}{A_0}}^{1} \frac{dx}{\sqrt{\Omega_{m0}(r) x^{-3} + \Omega_{\Lambda0}(r) x^2 + \Omega_{k0}(r)}},
\]

where we have defined \( x = A/A_0 \). Unfortunately, this integral is not always easy to be solved. However, we can find a particular solution for the case of a flat spatial geometry (\( \Omega_{k0}(r) = 0 \)). In this case, the above integral becomes:

\[
H_0(r) (t_o - t) = \int_{\frac{A}{A_0}}^{1} \frac{x dx}{\Omega_{m0}(r) \left[ 1 + \frac{\Omega_{\Lambda0}(r)}{\Omega_{m0}(r)} x^3 \right]}.
\]
By defining a new variable $y$ such that $x \equiv \left(\frac{\Omega_m}{\Omega_\Lambda_0}\right)^{1/3} y^{2/3}$, and noticing that
\[
\int \left(1 + y^2\right)^{-1/2} dy = \text{arcsinh} \ y,
\]
we obtain:
\[
H_0(r) (t_o - t) = \left[ \frac{2}{3\sqrt{\Omega_\Lambda_0(r)}} \text{arcsinh} \left( \frac{\sqrt{\Omega_\Lambda_0(r)}}{\Omega_m(r)} x^{3/2} \right) \right]^{1/3} A_0(r). \tag{2.20}
\]

Hence, by inverting the above relation, we have:
\[
A(t, r) = A_0(r) \left(\frac{\Omega_m(r)}{\Omega_\Lambda_0(r)}\right)^{1/3} \sinh \left[ \text{arcsinh} \left( \frac{\sqrt{\Omega_\Lambda_0(r)}}{\Omega_m(r)} \frac{3}{2} \sqrt{\Omega_\Lambda_0(r)} H_0(r) (t_o - t) \right) \right]^{3/2}. \tag{2.21}
\]

The case of pure dark matter can be recovered by considering the limit of small $\Omega_\Lambda_0$: indeed, by using the (small argument) expansions $\text{arcsinh} \ x \approx x$ and $\sinh \ x \approx x$, and the condition $\Omega_m = 1 - \Omega_\Lambda_0$, we obtain:
\[
A(t, r) \approx A_0(r) \left[ 1 - \frac{3}{2} H_0(r) (t_o - t) \right]^{2/3}. \tag{2.22}
\]

Let us finally stress three important aspects of the (exact) inhomogeneous solution that we have found by solving the integral (2.18) for a flat spatial geometry.

- For $t = t_o$ we easily obtain $A(t_o, r) = A_0(r)$, and we find that the integration function $A_0(r)$ is directly related to the radial distance. In particular, through a redefinition of the radial coordinate, we can always choose $A_0(r) = r$: in fact, considering the metric (2.1) and the solution (2.4) for $X(t, r)$, we can define a new radius $\tilde{r} = A_0(r) dr$. The spatial part of the line-element, at the present time $t = t_o$, in this case becomes
\[
d\sigma^2(t_o) = F(\tilde{r})^2 d\tilde{r}^2 + \tilde{r}^2 d\Omega^2.
\]

- Another possibility is to fix the arbitrary integration function by referring to the initial singularity of the LTB metric, generically present at the “big bang” time $t_{bb}$, and impose the condition $A(t_{bb}, r) = 0$. Consider, for simplicity, the solution (2.22) with $\Lambda = 0$: the condition $A(t_{bb}, r) = 0$ then implies $t_{bb}(r) = t_o - \frac{2}{3 H_0(r)}$. This shows that the time of the singularity is not the same at all points, but has an $r$-dependence which is fixed by the shape of $H_0(r)$. In the context of inhomogeneous geometries we have thus to take into account models in which the age of the Universe is different at different spatial positions.
• Because of the radial dependence of $H_0(r)$, it follows that $\Omega_{A0}$ is position dependent even if $\Lambda$ is a constant. In addition, the parameters $\Omega_{m0}$ and $\Omega_{k0}$ are position dependent even in the case of constant $H_0$. Indeed, they are proportional to $\alpha(r)$ and $k(r)$ which determine, respectively, the radial inhomogeneous distribution of the matter sources and of the spatial curvature.

Summarizing our discussion, we can say that the relevant free functions to be fixed in order to specify a particular model are three: the Hubble parameter $H_0(r)$ – or equivalently the time parameter $t_{bb}(r)$ – the mass parameter $\alpha(r)$ and the curvature parameter $k(r)$. Different choices of these parameters may correspond to really different physical situations: it has been shown, for instance, that pure (intrinsic) big bang inhomogeneities lead to decaying modes for structures [22]. At the same time, pure curvature inhomogeneities are compatible with growing modes for galaxies formation. For such a reason we will now address the problem of obtaining analytical solutions for models with a non-vanishing spatial curvature.

### 2.1.2 Curved CDM models

To obtain solutions for an LTB metric with non-vanishing spatial curvature let us first rewrite the integral (2.18) in the differential form:

$$H_0(r)dt = \frac{dx}{\sqrt{\Omega_{m0} x^{-1} + \Omega_{k0}}},$$  

(2.23)

(we have put $\Omega_{A0} = 0$, assuming a negligible contribution from a cosmological constant). Let us also define a conformal time coordinate $\eta$ such that $A(\eta, r) d\eta \equiv dt$, and recall that $x = A/A_0$. We obtain:

$$H_0(r)A_0(r) (\eta_0 - \eta) = \int_{\eta_0}^{\eta} \frac{dx}{A_0} \sqrt{\Omega_{m0}(r) x + \Omega_{k0}(r) x^2},$$

$$t_0 - t = \int_{\eta}^{\eta_0} A(\eta', r) d\eta'.$$

(2.24)

As before, by defining $y^2 = \frac{\Omega_{k0}}{\Omega_{m0}} x$, and noting that $\int (1 + y^2)^{-1/2} dy = \text{arcsinh } y$, we find that the first integral is equal to $$\left[ \frac{2}{\sqrt{\Omega_{k0}}} \frac{x}{\sqrt{\Omega_{m0}}} \frac{A}{A_0} \right].$$
Hence:

\[ A(\eta, r) = A_0 \frac{\Omega_{m0}}{\Omega_{k0}} \sinh \left[ \arcsinh \sqrt{\frac{\Omega_{k0}}{\Omega_{m0}}} - \frac{A_0 H_0 \sqrt{\Omega_{k0}}}{2} (\eta_o - \eta) \right]^2, \]

\[ H_0 (t_o - t) = \frac{\Omega_{m0}}{2 \Omega_{k0}^{3/2}} \left[ -A_0 H_0 \sqrt{\Omega_{k0}} (\eta_o - \eta) + \sinh \left( 2 \arcsinh \sqrt{\frac{\Omega_{k0}}{\Omega_{m0}}} \right) \right. \]

\[ \left. - \sinh \left( 2 \arcsinh \sqrt{\frac{\Omega_{k0}}{\Omega_{m0}}} - A_0 H_0 \sqrt{\Omega_{k0}} (\eta_o - \eta) \right) \right]. \quad (2.25) \]

Differently from the models discussed in the previous section, here we have analytic but implicit solutions in term of the synchronous time \( t \). We have to carefully take into account this point if we want to integrate the geodesic equations in term of the redshift \( z \).

Up to now we have discussed geometric properties of the LTB models which are in general well known. People usually refers to these properties when formulating inhomogeneous models with gravitational sources which can be describes as fluids, with particular equations of state. Before applying such results to the computation of cosmic distances, in the next two sections we will provide two possible – but less standard – example of solutions in the context of the LTB geometry.

The first example refers to a model in which electromagnetic fields are locally present. As a second example we will provide an interesting inhomogeneous solution where no fluido-dynamical description of the matter sources seems to be possible, considering the possibility of describing the sources with a model of fractal distribution.

# 2.2 LTB metric with electromagnetic sources

In order to introduce electromagnetic fields as possible gravitational sources, let us first calculate the electromagnetic energy-momentum tensor for the case of a (inhomogeneous) cosmological metric. In that particular context, it has been shown that the electromagnetic field tensor \( F_{\mu\nu} \) can be conveniently
referred to a given (possibly comoving) observer by defining [23]:

\[ F_{\mu \nu} = u_{\mu} E_{\nu} - u_{\nu} E_{\mu} + \epsilon_{\mu \nu \rho \sigma} B^{\rho} u^{\sigma}, \]

\[ * F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} = \epsilon^{\mu \nu \alpha \beta} u_{\alpha} E_{\beta} + u^{\mu} B^{\nu} - B^{\mu} u^{\nu}, \tag{2.26} \]

where \( u_{\mu} \) is the four-velocity of the observer, the vectors \( E_{\nu} = (0, \vec{E}) \) and \( B^{\rho} = (0, \vec{B}) \) parametrize, respectively, the electric and the magnetic fields measured by the given observer, \( \epsilon_{\mu \nu \rho \sigma} = \sqrt{-g} A_{\mu \nu \rho \sigma} \) is the covariant version of the totally antisymmetric tensor, \( \epsilon^{\mu \nu \alpha \beta} = A^{\mu \nu \rho \sigma} / \sqrt{-g} \) is the corresponding contravariant tensor, and \( A_{\mu \nu \rho \sigma} \) is the usual Levi-Civita symbol.

As prescribed by the principle of minimal coupling, the evolution of these fields is governed by the Maxwell equations in a curved space-time:

\[ \nabla^{\nu} F_{\mu \nu} = 4 \pi J^{\mu}, \]

\[ \nabla^{\nu} * F^{\mu \nu} = 0, \tag{2.27} \]

where \( J^{\mu} = (\rho, \vec{J}) \) is the electromagnetic four-current density. Since we are interested in the case of electrically uncharged matter, we put \( J^{\mu} = 0 \), and we can write the total electromagnetic contribution to the energy-momentum tensor as:

\[ T^{(EM)}_{\mu \nu} = F_{\mu \alpha} F_{\nu \beta} g^{\alpha \beta} - \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} g_{\mu \nu}. \tag{2.28} \]

Up to now, the introduced equations are completely general. However, since we are interested in isotropic and inhomogeneous models described by the LTB metric (2.1), we can consider, from now on, only the particular case in which \( E^{\nu} = \{0, E(t, r), 0, 0\} \) and \( B^{\rho} = \{0, B(t, r), 0, 0\} \) (as discussed in detail in [24]). In this metric background, and for a comoving observer with \( u^{\mu} = \{1, 0\} \), the electromagnetic field and its energy-momentum tensor can be written as follows:

\[ F_{\mu \nu} = \begin{pmatrix} 0 & X^2 E & 0 & 0 \\ -X^2 E & 0 & 0 & 0 \\ 0 & 0 & 0 & -X A^2 \sin \theta B \\ 0 & 0 & X A^2 \sin \theta B & 0 \end{pmatrix}, \]

\[ T^{(EM)}_{\mu \nu} = X^2 \frac{E^2 + B^2}{2} \text{diag} \left(1, -X^2, A^2, A^2 \sin^2 \theta\right), \tag{2.29} \]

or, equivalently, as follows:

\[ T^{(EM)}_{\mu \nu} = X^2 \frac{E^2 + B^2}{2} \text{diag} \left(-1, -1, 1, 1\right). \tag{2.30} \]

This energy-momentum tensor is different from that of a barotropic and perfect fluid, so that Eq. (2.12) does not apply. The tensor \( T_{\mu \nu} \), however,
is still diagonal so that, even by adding other perfect fluids sources, we find that the relation (2.4) keeps valid. The cosmological Einstein equations with electromagnetic and fluid sources are then given by:

\[
\frac{\dot{A}^2 + k}{A^2} + \frac{2\dot{A}\dot{A}'}{AA'} = 8\pi G \left[ \rho_m + \frac{A^2(E^2 + B^2)}{2(1 - k)} \right], \\
\frac{\dot{A}^2 + 2A\ddot{A} + k}{A^2} = 8\pi G \left[ \frac{A^2(E^2 + B^2)}{2(1 - k)} \right],
\]

where \(\rho_m\) represents the energy density of the fluid matter, and \(k = k(r)\).

The above equations have to be solved in combination with the Maxwell Eqs. (2.27) which, in our case, simply reduce to

\[
\partial_r \left( XA^2E \right) = 0, \\
\partial_t \left( XA^2E \right) = 0, \\
\partial_r \left( XA^2B \right) = 0, \\
\partial_t \left( XA^2B \right) = 0.
\]

It follows that:

\[
E^2 + B^2 = \frac{\gamma(r)}{4\pi GX^2A^4},
\]

where the integration constant has been normalized to \((4\pi G)^{-1}\) for later convenience. In the absence of charged sources, \(\gamma\) should be a constant. However, here we will take into account a possible radial dependence, \(\gamma(r)\), arising in the presence of a local nonzero charge density with \(J^i = 0\) and \(J^0(r) \neq 0\). The second of the two Einstein equations (2.31) thus reduces to:

\[
\dot{A}^2 + 2A\ddot{A} + k(r) = \frac{\gamma(r)}{A^2}.
\]

By applying the same formal manipulations as in the previous section, by defining \(-k(r) \equiv H_0^2(r)\Omega_{k0}(r)A_0^2(r), \alpha(r) \equiv H_0^2(r)\Omega_{m0}(r)A_0^2(r)\) and \(-\gamma(r) \equiv H_0^2(r)\Omega_{\gamma0}(r)A_0^2(r)\), and by imposing the constraint \(\Omega_{k0}(r) + \Omega_{\gamma0}(r) + \Omega_{m0}(r) = 1\), we find that our solution is given by the following integral:

\[
H_0(r) \left( t_o - t \right) = \int_0^1 \frac{dx}{\sqrt{\Omega_{k0}(r) + \Omega_{\gamma0}(r)x^{-1} + \Omega_{m0}(r)x^{-2}}}.
\]

This equation represents the starting point of the following discussion: it is possible, in fact, to obtain the scale factor \(A\) just by solving this integral and inverting the solution. However, the last step is not always simple: a particular solution for the spatially flat case has been given in [24] but, if \(\Omega_{k}(r) \neq 0\), there is no analytical way to perform the inversion in term of the synchronous time \(t\).

Here we will provide a general analytical but implicit solution in terms of the conformal time \(\eta\), such that \(dt \equiv A(\eta, r) d\eta = A_0(r) x d\eta\). By using
2.2. LTB METRIC WITH ELECTROMAGNETIC SOURCES

this time coordinate we can rewrite Eq. (2.34) as:

\[
H_0(r) (\eta_o - \eta) = \int_{A_0}^1 \frac{dx}{\sqrt{\Omega_{k0}(r) x^2 + \Omega_{m0}(r) x + \Omega_{\gamma0}(r)}}
\]

\[
t_o - t = \int_{\eta}^{\eta_o} A(\eta', r) d\eta'.
\] (2.35)

The first integral can be solved exactly as in the previous section. In fact, by introducing the auxiliary variable

\[
y^2 = \frac{\Omega_{k0}}{\sqrt{\Omega_{m0} - 4 \Omega_{k0} \Omega_{\gamma0}}} x^2 + \frac{\Omega_{m0}}{2 \sqrt{\Omega_{m0} - 4 \Omega_{k0} \Omega_{\gamma0}}} - \frac{\Omega_{m0}}{2 \Omega_{k0}} \left( \sqrt{\frac{\Omega_{m0}^2 - 4 \Omega_{k0} \Omega_{\gamma0}}{\Omega_{k0}^3}} \right) - \frac{H_0 \sqrt{\Omega_{k0}}}{2} (\eta_o - \eta)
\]

\[
\int_{\eta}^{\eta_o} A(\eta', r) d\eta' = 2 (\Omega_{k0})^{-1/2} \int (1 + y^2)^{-1/2} dy = 2 (\Omega_{k0})^{-1/2} \arcsinh y.
\]

This result, combined with Eqs. (2.35), gives the following solution:

\[
A(\eta, r) = A_0 \left\{ \frac{\sqrt{\Omega_{m0}^2 - 4 \Omega_{k0} \Omega_{\gamma0}}}{2 \Omega_{k0}} \times \right.
\]

\[
\cosh \left[ 2 \arcsinh \sqrt{\frac{2 \Omega_{k0} + \Omega_{m0}}{2 \sqrt{\Omega_{m0}^2 - 4 \Omega_{k0} \Omega_{\gamma0}}} - \frac{1}{2}} - \frac{H_0 \sqrt{\Omega_{k0}}}{2} (\eta_o - \eta) \right] - \frac{\Omega_{m0}}{2 \Omega_{k0}} \right\}
\]

\[
t_o - t = A_0 \left\{ - \frac{\Omega_{m0}}{2 \Omega_{k0}} (\eta_o - \eta) - \frac{\sqrt{\Omega_{m0}^2 - 4 \Omega_{k0} \Omega_{\gamma0}}}{H_0 \Omega_{k0}^{3/2}} \sinh \left[ - \frac{H_0 \sqrt{\Omega_{k0}}}{2} (\eta_o - \eta) \right] \times \right.
\]

\[
\cosh \left[ 2 \arcsinh \sqrt{\frac{2 \Omega_{k0} + \Omega_{m0}}{2 \sqrt{\Omega_{m0}^2 - 4 \Omega_{k0} \Omega_{\gamma0}}} - \frac{1}{2}} - \frac{H_0 \sqrt{\Omega_{k0}}}{2} (\eta_o - \eta) \right] \right\}.
\] (2.36)

We can check that this solution correctly reproduces the previous result (2.25) in the limit \( \Omega_{\gamma0} = 0 \). Also, it is important to stress that this is the first example in which we can exactly (even if implicitly) describe an inhomogeneous cosmological configuration including dark matter sources, local electromagnetic fields and spatial curvature.

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2.3 Fractal matter sources and LTB geometry

Taking into account the analysis of some astronomical catalogs, here we want to consider the possibility that inhomogeneous, but isotropic, metric backgrounds of the LTB type (like the solutions we found previously) may be compatible with a fractal distribution of matter sources.

Let us first present a short *excursus* on the problem we want to discuss. Matter distribution, in cosmology, is usually regarded as being the realization of a *stationary* stochastic point process. This is enough to satisfy the “weak” Copernican Principle, stating that there are no special points or special directions; however this does not necessarily imply spatial homogeneity. Spatially homogeneous, stationary stochastic processes satisfy the special and stronger case of the Copernican Principle described by Cosmological Principle. Indeed, isotropy around each point, together with the hypothesis that the matter distribution is a smooth function of position i.e., that this is analytical, implies spatial homogeneity. This is no longer the case for a non-analytic (i.e., non-smooth) structure, for which the obstacle to applying the FLRW solutions has in fact to do solely with the lack of spatial homogeneity.

Having this in mind, we want to approach the modelisation of the spatial inhomogeneities in a way that is more close to what we can learn from observations of galaxy structures. Indeed, the statistical analysis of the three-dimensional galactic surveys have shown that the galaxy distribution is characterized by power-law correlations in the locally observable portion of Universe [25, 26, 27]. More specifically, by choosing a specific point \( p \) from which we want to describe the distribution, it has been found that the average conditional number decays as \( \langle n(r) \rangle_p \sim r^{-\gamma} \), where \( \gamma = 0.9 \pm 0.1 \) for \( r \in [0.1, 20] \) Mpc/h, and \( \gamma = 0.2 \pm 0.1 \) for \( r \in [20, 100] \) Mpc/h [28]. However, whether or not correlations decay on scales \( r > 100 \) Mpc/h, and, on such scales, the distribution evolves towards uniformity, is still matter of considerable debate [29].

The power-law behavior of the conditional density can be interpreted as a galaxy distribution having fractal properties at small scales. In general, a fractal distribution is a non-analytical point distribution: that is, the average \( \langle n(r) \rangle_p \) decays as a power law. From a generic \( i \)-th position in the distribution, which in principle may also be different from the point \( p \), the (conditional) density decays as \( n_i(r) \), where \( \sigma_p(r) = \langle n_i(r)^2 \rangle - \langle n(r) \rangle_p^2 \approx \text{const.} \) with scale [30]. This situation is thus different from an analytical density profile characterized by a power-law decay of the density from the central point of the distribution [30]. However, if \( \sigma_p(r) < 1 \), as in the case of real galactic structures [28], we can neglect fluctuations around the average behavior and focus only on the power-law decay of the density. In practice, this means that we can approximate the discrete matter sources with a
density distribution

\[ \rho_d(r) = \sum_i m_i \delta(\vec{r} - \vec{r}_i) \approx \langle n(r) \rangle_p. \]  

(2.37)

where the position vector \( \vec{r} \) is referred to the point \( p \), where \( \vec{r}_i \) is the vector specifying the position of the \( i \)-th galaxy with mass \( m_i \).

This situation allows us the use a smooth LTB metric for describing the spatial decay of the density. However, we should consider an additional point: while the LTB model has a geometric center, we speculate that this metric applies to any point of the fractal structure when chosen as a center, so that, on the average, there is not any special point or direction. We are not able to quantify the perturbations neglected by making this assumption, but we can assume that, as long as the spatial fluctuations around the average keep limited, i.e. \( \sigma_p(r) < 1 \), this model provide us with a reasonable description of the local metric of a fractal object.

As discussed below, we do not need to assume that the fractal behavior extends up to an arbitrarily large scales, and that, for a moderate value of the homogeneity scale \( t_0 \) beyond which \( \langle n(r) \rangle_p \approx \text{const.} \), the modification of the magnitude-redshift relation due to the inhomogeneous and power-law behavior of the (conditional) density is enough to provide a best fit to the SN data without the need of introducing dark energy. Therefore, let us consider an energy-momentum tensor assuming the form:

\[ T^{\text{fractal}}_{\mu\nu} = \text{diag} \left(-\rho_d(r), 0\right). \]  

(2.38)

Because \( T^r_r \) is vanishing, we can consider the flat solution (2.22)

\[ A(t, r) = A_0(r) \left[1 - \frac{3}{2} H_0(r) (t_o - t) \right]^{2/3}, \]  

(2.39)

with:

\[ \alpha(r) = 2GM(r) = \int_{S^2_p(r)} \langle n(r) \rangle_p A'(t, r) A^2(t, r) 4\pi r^2 dr, \]  

(2.40)

where \( S^2_p(r) \) is a 3-D sphere of radius \( r \). Moreover, Eq. (2.16) with \( \Lambda = 0 \) and \( k(r) = 0 \) gives, for \( t = t_o \):

\[ 2GM(r) = A_0^3(r) H_0^2(r). \]  

(2.41)

In such a way, the present value of the Hubble parameter is intimately related to the mass according to:

\[ H_0(r) = \sqrt{\frac{2GM(r)}{A_0^3(r)}}. \]  

(2.42)
2.4. CONNECTIONS WITH PHYSICAL OBSERVABLES

For a pure fractal distribution we have $M(r) \sim r^D$, where $D = 3 - \gamma$ is the fractal dimension [30]. We assume that at large enough scales the distribution evolves towards a spatially homogeneous distribution, i.e. $D = 3$ for $r > l_0$. By choosing $A_0(r) = r$, we can then write the mass $M(r)$ as

$$M(r) = \frac{\pi^{D/2}}{\Gamma(D/2 + 1)} \Phi_0 \left( \frac{r}{l_0} \right)^D + \frac{4}{3} \pi \Psi_0 \left( \frac{r}{l_0} \right)^3,$$

where $\Gamma(x)$ is the Euler function, $D$ is the fractal exponent, $\Psi_0$ and $\Phi_0$ are amplitudes and $l_0$ is the transition length; in fact, taking in account that $D \leq 3$, for $r \ll l_0$ the fractal regime dominates, otherwise, for $r \gg l_0$, the homogeneous regime is the dominant one.

By defining

$$H^\text{ext}_0 \equiv \sqrt{\frac{8\pi G}{8 \pi G \Psi_0 l_0^3}},$$

we can rewrite the Hubble function (2.42) in a more intuitive way as:

$$H_0(r) = \sqrt{\frac{3}{4 \Gamma(D/2 + 1)} \left( \frac{\bar{H}_0^2 - (H^\text{ext}_0)^2}{4 \Gamma(D/2 + 1)} \left( \frac{r}{l_0} \right)^D + (H^\text{ext}_0)^2 \right)},$$

where $\bar{H}_0 \simeq 67$ Kms$^{-1}$Mpc$^{-1}$ is the local value of the present Hubble parameter, and where we require that, for $D = 3$, homogeneity is recovered ($H_0(r) = \bar{H}_0$). In such a way, $H^\text{ext}_0$ becomes the value of the Hubble constant once the transition to the homogenous regime has occurred.

In the last section of this chapter we will compare this result with present observational data, in order to get a possible numerical estimate of the fractal parameter $D$.

2.4 Connections with physical observables

As already stressed in Chapter 1, observational data directly provide us with the redshift $z$ and the luminosity distance $d_L$ (or the distance modulus $\mu$) of the astrophysical sources of known intrinsic luminosity. In this and in the following sections we will discuss the possibility of comparing those experimental results with the predictions of $d_L(z)$ obtained in the context of cosmological models based on the LTB geometry.
Let us recall, first of all, that in the LTB metric (2.1), and for a static observer located at the origin of the polar coordinate system, the angular distance (or area distance) $d_A$ of a source emitting radiation at time $t$, at a radial coordinate $r$, is simply given by:

$$d_A = A(t, r). \quad (2.46)$$

Let us denote with $k^\mu$ the four-momentum of the emitted light-like radiation, such that $g_{\mu\nu}k^\mu k^\nu = 0$. For a null radial geodesics we have $k^\mu = \{k^0, k^1, 0, 0\}$ so that $0 = g_{\mu\nu}k^\mu k^\nu = -k^0^2 + X^2 k^1^2$ or, equivalently, $k^\mu = k^0 \{1, X^{-1}, 0, 0\}$. From the general definition of the redshift parameter we thus immediately obtain $1 + z = k^0 k^0$. On the other hand, considering the time-like component of the geodesic equation, and using $\Gamma^t_{rr} = X \dot{X}$, we have:

$$0 = dk^0 + \Gamma^0_{11} k^1 dr = dk^0 + \frac{\Gamma^0_{rr}}{X^2} k^0 dt = dk^0 + k^0 \left(\frac{dX}{X}\right)_{r=\text{const}}, \quad (2.47)$$

from which $k^0 \sim X^{-1}$. Hence:

$$1 + z = \frac{X_o}{X_e}. \quad (2.48)$$

Both quantities $z$ and $d_A$ (or $d_L = (1 + z)d_A$) can then be written in terms of the LTB metric so that, once we have solved the corresponding Einstein equations, we can compare the predictions of the given model with the observational results.

However, differently from FLRW case, here we have a function $X$ which depends not only on $t$ but also on $r$, so that the redshift parameter cannot be directly related to the time parameter, as in the homogeneous case. From the physical point of view this is due to the fact that $z$ is measured along the observer’s past light-cone, which is geometrically defined by an algebraic relation which involves both $r$ and $t$. Due to the non-linear radial inhomogeneity, the radial dependence does not factorize, and analytically inverting Eq. (2.48) to get $t(z)$ is impossible, in general.

In last chapter it will be shown how this factorization is possible in a different (but fully inhomogeneous) case. For the moment we will obtain from the geodesic trajectory the equations determining $t(z)$ and $r(z)$, and will look for numerical solutions of such differential equations.
2.4. CONNECTIONS WITH PHYSICAL OBSERVABLES

2.4.1 Finding $t(z)$ and $r(z)$

Let us consider two light-like signals traveling towards the origin along a null radial geodesic ($ds^2 = 0$, $d\theta = d\phi = 0$), emitted from the source at the times $t$ and $t + \delta t$. By denoting the affine parameter with $\lambda$, we have that:

$$\frac{dt}{d\lambda} = -X(t, r) \frac{dr}{d\lambda},$$
$$\frac{d(t + \delta t)}{d\lambda} = -X(t + \delta t, r) \frac{dr}{d\lambda}. \quad (2.49)$$

Assuming that $\delta t$ is much smaller than the traveling time, we obtain, to first order,

$$\frac{d\delta t}{d\lambda} = -\delta t \dot{X}(t, r) \frac{dr}{d\lambda}. \quad (2.50)$$

Let us also recall that the redshift, by definition, satisfies $1 + z(\lambda) = \frac{\delta t_0}{\delta t(\lambda)}$, from which we get:

$$\frac{dz}{d\lambda} = -\frac{\delta t_0}{[\delta t(\lambda)]^2} \frac{d\delta t}{d\lambda} = (1 + z) \dot{X}(t, r) \frac{dr}{d\lambda}. \quad (2.51)$$

Therefore, the differential relations we need in order to compute $r(z)$ and $t(z)$ are given by:

$$\frac{dr}{dz} = \frac{1}{(1 + z)X(t, r)} = \frac{\sqrt{1 - k(r)}}{(1 + z)A'(t, r)},$$
$$\frac{dt}{dz} = \frac{dt}{dr} \frac{dr}{dz} = -\frac{X(t, r)}{(1 + z)X(t, r)} = -\frac{A'(t, r)}{(1 + z)A'(t, r)} \quad (2.52)$$

(for the last equalities we have used the result of Eq. (2.4), i.e. $X = F(r)A' = A'[1 - k(r)]^{-1/2}$). The luminosity distance can be finally computed as $d_L(z) = (1 + z)^2 A(t(z), r(z))$, and then compared with the observational data.

The set of Eqs. (2.52) can be directly solved by numerical methods in the case of the spatially flat models ($k(r) = 0$) since, for that models, the cosmological solutions can be explicitly expressed in terms of the synchronous time $t$. In the presence of spatial curvature we have shown in previous sections how to get analytical solutions. However, such solutions are given in terms of an auxiliary parameter, the conformal time $\eta$.

Following [31] we can then manipulate Eqs. (2.52) in order to obtain an equation also for $\eta(z)$. Let us define $a(t, r) \equiv \frac{A(t, r)}{r}$. The conformal time satisfies $d\eta = A(t, r)dt = \frac{a(t, r)}{r}dt$ so that, if we introduce a new variable $\tilde{\eta} \equiv \tau \eta$, we can express our solutions as:

$$A(t, r) = r a(\tilde{\eta}(t, r), r),$$
$$t(\tilde{\eta}, r) = \int a(\tilde{\eta}, r) d\tilde{\eta}, \quad (2.53)$$
from which:

\[ dt = a(\tilde{\eta}, r) \, d\tilde{\eta} + \left[ \int a'(\tilde{\eta}, r) \, d\tilde{\eta} \right] dr. \]

(2.54)

It follows that:

\[ \dot{\tilde{\eta}} = \frac{1}{a}, \quad \tilde{\eta}' = -\frac{t'}{a}, \quad \frac{dt}{dz} = a \frac{d\tilde{\eta}}{dz} + t' \frac{dr}{dz}, \]

(2.55)

and a straightforward computation gives:

\[ A' = a + r \, a' - r \, t' \frac{\partial a}{a} \equiv R_1(\tilde{\eta}, r), \]

(2.56)

\[ \dot{A}' = \frac{\partial a}{a} + r \frac{\partial a'}{a} - r \frac{a'}{a} \frac{\partial a}{a} + r \frac{t'}{a} \left( \frac{\partial a}{a} \right)^2 - r \frac{t'}{a} \frac{\partial^2 a}{a^2} \equiv R_2(\tilde{\eta}, r). \]

Eqs. (2.52) thus becomes:

\[ \frac{d\tilde{\eta}}{dz} = \frac{r}{A(\tilde{\eta}, r)} \frac{dt}{dz} - t'(\tilde{\eta}, r) \frac{dr}{dz}, \]

\[ = -\frac{r \, R_1(\tilde{\eta}, r) + A(\tilde{\eta}, r) \, t'(\tilde{\eta}, r) \, \sqrt{1-k(r)}}{(1+z) \, R_2(\tilde{\eta}, r)}, \]

\[ \frac{dr}{dz} = \frac{\sqrt{1-k(r)}}{(1+z) \, R_2(\tilde{\eta}, r)}. \]

(2.57)

In this way we are in the position of expressing theoretical predictions for the distances (or, better, for the received radiation flux) in terms of the redshift of the sources, in any isotropic but inhomogeneous scenario described by the LTB metric. It should be stressed that, \textit{a priori}, the above equations are not analytically solvable: a numerical integration is required, with appropriate specification of the initial conditions \( \tilde{\eta}(0) = 0 \) and \( r(0) = 0 \). In the last chapter of this work, however, we will introduce a different approach where no numerical integration is needed.

### 2.4.2 Off-center observers

The computations presented up to now are always referred to the case of observers located at the origin of the polar coordinate system (“on-center” observers). However, there are several papers, in the literature, discussing
the case of observers displaced from the symmetry center of the LTB geometry.

Following [32, 33], we thus consider the geodesic equation for a light-like signal (for instance, photons) received by an observer at a position different from the center. In this case we have $d\theta \neq 0$, but we can still assume $d\phi = 0$ because, thanks to the residual axial symmetry, we can align the observer’s displacement from the center in the direction the $z$ axis. The components of the null geodesic equation for the LTB metric, in this case, are given by:

$$
\frac{d^2t}{d\lambda^2} + X\dot{X} \left(\frac{dr}{d\lambda}\right)^2 + A\dot{A} \left(\frac{d\theta}{d\lambda}\right)^2 = 0
$$

$$
\frac{d^2r}{d\lambda^2} - \frac{X'}{X} \left(\frac{dr}{d\lambda}\right)^2 - \frac{AA'}{X^2} \left(\frac{d\theta}{d\lambda}\right)^2 + 2\dot{X} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0
$$

$$
\frac{d^2\theta}{d\lambda^2} + 2\frac{\dot{A}}{A} \frac{dt}{d\lambda} \frac{d\theta}{d\lambda} + 2\frac{A'}{A} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} = 0.
$$

The last equation can also be rewritten as $\frac{d}{d\lambda} \left( A^2 \frac{d\theta}{d\lambda} \right) = 0$, and implies that the quantity $J \equiv A^2 \frac{d\theta}{d\lambda}$ is constant along the trajectory: such a constant is physically related to the conservation of photon’s angular momentum.

The above system of second-order, coupled differential equations is expressed in terms of the affine parameter $\lambda$, but we are interested in using the redshift as our physical variable. To this purpose let us consider, as in the previous section, two photons traveling along a null geodesic ($ds^2 = 0$) with $d\phi = 0$, emitted from the source at the times $t$ and $t + \delta t$. We obtain the relations

$$
\left(\frac{dt}{d\lambda}\right)^2 = X^2(t, r) \left(\frac{dr}{d\lambda}\right)^2 + A^2(t, r) \left(\frac{d\theta}{d\lambda}\right)^2,
$$

$$
\left(\frac{dt}{d\lambda}\right)^2 + 2\frac{d\delta t}{d\lambda} \frac{dt}{d\lambda} = \left[ X^2(t, r) + 2\delta t \dot{X}(t, r) \right] \left(\frac{dr}{d\lambda}\right)^2,
$$

$$
+ \left[ A^2(t, r) + 2\delta t \dot{A}(t, r) \right] \left(\frac{d\theta}{d\lambda}\right)^2,
$$

where the second equation has been expanded up to first-order in $\delta t$ (assumed to be small with respect to the total traveling time). From their combination we obtain:

$$
\frac{1}{\delta t} \frac{d\delta t}{d\lambda} = \left[ \dot{X}(t, r) \left(\frac{dr}{d\lambda}\right)^2 + \dot{A}(t, r) \left(\frac{d\theta}{d\lambda}\right)^2 \right] \left(\frac{dt}{d\lambda}\right)^{-1}.
$$

On the other hand, from the definition of redshift, we have (see also Eq. (2.51)):

$$
\frac{dz}{d\lambda} = -\frac{\delta t(\lambda_0) \delta t}{\delta t(\lambda)^2} \frac{d\lambda}{d\lambda} = -\frac{1 + z \delta t}{\delta t(\lambda)}.
$$
which give us the final equation:

\[
\frac{dz}{d\lambda} = -(1 + z) \left[ \dot{X}(t, r) \left( \frac{dr}{d\lambda} \right)^2 + \dot{A}(t, r) \left( \frac{d\theta}{d\lambda} \right)^2 \right] \left( \frac{dt}{d\lambda} \right)^{-1} \equiv q. \tag{2.62}
\]

Eq. (2.62) solves our problem of expressing the coordinates \(t, r, \theta\) in terms of the redshift. But the solution is still in an implicit form. In order to get explicit expressions, let us introduce the new variables \(u \equiv \frac{dt}{d\lambda}\) and \(p \equiv \frac{dr}{d\lambda}\) so that, using the definition \(J = A^2 \frac{d\theta}{d\lambda}\), we can rewrite Eqs. (2.58) and (2.62) in the form:

\[
\begin{align*}
\frac{du}{d\lambda} + X \dot{X} p^2 + \frac{\dot{A}}{A^3} J^2 &= 0, \\
\frac{dp}{d\lambda} + \frac{X'}{X} p^2 - \frac{A'}{A^3 X^2} + 2 \frac{\dot{X}}{X} p u &= 0, \\
\frac{d\theta}{d\lambda} - \frac{J}{A^2} &= 0, \\
\frac{dz}{d\lambda} - \frac{1 + z}{u} \frac{du}{d\lambda} &= 0. \tag{2.63}
\end{align*}
\]

These equations are partially solved for \(u = u_o(1 + z)\), with constant \(u_o\). Remembering that (according to Eq. (2.62)) \(\frac{dt}{d\lambda} = \frac{dz}{d\lambda} \equiv q \frac{dt}{dz}\), and that \(\frac{dt}{dz} = u\), we can finally write the equations of interest as:

\[
\begin{align*}
\frac{dt}{dz} &= \frac{u_o(1 + z)}{q}, \\
\frac{dr}{dz} &= \frac{p}{q}, \\
\frac{d\theta}{dz} &= \frac{J}{qA^2}, \\
\frac{dp}{dz} &= \frac{1}{q} \left[ -\frac{X'}{X} p^2 + \frac{A'}{A^3 X^2} J^2 - 2 \frac{\dot{X}}{X} p u_o(1 + z) \right]. \tag{2.64}
\end{align*}
\]

As in the previous case of a central observer, even for off-center observers we have thus been able to write first-order differential equations for the coordinates of the geodesic trajectory, using the redshift as our differential variable. The initial conditions we have now to impose can be written as \(t(0) = t_o, r(0) = d, \theta(0) = 0\) and \(p(0) = \frac{\cos^2 \gamma}{X(t_o, d)}\), where \(\gamma\) is the angle of the incoming photon with respect to the \(z\) axis at the observer position, \(d\) is the radial coordinate of the observer, and \(\theta(0) = 0\) because of the alignment with \(z\) axis. The constants \(u_o\) and \(J\) remain to be specified. \(u_o\) can be chosen equal to \(-1\) while \(J\), due to its constancy, can be evaluated at the observer position as \(J = A(t_o, d) \sin \gamma\).

The system of equations (2.64) allows us to express any physical observable, defined in term of the metric component, as a function of \(z\). In
the subsequent discussion we will consider, in particular, the angular diame-
ter distance $d_A$, evaluated for an off-center observer characterized by the
angular parameter $\gamma$. Its expression is given by \cite{32}:

$$d_A = A(t, r) \sqrt{\frac{\sin \theta}{\sin \gamma}} \left[ \frac{X^2}{A^2} \left( \frac{\partial r}{\partial \gamma} \right)^2 + \left( \frac{\partial \theta}{\partial \gamma} \right)^2 \right]^{1/4}.$$  \hspace{1cm} (2.65)

This quantity can be evaluated using for instance the so-called observational
coordinates, which refer to a set of variables defined along the past light-
cone of the observer. However, in last chapter we will provide an alternative
derivation of such a quantity, adopting a similar (but qualitatively different)
coordinates system, useful to obtain exact (non-perturbative) expression for
the distance along the past light-cone of a given observer.

### 2.5 Data analysis

#### 2.5.1 Discriminating different models of $d_L(z)$

Let us now discuss the possibility of fitting the observational data with
a luminosity-redshift relation predicted in the context of an intrinsically
inhomogeneous geometry, and let us consider, as a particularly simple and
pedagogical example, a model of LTB type dominated by cold dark matter
sources \cite{34}. Neglecting the spatial curvature ($k(r) = 0$) and the possible
contribution of a cosmological constant, the corresponding LTB metric is
then specified by the solution (2.22) already derived in Sect. 2.1. Choosing,
in particular, the initial conditions $A_0(r) = r$ and $t_o = 0$, the solution
becomes

$$A(t, r) = r \left[ 1 + \frac{3}{2} H_0(r) t \right]^{2/3}.$$  \hspace{1cm} (2.66)

Our subsequent analysis, of course, can be similarly performed also for dif-
f erent, more complicated examples of LTB geometries.

In order to numerically integrate the set of differential equations (2.52),
and obtain $t(z)$ and $r(z)$, we have to choose a (possibly realistic) shape for
the radial function $H_0(r)$. To this purpose we will use the parametrization:

$$H_0(r) = \overline{H} + \Delta H e^{-r/r_V},$$  \hspace{1cm} (2.67)
2.5. DATA ANALYSIS

already suggested in [35] for a similar LTB context. Here $\bar{H}$, $\Delta H$ and $r_V$ are constant phenomenological parameters, and $\bar{H} + \Delta H = H_0$ corresponds to the locally measured value of the present Hubble constant. The distance $r_V$ represents, in our context, the typical distance scale above which inhomogeneity effects become rapidly negligible.

Let us consider, as the set of experimental data to be fitted by the theoretical predictions, the Union2 compilation of the Supernova Cosmology Project [36], which concerns redshift-magnitude measurements of 557 SNe of type Ia and provides, for each supernova, the observed distance modulus (with relative error) $\mu_{\text{obs}}(z_i) \pm \Delta \mu(z_i)$, $i = 1, \ldots, 557$, for redshift values ranging from $z_1 = 0.015$ to $z_{557} = 1.4$. The distance modulus $\mu(z)$ controls the difference between apparent and absolute magnitude, and is related to the luminosity distance by Eq. (1.32).

In the model we are considering, the luminosity distance of Eqs. (1.31), (2.46), with $H_0(r)$ given by Eq. (2.67), is characterized in principle by three independent parameters, and can be applied to fit the experimental data by allowing free variations of $\bar{H}$, $\Delta H$ and $r_V$. We have performed that exercise, and found that the resulting best fit provides for $H_0(0) \equiv \bar{H} + \Delta H$ a value very close to $70 \text{ Km s}^{-1}\text{Mpc}^{-1}$. We have thus chosen to concentrate the present discussion on a simpler, two-parameter fit of the data – which, in any case, is sufficiently accurate for our illustrative purpose – by imposing on our model the a priori constraint $\bar{H} + \Delta H = 70 \text{ Km s}^{-1}\text{Mpc}^{-1}$. In this way we can eliminate, for instance, $H_0$, and we can fit the experimental points $\mu_{\text{obs}}(z_i) \pm \Delta \mu(z_i)$ by performing a standard $\chi^2$ analysis, with

$$\chi^2 = \sum_{i=1}^{557} \left[ \frac{\mu_{\text{obs}}(z_i) - \mu(z_i, r_V, \Delta H)}{\Delta \mu(z_i)} \right]^2. \tag{2.68}$$

The theoretical values $\mu(z_i, r_V, \Delta H)$ can be determined, for each value of $z_i$, by numerically integrating Eqs. (2.52) (with $k(r) = 0$), and computing the corresponding $d_L(z_i) = (1 + z_i)^2A(t(z_i), r(z_i))$ as a function of the two parameters $r_V, \Delta H$. By minimizing the above $\chi^2$ expression we have found the best fit values

$$r_V = 3000 \pm 497 \text{ Mpc}, \quad \Delta H = 26.6 \pm 1.3 \text{ Km s}^{-1}\text{Mpc}^{-1}, \tag{2.69}$$

at a confidence level of 95%, and with a goodness of fit $\chi^2$/d.o.f. = 0.99. The result of the fit is graphically illustrated by the red curve plotted in the top panel of Fig. 2.1, superimposed to the full set of Union2 data (reported with error bars).

Consider now, for comparison, a fit of the same data performed in the context of a spatially flat FLRW geometry, with perfect fluid sources representing CDM and a cosmological constant $\Lambda$. Denoting with $\Omega_m$ and $\Omega_\Lambda$
the present critical fraction of dark matter and dark energy, we can express
the luminosity distance in the usual integral form as

\[ d_L(z) = \frac{1+z}{H_0} \int_0^z dx \left[ \Omega_m(1+x)^3 + \Omega_\Lambda \right]^{-1/2} \]  

(2.70)

(see Chap. 1). Proceeding as in the previous case, we will reduce the
number of parameters from 3 to 2 by imposing the same phenomenological
constraint as before, which in this case amounts to the condition \( H_0(\Omega_m + \Omega_\Lambda)^{1/2} = 70 \text{ Km s}^{-1} \text{ Mpc}^{-1} \). Using Eq. (2.70) to compute \( \mu(z_i, \Omega_m, \Omega_\Lambda) \),
and minimizing the corresponding \( \chi^2 \) expression, we obtain the best fit val-
ues \( \Omega_m = 0.27 \pm 0.01, \Omega_\Lambda = 0.71 \pm 0.03 \), at a confidence level of 95%, with
\( \chi^2/\text{d.o.f.} = 0.98 \). The result of the fit is illustrated by the blue curve on the
bottom panel of Fig. 2.1.

The luminosity-redshift relations of the two models of Fig. 2.1 are in
good agreement with the data, and in both cases the data points are fitted
at a comparable level of statistical accuracy. However, we can disclose an
important physical difference between the two fits if we subtract from the
distance modulus of the two models the distance modulus \( \mu_{\text{Milne}}(z) \) of a
linearly expanding (but globally flat) homogeneous Milne geometry (see e.g
[37]), namely if we consider the quantity

\[ \Delta(z) = \mu(z) - \mu_{\text{Milne}}(z) = 5 \log_{10} \left[ \frac{d_L(z)}{1 \text{ Mpc}} \right] - 5 \log_{10} \left[ \frac{z(2+z)}{2H_0 \text{ Mpc}} \right], \]  

(2.71)

where \( H_0 \) is given in units of Mpc\(^{-1}\). It is clear that positive or negative
values of \( \Delta \) correspond to luminosity distances which are – at a given fixed \( z \) – respectively larger or smaller than the reference values of the Milne model.

The case \( \Delta < 0 \) is typical of a decelerated Universe like that described by
the standard cosmological scenario, where, at the same fixed \( z \), the distances
are smaller (or the received fluxes of radiation, i.e. the apparent magnitudes,
are larger) than predicted by a linearly expanding model. The case \( \Delta > 0 \),
on the contrary, corresponds at the same \( z \) to larger distances (or smaller
radiation fluxes) than predicted by a linear expansion, and is only possible
if the model undergoes a period of “effective” accelerated expansion. In
this last case, the transition across the value \( \Delta = 0 \) defines an epoch –
characterized by the parameter \( z_{\text{acc}} \) such that \( \Delta(z_{\text{acc}}) = 0 \) – marking the
beginning of the cosmological phase directly imprinted by the kinematic
effects of the acceleration.

The plot of \( \Delta(z) \) is presented in Fig. 2.2 for three cases: the standard
CDM-dominated (always decelerated) model, and the two best-fit models of
Fig. 2.1 (corresponding to our example of inhomogeneous geometry and to
a typical example of homogeneous concordance cosmology). In the last two
cases we have plotted the central values of the fit (solid curves), as well as
Figure 2.1: The Hubble diagram of the Union2 dataset. The top panel illustrates the best-fit result for a two-parameter fit of our example of inhomogeneous geometry, with $\chi^2_{LTB}/d.o.f. = 0.99$. In the bottom panel we present the corresponding best-fit result for a homogeneous $\Lambda$CDM model, with $\chi^2_{\Lambda\text{CDM}}/d.o.f. = 0.98.$
the corresponding error bands at the 95% level of confidence (bounded by the dotted curves).

We can see from Fig. 2.2 that $\Delta(z)$ is always negative for the CDM model, as expected. For the other two models, instead, we have $\Delta(z) > 0$ in the redshift range $z < z_{\text{acc}}$ (because, as expected, a successful fit of the SNe data requires the presence of a phase describing – or mimicking – accelerated expansion). However, the values of $z_{\text{acc}}$ defined by the condition $\Delta(z_{\text{acc}}) = 0$ are largely different in the two models. We find, in particular,

$$z_{\text{acc}}^{\text{LTB}} = 1.07 \pm 0.06, \quad z_{\text{acc}}^{\Lambda\text{CDM}} = 1.43 \pm 0.10,$$

and this difference falls outside the error bands illustrated in Fig. 2.2 (it is also much larger than the experimental uncertainty affecting present redshift measurements). This suggests that a precise (near-future?) determination of this parameter could provide a clear physical discrimination among different models implementing statistically equivalent fits of SNe data.

![Figure 2.2](image-url)

**Figure 2.2:** The parameter $\Delta(z)$ of Eq. (2.71) for the two best-fit models of Fig. 2.1. In both cases we have shown the region allowed by the fit at the 95% C.L. (bounded by dotted lines). We have also reported (for comparison, and without error band) the case of the standard CDM model with $\Omega_m = 1$.

It should be mentioned, at this point, that in the computations of the error bands we have neglected the dispersion of data due to the possible presence of a cosmic background of stochastic perturbations: indeed, such a background may induce large errors at very small $z$, but in the range
2.5. DATA ANALYSIS

$z \sim 1$ (typical of $z_{acc}$) the induced errors are typically lying in the few-percent range [38, 39], hence are not expected to have a crucial impact on the results illustrated in Fig. 2.2. The same is expected to be true for the systematic errors – possibly slightly bigger than the previous ones, but in any case $\lesssim 10\%$ – induced on $z_{acc}^{LTB}$ (but not on $z_{acc}^{\Lambda CDM}$) by methods of SNe data reduction based on the assumption of standard homogeneous cosmology. Finally, we should note that a value of $z_{acc}$ compatible with that of the inhomogeneous model considered here could be reproduced also in a homogeneous $\Lambda CDM$ context, with realistic values of $\Omega_m$ and $\Omega_{\Lambda}$, but only at the price of introducing a large enough negative spatial curvature, with $\Omega_k \sim 0.1$ (for instance, a model with $\Omega_m = 0.3$, $\Omega_{\Lambda} = 0.6$, $\Omega_k = 0.1$ gives $z_{acc}^{\Lambda CDM} = 1.087$).

In order to stress the importance of the parameter $z_{acc}$ let us now consider another possible form of the phenomenological profile $H_0(r)$ appearing in the LTB solution, for instance the profile

$$H_0(r) = \overline{H} + \Delta H \tanh \left( \frac{r_0 - r}{2\Delta r} \right).$$

(2.73)

We can then explicitly check that different models are characterized by largely different values of $z_{acc}$ even within the same class of inhomogeneous geometries. By imposing, as before, the phenomenological constraint $H_0(0) = 70 \text{Km s}^{-1}\text{Mpc}^{-1}$ (in order to eliminate $\overline{H}$), we find that the new profile (2.73) provides indeed a satisfactory three-parameter fit of the Union2 data (see Fig. 2.3, top panel), with best fit values $r_0 = 2500 \pm 322$ Mpc, $\Delta r = 2387 \pm 170$ Mpc, $\Delta H = 37.5 \pm 2.8 \text{Km s}^{-1}\text{Mpc}^{-1}$, at a confidence level of 95%, with $\chi^2$/d.o.f. = 1.31. However, the corresponding value of $z_{acc}$ for this model (called LTB$^1$ in Fig. 2.3) is significantly different from that of the previous LTB model, and, most important, the behavior of $\Delta(z)$ is exactly the opposite of the standard one, for the range of $z$ of our interest (see Fig. 2.3, bottom panel). We have checked that the value of $\Delta(z)$, for LTB$^1$, turns back to the standard negative range only for $z > \sim 50$.

Let us finally comment on the possibility that an off-center position of the observer embedded in a spherically symmetric LTB geometry may significantly affect the determination of $z_{acc}^{LTB}$, thus providing obstructions to a precise discrimination between LTB-based and a more conventional (homogeneous) fit of the SNe data. Indeed, if the observer is located at a distance $r_0 \neq 0$ from the center of a spherically symmetry geometry, the corresponding luminosity distance $d_L$ (referred to the position $r_0$) is no longer isotropic but acquires an angular dependence, and this in turn induces an angular dispersion of the value of $z_{acc}$ which depends on $r_0$, and which obviously grows (in modulo) with the growth of $r_0$.

The angular (and then luminosity) distance of a source for off-center observers in a LTB geometry is given by Eq. (2.65). We have applied the
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Figure 2.3: The top panel illustrates the best-fit result for the model characterized by the hyperbolic profile of Eq. (2.73). The corresponding behavior of $\Delta(z)$, for the range of $z$ of our interest, is represented by the curve labelled LTB$_1$ reported in the bottom panel.
2.5. DATA ANALYSIS

set of Eqs. (2.64) to get the $z$ dependence of the various coordinates. We have considered, in particular, possible displacements from the centre in the range $r_0 \lesssim 10^{-2} r_V$, because – as discussed in [40] – higher values of $r_0$ would induce a dipole anisotropy too high to be compatible with present CMB observations.

The results of this exercise is illustrated in Fig. 2.4, where we have plotted the fractional variation $\Delta z_{\text{acc}}/z_{\text{acc}} \equiv [z_{\text{acc}}(r_0, \gamma) - z_{\text{acc}}(0)]/z_{\text{acc}}(0)$, for different values of $r_0$ up to $10^{-2} r_V$, for the LTB model characterized by the parameter $z_{\text{acc}}^{\text{LTB}}$ of Eq. (2.72). For the normalization of $\mu_{\text{Milne}}$ we have consistently used $H_0(r_0)$, but we have checked that using the fixed value $H_0 = 70 \text{Km s}^{-1}\text{Mpc}^{-1}$ simply rescales the zero of the difference $\Delta z_{\text{acc}}$, without affecting the overall amplitude of the dispersion. As shown in Fig. 2.4, the angular variation of $z_{\text{acc}}$ induced by $r_0 \neq 0$ is bounded to be at most at the one-percent level, and has thus a negligible impact on the results of Fig. 2.2.

![Figure 2.4](image_url)

**Figure 2.4:** The fractional variation of the parameter $z_{\text{acc}}$ as a function of the angular direction $\gamma$, for different values of the observer’s position $r_0$ ranging from 0 to $10^{-2} r_V$. The numerical labels of the curves are referred to the values of $r_0$, given in units of $10^{-2} r_V$.

In conclusion, we would like to stress again that the inhomogeneous model discussed in this section should not be intended as a realistic alternative to the successful concordance cosmology, but only as a pedagogical example to learn how to distinguish different fits of SNe data based on different geometrical schemes. To this purpose we have shown, in particular, that in this model the Universe enters the regime directly affected by the accelerated kinematics later than predicted by the $\Lambda$CDM scenario, i.e. $z_{\text{acc}}^{\text{LTB}} < z_{\text{acc}}^{\Lambda\text{CDM}}$. Hence, a precise determination of the transition epoch $z_{\text{acc}}$
(possibly through future extensions of the Hubble diagram to higher values of \( z \), or through indirect studies of the transfer function of primordial perturbations), could help us to physically discriminate among statistically equivalent fits.

### 2.5.2 Fractal distribution

An analysis similar to the previous one can be directly applied also to other physically interesting situations. In particular, to the fractal solution that we have found in Sec. 2.3. To this purpose, it is important to stress that in a pure fractal distribution the symmetry center is determined only after integration over the sphere, and the previous discussion concerning off-center observers is no longer relevant for our applications.

Let us consider the same type of matter-dominated, spatially flat LTB geometry as before, described by the solution (2.66), where we insert, however, the fractal profile of Eq. (2.45). The likelihood analysis is now referred to the parameters \( H_0 \), \( D \) and \( l_0 \), and we want to determine their best-fit values.

As before, we compare the observed values \( \mu_{\text{obs}}(z_i) \pm \Delta \mu(z_i) \) with the predicted values of the distance modulus \( \mu_{\text{th}}(z) \), and perform a standard \( \chi^2 \) analysis with

\[
\chi^2 = \sum_{i=1}^{557} \left[ \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i, D, l_0, H_0^{\text{ext}})}{\Delta \mu(z_i)} \right]^2,
\]

(2.74)

by requiring the minimization of \( \chi^2 \). The results are shown in Fig. 2.5, where the green curve corresponds to the best fit values for our model, i.e.:

\[
\begin{array}{c|c|c}
D & l_0 \text{ (Mpc)} & H_0^{\text{ext}} \text{ (km sec}^{-1}\text{ Mpc}^{-1}) \\
2.869 \pm 0.016 & 308 \pm 54 & 0.008 \pm 33
\end{array}
\]

with a \( \chi^2/\text{dof} = 1.19 \). This shows that, by introducing a fractal exponent significantly different from 3, we can obtain a satisfactory fit of the supernova data without introducing any dark energy contribution (note that the value \( H_0^{\text{ext}} = 0 \) corresponds to the case with no transition to the homogeneous regime).

Let us finally stress that the possibility of fitting the data with inhomogeneous and fractal solutions which – from a purely kinematic point of view – are of decelerated type, may suggest that the acceleration, at least in principle, might be an apparent effect of the assumed homogeneity.
2.5. DATA ANALYSIS

**Figure 2.5:** Luminosity distance for a very simple model of fractal Universe. Data from UNION2 catalog.

**Figure 2.6:** Log-Log plot for the distance modulus versus the redshift. The thick red curve corresponds to the case of the fractal model discussed in this section. The thin black curves refer to matter-dominated FLRW models with different values of $H_0^{\text{FLRW}}$, corresponding to the fractal parameter $H_0(r)$ evaluated at different distance scales, from $r = 1$ Mpc (bottom) to $r = 10^4$ Mpc (top).

In particular, as illustrated in Fig.2.6, our model (represented by the red curve) describes a smooth transition among different homogeneous FLRW,
CDM-dominated models with different values of $H_{0}^{FLRW}$ (black curves). Using the radial-dependent parameter $H_{0}(r)$ of Eq. (2.45), we can see that at low redshifts the fractal model corresponds to a FLRW model with $H_{0}^{FLRW} = H_{0}(r = 1 \text{ Mpc})$ but, at higher redshifts, the fractal LTB curve is better described by a FLRW model with $H_{0}^{FLRW} = H_{0}(r = 10000 \text{ Mpc})$. 
Chapter 3

From Bianchi-I type to a perturbed LTB metric

In the previous chapters we have discussed the important assumptions of the standard cosmological model concerning the homogeneity and isotropy of the background large-scale geometry. Nevertheless, on local scales the Universe is far from being exactly isotropic and homogeneous.

The problem of whether the background cosmic geometry, above a given (and large enough) scale, becomes fully homogeneous and isotropic – barring the possible presence of fluctuations of primordial inflationary origin – is certainly of fundamental importance, but we have at present no way to provide a decisive solution. Neither the observations of luminosity distances combined with galaxy number counts, nor the results concerning the Cosmic Microwave Background (CMB) radiation are enough to this purpose.

The question of whether only an isotropic and homogeneous background geometry is compatible with (and able to explain) the experimental data is, after all, a pertinent question because, with the standard assumption, it follows that more than the 96% of the content of our Universe must be “dark” and with no clear physical identification. The solution of the dark-energy/dark-matter puzzle is a central problem of modern cosmology.

Are there observables able to prove that the Universe becomes fully homogeneous and isotropic on large scales? Very interesting studies has been done in this direction [41, 42, 43]. The ΛCDM model of the Universe is remarkably successful, but we have important tensions between the model and the experimental data [44, 45]. On the other hand, dark energy is a problem not only for cosmology but also for fundamental physics. A detailed discussions on the dark energy is outside the scope of this work, but see for instance [46, 47] and references therein. In general there are many reasons to believe that the ΛCDM model presents various theoretical problems [48]. Also, we cannot reasonably expect that the presence of dark energy will be
tested in a near future through locally observable effects.

Up to now we have taken into account a particular type of inhomogeneity characterized by a radial symmetry, but we have not discussed isotropy. As is well known, the CMB radiation is highly isotropic, and this is considered as a strong evidence supporting the assumption that the background geometry is well described by the FLRW metric. This conclusion dates back to the results of the Ehlers-Gerz-Sachs theorem (EGS) [41] of 1968, also related to an earlier paper by Tauber and Weinberg [49] of 1961. But now, high precision cosmology is able to obtain more detailed information about our Universe, and we cannot so easily conclude from the properties of the CMB radiation that our region of space is fully isotropic [50].

Indeed, we have important observational evidences against an exact background isotropy [51], evidences which might be explained by a phase of anisotropic expansion during the past evolution of our Universe. In particular, there are aspects of the observed CMB distribution suggesting the presence of an anomalous plane-mirroring symmetry on large scales [52, 53]. These anomalous features are present in the set of seven-years WMAP data and in the Planck data, and seems to suggest that our present Universe might be affected by an intrinsic background anisotropy.

The anomalies present in the large-angle CMB distribution [54] can be separately listed as follows:

1) the alignment of the quadrupole and octupole moments [55, 56, 57, 58];
2) the large-scale asymmetry of the type discussed in [59, 60];
3) the existence of the so-called (and strange) “cold spot” [61];
4) the low quadrupole moment of the CMB temperature, which is largely suppressed with respect to the theoretical predictions, and which cannot be explained by the standard (inflationary) model. It could be due to a phase of ellipsoidal/Bianchi-I type anisotropic evolution of the Universe [62, 63, 64, 65].

A few years ago it has been shown [66, 67] that, if we don’t assume a priori homogeneity, then the presence of small CMB anisotropies implies that the geometry is not exactly FLRW, but is “almost” FLRW. Limits on the anisotropy and inhomogeneity can then be found by starting from the CMB observations. A cosmological model that takes into account this possibility, and stimulates interesting applications, is the anisotropic “Bianchi-I type” model, where small deviations from the FLRW geometry can be easily included, and possibly used to fit the CMB anomalies.

For these reasons we will adopt the Bianchi-I type metric in order to discuss a model of Universe where both inhomogeneities and anisotropies simultaneously coexist. This possibility has already been considered in different physical situations such as, for instance, when studying the role of the diffusion forces in governing the large-scale dynamics of an inhomoge-
neous and anisotropic Universe [68]. In this chapter we will compute the luminosity-distance for a LTB geometry “distorted” by the presence of a small Bianchi-I anisotropy, and the results will be compared with the available experimental data, as did in [69].

3.1 The Bianchi-I type metric

We shall consider a particular case of Bianchi-I type geometry which is still isotropic in the \( \{x, y\} \) plane, and is thus described by the line-element:

\[
ds^2 = -dt^2 + a^2(t) \left[ dx^2 + dy^2 \right] + b^2(t) dz^2 \tag{3.1}
\]

where \( t \) is the synchronous time as defined for the FLRW case, and \( \{x, y, z\} \) are spatial cartesian coordinates. The above metric differs from the FLRW one because the scale factor along the \( z \) axis is not equal to the one of the \( x-y \) plane (but \( a \) and \( b \) are functions of the time \( t \) only). The spatial curvature is vanishing like in the corresponding FLRW limit, obtained for \( b = a \).

Instead of working with \( a \) and \( b \), we can conveniently use \( a \) and the eccentricity parameter \( e(t) \equiv \sqrt{1 - \left( \frac{b}{a} \right)^2} \), so that Eq. (3.1) becomes:

\[
ds^2 = -dt^2 + a^2(t) \left\{ dx^2 + dy^2 + \left[ 1 - e^2(t) \right] dz^2 \right\} . \tag{3.2}
\]

When the anisotropy is small, and \( e^2 \) can be treated as a first order quantity, Eq. (3.2) can be seen as a particular case of a perturbed metric in the synchronous gauge. Hence, all the properties of that gauge also apply to the case we are considering. In addition, a residual gauge freedom can be used to fix \( e^2 = 0 \) at the present time \( t_0 \) (or, equivalently, \( b_0 = a_0 \)).

In order to write the Einstein’s equations let us start computing the Ricci tensor:

\[
R^t_t = 2 \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b}, \\
R^x_x = R^y_y = \frac{\dot{a} \dot{b}}{a b} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a}, \\
R^z_z = 2 \frac{\dot{a} \dot{b}}{a b} + \frac{\ddot{b}}{b} . \tag{3.3}
\]
The scalar curvature $R = R^\mu_\mu$ is thus given by:

$$R = 2 \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} + 2 \frac{\dot{a} \dot{b}}{ab} + \frac{\ddot{b}}{b} \right),$$

(3.4)

and the Einstein tensor $G^\nu_\mu = R^\nu_\mu - \delta^\nu_\mu R^2$ is:

$$G^t_t = - \left( \frac{\dot{a}}{a} \right)^2 - 2 \frac{\dot{a} \dot{b}}{ab}$$

$$G^x_x = G^y_y = - \frac{\dot{b}}{a b} - \frac{\dot{a}}{a} - \frac{\ddot{b}}{b}$$

$$G^z_z = - \left( \frac{\dot{a}}{a} \right)^2 - 2 \frac{\ddot{a}}{a}$$

(3.5)

We shall assume, as before, that the right-hand side of Einstein’s equations is given by the energy-momentum for a barotropic fluid with $T^\nu_\mu = \text{diag} \{ -\rho, p_x, p_y, p_z \}$. Consistently with the assumed anisotropy, the fluid pressure along the $z$ axis may differ from that of the $x$-$y$ plane. We shall thus put $T^\nu_\mu = \text{diag} \{ -\rho, p, p, p_z \}$, with an equation state

$$p = \gamma \rho \quad \text{and} \quad p_z = \gamma_z \rho,$$

(3.6)

where $\gamma$ and $\gamma_z$ are constants. Note that this choice is appropriate to describe special cases of anisotropic sources such as, in particular, cosmic strings ($\gamma = 0$ and $\gamma_z = -1$), domain walls ($\gamma = -1$ and $\gamma_z = 0$), and so on. However, such sources will not be considered here: in fact, their presence is possibly relevant to explain the generation of anisotropy in the early stages of the cosmological evolution, while we are concentrating our analysis on the late-time evolution, where matter structures and dark sources are the most important components.

### 3.2 A metric of LTB-Bianchi-I type

Let us introduce the metric which will be used throughout this chapter, and which describes an inhomogeneous and anisotropic geometry.
3.2. A METRIC OF LTB-BIANCHI-I TYPE

To this purpose we start writing the Bianchi-I type line-element (3.1) in terms of polar coordinates \( x = r \sin \theta \cos \phi \), \( y = r \sin \theta \sin \phi \) and \( z = r \cos \theta \):

\[
\begin{align*}
\frac{ds^2}{2} &= -dt^2 + [a^2(t) \sin^2 \theta + b^2(t) \cos^2 \theta] \, dr^2 + r^2 [a^2(t) \cos^2 \theta \\
&+ b^2(t) \sin^2 \theta] \, d\theta^2 + 2r [a^2(t) - b^2(t)] \sin \theta \cos \theta \, dr \, d\theta \\
&+ r^2 a^2(t) \sin^2 \theta \, d\phi^2.
\end{align*}
\]

(3.7)

This expression is more complicated than the corresponding cartesian one, but easier to be compared with the LTB metric. Let us also consider the new variables \( A_{\parallel} \), \( A_{\perp} \), defined by the correspondence:

\[
\begin{align*}
ra(t) &\rightarrow A_{\parallel}(t,r) \equiv A_{\parallel} \\
rb(t) &\rightarrow A_{\perp}(t,r) \equiv A_{\perp}.
\end{align*}
\]

(3.8)

With this choice we can easily obtain a generalized version of LTB-Bianchi-I type metric, expressed in polar coordinate, and written in the form:

\[
\begin{align*}
\frac{ds^2}{2} &= -dt^2 + \left( A_{\parallel}^2 \sin^2 \theta + A_{\perp}^2 \cos^2 \theta \right) \, dr^2 + \left( A_{\parallel}^2 \cos^2 \theta \\
&+ A_{\perp}^2 \sin^2 \theta \right) \, d\theta^2 + \left( A_{\parallel}' - A_{\perp}' \right) \sin \theta \cos \theta \, dr \, d\theta \\
&+ A_{\parallel}^2 \sin^2 \theta \, d\phi^2.
\end{align*}
\]

(3.9)

By construction, the pure anisotropic case is obtained in the limit \( A_{\parallel} = ra \) and \( A_{\perp} = rb \). We also recover, from Eq. (3.9), the known isotropic and spatially flat cases previously considered:

\[
\begin{align*}
&\begin{cases}
A_{\parallel}(t,r) = A_{\perp}(t,r), & \text{LTB} \\
A_{\parallel}(t,r) = A_{\perp}(t,r) = ra(t), & \text{FLRW}.
\end{cases}
\end{align*}
\]

(3.10)

The more general metric (3.9) describes an inhomogeneous geometry with axial symmetry. In view of a possible perturbative approach, let us define the following quantity,

\[
\epsilon(t,r) = A_{\perp} - A_{\parallel},
\]

(3.11)

which controls the degree of cosmological anisotropy. With this definition we obtain:

\[
\begin{align*}
A_{\parallel}' &= A_{\parallel}' + \epsilon' \\
A_{\perp}'^2 &= A_{\parallel}'^2 + \epsilon^2 + 2 A_{\parallel}' \epsilon' \\
A_{\parallel}^2 &= A_{\parallel}^2 + \epsilon^2 + 2 A_{\parallel} \epsilon \\
(A_{\parallel}^2)'' &= \left( A_{\parallel}^2 \right)' + (\epsilon^2)' + 2 A_{\parallel}' \epsilon + 2 A_{\parallel} \epsilon' \\
\end{align*}
\]

(3.12)
and the metric (3.9) can be rewritten as

\[ ds^2 = -dt^2 + \left[A_\parallel^2 + (\epsilon^2 + 2 A_\parallel \epsilon') \cos^2 \theta \right] dr^2 \]
\[ + \left[A_\parallel^2 + (\epsilon^2 + 2 A_\parallel \epsilon) \sin^2 \theta \right] d\theta^2 \]
\[ - \left[(\epsilon')^2 + 2 A_\parallel' \epsilon + 2 A_\parallel \epsilon' \right] \sin \theta \cos \theta dr d\theta \]
\[ + A_\parallel^2 \sin^2 \theta d\phi^2 \]
\[ \equiv \left(g^{(LTB)}_{\mu\nu} + \Delta g^{(AN)}_{\mu\nu} \right) dx^\mu dx^\nu \]

(3.13)

where:

\[ g^{(LTB)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -A_\parallel^2 & 0 & 0 \\ 0 & 0 & -A_\parallel^2 & 0 \\ 0 & 0 & 0 & -A_\parallel^2 \sin^2 \theta \end{pmatrix} \]

(3.14)

and

\[ \Delta g^{(AN)}_{\mu\nu} = \begin{pmatrix} \Delta g^{(AN)}_{11} \\ \Delta g^{(AN)}_{12} \\ \Delta g^{(AN)}_{22} \\ \Delta g^{(AN)}_{33} \end{pmatrix} \]

(3.18)

The index (LTB) refers to the pure Lemaître-Tolman-Bondi part of the metric, while the index (AN) labels the anisotropic corrections.

We have thus decomposed our metric tensor as the sum of a LTB metric describing a spatially flat inhomogeneous geometry, and a metric describing the anisotropy controlled by the parameter \( \epsilon(r,t) \). But we could also perform the decomposition with respect to the metric variable \( A_\perp \), and obtain

\[ g_{\mu\nu} \equiv g^{(LTB)}_{\perp\mu\nu} + \Delta g^{(AN)}_{\perp\mu\nu} \]

(3.19)

where \( g^{(LTB)}_{\perp\mu\nu} \) is given by Eq. (3.14) with the substitution of \( A_\parallel \) with \( A_\perp \), and where

\[ \Delta g^{(AN)}_{\perp\mu\nu} = \begin{pmatrix} \Delta g^{(AN)}_{11} \\ \Delta g^{(AN)}_{12} \\ \Delta g^{(AN)}_{22} \\ \Delta g^{(AN)}_{33} \end{pmatrix} \]

(3.20)

Up to now we have not performed any approximation for this particular inhomogeneous and anisotropic geometry. The metric is written in a set of
coordinates which has a clear physical interpretation, and we are still allowed to fix initial conditions by requiring that \( A_\perp = A_\parallel = r \) at the present time \( t_0 \). In that case, the spatial coordinates directly coincide with the radial distances and angles as measured today by a given static observer.

### 3.3 Einstein’s equations

We shall now write the Einstein equations for the LTB-Bianchi-I metric introduced in the previous section. To this purpose we will suppose that the cosmological anisotropy is sufficiently small to satisfy the condition

\[
\epsilon(t, r) \ll A_\parallel(t, r) \\
\epsilon'(t, r) \ll A'_\parallel(t, r), \tag{3.23}
\]

so that we can then perform a first-order expansion, replacing \( \Delta g^{(AN)}_{\mu\nu} \) with \( \delta g^{(AN)}_{\mu\nu} \), where:

\[
\begin{align*}
\delta g^{(AN)}_{\parallel 11} &= -2A'_\parallel \epsilon' \cos^2 \theta \\
\delta g^{(AN)}_{\parallel 22} &= -2A_\parallel \epsilon \sin^2 \theta \\
\delta g^{(AN)}_{\parallel 12} &= 2A_\parallel \epsilon' \sin \theta \cos \theta. \tag{3.24}
\end{align*}
\]

The associated Christoffel connection, to first order in \( \epsilon \), is given by:

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \approx \\
\simeq \frac{1}{2} \left( g^{\alpha\rho}_{(LTB)} + \delta g^{\alpha\rho}_{(AN)} \right) \left[ \partial_\mu (g_{\nu\rho}^{(LTB)} + \delta g_{\nu\rho}^{(AN)}) + \partial_\nu (g_{\rho\mu}^{(LTB)} + \delta g_{\rho\mu}^{(AN)}) + \partial_\rho (g_{\mu\nu}^{(LTB)} + \delta g_{\mu\nu}^{(AN)}) \right], \tag{3.25}
\]

and, neglecting second-order terms in \( \epsilon^2 \), it becomes

\[
\Gamma^\alpha_{\mu\nu} \approx \frac{1}{2} g^{\alpha\rho}_{(LTB)} \left( \partial_\mu g_{\nu\rho}^{(LTB)} + \partial_\nu g_{\rho\mu}^{(LTB)} - \partial_\rho g_{\mu\nu}^{(LTB)} \right) + \\
+ \frac{1}{2} g^{\alpha\rho}_{(LTB)} \left( \partial_\mu \delta g_{\nu\rho}^{(AN)} + \partial_\nu \delta g_{\rho\mu}^{(AN)} - \partial_\rho \delta g_{\mu\nu}^{(AN)} \right) + \\
+ \frac{1}{2} \delta g^{\alpha\rho}_{(AN)} \left( \partial_\mu g_{\nu\rho}^{(LTB)} + \partial_\nu g_{\rho\mu}^{(LTB)} - \partial_\rho g_{\mu\nu}^{(LTB)} \right). \tag{3.26}
\]

3.3. EINSTEIN’S EQUATIONS

The first line of Eq. (3.26) just corresponds $\Gamma^{\alpha\mu\nu}_{(LTB)}$, the Christoffel connection for the metric tensor $g_{\mu\nu}^{(LTB)}$. It is convenient to define

$$\Sigma^{\alpha\mu\nu} \equiv \frac{1}{2} g^{\alpha\rho(AN)} \left( \partial_{\mu} g_{\nu\rho}^{(AN)} + \partial_{\nu} g_{\rho\mu}^{(AN)} - \partial_{\rho} g_{\mu\nu}^{(AN)} \right)$$ (3.27)

and

$$\Theta^{\alpha\mu\nu} \equiv \frac{1}{2} \delta g^{\alpha\rho(AN)} \left( \partial_{\mu} g_{\nu\rho}^{(LTB)} + \partial_{\nu} g_{\rho\mu}^{(LTB)} - \partial_{\rho} g_{\mu\nu}^{(LTB)} \right),$$ (3.28)

so that the the first-order Christoffel connection can be written as:

$$\Gamma^{\alpha\mu\nu} = \Gamma^{\alpha\mu\nu}_{(LTB)} + \Sigma^{\alpha\mu\nu} + \Theta^{\alpha\mu\nu}.$$ (3.29)

Let us now compute the Ricci tensor:

$$R^{\nu\alpha} = \partial_{\mu} \Gamma^{\alpha\mu\nu}_{va} - \partial_{\nu} \Gamma^{\alpha\mu\mu}_{va} + \Gamma^{\mu\nu}_{va} \Gamma^{\alpha\mu\rho}_{va} - \Gamma^{\mu\nu}_{\nu\rho} \Gamma^{\alpha\mu\rho}_{\mu\alpha},$$ (3.30)

with $\Gamma^{\mu\nu}_{va}$ given by Eq. (3.29). We have

$$R^{\nu\alpha}_{va} = \partial_{\mu} \left( \Gamma^{\mu\nu}_{va} + \Sigma^{\mu}_{va} + \Theta^{\mu}_{va} \right) +$$

$$- \partial_{\nu} \left( \Gamma^{\mu\mu}_{va} + \Sigma^{\mu}_{va} + \Theta^{\mu}_{va} \right) +$$

$$+ \left( \Gamma^{\mu\nu}_{va} + \Sigma^{\mu}_{va} + \Theta^{\mu}_{va} \right) \left( \Gamma^{\rho\mu}_{va} + \Sigma^{\rho}_{va} + \Theta^{\rho}_{va} \right) +$$

$$- \left( \Gamma^{\mu\nu}_{va} + \Sigma^{\mu}_{va} + \Theta^{\mu}_{va} \right) \left( \Gamma^{\rho\mu}_{va} + \Sigma^{\rho}_{va} + \Theta^{\rho}_{va} \right).$$ (3.31)

Neglecting the second order terms $\Sigma\Sigma$, $\Theta\Theta$, $\Sigma\Theta$ and $\Theta\Sigma$, and defining:

$$R^{(LTB)}_{va} = \partial_{\mu} \Gamma^{\mu\nu}_{va} - \partial_{\nu} \Gamma^{\mu\mu}_{va} +$$

$$+ \Gamma^{\mu\nu}_{va} \Gamma^{\rho\mu}_{va} - \Gamma^{\mu\nu}_{\nu\rho} \Gamma^{\rho\mu}_{\mu\alpha},$$ (3.32)

$$R^{(\Sigma)}_{va} = \partial_{\mu} \Sigma^{\mu}_{va} - \partial_{\nu} \Sigma^{\mu}_{va} + \Sigma^{\mu}_{\nu\rho} \Gamma^{\rho\mu}_{va} + \Gamma^{\mu\nu}_{va} \Sigma^{\rho}_{va} +$$

$$- \Gamma^{\mu\nu}_{va} \Sigma^{\rho}_{va} - \Sigma^{\rho}_{\nu\rho} \Gamma^{\rho\mu}_{va},$$ (3.33)

$$R^{(\Theta)}_{va} = \partial_{\mu} \Theta^{\mu}_{va} - \partial_{\nu} \Theta^{\mu}_{va} + \Theta^{\mu}_{\nu\rho} \Gamma^{\rho\mu}_{va} + \Gamma^{\mu\nu}_{va} \Theta^{\rho}_{va} +$$

$$- \Gamma^{\mu\nu}_{va} \Theta^{\rho}_{va} - \Theta^{\rho}_{\nu\rho} \Gamma^{\rho\mu}_{va},$$ (3.34)

we can then write the Ricci tensor, to first order in $\delta g$, as

$$R^{\nu\alpha}_{va} = R^{(LTB)}_{va} + R^{(\Sigma)}_{va} + R^{(\Theta)}_{va}.$$ (3.35)

Consider now the perturbed expression of the energy momentum tensor, that we model with an anisotropic energy density given in general by:

$$\rho_{\text{mat}}(r, t, \theta) = \rho_{\parallel\text{mat}}(t, r) \sin^2 \theta + \rho_{\perp\text{mat}}(t, r) \cos^2 \theta =$$

$$= \rho_{\parallel\text{mat}}(t, r) + \delta_{\text{mat}}(t, r) \cos^2 \theta,$$ (3.36)

where $\delta_{\text{mat}} \equiv \rho_{\perp\text{mat}} - \rho_{\parallel\text{mat}}$, where $\delta_{\text{mat}} \ll \rho_{\parallel\text{mat}}$. This means that we choose a perturbed density distribution with a planar symmetry, to be consistent
3.4. THE LUMINOSITY DISTANCE

with the anisotropic geometry we are considering. Also, we define the total source as

\[ T_{\mu}^{\nu} = T_{\mu}^{(LTB)} + \delta T_{\mu}^{(AN)}, \]

(3.37)

where \( T_{\mu}^{(LTB)} \) is the dust matter source of the LTB metric \( g_{\mu\nu}^{(LTB)} \), while \( \delta T_{\mu}^{(AN)} \) has only contributions from \( \delta \text{mat} \) (i.e. we are still using a pressure-less matter fluid, neglecting the effect of a different matter pressure along different directions). In any case, the particular form of matter perturbations is not crucial for the results of this chapter.

To be explicit, as in the case of the Ricci tensor, we have to first order

\[ T_{\nu\alpha} = (g_{\nu\mu}^{(LTB)} + \delta g_{\nu\mu}^{(AN)})(T_{\alpha}^{\mu}^{(LTB)} + \delta T_{\alpha}^{\mu}^{(AN)}) = \]

\[ \cong T_{\nu\alpha}^{(LTB)} + \delta g_{\nu\mu}^{(AN)}T_{\alpha}^{\mu}^{(LTB)} + g_{\nu\mu}^{(LTB)}\delta T_{\alpha}^{\mu}^{(AN)}, \]

(3.38)

and the Einstein equations become, for this model,

\[ R_{\nu\mu}^{(LTB)} + R_{\nu\mu}^{(LTB)}\delta g^{\nu\mu} + (R_{\mu\alpha}^{(\Sigma)} + R_{\mu\alpha}^{\Theta})g^{\nu\alpha}^{(LTB)} = \]

\[ 8\pi G \left[ T_{\nu}^{\mu}^{(LTB)} + \delta T_{\nu}^{\mu}^{(AN)} + \frac{1}{2} \delta_{\mu}^{\nu} (T^{(LTB)} + \delta T^{(AN)}) \right]. \]

(3.39)

The homogeneous, unperturbed part of the geometry satisfies the LTB equations \( R_{\mu}^{\nu(LTB)} = 8\pi G \left[ T_{\mu}^{\nu(LTB)} + \frac{1}{2} \delta_{\mu}^{\nu} T^{(LTB)} \right] \), while the other terms give us the contribution of the first-order anisotropic corrections, as it will be shown later.

3.4 The luminosity distance

In this section we will compute the luminosity distance \( d_L \) for a cosmological source embedded in the geometry described by the metric Eq. (3.13).

Let us start with the definition of \( d_L \) based on the already mentioned Etherington law:

\[ d_L = (1 + z)^2 d_A, \]

(3.40)

and on the differential relation

\[ d(\ln d_A) = \frac{1}{2} \nabla_{\alpha} p^{\alpha} d\lambda, \]

(3.41)

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3.4. THE LUMINOSITY DISTANCE

where $\lambda$ is a time-like affine parameter and $p^\alpha = dx^\alpha/d\lambda$ is the four-momentum of a signal emitted by the source (for instance, a Supernova) and reaching the observer. We need to compute the covariant divergence $\nabla_\alpha p^\alpha$:

$$\nabla_\alpha p^\alpha = \partial_\alpha p^\alpha + \Gamma^\alpha_{\beta\gamma} p^\beta = \partial_\alpha p^\alpha + \partial_\alpha \sqrt{-g} p^\alpha \sqrt{-g} p^\mu \Gamma^\alpha_{\beta\mu} p^\beta,$$

(3.42)

where

$$-g = A_1^2 (g_{11} g_{22} - g_{12}^2) \sin^2 \theta \equiv A_1^2 B^2(t, r, \theta) \sin^2 \theta,$$

(3.43)

and where we have defined

$$B(t, r, \theta) \equiv \sqrt{g_{11}(t, r, \theta) g_{22}(t, r, \theta) - g_{12}(t, r, \theta)}.$$

(3.44)

In this way we obtain:

$$\frac{\partial_0 \sqrt{-g}}{\sqrt{-g}} = \left(\frac{\partial_0 A_1 B + A_1 \partial_0 B}{A_1 B \sin \theta}\right) = \frac{\partial_0 A_1}{A_1} + \frac{\partial_0 B}{B},$$

$$\frac{\partial_1 \sqrt{-g}}{\sqrt{-g}} = \left(\frac{\partial_1 A_1 B + A_1 \partial_1 B}{A_1 B \sin \theta}\right) = \frac{\partial_1 A_1}{A_1} + \frac{\partial_1 B}{B},$$

(3.45)

and Eq. (3.42) can be rewritten as

$$\partial_0 p^0 + \partial_1 p^1 + \left(\frac{\partial_0 A_1}{A_1} + \frac{\partial_0 B}{B}\right) p^0 + \left(\frac{\partial_1 A_1}{A_1} + \frac{\partial_1 B}{B}\right) p^1 = \partial_0 p^0 + \partial_1 p^1 + \frac{1}{A_1} \frac{dA_1}{d\lambda} + \left(\frac{\partial_0 B}{B} p^0 + \frac{\partial_1 B}{B} p^1\right).$$

(3.46)

In the last equation we have used the general relation between partial derivatives with respect to the coordinates $x^\alpha$ and total derivatives with respect to $\lambda$. In fact, if $\Phi$ is a generic function of $x^\alpha(\lambda)$, we have:

$$\frac{d}{d\lambda} \Phi(x^\alpha(\tau)) = \frac{\partial}{\partial x^\beta} \frac{dx^\beta}{d\lambda} \equiv \partial_\beta \Phi p^\beta.$$  

(3.47)

For the function $B(r, t, \theta)$ we have, in general, $\frac{dB}{d\lambda} = \partial_0 B p^0 + \partial_1 B p^1 + \partial_2 B p^2$. Considering radial signals with $p^2 = d\theta/d\lambda = 0$ (and assuming that our parameter $\theta = \text{const}$ can appropriately characterize the signal propagation), we can write

$$B(r, t, \theta) \approx B(r(\lambda), t(\lambda), \theta) \Rightarrow \frac{dB}{d\lambda} \approx \partial_0 B p^0 + \partial_1 B p^1,$$

(3.48)
3.4. THE LUMINOSITY DISTANCE

and Eq. (3.42) becomes:

$$\nabla_\alpha p^\alpha = \partial_0 p^0 + \partial_1 p^1 + \frac{1}{A} \frac{dA}{d\lambda} + \frac{1}{B} \frac{dB}{d\lambda}. \quad (3.49)$$

Concerning the partial derivatives of $p^\alpha$ we now recall that, for the propagation along radial geodesics, we have

$$dp^0 + \Gamma_{00}^0 dx^0 p^0 + \Gamma_{10}^0 (dx^1 p^0 + dx^0 p^1) + \Gamma_{11}^0 dx^1 p^1 = 0$$
$$dp^1 + \Gamma_{00}^1 dx^0 p^0 + \Gamma_{10}^1 (dx^1 p^0 + dx^0 p^1) + \Gamma_{11}^1 dx^1 p^1 = 0, \quad (3.50)$$

from which:

$$\partial_0 p^0 = - (\Gamma_{00}^0 p^0 + \Gamma_{10}^0 p^1)$$
$$\partial_1 p^1 = - (\Gamma_{10}^1 p^0 + \Gamma_{11}^1 p^1). \quad (3.51)$$

In addition, for our metric, $\Gamma_{00}^0 = \Gamma_{10}^0 = 0$ and

$$\Gamma_{10}^1 = \frac{1}{2} g^{11} (\partial_1 g_{01} + \partial_0 g_{11} - \partial_t g_{10})$$
$$+ \frac{1}{2} g^{12} (\partial_1 g_{02} + \partial_0 g_{21} - \partial_2 g_{10})$$
$$= - (g^{11} X \partial_0 X + g^{12} F \partial_0 F)$$

$$\Gamma_{11}^1 = \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_t g_{11})$$
$$+ \frac{1}{2} g^{12} (\partial_1 g_{12} + \partial_1 g_{21} - \partial_2 g_{11})$$
$$= - (g^{11} X \partial_1 X + 2 g^{12} F \partial_1 F - X \partial_2 X), \quad (3.52)$$

where we have put

$$g_{11} \equiv X^2 \quad g_{22} \equiv Y^2 \quad g_{12} \equiv F^2. \quad (3.53)$$

In the limit of small anisotropy, where we can apply the first-order results of Eq (3.29), we have also:

$$\Gamma_{10}^1 \rightarrow \Gamma_{10}^{\text{LTB}} = \frac{\partial_0 \partial_1 A}{\partial_1 A}, \quad (3.54)$$
$$\Gamma_{11}^1 \rightarrow \Gamma_{11}^{\text{LTB}} = \frac{\partial_0^2 A}{\partial_1 A}, \quad (3.55)$$

where $\partial_0^2 \equiv \frac{\partial^2}{\partial r^2}$. We can thus express Eq.(3.49) as

$$\nabla_\alpha p^\alpha \approx - \left( \frac{\partial_0 \partial_1 A}{\partial_1 A} p^0 + \frac{\partial_0^2 A}{\partial_1 A} p^1 \right) + \frac{1}{A} \frac{dA}{d\lambda} + \frac{1}{B} \frac{dB}{d\lambda}. \quad (3.56)$$
3.5. THE DISTANCE IN TERMS OF THE REDSHIFT PARAMETER

where, taking into account Eq. (3.47), the first two terms of this equation can also be rewritten as \((\partial_1 A_\parallel)^{-1} d(\partial_1 A_\parallel)/d\lambda\).

By inserting this result into the initial Eq. (3.41) we are led to

\[
d(ln d_A) \approx \frac{1}{2} \left( \frac{1}{A_\parallel} \frac{dA_\parallel}{d\lambda} + \frac{1}{B} \frac{dB}{d\lambda} - \frac{1}{\partial_1 A_\parallel} \frac{d\partial_1 A_\parallel}{d\lambda} \right) d\lambda,
\]

(3.57)

and we finally obtain the angular distance \(d_A\) as:

\[
d_A(t, r, \theta) = \sqrt{A_\parallel(t, r) B(r, t, \theta) \frac{\partial_1 A_\parallel}{\partial_1 A_\parallel(t, r)}}.
\]

(3.58)

We can easily recover, in the isotropic limit, the LTB result of Chapter 2. In this limit, in fact, we have

\[
A_\parallel(t, r) \rightarrow A(t, r), \quad B(t, r, \theta) \rightarrow A(t, r) \partial_1 A_\parallel(t, r)
\]

(3.59)

and

\[
d_A(t, r, \theta) \rightarrow d_A^{(LTB)}(t, r) = A_\parallel(t, r).
\]

(3.60)

3.5 The distance in terms of the redshift parameter

In order to prepare a comparison with the experimental data, in this section we will discuss how to express the luminosity distance (3.58) completely in terms of the redshift parameter \(z\). Looking at the explicit form of \(d_L\),

\[
d_L = (1 + z)^2 \sqrt{\frac{A_\parallel(t, r) B(r, t, \theta)}{\partial_1 A_\parallel(t, r)}},
\]

(3.61)

it is clear that we need explicit (or at least implicit) relations for \(r(z)\) e \(t(z)\).

To this end let us consider (as already done in Chapter 2) the transmission of two light-like signals emitted by a static source at the times \(t\) and \(t + \delta t\), and received by a static observe after propagation along a null radial geodesic parametrized by the affine coordinate \(\lambda\). By differentiating
3.5. THE DISTANCE IN TERMS OF THE REDSHIFT PARAMETER

the redshift definition $1 + z(\lambda) = \delta t_o/\delta t(\lambda)$ one then obtains (see also Eq. (2.51) and Eq. (2.61))

$$\frac{dz(\lambda)}{d\lambda} = -\frac{\delta t_o}{\delta t(\tau)^2} \frac{d\delta t(\lambda)}{d\lambda} \equiv -\frac{1 + z(\lambda)}{\delta t(\lambda)} \frac{d\delta t(\lambda)}{d\lambda},$$  \hspace{1cm} (3.62)

from which:

$$\frac{d\delta t}{d\lambda} = -\frac{\delta t}{1 + z} \frac{dz}{d\lambda}. \hspace{1cm} (3.63)$$

On the other hand, along the considered trajectory ($ds = d\theta = d\phi = 0$) we have

$$dt^2 - X(r, t, \theta)^2 dr^2 = 0 \Rightarrow dt = -X(r, t, \theta) dr$$  \hspace{1cm} (3.64)
(recall the definition of the metric component $X = g_{11}$ given in Eq. (3.53)) so that:

$$\frac{dt}{d\lambda} = -X(t, r, \theta) \frac{dr}{d\lambda}$$  \hspace{1cm} (3.65)

$$\frac{d(t + \delta t)}{d\lambda} = -X(t + \delta t, r, \theta) \frac{dr}{d\lambda}.$$  \hspace{1cm} (3.66)

Expanding the above equation,

$$\frac{dt}{d\lambda} + \frac{d\delta t}{d\lambda} \approx -[X(r, t, \theta) + \delta t \partial_0 X(r, t, \theta)] \frac{dr}{d\lambda} \hspace{1cm} (3.67)$$

we are led to:

$$\frac{d\delta t}{d\lambda} \approx -\delta t \partial_0 X(t, r, \theta) \frac{dr}{d\lambda} = -\delta t \partial_0 X(t, r, \theta) \frac{dr}{dz} \frac{dz}{d\lambda}. \hspace{1cm} (3.68)$$

A comparison of Eqs. (3.63) and (3.68) thus provides the sought differential relation for $r(z)$:

$$\frac{dr}{dz} = \frac{1}{1 + z} \frac{1}{\partial_0 X(r, t, \theta)}. \hspace{1cm} (3.69)$$

In order to obtain a similar equation for $t(z)$, it is important to note that:

$$\frac{dt}{d\lambda} = \frac{dt}{dz} \frac{dz}{d\lambda} \quad \text{and} \quad \frac{dr}{d\lambda} = \frac{dr}{dz} \frac{dz}{d\lambda}. \hspace{1cm} (3.70)$$

In this way, by taking into account Eqs. (3.65) and (3.69) we obtain:

$$\frac{dt}{dz} = -\frac{1}{1 + z} \frac{X(r, t, \theta)}{\partial_0 X(r, t, \theta)}. \hspace{1cm} (3.71)$$

Putting all together, we can finally write a system of differential equations for the luminosity distance as a function of the redshift $z$ and of the
3.6. COMPARISON WITH EXPERIMENTAL DATA

The index $\theta$ is here to recall explicitly that the functions $r$ and $t$ are obtained by solving the above system of differential equations with an angle $\theta$ which is fixed, and taken as a constant parameter typical of the given light ray trajectory. We stress that this angular dependence does not disappear (unlike in the case of the LTB metric) because of the (small, but non-vanishing) anisotropy present in the background geometry that we are considering.

3.6 Comparison with experimental data

Let us now consider a possible comparison of the previous theoretical results with the existing experimental observations, in particular with the Union 2 data set concerning Supernovae of type Ia.

Assuming that the (small) axial anisotropy is superimposed to the the CDM dominated, spatially flat, LTB geometry already discussed in Chapter 2, we shall consider a LTB-Bianchi-I model described by the metric (3.9) with

\[
A_{\parallel}(t,r) = r \left( 1 + \frac{3}{2} H_{\parallel}(r) t \right)^{\frac{3}{2}},
\]

(3.73a)

\[
A_{\perp}(t,r) = r \left( 1 + \frac{3}{2} H_{\perp}(r) t \right)^{\frac{3}{2}}.
\]

(3.73b)

By analogy with the example illustrated in Sect. 2.5, we will assume the following parametrization of the radial inhomogeneity:

\[
H_{\parallel/\perp}(r) = H_{\parallel/\perp} + \Delta H_{\parallel/\perp} \exp \left( -\frac{r}{r_{\parallel/\perp}} \right).
\]

(3.74)
3.6. COMPARISON WITH EXPERIMENTAL DATA

The LTB model is thus recovered in the limit $H_\| = H_\perp$, $\Delta H_\| = \Delta H_\perp$ and $r_\| = r_\perp$.

On these parameters we have to impose the constraint derived from the present, local value ($t = 0$ and $r = 0$) of the Hubble constant, namely $67.3 \pm 1.2 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [70]. This gives $H_\| + \Delta H_\| = H_\perp + \Delta H_\perp = 67.3$, so that we have:

$$H_\| = 67.3 - \Delta H_\|$$

(3.75a)

$$H_\perp = 67.3 - \Delta H_\perp.$$  

(3.75b)

This means that the six parameters of our model are not all independent, but we have only four independent parameters: $\Delta H_\|$, $\Delta H_\perp$, $r_\|$ and $r_\perp$.

In addition, we have to remember that the anisotropy of our model is a first-order perturbation, and is described by the small quantity $\epsilon(t, r)$ satisfying the inequalities (3.23). In the limit $\epsilon \sim 0$, $\epsilon' \sim 0$, we have

$$A_\perp(t, r) \approx A_\| (t, r) \Rightarrow H_\| (r) \approx H_\perp (r),$$

(3.76)

and it can be shown that

$$\frac{\Delta H_\perp}{\Delta H_\|} \approx \frac{r_\perp}{r_\|} \equiv \alpha.$$  

(3.77)

It follows that we can safely restrict to a class of models characterized – as before – by three physical parameters only: for instance, $\Delta H_\|$, $r_\|$ and $\alpha$. The two parameters $\Delta H_\|$ and $r_\|$ can be varied in an arbitrary range of values, taking into account, however, that $\alpha \simeq 1$.

We have then numerically integrated the system of equations (3.72), and performed a best fit analysis of the experimental points of the Union 2 data set, by varying $\alpha$ in the range $[0.9, 1]$ (and averaging over a discrete set of angular directions). The best fit values of the parameters are shown in the following table, and correspond to $\chi^2$/d.o.f. = 0.95.

<table>
<thead>
<tr>
<th>$\Delta H_|$ [km (s Mpc)$^{-1}$]</th>
<th>$r_|$ [Gpc]</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.4</td>
<td>2.88</td>
<td>1.1</td>
</tr>
</tbody>
</table>

In Fig. 3.1 we have shown the Hubble diagram for the 557 Supernovae Ia of the UNION 2 catalog, and the best fit curve for the simple anisotropic and inhomogeneous model discussed in this section.
Figure 3.1: The best-fit curve for a three-parameter LTB-Bianchi-I model with axial anisotropy. Points are from the 557 Supernovae Ia of the Union2 catalog.
3.7 Conclusion

In this chapter we have studied simple examples of inhomogeneous and anisotropic backgrounds, and their possible compatibility with present large scale observations. The LTB-Bianchi-I models we have considered display important physical properties and, at the same time, are not too complicated, and can easily produce testable predictions.

We have found that this class of models can fit present Supernovae data at an acceptable level of statistical accuracy. This means that the large scale inhomogeneities and anisotropies may generate dynamical effects simulating dark energy, thus removing the need for a cosmological constant.

These models can also be intended as a first step towards more general and more realistic cases. For instance, models with “Swiss-cheese” or fractal inhomogeneities, and models with larger, non perturbative anisotropies. Also, these models should be compared with a larger class of experimental data.

To this purpose, we will introduce in the following chapters a more sophisticated system of coordinates, adapted to describe the propagation and the observation of signals traveling along the past light-cone in generally inhomogeneous and anisotropic cosmological backgrounds.
3.7. CONCLUSION
Chapter 4

Exact results in general inhomogeneous backgrounds

Most of the observables involved in cosmological measurements are directly related to photons: for instance, the radiation flux emitted by astrophysical sources, their redshift, the angular size of compact objects, and even the CMB temperature and polarizations. This is due to the fact that only photons are stable enough to travel along cosmological distances, no matter how much the source is far from the observer (and remote in time).

On the other hand, during their very long travel, photons interact with all kind of structures, so that they can directly “feel” the details of the matter distribution, and retain the imprint of the inhomogeneities possibly present on the largest scales accessible to cosmological observations. It should be recalled, in this connection, that the forthcoming era of observational cosmology will be characterized by a level of experimental accuracy for large-scale measurements much higher than the present one [71, 72].

It becomes important, in this context, to adopt a more powerful formalism able to evaluate the physical observables directly taking into account the photon propagation on the past light-cone of the observer, and automatically including all present homogeneities and anisotropies in a non perturbative way. This project was recently started and interesting results have already been achieved [73, 74, 75, 38, 39], with applications also to non linear effects (see, for instance [76, 77]).

In the following sections we will introduce such a formalism and we will show how it simplifies the computation of several observables. In particular, we will provide exact expressions for redshift, angular distance and weak lensing quantities, following [78, 79].
4.1 Geodesic Light-Cone coordinates

Let us introduce a new set of coordinates \( x^\mu = \{ \tau, w, \tilde{\theta}^a \} \), with \( a = 1, 2 \), where \( \tau \) is a time-like parameter which can be identified with the proper time of a static observer in the synchronous gauge, \( w \) is a null coordinate which is constant on the past light-cone hypersurface, and \( \tilde{\theta}^a \) are two angular coordinates.

Using these coordinates, we can write the line-element of the so-called Geodesic Light-Cone (GLC) gauge as:

\[
\begin{align*}
\text{ds}^2 &= \Upsilon^2 \text{dw}^2 - 2\Upsilon \text{dwd}\tau + \gamma_{ab} \left( \text{d}\tilde{\theta}^a - U^a \text{dw} \right) \left( \text{d}\tilde{\theta}^b - U^b \text{dw} \right) \\
\end{align*}
\]

where \( a, b = 1, 2 \). Our metric components are thus parametrized by the function \( \Upsilon \), by the two-dimensional vector \( U^a \) and by \( \gamma_{ab} \), a \( 2 \times 2 \) symmetric tensor. We have six arbitrary functions, depending in principle on all coordinates: this is because the additional four total degrees of freedom of the metric have been fixed by this gauge choice. Except for this gauge fixing, the metric is completely general and can exactly describe the geometry of arbitrarily inhomogeneous and anisotropic space-time manifolds.

It is important to stress, for later use, that with the above choice of coordinates we are still left with a residual gauge freedom, corresponding to transformations acting on a subset of coordinates. For instance we can define new angles as follows, by leaving \( \tau \) and \( w \) unchanged:

\[
\tau \rightarrow \tau; \quad w \rightarrow w; \quad \tilde{\theta}^a \rightarrow \tilde{\theta}^a(\tilde{\theta}^b, w).
\]

With this transformation, \( U^a \) and \( \gamma_{ab} \) are changed, but \( \Upsilon \) is not. All the null metric components keep vanishing, and the GLC gauge is then preserved. Let us notice that the above residual gauge fixing is independent of \( \tau \): this means that such a transformation redefines angles at all times. Comparing with another well known case in perturbation theory we note that a residual gauge freedom is also present in the synchronous gauge for the three spatial coordinates, but, in that case, the coordinate redefinition has to be done at a given time, so that it holds only at a specific point.

It will be useful, for our application, to write the metric and its inverse in a matrix form:

\[
\begin{align*}
g_{\mu\nu} &= 
\begin{pmatrix}
0 & -\Upsilon & \tilde{\theta} \\
-\Upsilon & \Upsilon^2 + U^2 & -U_b \\
\tilde{\theta}^T & -U^T_a & \gamma_{ab}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
g^{\mu\nu} &= 
\begin{pmatrix}
-1 & -\Upsilon^{-1} & -U^b/\Upsilon \\
-\Upsilon^{-1} & 0 & \tilde{\theta} \\
-(U^a)^T/\Upsilon & \tilde{\theta}^T & \gamma^{ab}
\end{pmatrix}
\end{align*}
\]

where the superscript \( T \) refers to the transposed vector and \( \tilde{\theta} = \{0, 0\} \).
4.1. GEODESIC LIGHT-CONE COORDINATES

For a better understanding of the geometrical meaning of these coordinates we may note that $\tau$ and $w$ have two fundamental properties. Indeed, let us consider the hypersurface defined by $w = \text{constant}$. We have $\partial_\mu w = \delta_\mu^w$, therefore:

$$\partial^\mu w \partial_\mu w = g^{\mu\nu} \partial_\nu w \partial_\mu w = g^{ww} = 0. \quad (4.4)$$

This means that this is a null hypersurface (which, as already stressed, corresponds to the past light-cone of an arbitrarily chosen observer). On the other hand, let us consider the vector $u_\mu = -\partial_\mu \tau = -\delta_\mu^\tau$. We can prove that it defines a geodesic flow, since $(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) = 0$. Indeed:

$$(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) = g^{\nu\alpha} \partial_\nu \delta_\alpha^\tau \nabla_\nu \delta_\mu^\tau = -g^{\nu\alpha} \Gamma_{\nu\mu}^\tau$$

because $g^{\tau\nu} g^{\nu\tau} = g^{(\nu \rho)\tau}$. However, using, the explicit form of the GLC metric:

$$g^{\nu\tau} g^{\nu\tau} \partial_\mu g_{\rho\nu} = (g^{\tau\nu})^2 \partial_\mu g_{\nu\nu} + g^{\tau\nu} g^{\nu\tau} \partial_\mu g_{\tau\alpha}$$

$$= \frac{\partial_\mu U^2}{\Upsilon^2} + \frac{U^a \partial_\mu U^b}{\Upsilon^2} - 2 \frac{U^a \partial_\mu U_a}{\Upsilon^2} = 0. \quad (4.6)$$

Let us finally discuss the four-momentum $k^\mu$ of a photon (or of a light-like signal), and show that a photon traveling at constant $w$ and $\tilde{\theta}_a$ is a viable solution of the null geodesic equations.

Consider the vector $k^\mu = k_0 \delta_\mu^\tau$, which is light-like, since:

$$g_{\mu\nu} k^\mu k^\nu = g_{\tau\tau} k_0^2 = 0, \quad (4.7)$$

and impose the geodesic equation, namely:

$$k^\nu \partial_\nu k^\mu + \Gamma_{\alpha\beta}^\mu k^\alpha k^\beta = 0. \quad (4.8)$$

Noticing that $\Gamma_{\tau\tau}^\mu = \Upsilon^{-1} \partial_\tau \Upsilon \delta_\mu^\tau$, we have:

$$0 = \delta_\mu^\nu k_0 \partial_\tau k_0 + \Gamma_{\tau\tau}^\mu k_0^2 = k_0^2 \delta_\tau^\mu \left( \frac{\partial_\tau k_0}{k_0} + \frac{\partial_\tau \Upsilon}{\Upsilon} \right). \quad (4.9)$$

This is always satisfied provided $k_0 = \omega \Upsilon^{-1}$, with $\omega$ an arbitrary constant. From now on, we will use the result $k^\mu = \omega \Upsilon^{-1} \delta_\mu^\tau$. 83
The importance of using a set of coordinates adapted to the past light-cone of the observer can be hardly overestimated. For instance, as shown in [75], this choice greatly simplifies the exact computation of well-defined and gauge invariant averages of physical observables on the light-cone, exactly where such observables are defined.

4.2 Einstein’s equations in the GLC gauge

Due to its high level of generality, the GLC gauge provides an interesting and useful tool for applications to cosmology (and, more generally, to gravitational physics) [39]. However, just because of its generality, it may be difficult, in practice, to perform exact computations. Indeed, we have to solve a system of second-order coupled non-linear partial differential equations to evaluate all the six functions of this gauge. And this is a non trivial task, even numerically.

We present here a very simple, but illustrative, example for the case in which we limit ourselves to a homogeneous and isotropic geometry. In spite of its pedagogical purpose, it is the first time (to the best of our knowledge) that the Einstein equations are explicitly given on the light-cone. This represents a crucial step towards a non-perturbative computation of the backreaction. Also, having well defined background solutions provides a good starting point for developing a perturbative approach based on light-cone variables.

As is well known, the Einstein equations can be written in two different ways

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}
\]

\[
R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)
\]

(4.10)

where \( R_{\mu\nu} \) is the Ricci tensor, \( R \) is the scalar curvature and \( T_{\mu\nu} \) is the energy-momentum tensor, with \( T \) its trace. The general expression for the Ricci tensor is:

\[
R_{\mu\nu} = \frac{\partial_{\alpha} \left( \sqrt{-g} \Gamma_{\mu\nu}^{\alpha} \right)}{\sqrt{-g}} - \frac{\partial_{\mu} \partial_{\nu} \sqrt{-g}}{\sqrt{-g}} + \frac{\partial_{\mu} \sqrt{-g}}{\sqrt{-g}} \frac{\partial_{\nu} \sqrt{-g}}{\sqrt{-g}} - \Gamma_{\mu\rho}^{\alpha} \Gamma_{\alpha\nu}^{\rho}.
\]

(4.11)
4.2. EINSTEIN’S EQUATIONS IN THE GLC GAUGE

In the GLC gauge we have \( \sqrt{-g} = \Upsilon \sqrt{\gamma} \) and we obtain:

\[
\sqrt{-g} \frac{\partial_{\nu} \sqrt{-g}}{\sqrt{-g}} - \frac{\partial_{\nu} \sqrt{-g}}{\sqrt{-g}} = \frac{\partial_{\mu} \Upsilon \partial_{\gamma} \Upsilon}{\Upsilon} - \frac{\partial_{\mu} \partial_{\gamma} \Upsilon}{\Upsilon} + \frac{1}{2} \frac{\partial_{\mu} \gamma \partial_{\nu} \gamma}{\gamma} - \frac{1}{2} \frac{\partial_{\mu} \partial_{\nu} \gamma}{\gamma}. \tag{4.12}
\]

It follows that

\[
R_{\mu\nu} = \frac{\partial_{\alpha} \left( \Upsilon \sqrt{\gamma} \Gamma_{\mu\nu}^{\alpha} \right)}{\Upsilon \sqrt{\gamma}} + \frac{\partial_{\mu} \Upsilon \partial_{\nu} \Upsilon}{\Upsilon} - \frac{\partial_{\mu} \partial_{\nu} \Upsilon}{\Upsilon} + \frac{1}{2} \frac{\partial_{\mu} \gamma \partial_{\nu} \gamma}{\gamma} - \frac{1}{2} \frac{\partial_{\mu} \partial_{\nu} \gamma}{\gamma} - \Gamma_{\mu\nu}^{\alpha \mu} \Gamma_{\rho}^{\alpha \rho}. \tag{4.13}
\]

The expression for the Christoffel symbols is given in Appendix A. The above terms are very complicated, and the simplest component of the Einstein equations is the \( \tau\tau \) component (because of \( g_{\tau\tau} = 0 \)), which is given by

\[
\frac{1}{2} \left[ \frac{\left( \partial_{\tau} \gamma \right)^{2}}{\gamma} - \frac{\partial_{\tau}^{2} \gamma}{\gamma} + \frac{\partial_{\tau} \gamma \partial_{\tau} \Upsilon}{\Upsilon} - \frac{1}{2} \gamma_{ab} \partial_{\tau} \gamma_{bc} \partial_{\tau} \gamma_{da} \right] = 8\pi G T_{\tau\tau}. \tag{4.14}
\]

The other components are much harder (and much longer) to be written explicitly, because of the additional contribution due to the scalar curvature. As anticipated, we will thus concentrate on the simplest cosmological model corresponding to the case of the FLRW geometry.

This case is defined by

\[
U^a = 0, \quad \Upsilon = \Upsilon(\tau) \quad \gamma_{ab} = \Upsilon^{2}(\tau) \left\{ \sin \left[ \sqrt{k} \left( w - \int \frac{d\tau}{\Upsilon(\tau)} \right) \right] \right\}^{2} \text{diag} \left( 1, \sin^{2} \theta^{1} \right)
\]

\[
= \Upsilon(\tau) \chi_{\tau}(\tau, w) \text{diag} \left( 1, \sin^{2} \theta^{1} \right) \tag{4.15}
\]

where \( k \) represents the constant curvature of the maximally symmetric spatial hypersurfaces. We can easily convince ourselves that the above choice of the GLC functions corresponds to the FLRW geometry (with a scale factor \( a(t) = \Upsilon(\tau) \)) by considering the coordinate transformations:

\[
t = \tau \quad , \quad r = w - \int \frac{d\tau}{\Upsilon(\tau)}. \tag{4.16}
\]

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4.2. EINSTEIN’S EQUATIONS IN THE GLC GAUGE

With the above choice we obtain that the only two independent Einstein equations are:

\[
2 \left[ \left( \frac{\partial_\tau \Upsilon}{\Upsilon} \right)^2 - \frac{\partial^2_\tau \Upsilon}{\Upsilon^2} + \frac{k}{\Upsilon^2} \right] = 8\pi G T_{\tau\tau}
\]
\[-2 \Upsilon \partial_\tau^2 \Upsilon - \partial_\tau \Upsilon^2 \right] = 8\pi G T_{\mu\nu}.
\]

(4.17)

and these simple equations can be solved in various particular cases. For instance, we can consider a perfect barotropic fluid source, with equation of state \( p = n\rho \), and with

\[
T_{\mu\nu} = \rho(\tau) [(1 + n) u_\mu u_\nu + n g_{\mu\nu}] = \rho(\tau) \hat{T}_{\mu\nu}.
\]

(4.18)

The explicit matrix form of the energy momentum tensor is given by:

\[
T_{\mu\nu} = \rho(\tau) \begin{pmatrix}
1 + n & -n \Upsilon & 0 & 0 \\
-n \Upsilon & n \Upsilon^2 & 0 & 0 \\
0 & 0 & n \Upsilon^2 \chi_k^2 & 0 \\
0 & 0 & 0 & n \Upsilon^2 \chi_k^2 \sin^2 \tilde{\theta}^1
\end{pmatrix}
\]

(4.19)

\[
T^{\mu\nu} = \rho(\tau) \begin{pmatrix}
1 & \Upsilon^{-1} & 0 & 0 \\
\Upsilon^{-1} & \frac{1+n}{\Upsilon^2} & 0 & 0 \\
0 & 0 & n \Upsilon^{-2} \chi_k^{-2} & 0 \\
0 & 0 & 0 & n \Upsilon^{-2} \chi_k^{-2} \sin^{-2} \tilde{\theta}^1
\end{pmatrix}
\]

(4.20)

and the Bianchi identity implies

\[
\frac{\partial_\tau \rho}{\rho} = -3(1 + n) \frac{\partial_\tau \Upsilon}{\Upsilon},
\]

(4.21)

so that \( \rho = \rho_0 (\Upsilon / \Upsilon_0)^{-3(1+n)} \).

This gives us the same solutions as those we usually obtain for the FLRW models (as obvious from our approach). In particular, we have \( \Upsilon \sim \tau^{2/3} \) for the spatially flat, matter dominated model, and \( \Upsilon \sim \exp \alpha \tau \) for a model dominated by a cosmological constant (\( \Lambda \)), with \( \alpha = \sqrt{8\pi G \Lambda / 3} \). This time dependence can be easily understood since the GLC time \( \tau \) exactly corresponds to the synchronous time \( t \) of the FLR geometry.
4.3 The Jacobi map and the angular distance

Let us now introduce a standard geometric method, known with the name of “Jacobi map”, useful to describe the propagation of a light beam in curved space.

To this purpose, instead of considering a single photon of momentum \( k^\mu \), let us take two nearby light-like trajectories \( x^\mu_1(\lambda) \) and \( x^\mu_2(\lambda) \), separated by an (infinitesimal) displacement vector \( \xi^\mu(\lambda) \equiv x^\mu_1 - x^\mu_2 \). As is well known, the evolution of such a vector is governed by the equation of geodesic deviation:

\[
\nabla^2 \xi^\mu = R^\lambda_\alpha^\mu_\beta^\nu k^\alpha k^\nu \xi^\beta,
\]

where \( R^\lambda_\alpha^\mu_\beta^\nu \) is the Riemann tensor, and \( \nabla_\lambda = k^\alpha \nabla_\alpha \). It is clear that solving this equation gives a lot of information about a generic bundle of light rays emitted from the source towards the observer. Let us show in detail how this information can be translated to the physical observables of our interest.

The variable \( \xi^\mu \) is a four vector, but the equation for the component of \( \xi^\mu \) aligned along \( k^\mu \) (which obeys the null orthogonality conditions) is trivially satisfied, so that we are left with only two components of \( \xi^\mu \) satisfying non-trivial equations. Without any lack of generality we can thus project \( \xi^\mu \) along the so-called Sachs basis \( \{ s^\mu_A \} \) \([80, 81]\), given by two parallely transported four-vectors \( s^\mu_A \) (where \( A = 1, 2 \) are flat indices) defined by the conditions \([82, 83]\):

\[
g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB}, \tag{4.23}
\]

\[
s^\mu_A u_\mu = 0 \tag{4.24}
\]

\[
s^\mu_A k_\mu = 0 \tag{4.25}
\]

\[
\Pi^\mu_\nu \nabla_\lambda s^\nu_A = 0 \quad \text{with} \quad \Pi^\mu_\nu = \delta^\mu_\nu - \frac{k^\mu k_\nu}{(u^\alpha k_\alpha)^2} - \frac{k^\mu u_\nu + u^\nu k_\mu}{u^\alpha k_\alpha}. \tag{4.26}
\]

Here \( \Pi^\mu_\nu \) is a projector on the two-dimensional space orthogonal to \( u_\mu \) (the four velocity of the given observer) and to \( n_\mu = u_\mu + (u^\alpha k_\alpha)^{-1}k_\mu \). This last vector satisfies \( n^\alpha n_\alpha = 1 \) and \( n^\alpha k_\alpha = 0 \). Indeed:

\[
n^\mu n_\mu = u^\mu u_\mu + \frac{k^\mu k_\mu}{(u^\alpha k_\alpha)^2} + 2 = 1
\]

\[
n_\mu k^\mu = u_\mu k^\mu + (u^\alpha k_\alpha)^{-1} k_\mu k^\mu = 0. \tag{4.27}
\]

Referring to this basis, and neglecting the terms proportional to \( k^\mu \), we can write the displacement vector in the form

\[
\xi^\mu = h^A s^\mu_A, \tag{4.28}
\]

and, by projecting Eq. (4.22) along \( s^\mu_A \), we obtain:

\[
\frac{d^2 \xi^A}{d\lambda^2} = R^A_B s^B \tag{4.29}
\]
where we have defined $d/d\lambda \equiv k^\mu \partial_\mu$ and $R_B^A \equiv R_{\alpha \beta \mu \nu} k^\alpha k^\nu s_B^\beta s_A^\mu$.

The great advantage of this approach is the reduction of a four-dimensional differential equation on a generic curved manifold to a two-dimensional one on a flat subspace. We can still manipulate Eq. (4.29) by defining:

$$\xi^A(\lambda_s) \equiv J^A_B(\lambda_s, \lambda_o) \left( \frac{k^\mu \partial_\mu \xi^B}{k^\nu u_\nu} \right)_o,$$

where $o$ and $s$ refer, respectively, to observer and source position. Since the last factor of the above equation is a constant (evaluated at the observer position), we can rewrite Eq. (4.29) as:

$$\frac{d^2}{d\lambda^2} J^A_B(\lambda, \lambda_o) = R^A_C J^C_B,$$

(4.31)

$J^A_B$ is a $2 \times 2$ matrix usually known as the Jacobi map. Its theoretical as well as observational importance is crucially due to the fact that this matrix is directly related to the angular distance by the simple equation

$$\det J^A_B(\lambda_s, \lambda_o) = d_A^2.$$

(4.32)

Let us recall, in particular, that the angular distance is defined by $d_A^2 = dS_s/d\Omega_o$, where $d\Omega_o$ is the infinitesimal solid angle subtending the source at the observer position, and $dS_s$ is the cross-sectional area element perpendicular to the light ray at the source position. Once Eq. (4.31) has been solved, we are thus in the position of obtaining theoretical predictions to be compared with observations.

In order to solve the equation for the Jacobi map we have to provide appropriate initial conditions, given by

$$J^A_B(\lambda_o, \lambda_o) = 0 \quad \text{and} \quad \frac{d}{d\lambda} J^A_B(\lambda_o, \lambda_o) = \delta^A_B (k^\nu u_\nu)_o.$$

(4.33)

These conditions can be easily understood by noting that $J = 0$ at the observer position implies that the corresponding separation vector $\xi^A$ is vanishing, namely that the photon trajectories converge to that point. The second condition imposes just the correct normalization that we have to choose to be consistent with (4.30).

The Jacobi map can be related to other important quantities for the study of light-ray propagation in inhomogeneous geometries.

Let us note, to this purpose, that the second-order Eq. (4.31) can be rewritten as a system of two first-order differential equations. In fact, by introducing the so-called “deformation matrix” $S^A_B \equiv \frac{dJ^A_B}{d\lambda} (J^{-1})^C_B$, we have:

$$R^A_C J^C_B = \frac{d}{d\lambda} \frac{dJ^A_B}{d\lambda} = \frac{dS^A_C}{d\lambda} J^C_B + S^C_B \frac{dJ^A_C}{d\lambda}$$

$$= \frac{dS^A_C}{d\lambda} J^C_B + S^A_C S^D_B J^D_B.$$

(4.34)
4.3. THE JACOBI MAP AND THE ANGULAR DISTANCE

so that, after renaming indices, we find:

\[ \frac{dJ^A}{d\lambda} = S^C_A J^C_B, \quad \frac{dS^A}{d\lambda} + S^C_A S^C_B = R^A_B. \] (4.35)

On the other hand, the matrix \( S \) can be parametrized as [84]:

\[ S^A_B = \hat{\theta} \delta^A_B + \left( \begin{array}{cc} \hat{\sigma}_1 & \hat{\sigma}_2 \\ \hat{\sigma}_2 & -\hat{\sigma}_1 \end{array} \right). \] (4.36)

As a consequence, the second equation Eq. (4.35) can be decomposed into the so-called Sachs equations by considering its trace and trace-free part:

\[ \frac{d\hat{\theta}}{d\lambda} + |\hat{\sigma}|^2 + \hat{\theta}^2 = \frac{1}{2} \text{tr} R^A_B = \Phi_{00}, \] (4.37)
\[ \frac{d\hat{\sigma}}{d\lambda} + 2\hat{\theta} \hat{\sigma} = \Psi_0. \] (4.38)

Here we have used the relation \( \text{tr} (S^A_C S^C_B) = 2 (\hat{\theta}^2 + |\hat{\sigma}|^2) \), where \( \hat{\sigma} \equiv \hat{\sigma}_1 + i\hat{\sigma}_2 \), and we have introduced the quantities \( \Phi_{00} \) and \( \Psi_0 \) called, respectively, Ricci focusing and Weyl focusing, defined by the decomposition

\[ R^A_B = \Phi_{00} \delta^A_B + \begin{pmatrix} \text{Re} \Psi_0 & \text{Im} \Psi_0 \\ \text{Im} \Psi_0 & -\text{Re} \Psi_0 \end{pmatrix}. \] (4.39)

Let us notice, for later use, that \( \hat{\theta} \) and \( \hat{\sigma} \) are related to the so-called optical scalars [84] by

\[ \hat{\theta} \equiv \frac{1}{2} \nabla_{\mu} k^{\mu} \quad (\text{expansion scalar}), \] (4.40)
\[ |\hat{\sigma}|^2 \equiv \frac{1}{2} \nabla_{\mu} k^{\nu} \nabla^\mu k^{\nu} - \hat{\theta}^2 \quad (\text{shear scalar}), \] (4.41)

where \( k^{\mu} \) is the 4-momentum of the considered light rays. Also, let us recall the well-known relation between the Riemann and the Weyl tensor:

\[ C_{\alpha \beta \mu \nu} \equiv R_{\alpha \beta \mu \nu} - g_{\alpha \mu} R_{\beta \nu} + g_{\beta \nu} R_{\alpha \mu} + \frac{1}{3} R g_{\alpha \mu} g_{\beta \nu}. \] (4.42)

By using the properties of the Sachs basis, Eqs. (4.23)-(4.26), and the condition \( k^{\alpha} k_\alpha = 0 \), we obtain

\[ R^A_B \equiv R_{\alpha \beta \mu \nu} k^{\alpha} k^{\mu} S^\beta_A S^\nu_B = -\frac{1}{2} R_{\mu \nu} k^{\alpha} k^{\mu} \delta^A_B + C_{\alpha \beta \mu \nu} k^{\alpha} k^{\mu} s^\beta_A s^\nu_B. \] (4.43)

It follows that

\[ \Phi_{00} = -\frac{1}{2} R_{\alpha \beta} k^{\alpha} k^{\beta}, \quad \Psi_0 = \frac{1}{2} C_{\alpha \beta \mu \nu} k^{\mu} \Sigma^{\beta} \Sigma^{\nu}, \] (4.44)

where \( \Sigma^{\mu} \equiv s^{\mu}_A + i s^{A}_B \). Using the Einstein equations we then obtain a direct connection between the Ricci focusing \( \Phi_{00} \) and the matter distribution which can be written as \( R_{\mu \nu} k^{\mu} k^{\nu} = 8\pi G T_{\mu \nu} k^{\mu} k^{\nu} \) (we have used \( k^{\mu} k_\mu = 0 \)).
4.4 Weak lensing: the general formalism

The Jacobi map is a useful formalism for many applications concerning light propagation and geometric optics in curved space-times. In this section we will consider, in particular, the so-called effects of “weak lensing”.

It well known that light, because of its energy content, interacts gravitationally with matter. Hence, the light ray trajectories are deflected by the cosmic gravitational fields and, in principle, the deviation from straight propagation can be arbitrarily large. This is known as as effect of “gravitational lensing”. In principle, such an effect can be strong enough to produce intersection among different light-cones, and the points where this intersections occurs are called “caustics”. In the context of this work, however, we will assume that this possibility does not occur, and we will consider small deflections only, working in the approximation of weak gravitational lensing.

Let us start by noting that, because of the gravitational deflection, the angular position of the source measured by the observer, $\bar{\theta}_A^o$, is different, in general, from the “real” one from which light signals have been emitted, $\bar{\theta}_A^s$. To describe the relation among these angles one usually defines the two-dimensional amplification matrix (or “lens mapping” matrix) as follows:

$$A_{AB} \equiv \frac{d \bar{\theta}_A^o}{d \bar{\theta}_B^s}. \quad (4.45)$$

The matrix can be conveniently decomposed in terms of a trace and a traceless part as follows

$$A = \begin{pmatrix} 1 - \kappa - \hat{\gamma}_1 & -\hat{\gamma}_2 + \hat{\omega} \\ -\hat{\gamma}_2 - \hat{\omega} & 1 - \kappa + \hat{\gamma}_1 \end{pmatrix}, \quad (4.46)$$

where $\kappa$ is the so-called dimensionless surface mass density (while $1 - \kappa$ is called convergence), $|\hat{\gamma}|^2 = \hat{\gamma}_1^2 + \hat{\gamma}_2^2$ is the shear and $\hat{\omega}$ is the vorticity. The determinant of this matrix defines the important quantity called the magnification $\mu = (\det A)^{-1}$. From the decomposition (4.46), we then have:

$$\mu^{-1} = \det A = (1 - \kappa - \hat{\gamma}_1) (1 - \kappa + \hat{\gamma}_1) + (\hat{\omega} - \hat{\gamma}_2) (\hat{\omega} + \hat{\gamma}_2) = (1 - \kappa)^2 - |\hat{\gamma}|^2 + \hat{\omega}^2. \quad (4.47)$$

In the case of a perturbative computation we find, to first order, $\mu^{-1} \approx 1 - 2 \kappa$. This approximation is largely adopted in the literature for the description of weak lensing effects. In that case, we have a simple relation between convergence and magnification. However, having in mind the possibility of obtaining non-perturbative, exact results, we will always refer in the following to Eq. (4.47) without approximations, so that, in the context of this work, magnification and convergence will always be considered different quantities.
4.4. WEAK LENSING: THE GENERAL FORMALISM

We are now in the position of relating the amplification matrix to the Jacobi matrix, guessing that the 2-dimensional space on which they are defined is the same space spanned by the Sachs basis. In such a case we can identify the angles $\tilde{\theta}_o^A$, $\tilde{\theta}_s^A$ as follows (similarly to what is done also in [85, 86, 87]):

$$
\tilde{\theta}_o^A = \left( \frac{k^\mu \partial_\mu \xi^A}{k^\mu u_\mu} \right)_o,
\tilde{\theta}_s^A = \left( \frac{\xi^A}{d_A} \right)_s,
$$

(4.48)

where $\xi^A$ is given in (4.30), while $d_A$ is the angular distance evaluated in a homogeneous and isotropic background. In such a way, the optical quantities parametrizing the matrix $A$ can be expressed as:

$$
\kappa = 1 - \frac{\text{tr} J_A^B}{2d_A},
\mu = \frac{d_A^2}{\det J_A^B},
\tilde{\omega} = \frac{|J_1^2 - J_2^2|}{2d_A},
$$

$$
|\tilde{\gamma}|^2 = \left( \frac{\text{tr} J_A^B}{2d_A} \right)^2 + \left( \frac{|J_1^2 - J_2^2|}{2d_A} \right)^2 - \frac{\det J_A^B}{d_A^2}.
$$

(4.49)

It should be noted that, according to Eq. (4.32), the magnification can also be written in the interesting form:

$$
\mu = \left( \frac{d_A}{\bar{d}_A} \right)^2.
$$

(4.50)

This clearly tell us that the magnification parameter $\mu$ takes into account how much deviations from a homogeneous reference geometry can affect our estimation of physical distances.

Using the Etherington relation (1.31) and the definition of flux, $F \sim d_L^{-2}$, we also have:

$$
\mu \left( \frac{d_L}{\bar{d}_L} \right)^2 = \frac{F}{\bar{F}},
$$

(4.51)

where $\bar{F}$ is the flux of a homogeneous and isotropic model. It follows that the greater the magnification, the closer (or, equivalently, the brighter) is the source, so that, for $\mu > 1$, the sources are characterized by a luminosity distance smaller than predicted by the homogeneous scenario. The opposite is true if $\mu < 1$. With this approach we have thus a quantitative estimate of the inhomogeneity effects, and a direct comparison with the corresponding homogeneous situation.

In the following sections we will express redshift, distances and lensing observables directly in the GLC gauge of Sec. 4.1, to obtain their exact, non perturbative expression in generally inhomogeneous cosmological geometries.
4.5 Redshift in the GLC gauge

The simplest example we can consider is the redshift observable. Let us first recall that, in the GLC gauge, the velocity field $u_\mu$ of a geodesic observer which is static in the synchronous gauge is given by $u_\mu = \partial_\mu \tau$, while the photon four momentum is simply $k_\mu = \partial_\mu w$. Hence $u_\mu k^\mu = g_{\mu\nu} \partial_\mu \tau \partial_\nu w = g^{\tau w} = -\Upsilon^{-1}$, and the general redshift definition (1.30) gives

$$1 + z = \frac{\Upsilon(\tau_o, w_o, \tilde{\theta}^a_o)}{\Upsilon(\tau_s, w_s, \tilde{\theta}^a_s)}.$$ \hspace{1cm} (4.52)

However, along a past light-cone trajectory the GLC coordinates $w$ and $\tilde{\theta}^a$ are constant (see Sect. 4.1), so that:

$$1 + z = \frac{\Upsilon(\tau_o, w_o, \tilde{\theta}^a_o)}{\Upsilon(\tau_s, w_o, \tilde{\theta}^a_s)}.$$ \hspace{1cm} (4.53)

As in the FLRW case, the redshift thus factorizes into two terms which depend on the source and observer position.

The definition of Eq. (4.53) directly relates a single source parameter, i.e. the emitting time $\tau_s$, to the corresponding redshift parameter $z_s$ in an exact way. This means that, to all orders in perturbation theory, the emitting time can be expressed as a function of the measured redshift, by inverting the above equation, as:

$$\tau_s = \tau_s (z_s, w_o, \tilde{\theta}^a_o).$$ \hspace{1cm} (4.54)

This exact relation implies that the two-dimensional surface which lies on the past light-cone of the given observer (specified by $w = w_o$), and which is associated with a constant redshift $z_s$, is also exactly defined in terms of the coordinate $\tau$ by the equation $\tau = \tau_s$ (and this property greatly simplifies the computation of light-cone averages, see eg. [38, 88]).

It is important to stress that, in other gauges, the redshift may depend on the radial coordinate of the source if the geometry is not homogeneous (see e.g. the LTB case of Eq. (2.48)): in that case, we have to solve the geodesic equations in order to get $x^\mu(z)$. This is no need for that in the GLC gauge. Indeed, since $w$ and $\tilde{\theta}^a$ are constant on the past light-cone, if we are interested in evaluating a physical observable $f(x^\mu)$ in term of the measured redshift and angular position, we can write, according to Eq. (4.54):

$$f(\tau, w, \tilde{\theta}^a_o) = f(\tau(\tau_s, w_o, \tilde{\theta}^a_o), w, \tilde{\theta}^a_o) \equiv f(z, w, \tilde{\theta}^a_o).$$ \hspace{1cm} (4.55)
4.6 Jacobi map in GLC gauge

In this section we shall obtain an expression for the Jacobi map in the GLC gauge, following work already reported in [78, 79]. We notice that, throughout this section, the upper or lower position of capital indices (A, B, C...) will be completely irrelevant, as those indices are defined in a flat (Euclidean) two-dimensional space.

Let us start by considering the explicit form of Eq. (4.25):
\[ 0 = g_{\mu\nu} s^\mu_A k^\nu = g_{\mu\tau} s^\mu_A \omega^{-1} = \omega s^\nu_A \]  (4.56)
from which we obtain that
\[ s^\mu_A = \{ s^\tau_A, 0, s^a_A \}. \]
On the other hand, \( s^A_\mu = g_{\mu\nu} s^\nu_A = g_{\mu\tau} s^\tau_A + g_{\mu a} s^a_A \), but \( g_{\tau\tau} \) and \( g_{\tau a} \) are both vanishing, so we get
\[ s^A_\mu = \{ 0, s^w_A, s^a_A \}. \]
It follows that:
\[ \delta_{AB} = g_{\mu\nu} s^\nu_A s^\mu_B = g_{\tau\tau} s^\tau_A s^\tau_B + g_{\tau a} (s^\tau_A s^a_B + s^a_A s^\tau_B) + g_{ab} s^a_A s^b_B, \]  (4.57)
since \( g_{\tau\tau} = g_{\tau a} = 0 \).

Let us now multiply Eq. (4.57) by \( s^C_A s^D_B \). Using the orthonormality condition \( s^a_A s^b_A = \delta^a_b \) we have \( s^A_C s^D_D \delta_{AB} = \gamma_{ab} s^a_A s^b_B s^C_A s^D_D \), namely:
\[ s^A_C s^D_D = \gamma_{ab}. \]  (4.58)
This gives the relation
\[ \det s^A_C \det s^B_D = (\det s)^2 = \det \gamma_{ab}. \]  (4.59)
which will be used below.

In addition, the projector \( \Pi^\mu_\nu \) can be written explicitly as:
\[ \Pi^\mu_\nu = \delta^\mu_\nu - \delta^\mu_\tau g_{\tau\nu} \frac{u^\tau}{u^\nu} - \delta^\mu_\mu \frac{u^\nu}{u^\nu} - g_{\nu\tau} \frac{u^\mu}{u^\tau}, \]  (4.60)
By projecting \( \nabla_\lambda s^\nu_A \), and noting that \( g_{\mu\nu} \) is always parallely transported, we get:
\[ \Pi^\mu_\nu \nabla_\lambda s^\nu_A = \nabla_\lambda s^\mu_A - \delta^\mu_\tau \frac{u^\tau}{u^\nu} \nabla_\lambda s^\nu_A - \delta^\mu_\mu \frac{u^\nu}{u^\nu} \nabla_\lambda s^\nu_A - \frac{u^\mu}{u^\tau} \nabla_\lambda s^\tau_A, \]  (4.61)
where we have used the result \( s^A_\tau = 0 \). By imposing the property (4.26) of the Sachs basis we thus obtain the condition \( \nabla_\lambda s^\mu_A - \delta^\mu_\tau \nabla_\lambda s^\tau_A = 0 \). For \( \mu = \tau \), this is an identity. The component \( \mu = w \) is also automatically satisfied since \( s^A_\tau = 0 \). The only non-trivial condition is:
\[ \nabla_\lambda s^\nu_A = 0, \]  (4.62)
which means that the angular part of the Sachs basis must be parallely transported along the photons trajectory. By using the explicit form of the GLC metric we obtain

\[ \partial_\tau s^a_A + \frac{1}{2} \gamma^{ac} \partial_\tau \gamma_{bc} s^b_A = 0 \quad \Rightarrow \]
\[ \partial_\tau s^B_a s^a_A + \frac{1}{2} \gamma^{ac} \partial_\tau \gamma_{ca} s^B_a = 0 \quad \Rightarrow \]
\[ \partial_\tau s^a_A s_B - \partial_\tau s^B_A s_a = 0, \quad (4.63) \]

which can be easily rewritten as \( \epsilon^{AB} \partial_\tau s^a_A s_B = 0 \), where \( \epsilon^{AB} \) is the two dimensional antisymmetric symbol.

The previous Eqs. (4.57) and (4.62) completely fix the Sachs basis, while the condition (4.63) can always be implemented by using a residual local rotational symmetry of the two basis vectors. Suppose, in fact, that

\[ \epsilon^{AB} \partial_\tau s^a_A s_B = \not{X} \neq 0. \quad (4.64) \]

Since the Sachs basis is defined up to a local two-dimensional rotation \( \Lambda^A_B \) acting on the flat indices, we can always introduce a rotated basis \( \tilde{s}^a_A = \Lambda^B_A s^a_B \) such that:

\[ \epsilon^{AB} \partial_\tau \tilde{s}^a_A \tilde{s}^b_B = X + \epsilon \partial_\tau \Lambda^C_A \Lambda^D_B \delta_{CD} = X - 2 \partial_\tau \alpha, \quad (4.65) \]

where \( \alpha \) is the local rotation angle. By solving this simple differential equation we can always get rid of \( X \) and thus obtain a suitable set of parallely transported vectors.

An exact knowledge of the Sachs basis in the GLC gauge is crucial for obtaining the exact solution for the Jacobi map. Let us show in particular, that if we take \( \xi^a \) to be constant along the null geodesics, the geodesic deviation equation is automatically satisfied, and the exact Jacobi map can be given explicitly.

By noticing that \( \Gamma_{\tau\tau^B} = \gamma^{-1} \partial_\tau \gamma^D \delta^b_b \) and that \( \Gamma_{\tau w} = 0 \), we can show, first of all, that for a constant \( \xi^a \) the l.h.s. of Eq. (4.22) reads

\[ \nabla^2_\lambda \xi^b = \left( \frac{\omega}{\Upsilon} \right)^2 \left( \partial_\tau \Gamma_{\tau a}^b + \Gamma_{\tau c}^a \Gamma^c b - \Gamma^c b \gamma^{-1} \partial_\tau \Upsilon \right) \xi^a. \quad (4.66) \]

On the other hand, because \( k^\mu \sim \delta^\mu_\tau \) and \( s^w_A = 0 \), the only non-vanishing components of the Riemann tensor in Eq. (4.22) are:

\[ R_{\tau a w}^b = \partial_\tau \Gamma_{\tau a}^b + \Gamma_{\tau c}^a \Gamma^c b - \Gamma^c b \gamma^{-1} \partial_\tau \Upsilon. \quad (4.67) \]

Contracting this identity with \( \xi^a \), and using Eq. (4.66), we clearly reproduce Eq. (4.22) for \( \mu = b \). In a similar way, we can show that the geodesic deviation equation is satisfied also by a constant value of the component \( \xi^w \).
As in the general case, the fact that $\xi^A_a$ are covariantly constant also implies Eq. (4.29) for the $\xi^A_a$. At this point, using $\xi^A = \xi^a s^A_a + \xi^w s^A_w = \xi^a s^A_a$, and taking into account that $\xi^a$ is a arbitrary constant vector, we can also easily prove that:

$$\frac{d^2 s^A_a}{d\lambda^2} = R_{aa\beta j} k^{\alpha} k^{\beta} s_j^A, \quad j = (w, a), \quad (4.68)$$

where the term with $j = w$ does not contribute. The basic result of Eq. (4.68) allows us to construct the Jacobi map using the following ansatz:

$$J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) C^a_B, \quad (4.69)$$

where $C^a_B$ is a $\lambda$-independent matrix that can be explicitly found by imposing the second initial condition of Eq. (4.33) (the first one is automatically satisfied, because $s^A_a(\lambda_o) = 0$).

The initial condition gives:

$$(u^\mu k^\nu)_o \delta^A_B = \frac{d}{d\lambda} J^A_B(\lambda_o, \lambda_o) = \frac{d}{d\lambda} s^A_a(\lambda_o) C^a_B, \quad (4.70)$$

from which we find:

$$C^a_B = (u^\tau k^\sigma)_o (k^\tau \partial_{\tau} s^A_a)^{-1} = (u^\tau k^\sigma)_o \left( \frac{k^\tau}{2} s^A_a \partial_{\tau} \gamma_{ca} \right)^{-1}, \quad (4.71)$$

where last equality follows from the condition of parallel transport (4.62). By using the $2 \times 2$ antisymmetric symbol $\epsilon^{ab}$ we can also write:

$$C^a_B = - \left( 2 u^\tau \frac{\epsilon^{ab} \partial_{\tau} \gamma_{bc} \epsilon^{cd}}{\det[\partial_{\tau} \gamma_{ij}]} s^B_d \right)_o, \quad (4.72)$$

where we have introduced the inverse of the matrix $\partial_{\tau} \gamma_{ab}$. The Jacobi map thus takes the form:

$$J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) \left[ 2 u^\tau \frac{\epsilon^{ac} \partial_{\tau} \gamma_{cd} \epsilon^{db}}{\det[\partial_{\tau} \gamma_{ij}]} s^B_d(\lambda_o) \right] \equiv s^A_a(\lambda) \Delta^{ab}(\lambda_o) s^B_b(\lambda_o). \quad (4.73)$$

By using Eq. (4.59), and following the definition (4.32), the corresponding angular distance is then given by:

$$d^2_A = \det J^A_B = \sqrt{\gamma_o} \frac{4(u^2)_o}{(\det[\partial_{\tau} \gamma_{ab}])_o} \sqrt{\gamma_o}, \quad (4.74)$$

where we have called $\gamma = \det \gamma_{ab}$. Let us briefly comment this results.

This is (to the best of our knowledge) the only way to obtain exact expressions for Jacobi map and angular distance in a fully inhomogeneous and anisotropic background geometry. The redshift can be explicitly given
4.6. JACOBI MAP IN GLC GAUGE

in the same framework and, thanks to the Etherington relation, the above sentence is true also for the luminosity distance.

In this approach we have not considered neither dynamical gravitational equations nor particular forms of the energy-momentum tensor: hence Eq. (4.74) represents a purely geometrical result. This means that our solution should hold even in scenarios based on generalized gravitational equations. Also, only in the GLC gauge the factorization of terms at the observer and terms at the source occurs at such a level of generality.

From a physical point of view, it is interesting to note the presence of the observer velocity $u_\tau$. This term correctly includes the aberration effects possibly due to the local velocity of the observer. The notion of peculiar velocity, however, is gauge dependent. For instance, its value can be zero in the synchronous gauge (the one sharing the time coordinate with the GLC gauge) and different from zero in the longitudinal one. These aspects will be discussed in detail in the last chapter.

As anticipated in Sect. 4.3, the Jacobi map can be rewritten in terms of first order equations. In particular, the deformation matrix of Eq. (4.35), in the GLC gauge, becomes

\[
S^A_B = \frac{dJ^A_C}{d\lambda} (J^{-1})^C_B = \frac{ds^A_a}{d\lambda} C^C_b (C^{-1})^C_b s^b_B = \frac{ds^A_a}{d\lambda} s^a_B
\]

where, in last equalities, we have used the parallel transport condition for the Sachs basis and the definition $k_\mu = \omega_\tau^\mu \delta_\mu^\tau$. The deformation matrix is given in terms the expression of $s^a_A$ but, differently from the Jacobi map, its value its completely fixed by quantities evaluated at the source: no observer term appears. Another important difference concerns the factor $\omega_\tau^{-1}$, which does not appear in the Jacobi map. We recall that this quantity is defined up to a constant, depending on the photon’s four-momentum normalization.

Given the deformation matrix, we can evaluate its components and, in particular, the corresponding optical scalars. Using the property $s^a_A s^b_B = \gamma^{ab}$, and the decomposition of Eq. (4.36), we get:

\[
\hat{\theta} = \frac{\text{tr}S^A_B}{2} = \omega \frac{\gamma^{ab} \dot{\gamma}_{ab}}{4\Upsilon} = \omega \frac{\dot{\gamma}}{4\Upsilon \gamma},
\]

\[
|\hat{\sigma}|^2 = \dot{\sigma}_1^2 + \dot{\sigma}_2^2 = \left(\frac{\text{tr}S^A_B}{2}\right)^2 - \det S^A_B
\]

\[
= \left(\frac{\omega \dot{\gamma}}{4\Upsilon \gamma}\right)^2 - \omega^2 \frac{\det \dot{\gamma}_{ab}}{4\Upsilon^2 \gamma},
\]

where the dot denotes differentiation with respect to $\tau$. We note that the optical scalars are independent on the local rotation which can still be ap-
plied to the Sachs basis. Also, these expressions perfectly agree with the general definition of Eq. (4.40), with the usual identification $k^a = \omega \gamma^{-1} \delta^a_\nu$.

Finally, starting from the definitions of Eq. (4.44), we can express the Ricci and Weyl focusing parameters:

$$\Phi_{00} = \frac{\omega^2}{4T^2} \left[ \gamma^{ab} \dot{\gamma}_{ab} - \frac{1}{\bar{\gamma}} \dot{\gamma} \gamma^{ac} \gamma^{cd} \dot{\gamma}_{db} - \frac{1}{2} \gamma^{ab} \dot{\gamma}_{ac} \gamma^{cd} \dot{\gamma}_{db} \right],$$

$$\text{Re}\Psi_0 = \frac{\omega^2}{4T^2} \left[ \dot{\gamma}_{ab} - \frac{1}{\bar{\gamma}} \dot{\gamma} \gamma_{ab} - \frac{1}{2} \dot{\gamma}_{ac} \gamma^{cd} \dot{\gamma}_{db} \right] \left( s^a_1 s^b_1 - s^a_2 s^b_2 \right),$$

$$\text{Im}\Psi_0 = \frac{\omega^2}{4T^2} \left[ \dot{\gamma}_{ab} - \frac{1}{\bar{\gamma}} \dot{\gamma} \gamma_{ab} - \frac{1}{2} \dot{\gamma}_{ac} \gamma^{cd} \dot{\gamma}_{db} \right] \left( s^a_1 s^b_2 + s^a_2 s^b_1 \right).$$

(4.78)

where we have used the condition $s^a_A s^b_A = \gamma^{ab}$. Let us notice that $\Phi_{00}$ does not depend on the particular expression of the basis vectors $s^a_A$, while $\text{Re}\Psi_0$ and $\text{Im}\Psi_0$ do depend on it. However, if we compute the modulus of $\Psi_0$, after some algebraic manipulations we find:

$$\gamma^{ac} \gamma^{bd} + \gamma^{ad} \gamma^{bc} - \gamma^{ab} \gamma^{cd} = (s^a_1 s^b_1 - s^a_2 s^b_2)(s^c_1 s^d_1 - s^c_2 s^d_2) + (s^a_1 s^b_2 + s^a_2 s^b_1)(s^c_1 s^d_2 + s^c_2 s^d_1),$$

(4.79)

and we obtain

$$|\Psi_0|^2 = \frac{\omega^4}{16T^4} \left[ \dot{\gamma}_{ab} - \frac{1}{\bar{\gamma}} \dot{\gamma} \gamma_{ab} - \frac{1}{2} \dot{\gamma}_{ac} \gamma^{cd} \dot{\gamma}_{db} + \frac{1}{2} \dot{\gamma}_{ac} \gamma^{cd} \dot{\gamma}_{db} \right] \left[ \dot{\gamma}_{cd} - \frac{1}{\bar{\gamma}} \dot{\gamma} \gamma_{cd} - \frac{1}{2} \dot{\gamma}_{ce} \gamma^{ef} \dot{\gamma}_{fg} \right],$$

(4.80)

so that also $|\Psi_0|^2$ does not depend on the particular expression for the Sachs basis.

It should be stressed, however, that even if the above results are independent on the proper coordinates expression of $s^a_A$, they have been obtained by using the parallel transport condition, hence this independence is weaker than we could believe. In any case, one can appreciate the great simplicity of these expressions compared to what one obtains, for general geometries, in other gauges. We may thus expect that the geodesic light-cone gauge is perfectly adapted to the computation of the lensing quantities, as we shall see in the next section.
4.7  Weak lensing in the GLC gauge

We want to conclude this chapter by providing the exact expression for the components of the amplification matrix introduced in Sect. 4.4.

Let us first prove that:

$$\epsilon_{AB} s^A_a s^B_b = \sqrt{\gamma} \epsilon_{ab}$$  \hspace{1cm} (4.81)

where $\epsilon_{AB}$ is the two-dimensional antisymmetric symbol in flat space and $\epsilon_{ab}$ is the same symbol in the curved space described by the GLC metric tensor. To this end, let us notice that

$$\epsilon_{AB} s^A_a (s^B_b + s^B_a) = \frac{1}{2} \left( \epsilon_{AB} s^A a s^B b - \epsilon_{BA} s^B a s^A b \right) = 0.$$  \hspace{1cm} (4.82)

We can then write:

$$\epsilon_{AB} s^A_a s^B_b = A \epsilon_{ab}.$$  \hspace{1cm} (4.83)

The function $a A$ can be fixed by evaluating the determinant of the previous expression, which gives $- (\det s^A_a)^2 = -A^2$, i.e. $A = \sqrt{\gamma}$.

Once this has been proved, we can explicitly compute the components of $A$. First of all the magnification $\mu$: given the definition of Eq. (4.49) we obtain, in the GLC gauge,

$$\mu = \frac{\bar{u} - \frac{2}{\sqrt{\gamma}} \frac{2}{\gamma} \left[ \det \gamma_{\alpha\beta} \right]_{\alpha \beta}}{4 \sqrt{\gamma} a} = \left( \frac{\bar{d}_A}{d_A} \right)^2.$$  \hspace{1cm} (4.84)

This expression, as well as all the other lensing quantities of Eq. (4.49), contains the angular distance of the flat homogeneous and isotropic case, $\bar{d}_A \equiv d_A(\lambda_a)$. This distance can be explicitly written in the GLC gauge as $\bar{d}_A = \bar{\Upsilon}(\tau) r$, where $r = w - \int \bar{\Upsilon}^{-1}(\tau) d\tau$ is the corresponding radial distance.

Let us now consider the other lensing quantities presented in Eq. (4.49), namely convergence, shear and vorticity. They all depend on $\bar{d}_A$ (defined above) and on some combination of the Jacobi map components. According to Eq. (4.73), we thus need the general expression of $s^A_a(\lambda)$ in the GLC gauge, and this expression depends on an arbitrary angle $\beta$ related to their local rotational freedom in the parallel transport condition.

We have no need to present such expression here, and we will concentrate instead on a combination of lensing quantities that does not depend on the angle $\beta$. Let us call $\Delta^{ab}$ the quantity $\Delta^{ab}(\lambda_a)$ of Eq. (4.73). The squared of
the convergence given in Eq. (4.49) then reads:

\[
(1 - \kappa)^2 = \frac{1}{4d^2} \left( J_1^1 + J_2^2 \right)^2 = \frac{1}{4d^2} \left( (J_1^1)^2 + (J_2^2)^2 + 2J_1^1J_2^2 \right)
\]

\[
= \frac{1}{4d^2} \left[ s_1^1 \Delta_o^b s_1^1 (\lambda_o) s_1^1 \Delta_o^d s_1^1 (\lambda_o) + s_1^2 \Delta_o^b s_1^2 (\lambda_o) s_1^2 \Delta_o^d s_1^2 (\lambda_o)
+ 2s_1^1 \Delta_o^b s_1^1 (\lambda_o) s_1^2 \Delta_o^d s_1^2 (\lambda_o) \right]
\]

\[
= \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ s_1^1 s_1^1 (s_1^2 s_2^2)_o + s_1^2 s_2^2 (s_1^1 s_2^1)_o - 2s_1^1 s_1^1 (s_1^2 s_2^2)_o \right].
\] (4.85)

In the same way, using Eqs. (4.46), (4.49) and (4.73), we find that:

\[
\omega^2 = \frac{1}{4d^2} \left( J_1^1 - J_2^2 \right)^2
\]

\[
= \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ s_1^1 s_1^2 (s_1^1 s_2^2)_o + s_1^2 s_1^1 (s_1^1 s_2^2)_o - 2s_1^1 s_1^2 (s_1^1 s_2^2)_o \right].
\] (4.86)

\[
\gamma_1^2 = \frac{1}{4d^2} \left( J_1^1 - J_2^2 \right)^2
\]

\[
= \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ s_1^1 s_1^2 (s_1^1 s_2^2)_o + s_1^2 s_1^1 (s_1^1 s_2^2)_o - 2s_1^1 s_1^2 (s_1^1 s_2^2)_o \right].
\] (4.87)

The combination of the above equations gives

\[
(1 - \kappa)^2 + \omega^2 = \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ s_1^1 s_1^2 (s_1^1 s_2^2)_o + 2s_1^2 s_2^2 (s_1^1 s_2^1)_o \right],
\]

\[
\gamma_1^2 + \gamma_2^2 = \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ s_1^1 s_1^2 (s_1^1 s_2^2)_o - 2s_1^1 s_1^2 (s_1^2 s_2^1)_o \right].
\] (4.88)

which, thanks to the identities:

\[
s_1^1 s_1^1 = \gamma_{ab} , \quad \epsilon_{AB} s_1^A s_2^B = \sqrt{\gamma} \epsilon_{ab} ,
\] (4.89)

can be simplified into the new expressions:

\[
(1 - \kappa)^2 + \omega^2 = \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ \gamma_{ab} (\gamma_{bd})_o + 2\sqrt{\gamma} s_1^1 s_1^2 \epsilon_{bd} \right],
\]

\[
\gamma_1^2 + \gamma_2^2 = \frac{1}{4d^2} \Delta_o^b \Delta_o^d \left[ \gamma_{ab} (\gamma_{bd})_o - 2\sqrt{\gamma} s_1^1 s_1^2 \epsilon_{bd} \right].
\]
4.7. WEAK LENSING IN THE GLC GAUGE

We can finally use

\[
\left( \Delta^{ab} \gamma_{bc} \Delta_{cd} \right) \gamma_{ad} = 4 u_\tau^2 \left( \frac{\gamma_{ab} \gamma_{bc} \gamma_{cd}}{(\det \gamma_{ab})^2} \right) \gamma \gamma_{ad}
\]

\[
s_a^1 \Delta^a \gamma_{bc} \Delta_{cd} s^2_d = \frac{4 u_\tau^2 \sqrt{\gamma}}{(\det \gamma_{ab})},
\]

(4.90)
to obtain:

\[
(1 - \kappa)^2 + \hat{\omega}^2 = \left( \frac{u_\tau}{\bar{d}_A} \right)^2 \left\{ \gamma \gamma_{ab} \gamma_{bc} \gamma_{cd} \left( \frac{\gamma_{ab}}{(\det \gamma_{ab})^2} \right) \gamma \gamma_{ad} + 2 \frac{\sqrt{\gamma}}{(\det \gamma_{ab})} \right\},
\]

\[
\hat{\gamma}_1^2 + \hat{\gamma}_2^2 = \left( \frac{u_\tau}{\bar{d}_A} \right)^2 \left\{ \gamma \gamma_{ab} \gamma_{bc} \gamma_{cd} \left( \frac{\gamma_{ab}}{(\det \gamma_{ab})^2} \right) \gamma \gamma_{ad} - 2 \frac{\sqrt{\gamma}}{(\det \gamma_{ab})} \right\}.
\]

(4.91)

One can then re-express the magnification \( \mu \) using the above results and the definition (4.47), namely \( \mu \equiv \left[ (1 - \kappa)^2 + \hat{\omega}^2 - |\hat{\gamma}|^2 \right]^{-1} \). Let us emphasize that the ratio \( \bar{d}_A/\tau \) is contained in all the components of the matrix \( \mathcal{A} \) and that, since the lensing effects do not depend on the observer’s motion, we can choose the unperturbed distance \( \bar{d}_A \) with the same value of the velocity parameter as the perturbed one, i.e. \( \bar{u}_\tau = u_\tau \). Namely, we can use:

\[
\left( \frac{\bar{d}_A}{\bar{u}_\tau} \right)^2 = \sqrt{\gamma} \left[ \frac{4 \sqrt{\gamma}}{(\det \bar{\gamma}_{ab})} \right],
\]

(4.92)

where \( (\ldots) \) denotes unperturbed quantities.

In this chapter we have shown how to obtain exact expression for several important physical observables, all directly related to the propagation of light-like signals on the past light-cone of the observer. This has been possible thanks to the formalism of the GLC gauge. If we want to provide model-dependent predictions, we have now to insert explicit solutions of the cosmological equations, which are difficult to obtain in the GLC gauge, but well known in other gauges. In the following chapter we will thus discuss transformations between the GLC gauge and other coordinate systems, more frequently used to solve the Einstein equations.
Chapter 5

From exact results to predictions

In the previous chapter we have introduced a new formalism able to provide exact results for light-cone observables, quite independently on the given cosmological model and of its geometrical properties. We have also stressed that, within this formalism, we can take into account the redshift dependence without solving the geodesic equations: all the observables, in principle, can be directly and analytically expressed in terms of the observed \( z \). This is a great advantage with respect to different computational approaches discussed in the preceding chapters.

If we could solve the Einstein equation in the GLC gauge, we would be able to compare our models with observations directly in this gauge, in an exact way. This is possible, in principle, but, up to now, very difficult to be implemented in practice, because of mathematical difficulties. Fortunately enough, we have other ways to proceed.

In particular, we can express the quantities computed in the GLC gauge in terms of another system of coordinates such that, in that new system, the corresponding system of differential equations satisfied by the metric takes a much simpler form, and can be easily solved. More precisely, if \( \tilde{x}^\mu \) are the GLC coordinates and \( x^\mu \) those of the other system, the metric \( g^{\alpha\beta} \) that we know to solve the Einstein equations must be related to the GLC metric \( g^{\mu\nu}_{\text{GLC}} \) of Eq. (4.3) by:

\[
g^{\mu\nu}_{\text{GLC}}(\tilde{x}) = g^{\alpha\beta}(x) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}.
\]  

(5.1)

As a simple illustrative example, let us consider the coordinates transformation connecting a FLRW metric to the GLC one. By defining the conformal time \( \eta = \int \frac{du}{a(t)} \), the sought transformation given by:

\[
\tau = t, \quad w = r + \eta, \quad \tilde{\theta}^a = \theta^a.
\]

(5.2)
If we start, for instance, with the metric $g^{\alpha\beta}$ which is the inverse of the metric (1.4) (with $K = 0$, for simplicity), and insert the transformation (5.2) into the general rule (5.1), we obtain the metric

$$
g^{\mu\nu}_{\text{GLC}} = \begin{pmatrix}
-1 & -a^{-1} & 0 & 0 \\
-a^{-1} & 0 & 0 & 0 \\
0 & 0 & r^{-2}a^{-2} & 0 \\
0 & 0 & 0 & r^{-2}a^{-2}\sin^{-2}\theta
\end{pmatrix}.
$$

(5.3)

This is exactly a metric of the GLC type given in Eq. (4.3), for the particular case of $U^a = 0$, $\Upsilon = a$ and $\gamma^{ab} = r^{-2}a^{-2}\text{diag}(1, \sin^{-2}\theta)$.

We can add few comments on the above transformations by noting that, as expected, the combination $r + \eta$ just exactly identifies the past light-cone of the observer defined, in conformal time, by $r = -\eta + \text{const}$. This relation is exact for the FLRW metric but changes in an inhomogeneous geometry.

In the following sections we will provide less trivial coordinate transformations between perturbed inhomogeneous metrics and the GLC one.

### 5.1 From the GLC gauge to the synchronous one

Following the procedure described above, the first non-trivial coordinate system we want to connect with the GLC gauge is the one describing a FLRW metric perturbed to first order by scalar fluctuations, and described in the Synchronous Gauge (SG).

The time parameter of this gauge coincides with the proper time of static, comoving observers, and provides a convenient time coordinate for parametrizing the cosmological dynamics. Using Cartesian spatial coordinates, the perturbed geometry we are considering is described, in this gauge, by the following line-element:

$$
ds^2_{SG} = -dt^2 + a^2(t) \left[ (1 - 2\psi)\delta_{ij} + D_{ij} E \right] dx^i dx^j ; \quad D_{ij} = \partial_i \partial_j - \frac{1}{3} \delta_{ij} \Delta_3,
$$

where $\Delta_3 = \nabla^2$ is the usual Laplacian operator in 3-dimensional Euclidean space and $D_{ij}$ is traceless divergence operator.

For a comparison with the corresponding metric in GLC form it is convenient to express the SG metric in terms of spherical polar coordinates.
5.1. FROM THE GLC GAUGE TO THE SYNCHRONOUS ONE

\{r, \theta, \phi\}. In that case we find that the above line-element becomes
\[
d s_{SG}^2 = -dt^2 + a^2(t) \left[ (1 - 2Z)dr^2 - 2S_a dr d\theta^a + h_{ab} d\theta^a d\theta^b \right],
\]
(5.5)
and a direct calculation gives
\[
Z = \psi - \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} - \frac{1}{3} \Delta_3 \right) E ; \quad S_a = - \left( \frac{\partial_r}{r} - \frac{1}{r} \right) \partial_a E,
\]
\[
h_{ab} = \gamma_{ab}^0 \left[ 1 - 2\psi - \left( \frac{1}{3} \Delta_3 - \frac{1}{r} \partial_r \right) E \right] + \nabla_a \partial_b E,
\]
(5.6)
where \( \gamma_{ab}^0 = r^2 \text{diag}(1, \sin^2 \theta) \), and where \( \nabla_a \) represents the covariant angular derivative. In matrix form we then have
\[
\gamma_{SG}^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & a^{-2}(1 + 2Z) & a^{-2} \gamma_{0b} S_b \\
0 & a^{-2} \gamma_{0b} S_b & a^{-2} h_{ab}
\end{pmatrix},
\]
(5.7)
where \( \gamma_{ab}^0 = r^2 \text{diag}(1, \sin^{-2} \theta) \) and
\[
h_{ab} = \gamma_{ab}^0 \left[ 1 + 2\psi + \left( \frac{1}{3} \Delta_3 - \frac{1}{r} \partial_r \right) E \right] - \gamma_{ac}^0 \gamma_{bd}^0 \nabla_c \partial_d E.
\]
(5.8)

If we are interested into a first-order approximation only, we can start with the previous transformation between GLC and unperturbed FLRW metric as the zero-th order form of the transformation, and then add to it the first-order contributions. By doing this, the sought relation \( \tilde{x}^\mu = x^\mu(x^\nu) \) is given by:
\[
\tau = t,
\]
\[
w = r + \eta + \frac{1}{2} \int_{\eta_+}^{\eta_-} dx Z(\eta, x, \theta^a) \equiv r + \eta + w^{(1)},
\]
\[
\tilde{\theta}^a = \theta^a + \frac{1}{2} \int_{\eta_+}^{\eta_-} dx \chi^a(\eta, x, \theta^a) \equiv \theta^a + \tilde{\theta}^{a(1)},
\]
(5.9)
where \( \eta_\pm = r \pm \eta \), \( \partial_\eta = \partial_+ + \partial_- \), \( \partial_r = \partial_+ - \partial_- \), \( \partial_\pm = \frac{\partial}{\partial \eta_\pm} = \frac{1}{2} (\partial_\eta \pm \partial_r) \) and
\[
\chi^a = S^a + \frac{1}{2} \gamma_{0c} \int_{\eta_+}^{\eta_-} dx \partial_c Z(\eta, x, \theta^a),
\]
(5.10)
and the GLC metric components can be written in term of the SG ones as:
\[
Y = a(\eta) \left[ 1 - \frac{1}{2} (\partial_+ + \partial_-) \int_{\eta_+}^{\eta_-} dx Z(\eta, x, \theta^a) \right],
\]
(5.11)
\[
U^a = \frac{1}{2} (\partial_+ + \partial_-) \int_{\eta_+}^{\eta_-} dx \chi^a(\eta, y, \theta^a),
\]
(5.12)
\[
a^2 \gamma^{ab} = h^{ab} + \left[ \frac{1}{2} \gamma_{0c} \int_{\eta_+}^{\eta_-} dx \partial_c \chi^b(\eta, x, \theta^a) + a \leftrightarrow b \right].
\]
(5.13)
5.1. FROM THE GLC GAUGE TO THE SYNCHRONOUS ONE

Note that now the light-cone condition \( w = \text{const} \) is no longer equivalent to \( r + \eta = \text{const} \), but there are corrections. Note also that the angular transformations, Eqs. (5.9) are obtained by requiring that two different set of angles coincides at the observer position, i.e. \( \tilde{\theta} = \theta^a \) for \( r = 0 \). Indeed, at the observer, we have that the integral limits of the first-order corrections coincide, \( \eta_+ = \eta_- = \eta \), while the integrand remains finite. To see this, and for later use, let us expand \( E \) around the observer position. Choosing for simplicity \( \eta_0 = 0 \), we obtain, up to second order,

\[
E(r, \eta, \theta) = E_0 + E^\eta \eta + E^i x^i + \frac{1}{2} E'' \eta^2 + \frac{1}{2} E_{ij} x^i x^j + E^x \eta + \ldots \tag{5.14}
\]

where \( x^i = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \) and all coefficients are constant. Using Eq. (5.14), and applying the useful relations \( \Delta^2 x_i = -2 x_i \) and \( \Delta^2 (x_i x_j) = 2 r^2 \delta_{ij} - 6 x_i x_j \), where \( \Delta^2 \) is the 2-dimensional angular Laplacian related to \( \Delta^3 \) by

\[
\Delta^3 = \frac{1}{r^2} \Delta^2 + (\partial_r^2 + \frac{2}{r} \partial_r), \tag{5.15}
\]

we obtain, for \( r \to 0 \):

\[
S^a = - \frac{1}{2} \gamma^{ab}_0 \partial_b \left[ E_{ij} x^i x^j \right] + C_1 \]

\[
\frac{1}{2} \gamma^{ab}_0 \int_{\eta_+}^{\eta_-} dx \partial_b Z(\eta, x, \theta^a) = \frac{1}{2} \gamma^{ab}_0 \partial_b \left[ E_{ij} x^i x^j \right] + C_2. \tag{5.16}
\]

Here \( C_1 \) and \( C_2 \) are \( r \)-independent constant terms, and we have used that \( \partial_a \psi \sim O(r) \) for \( r \to 0 \). Therefore the integrand in Eq. (5.9) is finite at \( r = 0 \) and, since the integration range goes to zero, it follows that \( \tilde{\theta}^a \to \theta^a \). This also implies, through Eq. (5.12), that \( U^a = 0 = \tilde{\theta}^a \) on the observer’s worldline, consistently with the fact that the angles coincide in the two coordinate systems (and that those of the synchronous gauge are obviously constant).

Now, in order to express the angular distance obtained in the GLC gauge (see Eq. (4.74)) in terms of the SG coordinates, we have to evaluate the derivative with respect to \( \tau \) as a combination of derivatives with respect to \( \{t, r, \theta, \phi\} \). Using the previous results we obtain

\[
\partial_\mu \tilde{x}^\nu = \left( a^{-1} + \frac{1}{2} \partial_\eta \int_{\eta_-}^{\eta_+} dx Z \quad 1 + \frac{1}{2} \partial_r \int_{\eta_-}^{\eta_+} dx Z \quad \frac{1}{2} \partial_b \int_{\eta_-}^{\eta_+} dx \chi^b \quad \frac{1}{2} \partial_r \int_{\eta_-}^{\eta_+} dx \chi^b \right) \cdot \left( \chi^{-1} \right)^\nu_\mu \tag{5.17}
\]

so \( \tilde{\partial}_\mu = \chi^\nu_\mu \partial_\nu \). We are interested, in particular, in \( \partial_\tau \). Recalling that \( 2 \partial_- = \partial_\eta - \partial_r \) we get:

\[
\partial_\tau = \frac{2}{a(\eta)} \left[ \partial_- - \partial_- w^{(1)} \partial_r - \partial_- \tilde{\theta}^{(1)} \partial_r \right]. \tag{5.18}
\]
This derivative has to be applied at the observer position, \( t = t_o \) and \( r = 0 \), to the components of the metric tensor. What we need, in particular, are the derivatives of \( \gamma_{ab} \equiv a^2 r^2 \text{diag} \left( 1, \sin^2 \theta \right) + \gamma_{ab}^{(1)} \), where \( \gamma_{ab}^{(1)} \) describes the first-order perturbations of the angular part of the GLC metric:

\[
(\partial_r \gamma_{ab})_o = \frac{2}{a(\eta_o)} \left\{ \text{diag} \left( 1, \sin^2 \theta \right) \left[ (\nu^2 a a' - r a^2) + \partial_r \gamma_{ab}^{(1)} \right] - 2a^2 r \partial_r w^{(1)} \right\} - \delta^i_b \delta^j_b a^2 r^2 \sin 2 \theta \partial_r \bar{\theta}^{t(1)} \bigg|_o.
\] (5.19)

Using this result, we find that the angular distance of Eq. (4.74) becomes

\[
(d_A^2)_{SG} = \frac{a^2 r^2 \sin \theta_s}{\sin \theta_o} \left\{ 1 + \left( \frac{3}{2} \partial_r^2 - \frac{1}{2} \Delta_3 \right) E_o - 2 \psi_s + \left( \frac{1}{6} \Delta_3 - \frac{1}{2} \partial_r^2 \right) E_s + \frac{1}{4} \int_{\eta'_o}^{\eta_o} dx \gamma_{ab}^{(0)} \partial_a \partial_b \left( \partial_r - \frac{1}{r} \right) E \right\} - \frac{1}{4} \int_{\eta'_o}^{\eta_o} dx \gamma_{ab}^{(0)} \int_{\eta'_o}^{x} dy \partial_a \partial_b \left[ \psi + \left( \frac{1}{6} \Delta_3 - \frac{1}{2} \partial_r^2 \right) E \right] \bigg|_o.
\] (5.20)

The above general expression appears to imply corrections to the simplest formula \( d_A^2 = \sqrt{\gamma_s / \sin \theta_o} \), used e.g. in [38, 88]. In fact, by using the expansion in Eq. (5.14), we obtain the following \( r \)-independent result

\[
\det (\partial_r s^B_{ab}(\lambda_o)) = \sin \theta_o \left[ 1 + \frac{1}{2} \Delta_2 \left( \frac{1}{2} E_{ij} x^i x^j \right) \right] = \sin \theta_o \left[ 1 + \frac{1}{2} \left( \delta^{ij} E_{ij} - \frac{3}{r^2} E_{ij} x^i x^j \right) \right],
\] (5.21)

instead of the simple result \( \sin \theta_o \).

However, as in the case of the GLC gauge, we can try to use the residual gauge symmetry of the SG to remove the correction. Taking indeed the time-independent coordinate transformation:

\[
x^i \to \bar{x}^i = x^i + \frac{1}{2} L^i_j x^j,
\] (5.22)

we can choose the constant matrix \( L^i_j \) so as to make the new \( D_{ij} \bar{E} \) vanish at a given time. This can be chosen to be the present observer’s time \( \eta_o = 0 \), which clearly corresponds to setting \( E_{ij} = 0 \). Therefore, the gauge freedom in the SG is weaker than the one in the GLC, where we were able to remove the correction all along the observer’s world-line. Actually, this is what we should expect since the GLC transformation involves a change both in \( \gamma_{ab} \) and in \( U^a \). Hence, in general, this residual gauge fixing implies non-vanishing \( U^a \) and thus prevents identifying the GLC angles with those in the SG all along the observer’s geodesic. This is why in the SG we cannot remove the correction at all times.
5.1. FROM THE GLC GAUGE TO THE SYNCHRONOUS ONE

5.1.1 Luminosity distance in a metric of Bianchi-type I

We have already discussed the Bianchi-type I metric in Chapter 3. We have seen that, when the anisotropy is small, we can regard the eccentricity parameter as a first-order correction to the FLRW background. Since the Bianchi and FLRW metric share the same synchronous time $t$, and since, in both cases, $g_{tt} = 0$, we can describe the eccentricity as a scalar perturbation in the synchronous gauge. In particular, the Bianchi I metric (3.2) can be written as a SG metric (5.5) with

$$
\psi = \frac{1}{6} e^2(t)
$$

$$
E = -\frac{1}{2} z^2 e^2(t) = -\frac{1}{2} r^2 \cos^2 \theta e^2(t)
$$

(5.23)

where $z = r \cos \theta$ and where

$$
\Delta_3 E = \partial^2_r E = -e^2(t).
$$

(5.24)

We use the residual gauge freedom to normalize the eccentricity at the observer position in such a way that $e(t_o) = 0$, for consistency. In this way we obtain, from the SG angular distance of Eq. (5.20),

$$
\left( d^2_A \right)_{\text{Bianchi}} = \frac{a^2 e^2 \sin \theta}{\sin \theta_o} \left\{ 1 - 2\psi_o + \left( \frac{1}{6} \Delta_3 - \frac{1}{2} \partial^2_r \right) E_o \right. \\
+ \frac{1}{2} \int \gamma^a b dx^a \partial_d \partial_b \left( \partial_r - \frac{1}{r} \right) E \\
- \frac{1}{4} \int \gamma^a b \int \gamma^c d dy \partial_a \partial_b \left[ \psi + \left( \frac{1}{6} \Delta_3 - \frac{1}{2} \partial^2_r \right) E \right] \right\}
$$

(5.25)

Combining the above equations we have

$$
\left( d^2_A \right)_{\text{Bianchi}} = \frac{a^2 e^2 \sin \theta}{\sin \theta_o} \left\{ 1 - \frac{1}{2} \sin^2 \theta e^2 \\
+ \frac{1}{2} \int \gamma^a b dx^a \partial_d \partial_b \left( -\frac{1}{2} r \cos^2 \theta e^2 \right) \\
- \frac{1}{4} \int \gamma^a b \int \gamma^c d dy \partial_a \partial_b \left( \frac{1}{2} \cos^2 \theta e^2 \right) \right\}.
$$

(5.26)
5.1. FROM THE GLC GAUGE TO THE SYNCHRONOUS ONE

Since $\partial_a \partial_b \cos^2 \theta = -2 \cos 2 \theta \delta^1_a \delta^1_b$ and $\gamma^1_1 = r^{-1}$, this can be rewritten as:

$$\left( \frac{d^2}{\rho^2} \right)_{\text{Bianchi}} = \frac{2 \rho^2 \sin \theta_e}{\sin \theta_o} \left\{ 1 - \frac{1}{2} \sin^2 \theta e^2 \right\}$$

$$+ \frac{1}{2} \int_{\eta_o}^{\eta} dx \frac{e^2}{r} \cos 2 \theta$$

$$+ \frac{1}{4} \int_{\eta_o}^{\eta} dx \frac{1}{r^2} \int_{\eta}^{x} dy \; e^2 \cos 2 \theta \right\}. \quad (5.27)$$

Recalling that $r = \frac{\eta_o - \eta}{2}$, we have also

$$\left( \frac{d^2}{\rho^2} \right)_{\text{Bianchi}} = \frac{2 \rho^2 \sin \theta_e}{\sin \theta_o} \left\{ 1 - \frac{1}{2} \sin^2 \theta e^2 \right\}$$

$$+ \int_{\eta_o}^{\eta} dx \left[ \int_{\eta}^{x} \left( \frac{\eta - \eta_o}{\eta_o} \right) \right]$$

$$\int_{\eta}^{x} dy \; e^2 \cos 2 \theta \right\}. \quad (5.28)$$

or, by integrating with respect to $t$:

$$\left( \frac{d^2}{\rho^2} \right)_{\text{Bianchi}} = \frac{2 \rho^2 \sin \theta_e}{\sin \theta_o} \left\{ 1 - \frac{1}{2} \sin^2 \theta e^2 \right\}$$

$$- \cos 2 \theta \left[ \int_{t_o}^{t} dt' \frac{1}{a(t')} \eta(t) - \eta(t') e^2(t') \right]$$

$$- \int_{t_o}^{t} dt' \frac{1}{a(t')} \frac{e^2(t')}{[\eta(t) - \eta(t')]^2} \int_{t'}^{t_o} dt'' a(t'') e^2(t'') \right\}. \quad (5.29)$$

In order to compute the redshift let us notice that, for the particular metric we are considering,

$$Z = -\frac{\partial^2 E}{2} = \frac{e^2 \cos^2 \theta}{2}. \quad (5.30)$$

From Eq. (5.11), we then have:

$$\Upsilon = a(\eta) \left[ 1 + \frac{1}{2} \partial_\eta \int_\eta^{\eta_o} d\eta' e^2(\eta') \cos^2 \theta \right]$$

$$\Upsilon = a(t) \left[ 1 - \frac{e^2(t) \cos^2 \theta}{2} \right], \quad (5.31)$$

and the redshift is then immediately given by:

$$1 + z = \frac{\Upsilon_o}{\Upsilon_e} = \frac{a(t_o)}{a(t)} \left[ 1 + \frac{e^2(t) \cos^2 \theta}{2} \right]. \quad (5.32)$$
5.2 FROM THE GLC GAUGE TO THE LONGITUDINAL ONE

where we have used again the normalization $e(t_o) = 0$. According to the Etherington relation, the luminosity distance is finally given by

\[
\begin{align*}
(d^2_L)_{\text{Bianchi}} &= (1 + z)^4 \left( d^2_A \right)_{\text{Bianchi}} \\
&= \frac{a_o^2 r_e^2 \sin \theta_e}{\sin \theta_o} \left( \frac{a_o}{a_e} \right)^4 \left\{ 1 + e^2 \left( 2 \cos^2 \theta - \frac{1}{2} \sin^2 \theta \right) \\
&- \cos 2 \theta \left[ \int_{t}^{t_0} dt' \frac{1}{a(t') \left( r + \eta(t) - \eta(t') \right) e^2(t')} \right] \\
&- \int_{t}^{t_0} dt' \frac{1}{a(t')} \left( r + \eta(t) - \eta(t') \right)^2 \int_{t'}^{t_0} dt'' \frac{1}{a(t'')} e^2(t'') \right\}.
\end{align*}
\]

(5.33)

It may be interesting to note that angular and luminosity distances for particular cases of Bianchi-type I metric have already been discussed in the literature (see e.g. [89, 90]). Those calculations always necessarily refer to the explicit solutions of Einstein’s equations for the model under consideration.

Before concluding, let us stress that, differently from the GLC gauge, there is no factorization in Eq. (5.33) between observer and source terms. Integrals related to propagation along the line of sight are present, just as expected. Also, the observer terms seem to contain aberration effects which have to be carefully interpreted. Indeed, since the SG geodesic observer and the GLC one share the same state of motion, this cannot be due to a relative peculiar velocity. These corrections can be eliminated by locally redefining perturbations: namely, by fixing the residual gauge freedom in the SG at the observer time so as to recover the state of no aberration.

In the next section, we will study another perturbed metric where this is no longer possible: the longitudinal gauge.

5.2 From the GLC gauge to the longitudinal one

Let us now discuss another interesting gauge where our free-falling observer is no longer static: the so-called Longitudinal gauge (LG). Considering a perturbed FLRW background, and neglecting vector and tensor contribu-
tions, the LG metric takes the following form:

\[
ds_{LG}^2 = a^2(\eta) \left[ -(1 + 2\phi)d\eta^2 + (1 - 2\psi)\delta_{ij}dx^i dx^j \right] \\
= a^2(\eta) \left[ -(1 + 2\phi)d\eta^2 + (1 - 2\psi)(dr^2 + r^2 d\Omega^2) \right],
\]

where \(\phi\) and \(\psi\) are scalar perturbations. In principle, we don’t need to make any assumption about their possible dynamical sources: this means that anisotropic stress (namely \(\phi \neq \psi\)) are also allowed.

In order to compute the area distance in terms of standard LG perturbations we have to transform the GLC quantities appearing in Eq. (4.74) to LG quantities. This has already been done in [38, 91] for the particular case of no-anisotropic stress (where \(\phi = \psi\)). Here we extend the analysis to the more general case (but only for the first order). The point is that we have to impose suitable boundary conditions and, as in the previous section, we impose that \(i\) the transformation is non singular around \(r = 0\), and that \(ii\) the two-dimensional spatial section \(r = \text{const}\) are locally parametrized at the observer position by standard spherical coordinates. However, unlike the case of the SG, for the LG these conditions can only be imposed at the observer’s space-time position (defined as \(\eta = \eta_o\) and \(r = 0\)) since, as a consequence of the dynamical motion of the LG free-falling observer, the observer is no longer at the origin \((r = 0)\) of our coordinates system for \(\eta \neq \eta_o\).

By considering the above properties of the LG coordinates we obtain:

\[
\tau = \int_{\eta_{in}}^{\eta} d\eta'/a(\eta') + a(\eta)P(\eta, r, \theta^a) \equiv \int_{\eta_{in}}^{\eta} d\eta'/a(\eta') + \tau^{(1)}
\]

\[
w = \eta_+ + Q(\eta_+, \eta_-, \theta^a) \equiv \eta_+ + w^{(1)}
\]

\[
\tilde{\theta}^a = \theta^a + \frac{1}{2} \int_{\eta_o}^{\eta_+} dx \left[ \gamma^{ab}_{\theta b} \dot{Q} \right] (\eta_+, x, \theta^a) \equiv \theta^a + \tilde{\theta}^{a(1)}
\]

where we have defined:

\[
P(\eta, r, \theta^a) = \int_{\eta_{in}}^{\eta} d\eta'/a(\eta') \phi(\eta', r, \theta^a)
\]

\[
Q(\eta_+, \eta_-, \theta^a) = \int_{\eta_o}^{\eta_+} dx \frac{1}{2} (\psi + \phi) (\eta_+, x, \theta^a).
\]

Here \(\eta_{in}\) represents an early enough time when the perturbation (or better the integrand) was negligible: this means that the integrals over all relevant perturbation scales are insensitive to the actual value of \(\eta_{in}\).

To first order we can then use Eqs. (5.35), (5.36) and (5.37) to compute
the non-trivial entries of the GLC metric of Eq. (4.3), and obtain:

\[ \Upsilon^{-1} = \frac{1}{a(\eta)} (1 + \partial_+ Q - \partial_r P), \quad (5.39) \]

\[ U^a = \partial_0 \tilde{\theta}^a(1) - \frac{1}{a} \gamma^a_\theta \partial_0 \tau(1), \quad (5.40) \]

\[ \gamma^{ab} = a^{-2} \left\{ \gamma^{ab}_0 (1 + 2\psi) + \left[ \gamma^{ac}_0 \partial_c \tilde{\theta}^b(1) + (a \leftrightarrow b) \right] \right\}. \quad (5.41) \]

We now follow the same procedure as in the previous section. We need to evaluate the determinant of \( \gamma \). From Eq. (5.41) we obtain:

\[ \gamma^{-1} = \det \gamma^{ab} = \left( a^2 r^2 \sin \theta \right)^{-2} \left[ 1 - 2a \eta + 2\partial_a \tilde{\theta}(1) \right]. \quad (5.42) \]

The other missing quantity contains the time derivative at the observer position, which can be evaluated by exactly the same method as the one followed in Sect. 5.1. Since we are interested in \( \partial_\tau \), we find to first order:

\[ \partial_\tau \tilde{x}^\mu = \begin{pmatrix}
\frac{a(\eta)}{1 + \partial_+ Q + \partial_- Q} & \frac{a(\eta)\partial_r P}{1 + \partial_+ Q - \partial_- Q} & \frac{a(\eta)\partial_0 P}{\partial_+ \tilde{\theta}^{(1)} + \partial_- \tilde{\theta}^{(1)}}
\end{pmatrix}. \quad (5.43) \]

By inverting the relation among derivatives, we find:

\[ \partial_\tau = \frac{2}{a(\eta)} \left( \partial_\tau - \frac{\partial_0 P}{a} \partial_- - 2\partial_- P \partial_\tau - \partial_0 Q \partial_\tau - \partial_0 \tilde{\theta}(1) \partial_\tau \right). \quad (5.44) \]

We can compare this preliminary result with Eq. (5.18). Since \( Q \) corresponds to \( w_1 \), the two expressions differ only by terms depending on \( P \), which, in turn, is related to the transformation’s law of \( \tau \). In particular, the non-vanishing terms of the first row in (5.43) imply that GLC and LG coordinates don’t share the same time. As a consequence, geodesic observers in the LG are no longer static, and aberration effects due to their velocity can appear. Indeed, the contribution of the the observer terms to the angular distance is given by:

\[ \sin \theta_o \det \left( \partial_\tau s^B_\eta(\lambda_o) \right) = 1 - 2\partial_0 P_o. \quad (5.45) \]

Before discussion the physical meaning of this results, let us combine together numerator and denominator in Eq. (4.74) to get:

\[ (d_2^2)_{LG} = \frac{a^2 r^2 \sin \theta_s}{\sin \theta_o} \left\{ 1 - 2 \int_{\eta_o}^{\eta_s} dy^a \frac{a(y^a)}{a(\eta_o)} \partial_\tau \phi \left( y^a, 0, \theta^a \right) - 2\psi_s \right\}
- \frac{1}{2} \int_{\eta_o}^{\eta_s} dx \gamma_0^{ab} \int_{\eta_o}^{x} dy \frac{1}{2} \partial_a \partial_b \left[ \psi(\eta_+, y, \theta^a) + \phi(\eta_++, y, \theta^a) \right]. \quad (5.46) \]
This result will be compared to the corresponding one in the SG (see Eq. (5.20)) in the following section.

Let us now extend to second order our discussion of the terms evaluated at the observer position. Putting, for simplicity, \( \phi = \psi \), the second-order generalization of the LG metric (5.34) can be written as

\[
ds^2_{PG} = a^2(\eta) \left[ -(1 + 2\Phi)d\eta^2 + (1 - 2\Psi)\delta_{ij}dx^idx^j \right]
\]

where \( \Phi \) and \( \Psi \) are scalar perturbations defined, to second order, as:

\[
\Phi \equiv \psi + \frac{1}{2}\psi^{(2)}, \quad \Psi \equiv \psi + \frac{1}{2}\psi^{(2)}, \quad (5.48)
\]

The generalized coordinates transformations, in such a case, are given by:

\[
\tau = \tau^{(0)} + \tau^{(1)} + \tau^{(2)} \quad (5.49)
\]

\[
\tau = \left( \int_{\eta}^{\eta'} d\eta a(\eta') \right) + a(\eta)P(\eta, r, \theta^a)
\]

\[
+ \int_{\eta}^{\eta'} d\eta' \frac{a(\eta')}{2} \left[ (\phi^{(2)} - \psi^2) + (\partial_r P)^2 + \gamma_{0b}^a \partial_a P \partial_b P \right] (\eta', r, \theta^a),
\]

\[
w = w^{(0)} + w^{(1)} + w^{(2)} \quad (5.50)
\]

\[
w = \eta_+ + Q(\eta_+, \eta_-, \theta^a)
\]

\[
+ \frac{1}{4} \int_{\eta_0}^{\eta_+} dx \left[ (\psi^{(2)} + \phi^{(2)} + 4\psi \partial_+ Q + \gamma_{0b}^a \partial_a Q \partial_b Q) (\eta_+, x, \theta^a) \right],
\]

\[
\tilde{\theta}^a = \tilde{\theta}^{a(0)} + \tilde{\theta}^{a(1)} + \tilde{\theta}^{a(2)} \quad (5.51)
\]

\[
= \theta^a + \frac{1}{2} \int_{\eta_0}^{\eta_+} dx \left[ \gamma_{0c}^a \partial_c Q \right] (\eta_+, x, \theta^a)
\]

\[
+ \int_{\eta_0}^{\eta_+} dx \left[ \gamma_{0c}^a \zeta_c + \psi \xi^a + \lambda^a \right] (\eta_+, x, \theta^a),
\]

where we have used the following shorthand notations:

\[
\zeta_c = \frac{1}{2} \partial_c w^{(2)}
\]

\[
= \frac{1}{8} \int_{\eta_0}^{\eta_+} du \partial_c \left[ (\psi^{(2)} + \phi^{(2)} + 4\psi \partial_+ Q + \gamma_{0f}^e \partial_e Q \partial_f Q) \right], \quad (5.52)
\]

\[
\xi^a = \partial_+ \tilde{\theta}^{a(1)} + 2\partial_+ \tilde{\theta}^{a(1)}
\]

\[
= \partial_+ \left( \frac{1}{2} \int_{\eta_0}^{\eta_+} du \left[ \gamma_{0c}^a \partial_c Q \right] + \left[ \gamma_{0c}^a \partial_c Q \right] \right), \quad (5.53)
\]

\[
\lambda^a = \partial_+ \tilde{\theta}^{a(1)} \left( \partial_+ \tilde{\theta}^{a(1)} (\eta_+, x, \theta^a) - \delta^a_0 \partial_+ Q \right)
\]

\[
= \frac{1}{4} \left( \gamma_{0c}^a \partial_c Q \right) \left( \int_{\eta_0}^{\eta_+} du \partial_d \left( \gamma_{0e}^c \partial_e Q \right) \right) - \frac{1}{2} \left( \partial_+ Q \gamma_{0ab} \partial_b Q \right) \quad (5.54)
\]
Following the previous method, we find that:

\[
\frac{\sin \theta}{\det (\partial_r \tau^B_0(\lambda_o))} = 1 + \left[ -2 \partial_r P + 2(\partial_r P)^2 + \gamma_0^{ab} \partial_a \partial_b P - 2 \psi \partial_r P \right]_o \\
- \int_{\eta_{io}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)} \partial_r \left[ \phi^{(2)} - \psi^2 + (\partial_r P)^2 + \gamma_0^{ab} \partial_a \partial_b P \right].
\]

This result can also be rewritten in the convenient form:

\[
\frac{\sin \theta}{\det (\partial_r \tau^B_0(\lambda_o))} = \left[ (1 - \partial_r P)^2 + \nabla_i P \nabla^i P - 2 \psi \partial_r P \right]_o \\
- \int_{\eta_{io}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)} \partial_r \left( \phi^{(2)} - \psi^2 + \nabla_i P \nabla^i P \right),
\]

where \( \nabla_i \) is the gradient operator in polar coordinates.

This contribution to \( d_A^2 \) can be matched, following [85], to the kinematic correction \( d\Omega / d\hat{\Omega} \), where \( d\Omega \) is the infinitesimal solid angle measured at \( o \) by our free-falling observer, while \( d\hat{\Omega} \) is the infinitesimal solid angle measured at \( o \) by another observer which is static in the considered gauge. Indeed, considering that our observer has a peculiar velocity \( \vec{v} \), we find that the infinitesimal solid angle \( d\Omega \) transforms under a Lorentz boost as follows:

\[
d\Omega = \frac{1 - v^2}{(1 - \vec{v} \cdot \vec{n})^2} d\hat{\Omega},
\]

where \( \vec{n} \) is the unit vector along the direction connecting the source to the observer. We then expect that the effect of the velocity on \( d_A^2 \) can be factorized (up to second order) as:

\[
d_A^2 = \left[ \frac{(1 - \vec{v} \cdot \vec{n})^2}{1 - v^2} \right]_o \left( \frac{\sqrt{\gamma_s}}{\sin \theta_o} \right) = \left[ 1 + v^2 - 2 \vec{v} \cdot \vec{n} + (\vec{v} \cdot \vec{n})^2 \right]_o \left( \frac{\sqrt{\gamma_s}}{\sin \theta_o} \right).
\]

On the other hand, since \( \tau \) plays the role of the effective gauge-invariant velocity potential, we can expand the spatial components of the perturbed velocity \( v_\mu \) of the LG (geodesic) observer as:

\[
v_i = -\partial_i \tau^{(1)} - \partial_i \tau^{(2)}, \quad v^i = -\frac{1}{a^2} \left[ \partial^i \tau^{(1)} + \partial^i \tau^{(2)} + 2 \psi \partial^i \tau^{(1)} \right],
\]

where \( \tau^{(1)} \) and \( \tau^{(2)} \) denote, respectively, the first- and second-order part of the coordinate transformation \( \tau = \tau(\eta, r, \theta^a) \) between LG and GLC gauge (see Eq.(5.50) for their explicit expressions). We can also expand the unit vector \( n_\mu \), in polar coordinates and to first order (which is enough), as:

\[
n^\mu = \left( 0, -\frac{1}{\gamma}(1 + \psi), 0, 0 \right), \quad n_\mu = \left( 0, -a(1 - \psi), 0, 0 \right).
\]
Inserting the above expressions for $\vec{v}$ and $\vec{n}$ into Eq. (5.58) we get, up to second order:

$$\left[ 1 + v^2 - 2\vec{v} \cdot \vec{n} + (\vec{v} \cdot \vec{n})^2 \right]_0 = \left[ (1 - \partial_r P)^2 + \nabla_i P \nabla^i P - 2\psi \partial_r P \right]_0 \quad (5.61)$$

This shows that the perturbative corrections to the background relation $\det(\partial_\tau s^B_b(\lambda_0)) = \sin \theta_0$, appearing in Eq. (5.56), can be exactly interpreted (to second order) as the effect of the peculiar velocity of our free-falling observer in the LG. More generally, such velocity coincides with the so-called gauge-invariant velocity perturbation [14].

As in the case of the SG, we may ask whether this peculiar-velocity effect can be removed by some further gauge fixing. The answer in this case is negative. First of all, there is no residual gauge symmetry in the LG [14]. Also, if we fix the GLC gauge in order to recover the uncorrected result (as explained in Sec. 3), the GLC angles will not coincide with those in the LG even at $\eta = \eta_0$ and the peculiar velocity correction should now be a consequence of the modified coordinate transformation connecting the two gauges.

An interesting property of the kinematic correction in Eq. (5.57) is that, upon integration over the whole solid angle, the result one obtains is always $4\pi$ quite independently of the peculiar velocity $v$. Therefore, if we are only interested in the averaged energy flux (i.e. in $\langle d^{-2}_L \rangle$) on constant-redshift surfaces, all perturbative contributions to Eq. (5.56) (that were missed in [38, 91]) simply drop out! This is no longer true if one computes correlations functions or dispersions around averaged values, or in case one wants to average quantities other than the flux. In those cases the contributions of the velocity corrections are nonvanishing, but they turn out to be numerically subleading with respect to the other contributions already considered in [38, 91].

In order to obtain the complete second-order expression of $d_A$ we need finally to combine numerator and denominator of Eq. (4.73), expressed in terms of PG quantities. The result of this long, but straightforward calculation is reported in [78], where the final form of the luminosity distance as a function of the redshift is presented.
5.3 Comparing synchronous and longitudinal gauge

In order to compare the previous results for \(d_A\) obtained in two different gauges, and check their physical equivalence (up to first order), let us consider the infinitesimal gauge transformation which connects the SG to the LG. An “infinitesimal” coordinate transformation can be parameterized, to first-order, by the generator \(\epsilon^\mu_{(1)}\) as:

\[
x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu_{(1)},
\]

where

\[
\epsilon^\mu_{(1)} = \left( \epsilon^0_{(1)}, \partial^i \epsilon_{(1)} + \epsilon^i_{(1)} \right).
\]

Under the associated gauge transformation (or local field reparametrization) – where, by definition, old and new fields are evaluated at the same space-time position – a tensor object changes, to first order, as

\[
T^{(1)} \rightarrow \tilde{T}^{(1)} = T^{(1)} - L_{\epsilon^{(1)}} T^{(0)},
\]

where \(L_{\epsilon^{(1)}}\) is the Lie derivative performed with respect to the vector \(\epsilon^\mu_{(1)}\).

Following [92] we then obtain the following relations between SG and LG quantities:

\[
\phi = -\frac{aE'}{2a} - \frac{E''}{2},
\]

\[
\bar{\psi} = \psi + \frac{aE'}{2a} + \frac{1}{6} \Delta_3 E,
\]

\[
(d_A^2)_{LG} = (d_A^2)_{SG} - \left[ \epsilon^\mu \partial_\mu (d_A^2)_{(0)}^{(0)} \right]_o - \left[ \epsilon^\mu \partial_\mu (d_A^2)_{(0)}^{(0)} \right]_s,
\]

where we denoted with \(\bar{\psi}\) the spatial trace of LG and the components of \(\epsilon^\mu_{(1)}\) are given by \(^1\)

\[
\epsilon^0_{(1)} = \frac{aE'}{2}, \quad \epsilon_{(1)} = \frac{1}{2} E, \quad \epsilon^i_{(1)} = 0,
\]

and where we have taken into account that \(d_A^2\) is a bi-scalar object.

By applying the above relations we can express the LG result completely in terms of the SG variables as

\[
(d_A^2)_{LG} = \frac{a^2 r^2 \sin \theta_o}{\sin \theta_o} \left[ 1 + \partial_r E'_o - 2\psi_s - \frac{a'}{a_s} E''_s - \frac{1}{3} \Delta_3 E_s \right. \\
\left. - \frac{1}{4} \int_{\eta_o}^{\eta_r} dx \gamma_0^{ab} \int_{\eta_o}^{\eta_r} dy \partial_a \partial_b \left( \psi + \frac{1}{6} \Delta_3 E - \frac{E''}{2} \right) \right].
\]

\(^1\)We consider only scalar fluctuations and consequently put \(\epsilon^i_{(1)} = 0\).
This has to be compared with the r.h.s of Eq. (5.67), which is given by

\[
(d^2_A)_{SG} - \left[ \epsilon^{\mu \nu} \partial_\mu \left( d^2_A \right)^{(0)} \right]_o - \left[ \epsilon^{\mu \nu} \partial_\mu \left( d^2_A \right)^{(0)} \right]_s = \frac{a^2_s}{r^2_s} \sin \theta_s \left[ 1 + \left( \frac{3}{2} \partial_r^2 E - \frac{1}{2} \Delta_3 E \right)_o \right.
\]
\[+ \left. -2 \ddot{\psi}_s + \left( \frac{1}{6} \Delta_3 - \frac{1}{2} \partial_r^2 \right) E_s + \frac{1}{2} \int_{\eta_+}^{\eta_-} dx \gamma_{0}^{ab} \partial_a \partial_b \left( \partial_r - \frac{1}{r} \right) E \right.
\]
\[- \frac{1}{4} \int_{\eta_+}^{\eta_-} dx \gamma_{0}^{ab} \int_{\eta_+}^{x} dy \partial_a \partial_b \left( \psi + \frac{1}{6} \Delta_3 E - \frac{1}{2} \partial_r^2 E \right) + \left( \frac{1}{2} \partial_r^2 \cot \theta \partial_\theta E \right)_o \left.ight.
\]
\[- \frac{a^2_s}{r^2_s} E'_s \left. - \frac{1}{r_s} \partial_r E_s - \left( \frac{1}{2} \partial_r^2 \cot \theta \partial_\theta E \right)_s \right] . \tag{5.70}
\]

At first sight Eq.(5.70) has no contributions to \( d^2_A \) from terms in \( E \) that are linear in \( x^i \) when expanded around the observer’s position. This is in clear contrast with the LG expression of Eq.(5.69) which, on the contrary, includes such terms (see, in particular, the contribution \( (\partial_r E')_o \) describing the peculiar velocity of the observer). Besides this, the two expressions appear to differ in many other respects. However, as we shall now see, the two expressions are essentially the same, apart from a small difference that can be made to disappear (at a given value of the observer’s time) by exploiting the residual gauge freedom of the SG.

Indeed, by using in the integral of the third line of Eq.(5.70) the identity \( \partial_r^2 = \partial_\eta^2 - 4 \partial_\eta (\partial_\eta - \partial_\eta) \), we find that the \( \partial_r^2 \) contribution exactly reproduces the corresponding terms of the double integral of Eq. (5.69). The remaining contributions from \( \partial_r^2 \) can be integrated once and added to the integral appearing in the second line of Eq. (5.70) to give:

\[
\left. + \frac{1}{2} \int_{\eta_+}^{\eta_-} dx \gamma_{0}^{ab} \partial_a \partial_b \left( \partial_r - \frac{1}{r} \right) E(\eta_+, x, \theta^a) \right.
\]
\[+ \left. (\partial_x - \partial_\eta) \left[ E(\eta_+, x, \theta^a) - E(\eta_+, \eta_+, \theta^a) \right] \right] . \tag{5.71}
\]

These integrals can be done explicitly, giving contributions both at the source and at the observer. In particular, we can split the total contribution in two parts: the first one, given by

\[
- \frac{1}{2} \left( \gamma_{0}^{ab} \partial_a \partial_b E \right)_s , \tag{5.72}
\]

nicely combines with all remaining source terms of Eq. (5.70) to reproduce all remaining source terms of Eq. (5.69). The second contribution, obtained when at least one lower boundary of the various integrals is considered, is
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given by:

\[
\frac{1}{2} \gamma_0^{ab} \partial_a \partial_b E(\eta, \eta, \theta^a) + \left[ \left( r \gamma_0^{ab} \right)_s - \left( r \gamma_0^{ab} \right)_o \right] \partial_a \partial_b \partial_+ E(\eta, \eta, \theta^a),
\]

(5.73)

where \( E(\eta, \eta, \theta^a) \) is the limiting value of \( E \) approaching the observer position along the light-cone. All these contributions have to be evaluated at the observer, except for the geometric prefactor \( \left( r \gamma_0^{ab} \right)_s \) of the second term, which is referred to the source position. In order to obtain a full agreement between the results for \( d_A \) in the two gauges (see Eq.(5.67)) we should thus verify the following equality:

\[
(\partial_r E')_o = \left( \frac{3}{2} \partial_r^2 - \frac{1}{2} \Delta_2 \right) E_o + \frac{1}{2} \left( r \gamma_0^{ab} \right)_o \left( r \gamma_0^{ab} \right)_o (\partial_a \partial_b \partial_+ E)_o,
\]

(5.74)

where we recall that the second and third terms on the r.h.s. have to be evaluated along the light-cone, while the first term does not depend on the limiting process followed to arrive at the observer’s position, see Eq. (5.21), and has a different physical origin.

It is convenient, at this point, to use the expansion of the SG variable \( E \) around the observer position. Note, however, that we have to set \( E_i = 0 \) since otherwise the gauge transformation in Eq. (5.68) becomes singular at \( r = 0 \) (\( \delta \theta \sim r^{-2} E_i x_j \rightarrow \infty \)). With this constraint, using Eq.(5.14), the expansion of the term on the l.h.s. of Eq. (5.74) gives:

\[
(\partial_r E')_o = \left( \frac{E_i x^i}{r} \right)_o,
\]

(5.75)

while the expansion of all terms on the r.h.s. gives:

\[
\frac{1}{2} E_{ij} \left( \frac{3 x^i x^j}{r^2} - \delta_{ij} \right)_o - \left[ \frac{1}{r^2} \left( E_i x^i \eta - \frac{1}{4} E_{ij} \Delta_2 x^i x^j \right) \right]_o
+ \frac{1}{2} \left[ \left( \frac{1}{r s} - \frac{1}{r} \right) \left( \partial_\theta^2 + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \left( \frac{E_{ij} x^i x^j}{r} \eta + \frac{E'_{ij} x^i}{r} \right) \right]_o
= \left( \frac{E'_{ij} x^i}{r} \right)_o - \frac{1}{2} \left[ \left( \partial_\theta^2 + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \left( E_{ij} x^i x^j \right) \right]_o.
\]

(5.76)

To obtain the last line we have used the explicit definition of \( \Delta_2 \) and the fact that the explicit time-dependent contributions to the above equation, proportional to \( E'_{ij} x^i \eta \), are generated by the expansion of terms approaching the observer along light-cone trajectories. For such terms we can safely replace \( \eta \) with \(-r\) (to first order), so that the last two contributions of the second line exactly cancel between themselves, while the time-dependent contribution of the first line exactly matches the result of Eq. (5.75). The only mismatch consists of the last term appearing in Eq. (5.76).
Let us finally exploit the already mentioned residual gauge freedom of the SG, considering the (time independent!) gauge transformation (valid close to the observer’s position) with generator

$$\epsilon^\mu_{(1)} = \left(0, \partial^k \left(\frac{x^i x^j}{4} E_{ij}\right)\right), \quad (5.77)$$

leading to

$$E(r, \eta, \theta) \rightarrow \tilde{E}(r, \eta, \theta) = E(r, \eta, \theta) - \frac{1}{2} x^i x^j E_{ij} , \quad (5.78)$$

$$\psi(r, \eta, \theta) \rightarrow \tilde{\psi}(r, \eta, \theta) = \psi(r, \eta, \theta) + \frac{1}{6} \delta^{ij} E_{ij} . \quad (5.79)$$

By applying such a (partial) gauge fixing we can set $E_{ij}$ to zero in the small-$x$, small-$\eta$ expansion of $\tilde{E}(r, \eta, \theta)$ and thus eliminate all terms of Eq. (5.76) except for the $E_i'$ contribution. Within this gauge choice we thus find full agreement for the expression of $d_A$ in the two gauges at one observer’s time.

We recall that precisely the same residual gauge fixing was used (see Sect. 5.1) to remove the anisotropy correction in the SG at one specific time, i.e. to remove anisotropy around the observer ($E_{ij} = 0$ at $\eta = \eta_o$). Since the LG corresponds to choosing shear-free equal-time hypersurfaces, it is hardly surprising that such a residual gauge transformation is necessary in order to recover agreement between the two gauges.

### 5.4 From GLC to LTB metrics

In the previous sections of this chapter we have considered coordinate transformation between GLC and perturbed FLRW metric backgrounds. In this section we will extend the same approach to the case of exact, non-perturbative geometries, considering in particular the one described by the LTB metric.

As already stressed, such a geometry is invariant for spatial rotations around its center at $r = 0$. An observer located at that position cannot detect anisotropy, but only a radial inhomogeneity. Any other observer displaced away from the center experiences instead the presence of anisotropy, related to the existence of a preferred spatial direction aligned along the axis connecting him/herself with the center $r = 0$. 

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Our purpose here is to express the observable quantities computed in the GLC gauge for the case of an off-center observer embedded into the LTB geometry. We will thus assume that the centers of the GLC and the LTB coordinates are displaced by a given distance \(d\) acting as the parameter controlling, in a non-perturbative way, the strength of the anisotropy contributions.

For the sake of simplicity, and with no loss of generality, let us also assume that the azimuthal angles in the GLC and LTB coordinate systems are equal, i.e. that \(\tilde{\theta}_2 = \phi\) (see Fig. 5.1). In that case, by applying elementary geometric arguments, we obtain

\[
\tau = t, \quad w = W(t, r, \theta), \quad \tilde{\theta}_1 = \arccos \left(\frac{r \cos \theta - d}{\sqrt{r^2 + d^2 - 2rd \cos \theta}}\right), \quad \tilde{\theta}_2 = \phi, \quad (5.80)
\]

where the identity \(\tau = t\), proved in [38], holds because of the synchronous gauge choice of the LTB metric. Here \(W(t, r, \theta)\) is an implicit function that must satisfy the conditions

\[
g^{w\omega}_{\text{GLC}} = g^{w\alpha}_{\text{GLC}} = 0, \quad \text{i.e. :}
\]

\[
-(\partial_t W)^2 + X^{-2} (\partial_r W)^2 + A^{-2} (\partial_\theta W)^2 = 0,
\]

\[
X^{-2} \partial_t W \partial_r \tilde{\theta}_1 + A^{-2} \partial_\theta W \partial_\theta \tilde{\theta}_1 = 0. \quad (5.81)
\]

Given that that \(\partial_r \tilde{\theta}_1 = \frac{d \sin \theta}{r^2 + d^2 - 2rd \cos \theta}\) and \(\partial_\theta \tilde{\theta}_1 = \frac{r(r - d \cos \theta)}{r^2 + d^2 - 2rd \cos \theta}\), we obtain:

\[
\begin{align*}
\partial_t W = & \frac{A^2 d \sin \theta}{\sqrt{A^2 d^2 \sin^2 \theta + r^2 X^2 (r - d \cos \theta)^2}}, \\
\partial_r W = & \frac{r (r - d \cos \theta) X^2}{\sqrt{A^2 d^2 \sin^2 \theta + r^2 X^2 (r - d \cos \theta)^2}}. \quad (5.82)
\end{align*}
\]

It follows that the components of the GLC metric are given by:

\[
\begin{align*}
\Upsilon &= \frac{1}{\partial_t W} \\
U^a &= \begin{pmatrix} 0 & 0 \end{pmatrix} \\
\gamma^{ab} &= \begin{pmatrix} \frac{A^2 d^2 \sin^2 \theta + r^2 X^2 (r - d \cos \theta)^2}{A^2 X^2 (d^2 + r^2 - 2rd \cos \theta)^2} & 0 \\ 0 & A^{-2} \sin^{-2} \theta \end{pmatrix} \quad , \quad (5.83)
\end{align*}
\]

and the derivative with respect to \(\tau\) becomes:

\[
\partial_{\tau} f = \partial_t - \frac{r (r - d \cos \theta) \partial_t + d \sin \theta \partial_\theta}{\sqrt{A^2 d^2 \sin^2 \theta + r^2 X^2 (r - d \cos \theta)^2}}. \quad (5.84)
\]

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5.4. FROM GLC TO LTB METRICS

Following [78], we are now in the position of writing the angular distance for a generic off-center observer in the LTB coordinates. Such a problem has already been studied in the literature, but has been solved in a different and implicit way [40, 34]. As it will be shown below, our result, on the contrary, is explicitly given in terms of the LTB coordinates.

What we need for defining the angular distance (see Eq. (4.74)) is the quantity to be evaluated at the source position, given by:

\[
\sqrt{\gamma} = \frac{A^2 X \left( r^2 + d^2 - 2r \cos \theta \right)}{\sqrt{A^2 d^2 \sin^2 \theta + r^2 X^2 \left( r - d \cos \theta \right)^2}} \sin \theta, \tag{5.85}
\]

Figure 5.1: Coordinate relation between the LTB radius \( r \) (red) and the observer one \( \tilde{r} \) (blue). We are imposing that the azimuthal angles are equal \( (\tilde{\theta}^2 = \phi) \).
and the quantity to be evaluated at the observer position, given by

$$
\left( \frac{4\sqrt{\gamma}}{\det g^{ab} \gamma_{ab}} \right)_o = \frac{A_0(d)}{d X_0(d) \cos \frac{\omega}{2}} \equiv \frac{A_0(d)}{d X_0(d) \sin \theta} \equiv G(\theta) \frac{\omega}{2} \cos \theta_o \frac{\omega}{2} = \frac{A_0(d)}{d X_0(d) \sin \theta}.
$$

Here we have assumed that the observer position is at \( t = t_o, r = d, (\theta, \phi) = (\theta_o, \phi_o) \) in the LTB coordinates, and we have used the third relation of Eqs. (5.80), with \( r = d \), to prove that \( \cos(\theta_o/2) = \sin \tilde{\theta} \) (\( \tilde{\theta} \) being by definition the angle seen by the observer in the GLC coordinates). The function \( \tilde{G}(\tilde{\theta}) \) is a function of \( \tilde{\theta} \) and of the derivatives of \( A(t, r) \) and \( X(t, r) \) such that \( \tilde{G}(\tilde{\theta}) \to 1 \) when \( \theta_o \to 0 \).

We fix the angular dependence of the LTB coordinates in such a way that for the observer angles \( (\theta_o, \phi_o) \equiv (0, 0) \), the observer term reduces to \( (\sin \tilde{\theta})^{-1} \) (in accordance with [75, 38, 39, 91, 88]) and Eqs. (4.92), (5.85), (5.86) lead to:

$$
d_A^2 = \frac{A^2 X (r^2 + d^2 - 2rd \cos \theta)}{\sqrt{A^2 d^2 \sin^2 \theta + r^2 X^2 (r - d \cos \theta)^2}} \frac{A_0(d)}{d X_0(d) \sin \theta} \sin \theta \sin \tilde{\theta}.
$$

Let us recall that in the LTB metric we can always exploit a residual gauge of freedom to fix \( A(t_o, r) \equiv A_0(r) = r \). Also, assuming that the geometry is spatially flat and imposing the off-diagonal components of the Einstein equations, we find that the \( g_{11} \) metric component satisfies the condition \( X(t, r) = \partial_r A(t, r) \).

In our case no peculiar velocity terms appear, due to the stillness of the chosen observer. Also, the anisotropy at the observer position can be seen as a pure geometric effect which disappears for a central observer. In the FLRW limit we have \( A(t, r) \to ra(t) \) and \( X(t, r) \to a(t) \), and Eq. (5.87) reduces to:

$$
d_A^2 = \frac{a^2 \sqrt{r^2 + d^2 - 2rd \cos \theta} \ r \sin \theta}{\sin \tilde{\theta}}.
$$

This expression appears to be very different from the usual one of the FLRW case. However, by noticing that we can re-express it in terms of the radial coordinate of the observer, \( \tilde{r} = \sqrt{r^2 + d^2 - 2rd \cos \theta} \), we have that \( r \sin \theta = \tilde{r} \sin \tilde{\theta} \) and we obtain \( d_A^2 = \tilde{r}^2 a(t)^2 \), exactly as expected. This is a good consistency check for the validity of our result (5.87).

Let us now evaluate, in the same context, the optical components of the amplification matrix, defined in Sect. 4.7. From Eq. (5.83) we may note that \( \gamma_{ab} \) is diagonal, and we can immediately conclude that \( \tilde{\omega} = 0 \), i.e. that no vorticity is present. The other lensing quantities become, in this LTB
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geometry:

\[
\mu = \left( \frac{\ddot{d}_A}{d_A} \right)^2 ,
\]

\[
(1 - \kappa)^2 = \left( \frac{u_{\tau^o}}{d_A} \right)^2 \left[ \gamma_{11} \left( \frac{\gamma_{11}}{(\gamma_{11})^2} \right)_o + \gamma_{22} \left( \frac{\gamma_{22}}{(\gamma_{22})^2} \right)_o \right] + 2 \sqrt{\gamma_{11} \gamma_{22}} \left( \frac{\sqrt{\gamma_{11} \gamma_{22}}}{\gamma_{11} \gamma_{22}} \right)_o ,
\]

\[
|\dot{\gamma}|^2 = \left( \frac{u_{\tau^o}}{d_A} \right)^2 \left[ \gamma_{11} \left( \frac{\gamma_{11}}{(\gamma_{11})^2} \right)_o + \gamma_{22} \left( \frac{\gamma_{22}}{(\gamma_{22})^2} \right)_o \right] - 2 \sqrt{\gamma_{11} \gamma_{22}} \left( \frac{\sqrt{\gamma_{11} \gamma_{22}}}{\gamma_{11} \gamma_{22}} \right)_o .
\]

(5.89)

In our context we have a static geodesic observer, characterized by \( u_{\tau^o} = 1 \). Using the expression of \( \ddot{d}_A \) given in Eq. (5.88) we thus obtain:

\[
\mu = \frac{r d X_0(d) a^2(t)}{A_0(d) A^2(t, r) X(t, r)} \times \sqrt{\frac{A^2(t, r) d^2 \sin^2 \theta + r^2 X^2(t, r)(r - d \cos \theta)^2}{d^2 + r^2 - 2rd \cos \theta} } ,
\]

\[
(1 - \kappa)^2 = \frac{A^2(t, r)}{4 \sin \theta d^2 r a^2(t) X_0^2(d)} \sqrt{d^2 - 2dr \cos \theta + r^2} \left[ d^2 \sin^2 \theta X_0^2(d) \right] + \frac{A_0^2(d) X^2(t, r) (d^2 - 2dr \cos \theta + r^2)^2}{d^2 \sin^2 \theta A^2(t, r) + r^2 X^2(t, r)(r - d \cos \theta)^2} + \frac{2d \sin \theta A_0(d) X_0(d) X(t, r) (d^2 - 2dr \cos \theta + r^2)}{\sqrt{d^2 \sin^2 \theta A^2(t, r) + r^2 X^2(t, r)(r - d \cos \theta)^2} } ,
\]

\[
|\dot{\gamma}|^2 = \frac{A^2(t, r)}{4 \sin \theta d^2 r a^2(t) X_0^2(d)} \sqrt{d^2 - 2dr \cos \theta + r^2} \left[ d^2 \sin^2 \theta X_0^2(d) \right] + \frac{A_0^2(d) X^2(t, r) (d^2 - 2dr \cos \theta + r^2)^2}{d^2 \sin^2 \theta A^2(t, r) + r^2 X^2(t, r)(r - d \cos \theta)^2} - \frac{2d \sin \theta A_0(d) X_0(d) X(t, r) (d^2 - 2dr \cos \theta + r^2)}{\sqrt{d^2 \sin^2 \theta A^2(t, r) + r^2 X^2(t, r)(r - d \cos \theta)^2} } .
\]

(5.90)

Note that, as expected from our choice of the observer angular coordinates \( \theta_o = 0 \), the obtained expressions are independent (by symmetry) of the azimuthal angle \( \phi \).
5.5 Off-center observer in LTB coordinates: CDM and ΛCDM models

In this section we will study the redshift dependence of the lensing quantities deduced in the previous section, for an observer located away from the central position of the LTB geometry.

We will consider explicit examples based on the LTB models that we have presented in Chapter 2. Let us recall, to this purpose, that solving the Einstein equations for the LTB metric and for a generic collection of non interacting perfect barotropic fluids one obtains [35] :

$$H^2(t, r) = H_0^2(r) \sum_n \Omega_{n0}(r) \left[ \frac{A_0(r)}{A(t, r)} \right]^{\alpha_n}$$

$$\sum_n \Omega_{n0}(r) = 1$$

$$X(t, r) = \frac{\partial_t A(t, r)}{\sqrt{1 - k(r)}},$$

(5.91)

where $H(t, r) \equiv \partial_t A(t, r)/A(t, r)$, $H_0(r) \equiv H(t_0, r)$ is the inhomogeneous Hubble function evaluated “today”, and $k(r)$ is a function that controls the curvature of the three-dimensional spatial hypersurfaces. The parameters $\Omega_{n0}(r) \equiv \Omega_n(t_0, r)$ and $\alpha_n$ are, respectively, the present value of the energy density in critical units and the power controlling the evolution of the $n$-th component of the fluid mixture (see also Chapter 2). From now on we will consider, for simplicity, only a flat model with $k(r) = 0$.

We can also define a mass parameter $M_0(r)$, related to the present matter density $\rho_0(r)$ by $\rho_0(r) \equiv M_0(r)/(4\pi A_0^3(r)/3)$, where, according to the second of Eqs. (5.91):

$$2G\rho_0(r) = \frac{3}{4\pi} \Omega_{m0}(r) H_0^2(r) = \frac{3}{4\pi} \left[ 1 - \sum_{n \neq \text{matter}} \Omega_{n0}(r) \right] H_0^2(r),$$

(5.92)

and where the matter density $\rho(t, r)$ satisfies $\rho(t, r) A(t, r)^3 = \rho_0(r) A_0(r)^3$.

In the following we will consider two particular solution only: the CDM case, with $\Omega_{m0}(r) = 1$, and the ΛCDM one, with $\Omega_{m0}(r) = 1 - \Omega_{\Lambda 0}(r)$.

Let us add that we can also conveniently define $\Omega_{\Lambda 0}(r) = \Omega_{\Lambda 0}(H_0^2/H_0^2(r))$, where $\Omega_{\Lambda 0}$ and $H_0$ represent the constant values of the corresponding homogeneous limit, and will be interpreted here as the background quantities that we recover at $r \to \infty$. Hence, for our purpose, $H_0(r)$ completely takes into account the radial profile of the energy density distribution, and is enough for a directly study the under/overdensity effects we want to consider.
5.5. OFF-CENTER OBSERVER IN LTB COORDINATES: CDM AND ΛCDM MODELS

We will choose, in particular, the following ansatz for $H_0(r)$:

$$H_0(r) = H_0 \left( 1 - \frac{H_0^2 - H_m^2}{H_0^2} \frac{\tanh \left( \frac{d-r_0}{2 \Delta r} \right) - \tanh \left( \frac{r-r_0}{2 \Delta r} \right)}{\tanh \left( \frac{d-r_0}{2 \Delta r} \right) + \tanh \left( \frac{r_0}{2 \Delta r} \right)} \right)^{1/3} \tag{5.93}$$

where $r_0$ is the typical radius of the under/overdensity region, and $d$ is the distance between the observer and the center of the inhomogeneity that appears in Eq. (5.80). This distance is assumed to be much larger than the under/overdensity radius, i.e. $d \gg r_0$, in such a way to have $\lim_{r \to \infty} H_0(r) = H_0$ from Eq. (5.93). Also, $\Delta r$ controls the distance scale of transition from the inhomogeneity bubble to the background, and it is assumed to satisfy $\Delta r \ll r_o \ll d$. Finally, $H_0$ is the background value of the Hubble constant, i.e. about 70 km s$^{-1}$ Mpc$^{-1}$, and $H_m$ is the Hubble constant at the center of the inhomogeneous region.

Using the ansatz of Eq. (5.93) into Eq. (5.92) we can express the density of this model in LTB coordinates and we obtain that the matter density of the background is proportional to $H_0^2$, while the density inside the bubble is proportional to $H_m^2$ (for a sharp transition $\Delta r \ll r_o$, as assumed here). Hence, choosing an under (over) density model means choosing $H_m$ lower (greater) than $H_0$.

We can now discuss these general aspects in two particular models: the CDM and ΛCDM models. In Chapt. 2, we have seen that the inhomogeneous scale factor is given, in these two cases, by:

- **inhomogeneous CDM model:**

  $$A(t, r) = r \left[ 1 + \frac{3}{2} H_0(r) t \right]^{2/3}, \tag{5.94}$$

  where $\Omega_{m0}(r) = 1$ (hence $\Omega_{A0}(r) = 0$), and where we have chosen $t_0 = 0$. In such a context the matter density is given by $\rho_0(r) = 3H_0^2(r)/8\pi G$.

- **inhomogeneous ΛCDM model:**

  $$A(t, r) = r \left[ \frac{1 - \Omega_{A0}(r)}{\Omega_{A0}(r)} \right]^{1/3} \left( \sinh \left[ \arcsinh \sqrt{\frac{\Omega_{A0}(r)}{1 - \Omega_{A0}(r)}} \right] \right)^{2/3}$$

  $$+ \frac{3}{2} \sqrt{\Omega_{A0}(r)} H_0(r) t \right)^{2/3}, \tag{5.95}$$

  where $t_0 = 0$, $\Omega_{A0}(r) + \Omega_{m0}(r) = 1$ and $H_0^2 \Omega_{A0} = H_m^2(r) \Omega_{A0}(r)$. This last condition is due to the fact that $\Lambda$ is constant. We can rewrite
this solution in terms of the parameters $H_0(r)$, $\Omega_{\Lambda 0}$, $H_0$, namely:

$$
A(t, r) = r \left[ \frac{H_0^2(r)}{\Omega_{\Lambda 0} H_0^2} - 1 \right]^{1/3} \left( \sinh \left[ \arcsinh \left( H_0 \sqrt{\frac{\Omega_{\Lambda 0}}{H_0^2(r) - \Omega_{\Lambda 0} H_0^2}} \right) \right] \right) + \frac{3}{2} \sqrt{\Omega_{\Lambda 0} H_0 t} \right)^{2/3}.
$$

(5.96)

Following the present experimental results, we can choose for these parameters the background values $H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}$ and $\Omega_{\Lambda 0} = 0.68$. For the matter density we obtain $8\pi G \rho_0(r) = 3 \left( H_0^2(r) - \Omega_{\Lambda 0} H_0^2 \right)$. This means that the radial dependence is the same as in the CDM model.

With the above choices the shape of the inhomogeneities is entirely controlled, in both cases, by the choice of the parameters $r_0$, $\Delta r$ and $H_{in}$ appearing inside $H_0(r)$ (see Eq. (5.93)). As an illustrative example, we will choose an under/overdensity located at $d = 10$, $d = 100$ and $d = 1000$ Mpc from us, with a radius $r_0 = 1$ Mpc and a transition shell of size $\Delta r = 0.1$ Mpc. Also, we define a density contrast $\delta(r)$, relative to the background density $\rho_{BG} = 3 \Omega_{m0} H_0^2 / 8\pi G = 3(1 - \Omega_{\Lambda 0}) H_0^2 / 8\pi G$, and defined by:

$$
\delta(r) = \frac{\rho_0(r) - \rho_{BG}}{\rho_{BG}} = \frac{H_0^2(r) - H_0^2}{H_0^2 (1 - \Omega_{\Lambda 0})},
$$

(5.97)

which can be applied to both models (by imposing $\Omega_{\Lambda 0} = 0$ in the CDM case). In such a way the maximum contrast is obtained at the center of the bubble, where $\delta_{\text{max}} = \delta(0) = H_0^2 - H_0^2 (1 - \Omega_{\Lambda 0})$. In particular, if we consider a variation of $2 \text{ km s}^{-1} \text{Mpc}^{-1}$ for $H_0$, we find a maximum density contrast $\delta_{\text{max}}$ ranging from $-0.056$ for the underdensity to $0.058$ for the overdensity in the CDM scenario; in the $\Lambda$CDM scenario, instead, we have values of $\delta_{\text{max}}$ ranging from $-0.176$ to $0.181$, respectively, for under and over densities.

We have analyzed the effects of these different configurations on the behaviour of the variables $\mu$ (the lensing magnification) and $\Delta m = 5 \log_{10}(d_A/d_{\tilde{A}})$ (i.e. the difference between the distance modulus of the LTB metric and the background homogeneous metric). We have computed these quantities for a source placed along the vertical axis (i.e. $\theta = \pi = \tilde{\theta}$), and plotted them versus the redshift $z$. To this purpose we have numerically solved the geodesic equation with LTB coordinates using Eqs. (2.64). Our results are presented in Figs. 5.2, 5.3 and 5.4, where the solid lines refers to the CDM model while the dotted lines refer to the $\Lambda$CDM case.

We can note that deviation from homogeneity produce effects of the same order of magnitude in both models. Also, for $d = 10$ Mpc, the effects exactly
5.5. OFF-CENTER OBSERVER IN LTB COORDINATES: CDM AND ΛCDM MODELS

Figure 5.2: On the left side the magnification $\mu$ is plotted for under and over density at a distance of 10 Mpc from the observer. On the right side, the same plot for the difference of the distance modulus $\Delta m = 5 \log_{10}(d_A/\bar{d}_A)$. Solid lines refer to the CDM model, dotted lines refer to the ΛCDM model.

appear at the same redshift for both model (Figs. 5.2). This means that the effects due to inhomogeneities, in this case, are rather insensitive to the value of $\Lambda$, at least for small distances. However, when the inhomogeneous region is placed at $d = 100$ Mpc, differences as a function of redshift appear.

This difference becomes more evident if we consider an inhomogeneous region at $d = 1000$ Mpc, and we can see that the greater the distance, the higher is the correction. Indeed, at $d = 10$ Mpc the correction induced on the distance modulus is about 0.015% (see the peaks in Figs. 5.2), while at $d = 100$ Mpc the maximum $\Delta \mu$ is $\sim 0.15\%$ (see the maxima in Figs. 5.3); for $d = 1000$ Mpc the deviation from the homogeneous prediction is about 1.5%.

It is also important to stress that the redshift required for corrections due to large scale inhomogeneities in the ΛCDM model is lower than the analogous case for the CDM model. We can interpret these results in the following way: the possible presence of large scale inhomogeneities ($d \geq 500$ Mpc) can produce non negligible corrections to the distance modulus. Even if the amount of corrections is of the same level for both models, the CDM scenario predicts that they can appear later than in the ΛCDM case.
We finally present the results of our LTB example for a source placed at a position which is not aligned with the vertical axis (defining the direction of the displacement of the observer from the center of the bubble). We have thus studied the redshift evolution of $\mu$, $\sin \theta (1 - \kappa)^2$ and $\sin \theta |\hat{\gamma}|^2$ for different values of the angle $\tilde{\theta}$, in particular for $\tilde{\theta} = \pi - \arcsin(\{10, 2, 1, 0.5, 0\} \times r_o/d)$. The convergence and shear quantities have been multiplied by $\sin \theta$ in order to get rid of the coordinate divergence at $\tilde{\theta} = \pi = \theta$ (see Eq. (5.90)). The corresponding plots have been obtained by solving the geodesic equation, as before, and are shown in Fig. 5.5 (respectively in thin, thick, dotted and dashed lines).

We can interpret these curves in the following way. The thin solid curves correspond to the angle far away from the bubble ($\tilde{\theta} = \pi - \arcsin(10r_o/d)$) and at its center ($\tilde{\theta} = \pi$). The effect of the bubble is practically inexistent in the first case, and the values are very close to $\mu = 1$, $\kappa = 0$ and $|\hat{\gamma}| = 0$, as expected for the homogeneous case. The second case shows the maximal effect produced by the bubble, because the photons go through the longest part of it. For the dashed line what is involved is roughly the half-radius
of the bubble ($\tilde{\theta} = \pi - \arcsin(r_o/2d)$) and this case is quite similar to the bubble center, as the density profile described by $H_0(r)$ is already reaching its central value for such an angle.

It is important to stress that in Fig. 5.5 we have a magnification of different sign for the line of sight pointing at the border of the inhomogeneous region ($\tilde{\theta} = \pi - \arcsin(r_o/d)$, dotted curve). We can deduce that in these regions (“before” and “after” the bubble w.r.t. the observer) the non-linear effects become very important, so large deviations from the perturbative corrections appear and our exact results take such a situation into account. The importance of the exact non-linear result has already been noticed by [93]. However, the shapes of the curves are different. We can argue that these differences are due to the particular choice of the inhomogeneity profile in Eq. (5.93) and to our choice of flat spatial-hypersurfaces. Finally, one can show that the deviation from the homogeneous scenario, outside the bubble, is equally shared between the shear and the convergence, in such a way that we have no effect on the magnification. It has been checked that all the curves in Fig. 5.5 satisfy the relation $|\hat{\gamma}|^2 = (1 - \kappa)^2 - \mu^{-1}$.

Figure 5.4: On the left side the magnification $\mu$ is plotted for under and over density at a distance of 1000 Mpc from the observer. On the right side, the same plot for the difference of the distance modulus $\Delta m = 5 \log_{10}(d_A/\bar{d}_A)$. Solid lines refer to the CDM model, dotted lines refer to the $\Lambda$CDM model.
Figure 5.5: Redshift evolution of convergence, shear and magnification (from top to bottom) for 5 different angles of observation: \( \theta = \pi \) and \( \pi - \arcsin(10 r_o / d) \) (thin), \( \pi - \arcsin(2 r_o / d) \) (thick), \( \pi - \arcsin(r_o / d) \) (dotted) and \( \pi - \arcsin(r_o / 2d) \) (dashed). The case represented here is the one of an underdensity with \( r_o = 1 \) Mpc located at \( d = 10 \) Mpc and with a limit background geometry of \( \Lambda \)CDM type.
We can conclude this section by noting that the plots presented here are the consequence of the exact and explicit derivation of the lensing quantities given in Sec. 5.4. To the best of our knowledge, our approach is the first to propose such an explicit result for off-center observers in a class of LTB models, and this is a consequence of the great flexibility of the geodesic light-cone coordinates. Introducing models with different parameters and density profiles could be used also for more precise and realistic cosmological applications.
5.5. OFF-CENTER OBSERVER IN LTB COORDINATES: CDM AND ACDM MODELS
Summary and conclusions

This thesis has been mainly devoted to understanding how much the presence of (small) deviations from homogeneity and isotropy in the large scale space-time geometry can effect the physical observables describing the propagation of light-like signals. Our attention has been focused, in particular, on the frequency redshift, the luminosity distance and the weak lensing effects related to photon propagation. As already stressed, all these quantities are intrinsically connected among themselves, and closely linked to the observed radiation flux emitted by a given source.

We started, in Chapter 1, by introducing the standard cosmological model based on the assumption of homogeneity and isotropy, and described, geometrically, by the well known Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We have derived the corresponding cosmological equations provided by the theory of General Relativity, and discussed their possible solutions for the case of perfect fluid sources. At the end of the chapter we have briefly illustrated the gauge invariant approach to the theory of cosmological scalar perturbations, useful to correctly describe the properties of the Cosmic Microwave Background and take into account its anisotropies.

In Chapter 2, we have introduced the simplest non-perturbative inhomogeneous generalization of the FLRW metric, i.e. the Lemaître-Tolman-Bondi (LTB) metric. This metric describes a radially inhomogeneous geometry which is still isotropic around one special point, chosen as the center of the polar coordinate system. In that context we have illustrated various possible configurations of matter sources, and the corresponding cosmological solutions with and without spatial (inhomogeneous) curvature. In particular, we have presented two new cases: a model dominated by a inhomogeneous cosmic electromagnetic field, and a model characterized by a fractal distribution of CDM sources. In both cases we have obtained exact analytical solutions for the time evolution of the LTB geometry and of the gravitational sources.

After discussing how to define redshift and distances in the generalized geometry of LTB type, we have compared the predictions of the previous models with the Supernovae data provided by the Union2 catalog. Performing a best fit analysis we have found that models with inhomogeneous and
fractal sources, for appropriate values of their parameters, can be in principle statistically acceptable at the same level of confidence as the homogeneous model with dark energy sources. In order to discriminate among physically different – but statistically indistinguishable – models we have then suggested a new possible method based on the measurement of the so-called $z_{\text{acc}}$ parameter, marking the beginning of the epoch of “effective” accelerated expansion. We have also shown that this parameter has the virtue of being very insensitive to a possible displacement of the observer away from the center of the LTB geometry.

Chapter 3 takes into account a possible generalization of the LTB metric, obtained by adding anisotropic degrees of freedom. The anisotropy is treated perturbatively, and the new contributions to Einstein’s equation are derived to first order in the perturbation of the metric and of the energy-momentum of the perfect fluid sources. A generalized luminosity-redshift relation is obtained and compared, as before, with the Union2 observational data. The results of the best-fit analysis show that the presence of a small background anisotropy cannot be excluded by present observations.

In Chapter 4 we have illustrated a recently proposed formalism, suitable to include in an exact (non-perturbative) way all (possibly non-linear) inhomogeneity corrections affecting the propagation of light-like signals and the determination of the associated observables: the so-called geodesic light-cone (GLC) gauge. In the particular frame specifying this gauge the time-like coordinate coincides with the proper time of the synchronous gauge, two spatial coordinates coincide with the polar angles referred to the observer position, and the remaining spatial coordinate is the one which non-perturbatively identifies the past light-cone of the observer.

In that chapter we have discussed the geometric formalism leading to exact expressions for the luminosity-redshift relation and for the computation of the so-called weak lensing effects, induced by the large scale distribution of inhomogeneous cosmic matter. The method is completely geometric, and the results can be easily extended, in principle, to generalized theories of gravity (such as metric-affine gravity). The results for the angular distance, in particular, lead to a nicely factorized expression separating the observer terms from the source contributions, and automatically include the aberration effects due to a possible peculiar velocity of the observer, in perfect agreement with the laws of special relativity.

Finally, Chapter 5 is devoted to specific applications of the results obtained in the previous chapter. In particular, the angular distance for a perturbed cosmological background is expressed, up to first order, both in the synchronous and longitudinal gauge, and their consistency is checked. Since the longitudinal and GLC gauges don’t share the same time coordinate, a relative velocity appears once the GLC results are given in terms of variables of the longitudinal gauge. As already stressed, the relativis-
tic aberration associated to this effect is perfectly taken in account in this formalism.

We have also computed the coordinate transformation connecting the GLC gauge to the LTB metric. In particular, we have considered the non-trivial case of an off-center observer: in this case the vertex of the past light-cone of the given observer does not coincide with the center of LTB metric. Nevertheless, we have been able to derive an exact expression for the angular distance measured by the off-center observer: the obtained result appears in explicit form, and improves the implicit results previously obtained in Chapter 2 with different methods.

We have finally obtained, in this context, general expressions for weak lensing quantities. Of course, because of the displacement between the GLC observer and the LTB one, an anisotropy axis may be generated. However, differently from Chapter 3, the considered inhomogeneities are far from the observer, and the anisotropic corrections are not relevant like the large-scale ones. To conclude, we have provided a specific ansatz for the possible size of the inhomogeneous regions affecting the lensing magnification, and obtained a large range of distances, from 10 to 1000 Mpc, both in the CDM and ΛCDM models.

In conclusion we have presented, with this thesis, non-trivial solutions describing inhomogeneous cosmological configurations of particular physical interest (such as local distributions of electromagnetic energy density and fractal dark matter distributions). We have discussed their possible compatibility with the fit of the Supernovae data, with the possible inclusion of small anisotropic corrections (such as those due to an observer located away from the preferred center of symmetry of the given geometry). We have finally presented and discussed in detail the formalism of the GLC gauge, and applied this method to obtain exact, non perturbative results for physical observables of cosmological relevance (such as the luminosity distance and the lensing magnification). The GLC results can then be transformed to other gauges, and this procedure provides a consistent framework to take into account the physical effects of inhomogeneities on cosmological observables, thus answering the question posed at the end of the Introduction of this thesis.
Appendix A

Christoffel’s symbols in the GLC gauge

Here we report the Christoffel’s symbol for the GLC gauge, already given in [78]:

\[
\Gamma^\rho_{\tau\tau} = \delta^\rho_\tau \frac{\partial \mathcal{Y}}{\mathcal{Y}} \quad \Gamma^\tau_{\omega\omega} = -\frac{1}{2\mathcal{Y}} \partial_\omega U^2 + \mathcal{Y} \partial_\tau \mathcal{Y} + \frac{1}{2} \partial_\tau U^2 + \frac{U_c}{\mathcal{Y}} \left( \partial_\omega U_c + \mathcal{Y} \partial_\tau + \frac{1}{2} \partial_\tau U^2 \right)
\]

\[
\Gamma^\omega_{\tau\tau} = \frac{\partial \mathcal{Y}}{\mathcal{Y}} + \partial_\tau \mathcal{Y} + \frac{1}{2} \partial_\tau U^2
\]

\[
\Gamma^a_{\omega\omega} = U^a \left( \frac{\partial \mathcal{Y}}{\mathcal{Y}} + \partial_\tau \mathcal{Y} + \frac{1}{2} \partial_\tau U^2 \right) - \gamma^{ad} \left( \partial_\omega U_d + \mathcal{Y} \partial_d \mathcal{Y} + \frac{1}{2} \partial_d U^2 \right)
\]

\[
\Gamma^\tau_{ab} = \frac{1}{2\mathcal{Y}} \left( \mathcal{Y} \partial_\tau \gamma_{ab} + \partial_a U_b + \partial_b U_a + \partial_\omega \gamma_{ab} - U^c (\partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab}) \right)
\]

\[
\Gamma^\omega_{ab} = \frac{U_c}{2\mathcal{Y}} \gamma^{cd} \left( \partial_a \gamma_{bd} + \partial_b \gamma_{da} - \partial_d \gamma_{ab} \right)
\]

\[
\Gamma^\tau_{\tau\omega} = \frac{U^a}{2\mathcal{Y}} \left( \partial_\tau U_a - \partial_a \mathcal{Y} \right) - \frac{1}{2} \frac{\partial_\tau U^2}{\mathcal{Y}}
\]

\[
\Gamma^\tau_{\tau a} = \frac{1}{2\mathcal{Y}} \left( \partial_\tau U_a + \partial_a \mathcal{Y} - U^b \partial_\tau \gamma_{ab} \right)
\]

\[
\Gamma^a_{\tau\omega} = \frac{\gamma^{bc} \partial_\tau \gamma_{ca}}{2} \Gamma^w_{\tau a} = 0
\]

\[
\Gamma^b_{\omega\omega} = \frac{U^b}{2\mathcal{Y}} \left( \partial_b \mathcal{Y} - \partial_\tau U_b \right) + \frac{\gamma^{bc}}{2} \left( \partial_\omega \gamma_{ac} + \partial_c U_a - \partial_a U_c \right)
\]

\[
\Gamma^b_{a\tau} = \frac{\gamma^{bc} \partial_\tau \gamma_{ca}}{2} \Gamma^w_{\tau a} = 0
\]

\[
\Gamma^\tau_{\omega a} = -\frac{\partial_\omega \mathcal{Y}}{2} - \frac{\partial_a U_a}{2} - \frac{\partial_\tau U^2}{2\mathcal{Y}} \left( \partial_\omega \gamma_{ab} + \partial_b U_a - \partial_a U_b \right)
\]

\[
\Gamma^a_{\tau\omega} = \frac{\gamma^{ab}}{2} \left( \partial_b \mathcal{Y} - \partial_\tau U_b \right)
\]

(A.1)
Bibliography


